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Unital versions of the higher order peak algebras

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Abstract. We construct unital extensions of the higher order peak algebras defined by Krob and the third author in [Ann. Comb. 9 (2005), 411–430], and show that they can be obtained as homomorphic images of certain subalgebras of the Mantaci-Reutenauer algebras of type B . This generalizes a result of Bergeron, Nyman and the first author [Trans. AMS 356 (2004), 2781–2824].

Résumé. Nous construisons des extensions unitaires des algèbres de pics d'ordre supérieur définies par Krob et le troisième auteur dans [Ann. Comb. 9 (2005), 411–430], et nous montrons qu'elles peuvent être obtenues comme images homomorphes de certaines sous-algèbres des algèbres de Mantaci-Reutenauer de type B . Ceci généralise un résultat dû à Bergeron, Nyman et au premier auteur [Trans. AMS 356 (2004), 2781–2824].

Keywords: Descent algebras, Noncommutative symmetric functions, Peak algebras

1 Introduction

A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ is an index i such that $\sigma(i) > \sigma(i+1)$. A descent is a *peak* if moreover $i > 1$ and $\sigma(i) > \sigma(i-1)$. The sums of permutations with a given descent set span a subalgebra of the group algebra, the *descent algebra* Σ_n . The *peak algebra* \mathring{P}_n of \mathfrak{S}_n is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit. Descent algebras can be defined for all finite Coxeter groups [19]. In [2], it is shown that the peak algebra of \mathfrak{S}_n can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group B_n .

The direct sum of the peak algebras turns out to be a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with \mathbf{Sym} , the Hopf algebra of noncommutative symmetric functions [9]. As explained in [5], it turns out that a fair amount of results on the peak algebras can be deduced from the case $q = -1$ of a q -identity of [11]. Specializing q to other roots of unity, Krob and the third author introduced and studied *higher order peak algebras* in [12]. Again, these are non-unital, and it is natural to ask whether the construction of [2] can be extended to this case.

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We will show that this is indeed possible. We first construct the unital versions of the higher order peak algebras by a simple manipulation of generating series. We then show that they can be obtained as homomorphic images of the *Mantaci-Reutenauer algebras* of type B . Hence no Coxeter groups other than B_n and \mathfrak{S}_n are involved in the process; in fact, the construction is related to the notion of superization, as defined in [16], rather than to root systems or wreath products.

2 Notations and background

2.1 Noncommutative symmetric functions

We will assume familiarity with the notations of [9] and with the main results of [12]. We recall a few definitions for the convenience of the reader.

The Hopf algebra of noncommutative symmetric functions is denoted by \mathbf{Sym} , or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet A . Linear bases of \mathbf{Sym}_n are labelled by compositions $I = (i_1, \dots, i_r)$ of n (we write $I \vDash n$). The noncommutative complete and elementary functions are denoted by S_n and Λ_n , and $S^I = S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I . The *descent set* of I is $\text{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}$. The *descent composition* of a permutation $\sigma \in \mathfrak{S}_n$ is the composition $I = D(\sigma)$ of n whose descent set is the descent set of σ .

Recall from [8] that for an infinite totally ordered alphabet A , $\mathbf{FQSym}(A)$ is the subalgebra of $\mathbb{C}\langle A \rangle$ spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\text{std}(w)=\sigma} w, \quad (1)$$

that is, the sum of all words in A^n whose standardization is the permutation $\sigma \in \mathfrak{S}_n$. The noncommutative ribbon Schur function $R_I \in \mathbf{Sym}$ is then

$$R_I = \sum_{D(\sigma)=I} \mathbf{G}_\sigma. \quad (2)$$

This defines a Hopf embedding $\mathbf{Sym} \rightarrow \mathbf{FQSym}$. The Hopf algebra \mathbf{FQSym} is self-dual under the pairing $(\mathbf{G}_\sigma, \mathbf{G}_\tau) = \delta_{\sigma, \tau^{-1}}$ (Kronecker symbol). Let $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, so that $\{\mathbf{F}_\sigma\}$ is the dual basis of $\{\mathbf{G}_\sigma\}$. The *internal product* $*$ of \mathbf{FQSym} is induced by composition \circ in \mathfrak{S}_n in the basis \mathbf{F} , that is,

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma \circ \tau} \quad \text{and} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau \circ \sigma}. \quad (3)$$

Each subspace \mathbf{Sym}_n is stable under this operation, and anti-isomorphic to the descent algebra Σ_n of \mathfrak{S}_n . For $f_i \in \mathbf{FQSym}$ and $g \in \mathbf{Sym}$, we have the splitting formula

$$(f_1 \dots f_r) * g = \mu_r \cdot (f_1 \otimes \dots \otimes f_r) *_r \Delta^r g, \quad (4)$$

where μ_r is r -fold multiplication, and Δ^r the iterated coproduct with values in the r -th tensor power.

2.2 The Mantaci-Reutenauer algebra of level 2

We denote by \mathbf{MR} the free product $\mathbf{Sym} \star \mathbf{Sym}$ of two copies of the Hopf algebra of noncommutative symmetric functions [14]. That is, \mathbf{MR} is the free associative algebra on two sequences (S_n) and $(S_{\bar{n}})$ ($n \geq 1$). We regard the two copies of \mathbf{Sym} as noncommutative symmetric functions on two auxiliary

alphabets: $S_n = S_n(A)$ and $S_{\bar{n}} = S_n(\bar{A})$. We denote by $F \mapsto \bar{F}$ the involutive automorphism which exchanges S_n and $S_{\bar{n}}$. The bialgebra structure is defined by the requirement that the series

$$\sigma_1 = \sum_{n \geq 0} S_n \text{ and } \bar{\sigma}_1 = \sum_{n \geq 0} S_{\bar{n}} \quad (5)$$

are grouplike. The internal product of \mathbf{MR} can be computed from the splitting formula and the conditions that σ_1 is neutral, $\bar{\sigma}_1$ is central, and $\bar{\sigma}_1 * \sigma_1 = \sigma_1$.

In [15], an embedding of \mathbf{MR} in the Hopf algebra \mathbf{BFQSym} of free quasi-symmetric functions of type B (spanned by colored permutations) is described. Under this embedding, left $*$ -multiplication by $\Lambda_n = \mathbf{G}_{n \ n-1 \dots 2, 1}$ corresponds to right multiplication by $n \ n-1 \dots 2, 1$ in the group algebra of B_n . This implies that left $*$ -multiplication by λ_1 is an involutive anti-automorphism of \mathbf{BFQSym} , hence of \mathbf{MR} .

2.3 Noncommutative symmetric functions of type B

The hyperoctahedral analogue \mathbf{BSym} of \mathbf{Sym} , defined in [6], is the right \mathbf{Sym} -module freely generated by another sequence (\tilde{S}_n) ($n \geq 0$, $\tilde{S}_0 = 1$) of homogeneous elements, with $\tilde{\sigma}_1$ grouplike. This is a coalgebra, but not an algebra. It is endowed with an internal product, for which each homogeneous component \mathbf{BSym}_n is anti-isomorphic to the descent algebra of B_n .

3 Solomon descent algebras of type B

3.1 Descents in B_n

The hyperoctahedral group B_n is the group of signed permutations. A signed permutation can be denoted by $w = (\sigma, \epsilon)$ where σ is an ordinary permutation and $\epsilon \in \{\pm 1\}^n$, such that $w(i) = \epsilon_i \sigma(i)$. If we set $w(0) = 0$, then, $i \in [0, n-1]$ is a descent of w if $w(i) > w(i+1)$. Hence, the descent set of w is a subset $D = \{i_0, i_0 + i_1, \dots, i_0 + i_1 + \dots + i_{r-1}\}$ of $[0, n-1]$. We then associate to D a so-called type- B composition (a composition whose first part can be zero) $(i_0 - 0, i_1, \dots, i_{r-1}, n - i_{r-1})$. The sum of all signed permutations whose descent set is contained in D is mapped to $\tilde{S}^I := \tilde{S}_{i_0} S^{I'}$ by Chow's anti-isomorphism [6], where $I' = (i_1, \dots, i_r)$.

3.2 Noncommutative supersymmetric functions

An embedding of \mathbf{BSym} as a sub-coalgebra and sub- \mathbf{Sym} -module of \mathbf{MR} can be deduced from [14]. To describe it, let us define, for $F \in \mathbf{Sym}(A)$,

$$F^\sharp = F(A|\bar{A}) = F(A - q\bar{A})|_{q=-1} \quad (6)$$

(the supersymmetric version of F). The superization of $F \in \mathbf{Sym}(A)$ can also be given by

$$F^\sharp = F * \sigma_1^\sharp. \quad (7)$$

Indeed, σ_1^\sharp is grouplike, and for $F = S^I$, the splitting formula gives

$$(S_{i_1} \cdots S_{i_r}) * \sigma_1^\sharp = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\sharp \otimes \cdots \otimes \sigma_1^\sharp)] = S^{I^\sharp}. \quad (8)$$

We have

$$\sigma_1^\sharp = \bar{\lambda}_1 \sigma_1 = \sum \Lambda_i S_j. \quad (9)$$

The element $\bar{\sigma}_1$ is central for the internal product, and

$$\bar{\sigma}_1 * F = \bar{F} = F * \bar{\sigma}_1. \quad (10)$$

Hence,

$$\bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1 \bar{\sigma}_1 =: \sigma_1^\flat. \quad (11)$$

The basis element \tilde{S}^I of **BSym**, where $I = (i_0, i_1, \dots, i_r)$ is a type B -composition, can be embedded as

$$\tilde{S}^I = S_{i_0}(A) S^{i_1 i_2 \dots i_r}(A | \bar{A}). \quad (12)$$

We will identify **BSym** with its image under this embedding.

3.3 A proof that **BSym** is $*$ -stable

We are now in a position to understand why **BSym** is a $*$ -subalgebra of **MR**. The argument will be extended below to the case of unital peak algebras. Let $F, G \in \mathbf{Sym}$. We want to understand why $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in **BSym**. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G_{(1)}^\sharp) (F^\sharp * \sigma_1 G_{(2)}^\sharp). \quad (13)$$

We now only have to show that each term $F^\sharp * \sigma_1 G_{(2)}^\sharp$ is in **Sym** $^\sharp$. We may assume that $F = S^I$, and for any $G \in \mathbf{Sym}$,

$$S^I * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S_{i_1}^\sharp \otimes \dots \otimes S_{i_r}^\sharp) * (\sigma_1 G_{(1)}^\sharp \otimes \dots \otimes \sigma_1 G_{(r)}^\sharp)] \quad (14)$$

so that it is sufficient to prove the property for $F = S_n$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_1 \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_1 * \sigma_1 G_{(1)}^\sharp) (\sigma_1 G_{(2)}^\sharp) \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * \sigma_1 G_{(1)}^\sharp) \cdot \sigma_1 \cdot G_{(2)}^\sharp \end{aligned} \quad (15)$$

Now,

$$\lambda_1 * \sigma_1 G_{(1)}^\sharp = (\lambda_1 * G_{(1)}^\sharp) (\lambda_1 * \sigma_1) = (\lambda_1 * G_{(1)}^\sharp) \lambda_1, \quad (16)$$

since λ_1 is an anti-automorphism. We then get

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= \sum_{(G)} (\bar{\sigma}_1 * ((\lambda_1 * G_{(1)}^\sharp) \lambda_1) \cdot \sigma_1 \cdot G_{(2)}^\sharp) \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * G_{(1)}^\sharp) \cdot (\bar{\sigma}_1 * \lambda_1) \sigma_1 \cdot G_{(2)}^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_1 * G_{(1)}^\sharp) \cdot \sigma_1^\sharp \cdot G_{(2)}^\sharp \end{aligned} \quad (17)$$

Now, the result will follow if we can prove that $\bar{\lambda}_1 * G^\sharp$ is in \mathbf{Sym}^\sharp for any $G \in \mathbf{Sym}$.

For $G = S^I$,

$$\bar{\lambda}_1 * S^{I\sharp} = \lambda_1 * \bar{\sigma}_1 * S^I * \sigma_1^\sharp = \lambda_1 * S^I * \bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1 * S^I * \sigma_1^\flat. \quad (18)$$

Since left $*$ -multiplication by λ_1 in an anti-automorphism, we only need to prove that $\lambda_1 * S_n^\flat$ is of the form G^\sharp . And indeed,

$$\begin{aligned} \lambda_1 * S_n^\flat &= \sum_{i+j=n} \lambda_1 * (\Lambda_i S_j) \\ &= \sum_{i+j=n} (\lambda_1 * S_j)(\lambda_1 * \Lambda_i) \\ &= \sum_{i+j=n} \Lambda_{\bar{j}} S_i = S_n^\sharp. \end{aligned} \quad (19)$$

This concludes the proof that \mathbf{BSym} is a $*$ -subalgebra of \mathbf{BFQSym} .

4 Unital versions of the higher order peak algebras

As shown in [5], much of the theory of the peak algebra can be deduced from a formula of [11] for $R_I((1-q)A)$, in the special case $q = -1$. In [12], this formula was studied in the case where q is an arbitrary root of unity, and higher order analogs of the peak algebra were obtained. In [2], it was shown that the classical peak algebra can be extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of type B . In this section, we construct unital extensions of the higher order peak algebras.

Let q be a primitive r -th root of unity. All objects introduced below will depend on q (and r), although this dependence will not be made explicit in the notation. We denote by θ_q the endomorphism of \mathbf{Sym} defined by

$$\tilde{f} = \theta_q(f) = f((1-q)A) = f(A) * \sigma_1((1-q)A). \quad (20)$$

We denote by $\mathring{\mathcal{P}}$ the image of $\tilde{\mathcal{P}}$ and by \mathcal{P} the right $\mathring{\mathcal{P}}$ -module generated by the S_n for $n \geq 0$. Note that $\mathring{\mathcal{P}}$ is by definition a left $*$ -ideal of \mathbf{Sym} .

Theorem 4.1 \mathcal{P} is a unital $*$ -subalgebra of \mathbf{Sym} . Its Hilbert series is

$$\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \dots - t^r}. \quad (21)$$

Proof – Since the internal product of homogeneous elements of different degrees is zero, it is enough to show that, for any $f, g \in \mathbf{Sym}$, $\sigma_1 \tilde{f} * \sigma_1 \tilde{g}$ is in \mathcal{P} . Thanks to the splitting formula,

$$\begin{aligned} \sigma_1 \tilde{f} * \sigma_1 \tilde{g} &= \mu[(\sigma_1 \otimes \tilde{f}) * \sum_{(g)} \sigma_1 \tilde{g}_{(1)} \otimes \sigma_1 \tilde{g}_{(2)}] \\ &= \sum_{(g)} (\sigma_1 \tilde{g}_{(1)}) (\tilde{f} * \sigma_1 \tilde{g}_{(2)}). \end{aligned} \quad (22)$$

Thus, it is enough to check that $\tilde{f} * \sigma_1 \tilde{h}$ is in $\mathring{\mathcal{P}}$ for any $f, h \in \mathbf{Sym}$. Now,

$$\tilde{f} * \sigma_1 \tilde{h} = f * \sigma_1((1-q)A) * \sigma_1 \tilde{h}, \quad (23)$$

and since $\mathring{\mathcal{P}}$ is a \mathbf{Sym} left $*$ -ideal, we only have to show that $\sigma_1((1-q)A) * \sigma_1 \tilde{h}$ is in $\mathring{\mathcal{P}}$. One more splitting yields

$$\begin{aligned} \sigma_1((1-q)A) * \sigma_1 \tilde{h} &= (\lambda_{-q} \sigma_1) * \sigma_1 \tilde{h} \\ &= \mu[(\lambda_{-q} \otimes \sigma_1) * \sum_{(h)} \sigma_1 \tilde{h}_{(1)} \otimes \sigma_1 \tilde{h}_{(2)}] \\ &= \sum_{(h)} (\lambda_{-q} * \sigma_1 \tilde{h}_{(1)}) (\sigma_1 \tilde{h}_{(2)}) \\ &= \sum_{(h)} (\lambda_{-q} * \tilde{h}_{(1)}) \lambda_{-q} \sigma_1 \tilde{h}_{(2)} \end{aligned} \quad (24)$$

(since left $*$ -multiplication by λ_{-q} is an anti-automorphism, namely the composition of the antipode and q^{degree}). The first parentheses $(\lambda_{-q} * \tilde{h}_{(1)})$ are in $\mathring{\mathcal{P}}$ since it is a left $*$ -ideal. The middle term is $\sigma_1((1-q)A)$, and the last one is in $\mathring{\mathcal{P}}$ by definition.

Recall from [12, Prop. 3.5] that the Hilbert series of $\mathring{\mathcal{P}}$ is

$$\sum_{n \geq 0} \dim \mathring{\mathcal{P}}_n t^n = \frac{1 - t^r}{1 - t - t^2 - \dots - t^r}. \quad (25)$$

From [12, Lemma 3.13 and Eq. (3.9)], it follows that $S_n \notin \mathring{\mathcal{P}}$ if and only if $n \equiv 0 \pmod{r}$, so that the Hilbert series of \mathcal{P} is

$$\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \dots - t^r}. \quad (26)$$

■

5 Back to the Mantaci-Reutenauer algebra

The above proofs are in fact special cases of a master calculation in the Mantaci-Reutenauer algebra, which we carry out in this section.

Let q be an arbitrary complex number or an indeterminate, and define, for any $F \in \mathbf{MR}$,

$$F^\sharp = F * \sigma_1(A - q\bar{A}) = F * \sigma_1^\sharp. \quad (27)$$

Since σ_1^\sharp is grouplike, it follows from the splitting formula that

$$F \mapsto F^\sharp \quad (28)$$

is an automorphism of \mathbf{MR} for the Hopf structure. In addition, it is clear from the definition that it is also an endomorphism of left $*$ -modules. We refer to it as the \sharp transform.

We now define

$$\mathring{Q} = \mathbf{MR}^\sharp, \quad (29)$$

the image of the \sharp transform. Since the latter is an endomorphism of Hopf algebras and of left $*$ -modules, \mathring{Q} is both a Hopf subalgebra of \mathbf{MR} and a left $*$ -ideal. When q is a root of unity, its image under the specialization $\bar{A} = A$ is the non-unital peak algebra \mathring{P} of Section 4 (and for generic q , it is \mathbf{Sym}).

Let \mathcal{Q} be the right \mathring{Q} -module generated by the S_n , for all $n \geq 0$. Clearly, the identification $\bar{A} = A$ maps \mathcal{Q} onto \mathcal{P} , the unital peak algebra of Section 4.

Theorem 5.1 *\mathcal{Q} is a $*$ -subalgebra of \mathbf{MR} , containing \mathring{Q} as a left ideal.*

Proof – Let $F, G \in \mathbf{MR}$. As above, we want to show that $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in \mathcal{Q} . Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)}) (F^\sharp * \sigma_1 G^\sharp_{(2)}) \quad (30)$$

and we only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in \mathring{Q} . We may assume that $F = S^I$, where I is now a bicolored composition, and for any $G \in \mathbf{MR}$,

$$S^I * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^\sharp_{i_1} \otimes \cdots \otimes S^\sharp_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})] \quad (31)$$

so that it is sufficient to prove the property for $F = S_n$ or $S_{\bar{n}}$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_{-q} \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_{-q} 1 * \sigma_1 G^\sharp_{(1)}) (\sigma_1 G^\sharp_{(2)}) \\ &= \sum_{(G)} (\bar{\lambda}_{-q} * G^\sharp_{(1)}) \cdot \sigma_1^\sharp \cdot G^\sharp_{(2)} \end{aligned} \quad (32)$$

which is in \mathring{Q} , since it is a subalgebra and a left $*$ -ideal, and similarly,

$$\begin{aligned} \bar{\sigma}_1^\sharp * \sigma_1 G^\sharp &= (\lambda_{-q} \bar{\sigma}_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\lambda_{-q} * \sigma_1 G^\sharp_{(1)}) (\bar{\sigma}_1 \bar{G}^\sharp_{(2)}) \\ &= \sum_{(G)} (\lambda_{-q} * G^\sharp_{(1)}) \cdot \bar{\sigma}_1^\sharp \cdot \bar{G}^\sharp_{(2)} \end{aligned} \quad (33)$$

is also in \mathring{Q} . ■

The various algebras introduced in this paper and their interrelationships are summarized in the following diagram.

$$\begin{array}{ccccccc} \mathring{Q} & \subseteq & \mathcal{Q} & \subseteq & \mathbf{MR} & \subseteq & \mathbf{BFQSym} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathring{P} & \subseteq & \mathcal{P} & \subseteq & \mathbf{Sym} & \subseteq & \mathbf{FQSym} \end{array} \quad (34)$$

Note that in the special case $q = -1$, by the results of Section 3.3, \mathcal{Q}_n is the (Solomon) descent algebra of B_n , \mathcal{Q} is isomorphic to \mathbf{BSym} , and \mathcal{P} is the unital peak algebra of [2].

6 Further developments

6.1 Inversion of the generic \sharp transform

For generic q , the endomorphism (27) of \mathbf{MR} is invertible; therefore

$$\mathring{\mathcal{Q}} \sim \mathbf{MR}. \quad (35)$$

The inverse endomorphism of \mathbf{MR} arises from the transformation of alphabets

$$A \mapsto (q\bar{A} + A)/(1 - q^2), \quad (36)$$

which is to be understood in the following sense:

$$\sigma_1 \left(\frac{q\bar{A} + A}{1 - q^2} \right) := \prod_{k \geq 0} \sigma_{q^{2k+1}}(\bar{A}) \sigma_{q^{2k}}(A). \quad (37)$$

Indeed,

$$\begin{aligned} \sigma_1 \left(\frac{q\bar{A} + A}{1 - q^2} \right) * \sigma_1(A - q\bar{A}) &= \prod_{k \geq 0} \sigma_{q^{2k+1}}(\bar{A} - qA) \sigma_{q^{2k}}(A - q\bar{A}) \\ &= \prod_{k \geq 0} \lambda_{-q^{2k+2}}(A) \sigma_{q^{2k+1}}(\bar{A}) \lambda_{-q^{2k+1}}(\bar{A}) \sigma_{q^{2k}}(A) \\ &= \sigma_1(A). \end{aligned} \quad (38)$$

By normalizing the term of degree n in (37), we obtain B_n -analogs of the q -Klyachko elements defined in [9]:

$$K_n(q; A, \bar{A}) := \prod_{i=1}^n (1 - q^{2i}) S_n \left(\frac{q\bar{A} + A}{1 - q^2} \right) = \sum_{I \models n} q^{2 \text{maj}(I)} R_I(q\bar{A} + A). \quad (39)$$

This expression can be completely expanded on signed ribbons. From the expression of R_I in \mathbf{FQSym} , we have

$$R_I(\bar{A} + A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(\bar{A} + A) \quad (40)$$

where $\bar{A} + A$ is the ordinal sum. If we order \bar{A} by

$$\bar{a}_1 < \bar{a}_2 < \dots < \bar{a}_k < \dots \quad (41)$$

then, arguing as in [16], we have

$$\mathbf{G}_\sigma(\bar{A} + A) = \sum_{\text{std}(\tau, \epsilon) = \sigma} \mathbf{G}_{\tau, \epsilon} \quad (42)$$

so that

$$R_I(\bar{A} + A) = \sum_{\rho(J)=I} R_J \quad (43)$$

where for a signed composition $J = (J, \epsilon)$, the unsigned composition $\rho(J)$ is defined as the shape of $\text{std}(\sigma, \epsilon)$, where σ is any permutation of shape J .

Replacing \bar{A} by $q\bar{A}$, one obtains the expansion of the q -Klyachko elements of type B :

$$K_n(q; A, \bar{A}) = \sum_J q^{\text{bmaj}(J)} R_J \quad (44)$$

where

$$\text{bmaj}(J) = 2 \text{maj}(\rho(J)) + |\epsilon|, \quad (45)$$

where $|\epsilon|$ is the number of minus signs in ϵ .

For example,

$$K_2(q) = R_2 + q^2 R_{\bar{2}} + q^2 R_{11} + q^3 R_{1\bar{1}} + q R_{\bar{1}1} + q^4 R_{\bar{1}\bar{1}}. \quad (46)$$

$$\begin{aligned} K_3(q) = & R_3 + q^3 R_{\bar{3}} + q^4 R_{21} + q^5 R_{2\bar{1}} + q^2 R_{\bar{2}1} + q^7 R_{\bar{2}\bar{1}} + q^2 R_{12} + q^4 R_{1\bar{2}} \\ & + q R_{\bar{1}2} + q^5 R_{\bar{1}\bar{2}} + q^6 R_{111} + q^7 R_{11\bar{1}} + q^3 R_{1\bar{1}1} + q^8 R_{1\bar{1}\bar{1}} \\ & + q^5 R_{\bar{1}11} + q^6 R_{\bar{1}\bar{1}1} + q^4 R_{\bar{1}\bar{1}\bar{1}} + q^9 R_{\bar{1}\bar{1}\bar{1}\bar{1}}. \end{aligned} \quad (47)$$

This major index of type B is the flag major index defined in [1].

Following [1] and considering the signed composition (where ϵ is encoded as boolean vector for readability)

$$J = (2, 1, 1, \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) = (2113124122, 00001111110000100000) \quad (48)$$

we can take the smallest permutation of shape $(2, 1, 1, 3, 1, 2, 4, 1, 2, 2)$, which is

$$\alpha = 15432698711101213161514181719 \quad (49)$$

sign it according to ϵ , which yields

$$1543\bar{2}\bar{6}\bar{9}\bar{8}\bar{7}\bar{1}\bar{1}10121316\bar{1}\bar{5}14181719 \quad (50)$$

whose standardized is

$$81110912543612131416715181719 \quad (51)$$

and has shape $\rho(J) = (2, 1, 1, 3, 1, 6, 3, 2)$. The major index of $\rho(J)$ is 55, the number of minus signs in ϵ is 7, so $\text{bmaj}(J) = 2 \times 55 + 7 = 117$.

The major index of type B can be read directly on signed compositions without reference to signed permutations as follows: one can get $\rho(J)$ by first adding the absolute values of two consecutive parts if the left one is signed and the second one is not, then remove the signs and proceed as before.

A different solution consists in reading the composition from right to left, then associate weight 0 (resp. 1) to the rightmost part if it is positive (resp. negative) and then proceed left by adding 2 to the weight if the two parts are of the same sign and 1 if not. Finally, add up the product of the absolute values of the parts with their weight.

For example, with the same J as above we have the following weights:

$$\begin{aligned} J = & (2, 1, 1, \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) \\ \text{weights} : & 14\ 12\ 10\ 9\ 7\ 5\ 4\ 3\ 2\ 0 \end{aligned} \quad (52)$$

so that we get $2 \cdot 14 + 1 \cdot 12 + 1 \cdot 10 + 3 \cdot 9 + 1 \cdot 7 + 2 \cdot 5 + 4 \cdot 4 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 = 117$.

This technique generalizes immediately to colored compositions with a fixed number c of colors $0, 1, \dots, c-1$: the weight of the rightmost cell is its color and the weight of a part is equal to the sum of the weight of the next part and the unique representative of the difference of the colors of those parts modulo c belonging to the interval $[1, c]$.

6.2 Generators and Hilbert series

For $n \geq 0$, let

$$S_n^\pm = S_n(A) \pm S_n(\bar{A}), \quad (53)$$

and denote by \mathcal{H}_n the subalgebra of \mathbf{MR} generated by the S_k^\pm for $k \leq n$. For $n \geq 0$, we have

$$(S_n^\pm)^\sharp \equiv (1 \mp q^n) S_n^\pm \pmod{\mathcal{H}_{n-1}}, \quad (54)$$

so that the $(S_n^\pm)^\sharp$ such that $1 \mp q^n \neq 0$ form a set of free generators in \mathbf{MR}^\sharp .

Conjecture 6.1 *If r is odd, a basis of \mathbf{MR}^\sharp will be parametrized by colored compositions such that parts of color 0 are not $\equiv 0 \pmod{r}$ and parts of color 1 are arbitrary. The Hilbert series is then*

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \dots + t^r)}. \quad (55)$$

If r is even, there is the extra condition that parts of color 1 are not $\equiv r/2 \pmod{r}$. The Hilbert series is then

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \dots + t^r) + t^{r/2}}. \quad (56)$$

For example,

$$H_2(t) = 1 + t + 2t^2 + 4t^3 + 8t^4 + 16t^5 + 32t^6 + 64t^7 + 128t^8 + O(t^9) \quad (57)$$

$$H_3(t) = 1 + 2t + 6t^2 + 17t^3 + 50t^4 + 146t^5 + 426t^6 + 1244t^7 + 3632t^8 + O(t^9) \quad (58)$$

$$H_4(t) = 1 + 2t + 5t^2 + 14t^3 + 38t^4 + 104t^5 + 284t^6 + 776t^7 + 2120t^8 + O(t^9) \quad (59)$$

If these conjectures are correct, the Hilbert series of the right \mathbf{MR}^\sharp -modules generated by the S_n are respectively

$$\frac{1}{1 - 2(t + t^2 + \dots + t^r)}, \quad (60)$$

or

$$\frac{1}{1 - 2(t + t^2 + \dots + t^r) + t^{r/2}}. \quad (61)$$

according to whether r is odd or even.

The cases $r = 1$ and $r = 2$ are easily proved as follows. Assume first that $q = 1$. Set

$$f = 1 + (\sigma_1^+)^\sharp = (\sigma_1 + \lambda_{-1})(A - \bar{A}), \quad (62)$$

$$g = (\sigma_1^-)^\sharp - 1 = (\sigma_1 - \lambda_{-1})(A - \bar{A}). \quad (63)$$

Then, $f^2 = g^2 + 4$, so that

$$f = 2 \left(1 + \frac{1}{4} g^2 \right)^{\frac{1}{2}} \quad (64)$$

which proves that the $(S_n^+)^\sharp$ can be expressed in terms of the $(S_m^-)^\sharp$.

Similarly, for $q = -1$, one can express

$$f = \sum_{n \geq 1} (S_{2n}^+)^\sharp + \sum_{n \geq 0} (S_{2n+1}^-)^\sharp \quad (65)$$

in terms of

$$g = \sum_{n \geq 1} (S_{2n}^-)^\sharp + \sum_{n \geq 0} (S_{2n+1}^+)^\sharp \quad (66)$$

since, as is easily verified,

$$(f+2)^2 = g^2 + 4, \text{ i.e., } f = -2 + 2 \left(1 + \frac{1}{4}g^2\right)^{\frac{1}{2}}. \quad (67)$$

Apparently, this approach does not work anymore for higher roots of unity.

7 Appendix: monomial expansion of the $(1 - q)$ -kernel

The results of [16, 7] allow us to write down a new expansion of $S_n((1 - q)A)$, in terms of the monomial basis of [4]. The special case $q = 1$ gives back a curious expression of Dynkin's idempotent, first obtained in [3].

Let σ be a permutation. We then define its *left-right minima* set $\text{LR}(\sigma)$ as the values of σ that have no smaller value to their left. We will denote by $\text{lr}(\sigma)$ the cardinality of $\text{LR}(\sigma)$. For example, with $\sigma = 46735182$, we have $\text{LR}(\sigma) = \{4, 3, 1\}$, and $\text{lr}(\sigma) = 3$.

Let us now decompose $S_n((1 - q)A)$ on the monomial basis \mathbf{M}_σ (see [4]) of \mathbf{FQSym} . Thanks to the Cauchy formula of \mathbf{FQSym} [7], we have

$$S_n((1 - q)A) = \sum_{\sigma} \mathbf{S}^\sigma (1 - q) \mathbf{M}_\sigma(A), \quad (68)$$

where \mathbf{S} is the dual basis of \mathbf{M} . Given the transition matrix between \mathbf{M} and \mathbf{G} , we see that

$$\mathbf{S}^\sigma = \sum_{\tau \leq \sigma^{-1}} \mathbf{F}_\tau, \quad (69)$$

where \leq is the right weak order, e.g., $\mathbf{S}^{312} = \mathbf{F}_{123} + \mathbf{F}_{213} + \mathbf{F}_{231}$. Thanks to [16], we know that $\mathbf{F}_\sigma(1 - q)$ is either $(-q)^k$ if $\text{Des}(\sigma) = \{1, \dots, k\}$ or 0 otherwise. Let us define *hook permutations* of hook k the permutations σ such that $\text{Des}(\sigma) = \{1, \dots, k\}$. Now, $\mathbf{S}^\sigma(1 - q)$ amounts to compute the list of *hook permutations* smaller than σ . Note that hook permutations are completely characterized by their left-right minima. Moreover, if τ is smaller than σ in the right weak order, then $\text{LR}(\tau) \subset \text{LR}(\sigma)$.

Hence all hook permutations smaller than a given permutation σ belong to the set of hook permutations with left-right minima in $\text{LR}(\sigma)$. Since by elementary transpositions decreasing the length, one can get from σ to the hook permutation with the same left-right minima and then from this permutation to all the others, we have:

Theorem 7.1 *Let n be an integer. Then*

$$S_n((1 - q)A) = \sum_{\sigma \in \mathfrak{S}_n} (1 - q)^{\text{lr}(\sigma)} \mathbf{M}_\sigma. \quad (70)$$

■

In the particular case $q = 1$, we recover a result of [3]:

$$\Psi_n = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=1}} \mathbf{M}_\sigma, \quad (71)$$

where Ψ_n is the noncommutative power sum associated with Dynkin's idempotent [11, Prop. 5.2].

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