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PSEUDO-HOLOMORPHIC FUNCTIONS AT THE CRITICAL EXPONENT

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Abstract. We study Hardy classes on the disk associated to the equation $\bar\partial w = \alpha\bar w$ for $\alpha \in L^r$ with $2 \le r < \infty$. The paper seems to be the first to deal with the case $r = 2$. We prove an analog of the M. Riesz theorem and a topological converse to the Bers similarity principle. Using the connection between pseudo-holomorphic functions and conjugate Beltrami equations, we deduce well-posedness on smooth domains of the Dirichlet problem with weighted L^p boundary data for 2-D isotropic conductivity equations whose coefficients have logarithm in $W^{1,2}$. In particular these are not strictly elliptic. Our results depend on a new multiplier theorem for $W_0^{1,2}$-functions.

1. Introduction

Pseudo-holomorphic functions of one complex variable, *i.e.* solutions to a $\bar\partial$ equation whose right-hand side is a real linear function of the unknown variable, are perhaps the simplest generalization of holomorphic functions. They received early attention in [41, 11] and extensive treatment in [6, 42] when the coefficients are L^r-summable, $r > 2$ While [6] takes on a function-theoretic viewpoint, [42] dwells on integral equations and leans on applications to geometry, elasticity and hydrodynamics. Recent developments and applications to various boundary value problems can be found in [31, 43, 15]. Hardy classes for such functions were introduced in [35] and subsequently considered in [27, 28, 29, 5] in the range of exponents $1 < p < \infty$, see [14, 30, 16, 4] for further generalizations to multiply connected domains. The connection between pseudo-holomorphic functions and conjugate Beltrami equations makes pseudo-holomorphic Hardy classes a convenient framework to solve Dirichlet problems with L^p boundary data for isotropic conductivity equations [5, 14, 4]. These are also instrumental in [17, 18, 19, 16] to approach certain inverse boundary problems.

As reported in [7], I. N. Vekua stressed on several occasions an interest in developing the theory for L^r coefficients when $1 < r \le 2$. However, solutions then need no longer be continuous which has apparently been an obstacle to such extensions, see [7, 36] for classes of coefficients that ensure such continuity. The present paper seems to be the first to deal with the critical exponent $r = 2$. We develop a theory of pseudo-holomorphic Hardy spaces on the disk in the range $1 < p < \infty$, prove existence of L^p boundary

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values, and give an analog of the M. Riesz theorem in this context. As a byproduct, we obtain a Liouville-type theorem.We also develop a topological parametrization by holomorphic Hardy functions which is new even for $r > 2$. We apply our result to well-posedness of the Dirichlet problem with weighted L^p boundary data for 2-D conductivity equations whose coefficients have logarithm in $W^{1,2}$. In particular these are not bounded away from zero nor infinity and no strict ellipticity prevails, which makes for results of a novel type. Accordingly, solutions may be locally unbounded.

As in previous work on pseudo-holomorphic functions, we make extensive use of the Bers similarity principle, but in our case it requires a thorough analysis of smoothness and boundedness properties of exponentials of $W^{1,2}$ functions which is carried out in a separate appendix. There we prove a theorem, one of the main technical results of the paper, asserting that the exponential of a $W_0^{1,2}$ function in the disk is a multiplier from the space of functions with L^p maximal function on the unit circle to the space of functions satisfying a Hardy condition of order p on the unit disk. This would have higher dimensional analogs, but we make no attempt at developing them and stick to dimension 2 throughout the paper.

In Section 2 we introduce main notations and discuss numerous facts on Sobolev spaces we use later on. In Section 3 we formulate the classical similarity principle (factorization) for pseudo-holomorphic functions. A converse statement is given in Section 4. Section 5 is devoted to pseudo-holomorphic Hardy spaces; we give there a topological converse to the similarity principle. In Section 6 we obtain a generalization of the M. Riesz theorem on the conjugate operator. Section 7 contains an application of our results to the conductivity equation with exp-Sobolev coefficients. Finally, several technical results and a multiplier theorem are contained in the appendix, Section 8.

2. Notations and preliminaries

Let $\mathbb{C} \sim \mathbb{R}^2$ be the complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We designate by $\mathbb{T}_{\xi,\rho}$ and $\mathbb{D}_{\xi,\rho}$ respectively the circle and the open disk centered at ξ of radius ρ. We simply write $\mathbb{T}_\rho$, $\mathbb{D}_\rho$ when $\xi = 0$, and if $\rho = 1$ we omit the subscript. If f is a function on $\mathbb{D}_\rho$, we often denote by f_ρ the function on $\mathbb{D}$ defined by $f_\rho(\xi) := f(\rho\xi)$. Given $\xi \in \mathbb{T}$ and $\gamma \in (0, \pi/2)$, we let $\tilde{\Gamma}_{\xi,\gamma}$ indicate the open cone with vertex ξ and opening 2γ, symmetric with respect to the line $(0,\xi)$. We define $\Gamma_{\xi,\gamma} = A_{\xi,\gamma} \cup \bar{\mathbb{D}}_{\sin\gamma}$, where $A_{\xi,\gamma}$ is the bounded component of $\tilde{\Gamma}_{\xi,\gamma} \setminus \bar{\mathbb{D}}_{\sin\gamma}$.

A complex-valued function f on $\mathbb{D}$ has non-tangential limit ℓ at ξ if $f(z)$ tends to ℓ as $z \to \xi$ inside $\Gamma_{\xi,\gamma}$ for every γ. The non-tangential maximal function of f (with opening 2γ) is the real-valued map $\mathcal{M}_\gamma f$ on $\mathbb{T}$ given by

$$\mathcal{M}_\gamma f(\xi) := \sup_{z \in \mathbb{D} \cap \Gamma_{\xi,\gamma}} |f(z)|, \qquad \xi \in \mathbb{T}. \tag{2.1}$$

For $E \subset \mathbb{C}$ and f a function on a set containing E, we let $f_{|E}$ indicate the restriction of f to E. We put $|E|$ for the planar Lebesgue measure of E, as no confusion can arise with complex modulus. The differential of that measure is denoted interchangeably by

$$dm(z) = dx\,dy = (i/2)\ dz \wedge d\overline{z}, \qquad z = x + iy.$$

When $\Omega \subset \mathbb{C}$ is an open set, we denote by $\mathcal{D}(\Omega)$ the space of C^∞-smooth complex-valued functions with compact support in Ω, equipped with the usual topology[1]. Its dual $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. For $p \in [1, \infty]$, we let $L^p(\Omega)$ and $W^{1,p}(\Omega)$ be the usual Lebesgue and Sobolev spaces with respect to dm; we sometimes consider their subspaces of real-valued functions $L^p_{\mathbb{R}}(\Omega)$ and $W^{1,p}_{\mathbb{R}}(\Omega)$. The space $W^{1,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose first distributional derivatives lie in $L^p(\Omega)$, with the norm:

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\partial f\|_{L^p(\Omega)} + \|\overline{\partial} f\|_{L^p(\Omega)}.$$

Here ∂ and $\overline{\partial}$ stand for the usual complex derivatives:

$$\partial f := \partial_z f = \frac{1}{2}(\partial_x - i\partial_y)f \text{ and } \overline{\partial} f := \partial_{\overline{z}} f = \frac{1}{2}(\partial_x + i\partial_y)f, \qquad z = x + iy.$$

Setting $\nabla f := (\partial_x f, \partial_y f)$ to mean the ($\mathbb{C}^2$-valued) gradient of f, observe that the pointwise relation $\|\nabla f\|^2_{\mathbb{C}^2} = 2|\partial f|^2_2 + 2|\bar{\partial} f|^2_2$ holds. Note also the identities $\overline{\partial f} = \overline{\partial}\ \overline{f}$ and $\Delta = 4\partial\bar{\partial}$, where Δ is the Euclidean Laplacian. By Weyl's lemma [20, Theorem 24.9], the distributions $u \in \mathcal{D}'(\Omega)$ such that $\Delta u = 0$ are exactly the harmonic functions on Ω. Subsequently, the distributions $\psi \in \mathcal{D}'(\Omega)$ such that $\bar{\partial}\psi = 0$ are exactly the holomorphic functions on Ω. The space $\mathcal{D}(\mathbb{R}^2)$ is dense in $W^{1,p}(\mathbb{R}^2)$ for $p \in [1, \infty)$, and in general we let $W^{1,p}_0(\Omega)$ indicate the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. The space $W^{1,\infty}(\Omega)$ identifies with Lipschitz-continuous functions on Ω [40, Section V.6.2].

We also introduce the spaces $L^p_{loc}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$ of distributions whose restriction to any relatively compact open subset $\Omega_0 \subset \Omega$ lies in $L^p(\Omega_0)$ and $W^{1,p}(\Omega_0)$ respectively. They are topologized by the family of semi-norms $\|f_{\Omega_n}\|_{L^p(\Omega_n)}$ and $\|f_{\Omega_n}\|_{W^{1,p}(\Omega_n)}$, where $\{\Omega_n\}$ is a sequence of relatively compact open subsets exhausting Ω.

Below we indicate some properties of Sobolev functions, most of them standard. They are valid on bounded Lipschitz domains (*i.e.* domains Ω whose boundary $\partial\Omega$ is locally isometric to the graph of a Lipschitz function).

- For $1 \le p \le \infty$, every $f \in W^{1,p}(\Omega)$ is the restriction to Ω of some $\tilde{f} \in W^{1,p}(\mathbb{R}^2)$. In fact, there is a continuous linear map

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2) \text{ such that } (Ef)_{|\Omega} = f \tag{2.2}$$

[1] *i.e.* the inductive topology of subspaces $\mathcal{D}_K$ consisting of functions supported by the compact set K, each $\mathcal{D}_K$ being topologized by uniform convergence of all derivatives [38, Section I.2].

(the extension theorem [12, Proposition 2.70]). When $\Omega = \mathbb{D}_\rho$, we may simply put $(Ef)_{|\mathbb{C}\setminus\mathbb{D}_\rho}(z) = \varphi(z)f(\rho^2/\bar{z})$, where $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi_{|\mathbb{D}_\rho} \equiv 1$. The extension theorem entails that smooth functions on $\overline{\Omega}$ are dense in $W^{1,p}(\Omega)$ when $1 \le p < \infty$.

- For $p > 2$, $W^{1,p}(\Omega)$ embeds continuously in the space of Hölder-smooth functions with exponent $1-2/p$ on Ω, in particular functions in $W^{1,p}(\Omega)$ extend continuously to $\overline{\Omega}$, and $W^{1,p}(\Omega)$ is an algebra where multiplication is continuous and derivatives can be computed by the chain rule. For $1 \le p < 2$ the embedding is in $L^{p^*}(\Omega)$ with $p^* = 2p/(2-p)$, while $W^{1,2}(\Omega)$ is embedded in all $L^\ell(\Omega)$, $\ell \in [1,\infty)$ (the Sobolev embedding theorem [1, Theorems 4.12,4.39]).
- For $p \le 2$ the embedding $W^{1,p}(\Omega) \to L^\ell(\Omega)$ is compact when $\ell \in [1, p^*)$ (the Rellich–Kondrachov theorem [1, Theorem 6.3]); $p^* = \infty$ for $p = 2$.
- If $g \in \mathcal{D}'(\Omega)$ has derivatives in $L^p(\Omega)$ for some $p \in [1,\infty)$, then $g \in W^{1,p}(\Omega)$ [12, Theorem 6.74][2]. Moreover, there exists $C = C(\Omega, p)$ such that

$$\|g - g_\Omega\|_{L^p(\Omega)} \le C\big(\|\partial g\|_{L^p(\Omega)} + \|\bar\partial g\|_{L^p(\Omega)}\big), \quad \text{with } g_\Omega := \frac{1}{|\Omega|}\int_\Omega g\,dm \tag{2.3}$$

(the Poincaré inequality [44, Theorem 4.2.1]). Let $C_1 = C_1(p)$ be a number for which (2.3) holds for $\Omega = \mathbb{D}$; it is easily seen by homogeneity that if $\xi \in \mathbb{C}$, $\rho > 0$, and $g \in W^{1,p}(\mathbb{D}_{\xi,\rho})$, then

$$\Big(\frac{1}{|\mathbb{D}_{\xi,\rho}|}\int_{\mathbb{D}_{\xi,\rho}} |g - g_{\mathbb{D}_{\xi,\rho}}|^p\,dm\Big)^{1/p} \le C_1\rho^{1-2/p}\big(\|\partial g\|_{L^p(\mathbb{D}_{\xi,\rho})} + |\bar\partial g\|_{L^p(\mathbb{D}_{\xi,\rho})}\big). \tag{2.4}$$

In particular, if $p = 2$ and $\partial g, \bar\partial g \in L^2(\Omega)$, then the right hand side of (2.4) is bounded and arbitrarily small as $\rho \to 0$, thereby asserting that g lies in $VMO(\Omega)$, the space of functions with vanishing mean oscillation on Ω [10].
- $W^{1,p}(\Omega)$-functions need not be continuous nor even locally bounded when $p \le 2$; however, if $p > 1$, their non-Lebesgue points form a set of Bessel $B_{1,p}$-capacity zero [44, Theorem 3.10.2]. Such sets are very thin: not only do they have measure zero but also their Hausdorff $H^{2-p+\varepsilon}$-dimension is zero for each $\varepsilon > 0$ [44, Theorem 2.6.16]. When speaking of pointwise values of $f \in W^{1,p}(\Omega)$, we pick a representative such that $f(z) = \lim_{\varepsilon\to 0} f_{\mathbb{D}_{z,\varepsilon}}$ outside a set of $B_{1,p}$-capacity zero. At such a z, f is said to be *strictly defined*.
- If $L^\lambda(\partial\Omega)$ is understood with respect to arclength, then $W^{1,\lambda}(\partial\Omega)$ is naturally defined using local coordinates since any Lipschitz-continuous change of variable preserves Sobolev classes [44, Theorem 2.2.2]. Each $f \in W^{1,p}(\Omega)$ with $1 < p \le \infty$ has a trace on $\partial\Omega$

[2]The proof given there for bounded C^1-smooth Ω carries over to the Lipschitz case.

(denoted again by f or sometimes by $\mathrm{tr}_{\partial\Omega} f$ for emphasis), which lies in the Sobolev space $W^{1-1/p,p}(\partial\Omega)$ of non-integral order[3]. The latter is a real interpolation space between $L^p(\partial\Omega)$ and $W^{1,p}(\partial\Omega)$, with the norm given by [1, Theorem 7.47]:

$$\|g\|_{W^{1-1/p,p}(\partial\Omega)} = \|g\|_{L^p(\partial\Omega)}+\Big(\int_{\partial\Omega\times\partial\Omega}\frac{|g(t)-g(t')|^p}{\big(\Lambda(t,t')\big)^p}\,d\Lambda(t)d\Lambda(t')\Big)^{1/p}, \quad (2.5)$$

where $\Lambda(t,t')$ indicates the length of the arc (t,t') on $\partial\Omega$. Note that $|t-t'|\sim\Lambda(t,t')$ since $\partial\Omega$ is Lipschitz. The trace operator defines a continuous surjection from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega)$ [24, Theorem 1.5.1.3]. The pointwise definition of $\mathrm{tr}_{\partial\Omega} f$ Λ-*a.e.* is based on the extension theorem and the fact that non-Lebesgue points of Ef (see (2.2)) have Hausdorff H^1-measure zero [44, Remark 4.4.5]. Of course $\mathrm{tr}_{\partial\Omega} f$ coincides with the restriction $f_{|\partial\Omega}$ whenever f is smooth on $\overline{\Omega}$. The subspace of functions with zero trace is none but $W_0^{1,p}(\Omega)$.

Since the integral in the right hand side of (2.5) does not change if we add a constant to g, it follows from (2.3) by the continuity of the trace operator that

$$\Big(\int_{\partial\Omega\times\partial\Omega}\frac{|g(t)-g(t')|^p}{\big(\Lambda(t,t')\big)^p}\,d\Lambda(t)d\Lambda(t')\Big)^{1/p}\le C\Big(\|\partial g\|_{L^p(\Omega)}+\|\bar\partial g\|_{L^p(\Omega)}\Big), \quad (2.6)$$

where the constant C depends on Ω and p.

A variant of the Poincaré inequality involving the trace is as follows: whenever $E\subset\partial\Omega$ has arclength $\Lambda(E)>0$, there is $C>0$ depending only on p, Ω and E such that

$$\Big\|g-\int_E \mathrm{tr}_{\partial\Omega}\, g\Big\|_{L^p(\Omega)}\le C\big(\|\partial g\|_{L^p(\Omega)}+\|\bar\partial g\|_{L^p(\Omega)}\big). \quad (2.7)$$

This follows immediately from the continuity of the trace operator, the Rellich–Kondrachov theorem, and [44, Lemma 4.1.3].

- For $p\in(1,\infty)$ the trace operator has a continuous section [24, Theorem 1.5.1.3], that is, for each $\psi\in W^{1-1/p,p}(\partial\Omega)$, there is $g\in W^{1,p}(\Omega)$ such that

$$\|g\|_{W^{1,p}(\Omega)}\le C\|\psi\|_{W^{1-1/p,p}(\partial\Omega)}, \qquad \mathrm{tr}_{\partial\Omega}\, g=\psi, \quad (2.8)$$

with $C=C(\Omega,p)$. If we assume that Ω is C^1-smooth and not just Lipschitz, then the function g in (2.8) can be chosen to be harmonic in Ω (elliptic regularity theory [26, p.165 & Theorem 1.3])[4].

[3] We leave out the case $p=1$ where the trace is merely defined in $L^1(\partial\Omega)$. The space $W^{1-1/p,p}(\partial\Omega)$ coincides with the Besov space $B_p^{1-1/p,p}(\partial\Omega)$, but we need not introduce Besov spaces here.

[4] In fact, elliptic regularity holds for $1<p<\infty$ as soon as $\partial\Omega$ is locally the graph of a function with VMO derivative [33, Theorem 1.1]. If $\partial\Omega$ is only Lipschitz-smooth, then the range of p has to be restricted in a manner that depends on the Lipschitz constant, see [26, 33].

- The non-integral version of the Sobolev embedding theorem [1, Theorem 7.34] asserts that $W^{1-1/\beta,\beta}(\partial\Omega)$ embeds continuously in $L^{\beta/(2-\beta)}(\partial\Omega)$ if $1<\beta<2$, while $W^{1/2,2}(\partial\Omega)$ embeds in $L^{\ell}(\partial\Omega)$ for all $\ell\in[1,\infty)$. The corresponding generalization of the Rellich–Kondrachov theorem [12, Theorem 4.54] is as follows: if $1<\beta\leq 2$, then $W^{1-1/\beta,\beta}(\partial\Omega)$ embeds compactly in $L^{\ell}(\partial\Omega)$ for $\ell<\beta/(2-\beta)$.
- When $p\in(2,\infty)$, the nonlinear map $f\mapsto e^f$ is bounded and continuous from $W^{1,p}(\Omega)$ into itself: this follows from the Taylor expansion of exp because $W^{1,p}(\Omega)$ is an algebra. When $p=2$ this property no longer holds, but still $f\mapsto e^f$ is continuous and bounded from $W^{1,2}(\Omega)$ into $W^{1,q}(\Omega)$ for each $q\in[1,2)$; in particular $\mathrm{tr}_{\partial\Omega}e^f=e^{\mathrm{tr}_{\partial\Omega}f}$ exists in $W^{1-1/q,q}(\partial\Omega)$ for $1<q<2$. This is the content of Proposition 8.4 that we could not locate in the literature.
- We use at some point the Sobolev space $W^{2,p}(\Omega)$ of functions in $L^p(\Omega)$ whose first distributional derivatives lie in $W^{1,p}(\Omega)$, equipped with the norm:

$$\|f\|_{W^{2,p}(\Omega)}=\|f\|_{L^p(\Omega)}+\|\partial f\|_{W^{1,p}(\Omega)}+\|\overline{\partial}f\|_{W^{1,p}(\Omega)}.$$

  When $p\leq 2$, the Rellich–Kondrachov theorem implies that $W^{2,p}(\Omega)$ is compactly embedded in $W^{1,\ell}(\Omega)$ for $\ell\in[1,p^*)$.

Given a bounded domain Ω and $h\in L^p(\Omega)$, $1<p<\infty$, let $\tilde{h}$ denote the extension of h by 0 off Ω. The Cauchy integral operator applied to $\tilde{h}$ defines a function $\mathcal{C}(h)\in W^{1,p}_{loc}(\mathbb{R}^2)$ given by

$$\mathcal{C}(h)(z)=\frac{1}{\pi}\int_\Omega\frac{h(t)}{z-t}dm(t)=\frac{1}{2\pi i}\int_\Omega\frac{h(\xi)}{\xi-z}d\xi\wedge d\xi,\qquad z\in\mathbb{C}.\tag{2.9}$$

Indeed, $\mathcal{C}(h)$ lies in $L^1_{loc}(\mathbb{C})$ by Fubini's theorem. Furthermore, $z\mapsto 1/(\pi z)$ is a fundamental solution of the $\bar\partial$ operator and it follows that $\bar\partial\mathcal{C}(h)=\tilde{h}$ in the sense of distributions. In another connection (see [2, Theorem 4.3.10] and the remark thereafter), the complex derivative $\partial\mathcal{C}(h)$ is given by the singular integral

$$\mathcal{B}(h)(z):=\lim_{\varepsilon\to 0}-\frac{1}{\pi}\int_{\Omega\setminus D(z,\varepsilon)}\frac{h(\xi)}{(z-\xi)^2}dm(\xi),\qquad z\in\mathbb{C},\tag{2.10}$$

which is the so-called Beurling transform of $\tilde{h}$. By a result of Calderòn and Zygmund (see [2, Theorem 4.5.3]) this transform maps $L^p(\mathbb{C})$ continuously into itself, and altogether we conclude that $\mathcal{C}(h)\in W^{1,p}_{loc}(\mathbb{C})$, as announced. The discussion above shows in particular that $\varphi:=\mathcal{C}(h)_{|\Omega}$ lies in $W^{1,p}(\Omega)$, and that

$$\|\partial\varphi\|_{L^p(\Omega)}+\|\bar\partial\varphi\|_{L^p(\Omega)}=\|\mathcal{B}(h)_{|\Omega}\|_{L^p(\Omega)}+\|h\|_{L^p(\Omega)}\leq c\|h\|_{L^p(\Omega)},$$

where c depends only on p. In addition, it is a consequence of Fubini's theorem that $\|\varphi\|_{L^p(\Omega)}\leq 6\,\mathrm{diam}\,\Omega\,\|h\|_{L^p(\Omega)}$ [2, Theorem 4.3.12]. Therefore, we have

$$\|\mathcal{C}(h)\|_{W^{1,p}(\Omega)} \leq C\|h\|_{L^p(\Omega)}, \tag{2.11}$$

where C depends only on p and Ω. Moreover, if $\Omega \subset \mathbb{D}_R$, then $\mathcal{C}(h)$ coincides on Ω with the convolution of $\tilde{h}$ with $z \mapsto \chi_{\mathbb{D}_{2R}}(z)/z$, where χ_E denotes the characteristic function of a set E. Therefore $\partial\mathcal{C}(\varphi)_{|\Omega} = \mathcal{C}(\partial\varphi)_{|\Omega}$ whenever $\varphi \in \mathcal{D}(\Omega)$, and by density argument it follows that

$$\|\mathcal{C}(h)\|_{W^{2,p}(\Omega)} \leq C\|h\|_{W^{1,p}(\Omega)}, \qquad h \in W_0^{1,p}(\Omega), \tag{2.12}$$

for $p \in (1,\infty)$ and some $C = C(p,\Omega)$.
Properties of the Cauchy transform make it a basic tool to integrate $\bar{\partial}$-equations in Sobolev classes. In this connection, we record the following facts.

- Given a bounded open set $\Omega \subset \mathbb{C}$ and $a \in L^p(\Omega)$ with $p \in (1,\infty)$, a distribution $A \in \mathcal{D}'(\Omega)$ satisfies $\bar{\partial}A = a$ if and only if $A = \mathcal{C}(a) + \Phi$ where Φ is holomorphic in Ω. This follows from the relation $\bar{\partial}\mathcal{C}(a) = a$ and Weyl's lemma. By (2.11), A belongs to $W^{1,p}(\Omega)$ if and only if Φ does. By localization, it follows that if $f \in L^1_{loc}(\Omega)$ satisfies $\bar{\partial}f \in L^p_{loc}(\Omega)$, then $f \in W^{1,p}_{loc}(\Omega)$.
- Given a bounded C^1-smooth simply connected domain $\Omega \subset \mathbb{C}$ and $a \in L^p(\Omega)$ with $p \in (1,\infty)$, for every $\psi \in W^{1-1/p,p}(\partial\Omega)$, $\lambda \in \mathbb{R}$, $\theta_0 \in \mathbb{R}$, there exists a unique $A \in W^{1,p}(\Omega)$ such that $\bar{\partial}A = a$ with $\mathrm{tr}_{\partial\Omega}\,\mathrm{Re}\,(e^{i\theta_0}A) = \psi$, and $\int_{\partial\Omega}\mathrm{Im}\,(e^{i\theta_0}A) = \lambda$. Moreover, there exists C depending only on p and Ω such that

  $$\|A\|_{W^{1,p}(\Omega)} \leq C\big(\|a\|_{L^p(\Omega)} + \|\psi\|_{W^{1-1/p,p}(\partial\Omega)} + |\lambda|\big). \tag{2.13}$$

  To see this, it suffices, in view of (2.11) and the previous remark, to consider the case $a = 0$. Clearly, we may also assume that $\theta_0 = 0$. By elliptic regularity, there is a unique $u \in W^{1,p}_{\mathbb{R}}(\Omega)$, harmonic in Ω and such that $\mathrm{tr}_{\partial\Omega}u = \psi$. Moreover, u satisfies $\|u\|_{W^{1,p}(\Omega)} \leq C\|\psi\|_{W^{1-1/p,p}(\partial\Omega)}$. As Ω is simply connected, integrating the conjugate differential yields a so-called harmonic conjugate to u, that is a real-valued harmonic function v, such that $A := u+iv$ is holomorphic in Ω. Since u and v are real, the Cauchy–Riemann equations give $|\partial v| = |\bar{\partial}v| = |\partial u|$. Hence, we have $v \in W^{1,p}_{\mathbb{R}}(\Omega)$. Clearly v is unique up to an additive constant, and if $\int_{\partial\Omega} v = \lambda$ we deduce from (2.7) that $\|v\|_{W^{1,p}(\Omega)} \leq C_1\|u\|_{W^{1,p}(\Omega)} + c_1|\lambda|$ so that (2.13) holds (with $a = 0$), as desired.

When $h \in L^2(\mathbb{C})$ has unbounded support, definition (2.9) of the Cauchy transform is no longer suitable. Instead, one renormalizes the kernel and defines

$$\mathcal{C}_2(h)(z) := \frac{1}{\pi}\int_{\mathbb{R}^2}\Big(\frac{1}{z-t} + \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}(t)}{t}\Big)h(t)\,dm(t), \qquad z \in \mathbb{C}. \tag{2.14}$$

Since $h \in L^2(\mathbb{C})$, the integral in (2.14) converges for a.e. $z \in \mathbb{C}$ by Fubini's theorem and the Schwarz inequality. In fact, the function $\mathcal{C}_2(h)$ belongs to the space $VMO(\mathbb{C})$ [2, Theorem 4.3.9]. Furthermore, $\bar{\partial}\mathcal{C}_2(h) = h$ and $\partial\mathcal{C}_2(h) = \mathcal{B}(h)$ [2, Theorem 4.3.10]. In particular, $\mathcal{C}_2(h)$ lies in $W^{1,2}_{loc}(\mathbb{C})$ and the map $h \mapsto \mathcal{C}_2(h)$ maps $L^2(\mathbb{C})$ continuously into $W^{1,2}_{loc}(\mathbb{C})$.
In Section 8.1 we prove the following estimate, valid for some absolute constant C:

$$\frac{\|\mathcal{C}_2(h)\|_{L^2(\mathbb{D}_R)}}{R} \le C\big(1 + (\log R)^{1/2}\big)\|h\|_{L^2(\mathbb{D}_R)}, \qquad R \ge 1. \tag{2.15}$$

Hereafter, all classes of functions we consider are embedded in $L^p_{loc}(\Omega)$ for some $p \in (1, +\infty)$, and solutions to differential equations are understood in the distributional sense.
On the disk, we often use the elementary fact that if $f \in W^{1,p}(\mathbb{D})$, then f_ρ converges to f in $W^{1,p}(\mathbb{D})$ as $\rho \to 1^-$.

Here and later on we use the same symbols (like C) to denote different constants.

3. Pseudo-holomorphic functions

Pseudo-holomorphic functions on an open set $\Omega \subset \mathbb{C}$ are those functions Φ that satisfy an equation of the form

$$\bar{\partial}\Phi(z) = a(z)\overline{\Phi(z)} + b(z)\Phi(z), \qquad z \in \Omega. \tag{3.1}$$

We restrict ourselves to the case where Ω is bounded and $a, b \in L^r(\Omega)$ for some $r \in [2, \infty)$. Accordingly, we only consider solutions Φ which belong to $L^\gamma_{loc}(\Omega)$ for some $\gamma > r/(r-1)$, so that, by Hölder's inequality, the right hand side of (3.1) defines a function in $L^\lambda_{loc}(\Omega)$ for some $\lambda > 1$. As a consequence, Φ belongs to $W^{1,\lambda}_{loc}(\Omega)$.
Let $B \in W^{1,r}(\Omega)$ be such that $\bar{\partial}B = b$. A simple computation (using Proposition 8.4 if $r = 2$) shows that Φ satisfies (3.1) if and only if $w := e^{-B}\Phi$ satisfies

$$\bar{\partial}w = \alpha\overline{w}, \tag{3.2}$$

where $\alpha := ae^{-2i\mathrm{Im}B}$ has the same modulus as a. Note (again from Proposition 8.4 for $r = 2$) that $w \in W^{1,\lambda'}_{loc}(\Omega)$ for some $\lambda' > 1$. Therefore, by the Sobolev embedding theorem, w lies in $L^{\gamma'}_{loc}(\Omega)$ for some $\gamma' > 2$, and so equation (3.2) is a simpler but equivalent form of (3.1) which is the one we shall really work with.
We need a factorization principle which goes back to [41], and was called by Bers the similarity principle (similarity to holomorphic functions, that is). It was extensively used in all works mentioned above. We provide a proof because we include the case $r = 2$ and discuss normalization issues when Ω is smooth.

Lemma 3.1 (Bers Similarity principle)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain, $\alpha \in L^r(\Omega)$ for some $r \in [2,\infty)$, and $w \in L^\gamma_{loc}(\Omega)$ be a solution to (3.2) with $\gamma > r/(r-1)$. Then*

(i) *The function w admits a factorization of the form*
$$w = e^s F, \qquad z \in \Omega, \tag{3.3}$$
where F is holomorphic in Ω, $s \in W^{1,r}(\Omega)$ with
$$\|s\|_{W^{1,r}(\Omega)} \le C\|\alpha\|_{L^r(\Omega)}, \tag{3.4}$$
and C depends only on r and Ω.

(ii) *Assume in addition that Ω is C^1-smooth. If $w \not\equiv 0$ and we fix some $\psi \in W^{1-1/r,r}_{\mathbb{R}}(\partial\Omega)$, $\lambda \in \mathbb{R}$, and $\theta_0 \in \mathbb{R}$, then s can be uniquely chosen in (3.3) so that $tr_{\partial\Omega}\mathrm{Re}(e^{i\theta_0}s) = \psi$ and $\int_{\partial\Omega} \mathrm{Im}\,(e^{i\theta_0}s) = \lambda$. In this case, there is a constant C depending only on r and Ω such that*
$$\|s\|_{W^{1,r}(\Omega)} \le C\big(\|\alpha\|_{L^r(\Omega)} + \|\psi\|_{W^{1-1/r,r}(\partial\Omega)} + |\lambda|\big). \tag{3.5}$$

(iii) *Either $w \equiv 0$ or $w \neq 0$ a. e. on Ω[5]. Moreover, $w \in W^{1,r}_{loc}(\Omega)$ if $r > 2$ and $w \in W^{1,q}_{loc}(\Omega)$ for all $q \in [1,2)$ if $r = 2$.*

Proof. We pointed out already that $w \in W^{1,\ell}_{loc}(\Omega)$ for some $\ell > 1$. Set by convention $\overline{w(\xi)}/w(\xi) = 0$ if $w(\xi) = 0$, and let $s := \mathcal{C}(\alpha\bar{w}/w)_{|\Omega}$. Then $s \in W^{1,r}(\Omega)$ with $\bar\partial s = \alpha\overline{w}/w$, and (2.11) yields (3.4). To show that $F = e^{-s}w$ is in fact holomorphic, we compute
$$\bar\partial(e^{-s}w) = e^{-s}\left(-\bar\partial s\, w + \bar\partial w\right) = e^{-s}\left(-\frac{\alpha\bar{w}}{w}w + \alpha\bar{w}\right) = 0,$$
where the use of the Leibniz and the chain rules is justified by Proposition 8.4 if $r = 2$. This proves (i).
Since s is finite a.e. on Ω (actually outside of a set of $B_{1,2}$-capacity zero), e^s is a.e. nonzero and so is w unless the holomorphic function F is identically zero. If $r > 2$, then $e^s \in W^{1,r}(\mathbb{D})$, and since F is locally smooth we get that $w \in W^{1,r}_{loc}(\Omega)$; if $r = 2$, it follows from Proposition 8.4 that $e^s \in W^{1,q}(\Omega)$ for all $q \in [1,2)$, and thus $w = e^s F$ lies in $W^{1,q}_{loc}(\Omega)$. This proves (iii).
Finally, if Ω is C^1-smooth and $w \not\equiv 0$ (hence $w \neq 0$ a.e. by the above argument), there exists a unique $s \in W^{1,r}(\Omega)$ satisfying the equations $\bar\partial s = \alpha\overline{w}/w$, $\mathrm{tr}_{\partial\Omega}\mathrm{Re}(e^{i\theta_0}s) = \psi$, $\int_{\partial\Omega}\mathrm{Im}\,(e^{i\theta_0}s) = \lambda$, and (2.13) yields (3.5). Moreover, if (3.3) holds for some $s \in W^{1,r}(\Omega)$ and some holomorphic F, we find upon differentiating that $\bar\partial s = \alpha\overline{w}/w$, therefore factorization (3.3) is unique with the aforementioned conditions. This proves (ii). □

A weak converse to the similarity principle is as follows: if $s \in W^{1,r}(\Omega)$ and F is holomorphic on Ω, then $w = e^s F$ satisfies (3.2) with $\alpha := \bar\partial s\, e^s F/(e^{\bar{s}}\bar{F}) \in$

[5] In fact, more is true: if $r > 2$, then e^s never vanishes and w has at most countably many zeros, namely those of F. If $r = 2$, w is strictly defined and nonzero outside a set of Bessel $B_{1,2}$-capacity zero (containing the zeros of F and the non Lebesgue points of s).

$L^r(\Omega)$. This remark shows that, in general, we cannot expect solutions of (3.2) to lie in $L^\infty_{loc}(\Omega)$ when $r = 2$.

4. Holomorphic parametrization

When $r > 2$, it follows from [42, Theorem 3.13] that for each holomorphic function F on Ω and each $\alpha \in L^r(\Omega)$, there is $\Phi \in W^{1,r}(\Omega)$ such that $w := \Phi F$ satisfies (3.2). In this section we improve this assertion to a strong converse of the similarity principle, valid for $2 \le r < \infty$, which leads to a parametrization of pseudo-holomorphic functions by holomorphic functions. We state the result for the disk, which is our focus in the present paper, but we mention that it carries over at once to Dini-smooth[6] simply connected domains, granted the conformal invariance of equation (3.2) pointed out in [4, Section 3.2].

Theorem 4.1. *Let $\alpha \in L^r(\mathbb{D})$ for some $r \in [2,\infty)$, and let $F \not\equiv 0$ be holomorphic on $\mathbb{D}$. Choose $\psi \in W_{\mathbb{R}}^{1-1/r,r}(\mathbb{T})$, and $\lambda \in \mathbb{R}$. Then there exists a unique $s \in W^{1,r}(\mathbb{D})$ such that $w = e^s F$ is a solution of* (3.2) *with $tr_{\mathbb{T}}\mathrm{Im}\, s = \psi$ and $\int_{\mathbb{T}} \mathrm{Re}\, s = \lambda$. Moreover,* (3.5) *holds with some C depending only on r.*

From the proof of the theorem, we obtain also the following variant thereof.

Corollary 4.2. *Theorem 4.1 remains valid if, instead of $tr_{\mathbb{T}}\mathrm{Im}\, s = \psi$ and $\int_{\mathbb{T}} \mathrm{Re}\, s = \lambda$, we prescribe $tr_{\mathbb{T}}\mathrm{Re}\, s = \psi$ and $\int_{\mathbb{T}} \mathrm{Im}\, s = \lambda$.*

Before establishing Theorem 4.1, we need to take a closer look at pairs s, F for which (3.3) and (3.2) hold. We do this in the following subsection.

4.1. Arguments of pseudo-holomorphic functions.

Let $w \in L^\gamma_{loc}(\mathbb{D})$ satisfy (3.2), $\gamma > r/(r-1)$, and consider factorization (3.3) provided by Lemma 3.1. Locally around points where F does not vanish, w has a Sobolev-smooth argument, unique modulo $2\pi\mathbb{Z}$, which is given by $\arg w = \arg F + \mathrm{Im}\, s$. Since $\log F$ is harmonic and $\bar\partial s = \alpha\bar w/w$, we deduce that around such points $\Delta \log w = 4\partial(\alpha e^{-2i\arg w})$. In particular, $\arg w$ satisfies the nonlinear (yet quasilinear) equation $\Delta \arg w = 4\,\mathrm{Im}(\partial(\alpha e^{-2i\arg w}))$, and then $\log|w|$ is determined by $\arg w$ up to a harmonic function that turns out to be completely determined by (3.2). The lemma below dwells on this observation but avoids speaking of $\arg F$ (which may not be globally defined if F has zeros).

Lemma 4.3. *Let $\alpha \in L^r(\mathbb{D})$ for some $r \in [2,\infty)$ and let F be a non identically zero holomorphic function in $\mathbb{D}$. If we set $\beta := \alpha\overline{F}/F$, then a function $s \in W^{1,r}(\mathbb{D})$ is such that $w := e^s F$ satisfies* (3.2) *if and only*

[6] A domain is Dini-smooth if its boundary has a parametrization with Dini-continuous derivative. Conformal maps between such domains have derivatives that extend continuously up to the boundary.

if $\bar\partial s = \beta e^{-2i\mathrm{Im}\, s}$. *This is equivalent to saying that* $s = \varphi_1 + i\varphi_2$ *where* $\varphi_1, \varphi_2 \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ *satisfy the relations*

$$\Delta\varphi_2 = 4\,\mathrm{Im}\big(\partial\left(\beta e^{-2i\varphi_2}\right)\big), \tag{4.1}$$

$$\varphi_1 = \mathrm{Re}\,\mathcal{C}\left(\beta e^{-2i\varphi_2}\right) + v, \tag{4.2}$$

where v *is a harmonic conjugate to the harmonic function* $u \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ *such that* $tr_{\mathbb{T}} u = tr_{\mathbb{T}}\,\mathrm{Im}\left(\mathcal{C}(\beta e^{-2i\varphi_2})\right) - tr_{\mathbb{T}}\varphi_2$.

Proof. Using Proposition 8.4 to justify the computation in case $r = 2$, we find that $s \in W^{1,r}(\mathbb{D})$ with $w = e^s F$ satisfies (3.2) if and only if $\bar\partial s - \beta e^{\bar s - s} = 0$. With the notation $\varphi_1 := \mathrm{Re}\, s$ and $\varphi_2 := \mathrm{Im}\, s$ this is equivalent to

$$\bar\partial\varphi_1 = \beta\exp\left(-2i\varphi_2\right) - i\bar\partial\varphi_2, \qquad \varphi_1, \varphi_2 \in W^{1,r}_{\mathbb{R}}(\mathbb{D}). \tag{4.3}$$

Solving this $\bar\partial$-equation for φ_1 using the Cauchy operator, we can rewrite (4.3) as

$$\varphi_1 = \mathcal{C}\left(\beta e^{-2i\varphi_2}\right) - i\varphi_2 + A, \qquad \varphi_1, \varphi_2 \in W^{1,r}_{\mathbb{R}}(\mathbb{D}), \tag{4.4}$$

where A is holomorphic in $\mathbb{D}$. Since $\beta e^{-2i\varphi_2} \in L^r(\mathbb{D})$ we obtain that $\mathcal{C}(\beta e^{-2i\varphi_2}) \in W^{1,r}(\mathbb{D})$, hence φ_1, φ_2 belong to $W^{1,r}(\mathbb{D})$ if and only if A does. Therefore, given $\varphi_2 \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$, equation (4.4) gives rise to a real-valued φ_1 in $W^{1,r}(\mathbb{D})$ if and only if the holomorphic function A lies in $W^{1,r}(\mathbb{D})$ and satisfies the relation

$$-\mathrm{Im}\,\mathcal{C}\left(\beta e^{-2i\varphi_2}\right) + \varphi_2 = \mathrm{Im}\, A. \tag{4.5}$$

By the discussion after (2.13) such an A exists if and only if the left hand side of (4.5) is harmonic; since Δ commutes with taking the imaginary part, this condition amounts to

$$\Delta\varphi_2 - 4\mathrm{Im}\Big(\partial\bar\partial\mathcal{C}(\beta e^{-2i\varphi_2})\Big) = \Delta\varphi_2 - 4\mathrm{Im}\Big(\partial(\beta e^{-2i\varphi_2})\Big) = 0$$

which is (4.1). Then, by (4.5), $\mathrm{Im}\, A$ is the harmonic function $h \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ having trace $\mathrm{tr}_{\mathbb{T}}\varphi_2 - \mathrm{tr}_{\mathbb{T}}\mathrm{Im}\,\mathcal{C}(\beta e^{-2i\varphi_2}) \in W^{1-1/r,r}(\mathbb{T})$. Subsequently $\mathrm{Re}\, A = \mathrm{Im}\,(iA)$ must be a harmonic conjugate to $-h = u$, and taking real parts in (4.4) yields (4.2). □

4.2. **Proof of Theorem 4.1.**

4.2.1. *Existence part.* In this subsection, we prove existence of s in the conditions of Theorem 4.1. Note that (3.5) will automatically hold by Lemma 3.1 (ii) applied with $\theta_0 = -\pi/2$. Let $A \in W^{1,r}(\mathbb{D})$ be holomorphic in $\mathbb{D}$ with $\mathrm{tr}_{\mathbb{T}}\mathrm{Re}\, A = \psi$ and $\int_{\mathbb{T}} \mathrm{Im}\, A = -\lambda$. Writing $e^s F = e^{s-iA}(e^{iA}F)$, we see that we may assume $\psi = 0$ and $\lambda = 0$ upon replacing F by $e^{iA}F$. In addition, upon changing α by $\alpha\overline{F}/F$, we can further suppose that $F \equiv 1$ thanks to (4.1) and (4.2).

We first deal with the case $r = 2$ and begin with fairly smooth α, say $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$. Consider the following (non-linear) operator G_α acting on $\varphi \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$:

$$G_\alpha(\varphi)(z) := -\frac{2}{\pi}\int_{\mathbb{D}} \log\left|\frac{1-\bar{z}t}{z-t}\right| \mathrm{Im}\Big(\partial\big(\alpha(t)e^{-2i\varphi(t)}\big)\Big)\, dm(t), \quad z \in \mathbb{D}. \tag{4.6}$$

Since $|e^{-2i\varphi}| = 1$ and $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$, we get from Proposition 8.4 that $\partial(\alpha e^{-2i\varphi}) \in L^2(\mathbb{D})$, therefore the above integral exists for every $z \in \mathbb{C}$ by the Schwarz inequality. In fact, $G_\alpha(\varphi)$ is the Green potential of $4\mathrm{Im}\big(\partial\big(\alpha e^{-2i\varphi}\big)\big)$ in $\mathbb{D}$, that is, its distributional Laplacian is $4\mathrm{Im}\big(\partial\big(\alpha e^{-2i\varphi}\big)\big)$ and its value on $\mathbb{T}$ is zero, compare to [2, Section 4.8.3]. To prove existence of s subject to the conditions $\psi = 0$, $\lambda = 0$, and $F \equiv 1$, it suffices by Lemma 4.3 to verify that G_α has a fixed point in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$. First, we check that G_α is compact from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into itself, meaning that it is continuous and maps bounded sets to relatively compact ones.

Lemma 4.4. *If $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$, then the operator G_α is bounded and continuous from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into $W^{2,2}_{\mathbb{R}}(\mathbb{D})$ and it is compact from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into itself.*

Proof. To prove the boundedness and continuity of G_α from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into $W^{2,2}_{\mathbb{R}}(\mathbb{D})$, observe from (8.17) and the dominated convergence theorem that the map $\varphi \mapsto \mathrm{Im}\left(\partial\left(\alpha e^{-2i\varphi}\right)\right)$ is bounded and continuous from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into $L^2_{\mathbb{R}}(\mathbb{D})$. Therefore it suffices to prove the boundedness from $L^2_{\mathbb{R}}(\mathbb{D})$ into $W^{2,2}_{\mathbb{R}}(\mathbb{D})$ of the linear potential operator:

$$P(\psi) := -\frac{1}{2\pi}\int_{\mathbb{D}} \log\left|\frac{1-\bar{z}t}{z-t}\right| \psi(t)\, dm(t).$$

The latter is a consequence of properties of the Cauchy and Beurling transforms listed in Section 2 [2, Section 4.8.3]. Compactness of G_α from $L^2(\mathbb{D})$ into $W^{1,2}(\mathbb{D})$ now follows from compactness of the embedding of $W^{2,2}(\mathbb{D})$ into $W^{1,2}(\mathbb{D})$ asserted by the Rellich–Kondrachov theorem. $\square$

Since G_α is compact on $W^{1,2}_{\mathbb{R}}(\mathbb{D})$, a sufficient condition for it to have a fixed point is given by the Leray–Schauder theorem [23, Theorem 11.3]: there is a number M for which the *a priori* estimate $\|\varphi\|_{W^{1,2}(\mathbb{D})} \le M$ holds whenever $\varphi \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$ and $\varepsilon \in [0,1]$ satisfy

$$\varphi = \varepsilon G_\alpha(\varphi). \tag{4.7}$$

Now, if (4.7) is true, then (4.1) is satisfied with $\beta = \varepsilon\alpha$ and φ instead of φ_2. Therefore by Lemma 4.3, there exist $\varphi_{1,\varepsilon} \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$ and $s_\varepsilon := \varphi_{1,\varepsilon} + i\varphi$ such that

$$\bar{\partial} e^{s_\varepsilon} = \varepsilon\alpha\,\overline{e^{s_\varepsilon}}.$$

Applying Lemma 3.1 (ii) with $\Omega = \mathbb{D}$, $F \equiv 1$, $s = s_\varepsilon$, $\psi \equiv 0$, $\theta_0 = -\pi/2$ and $\lambda = 0$, we get from (3.5) that for some absolute constant C

$$\|\varphi\|_{W^{1,2}(\mathbb{D})} \le \|s_\varepsilon\|_{W^{1,2}(\mathbb{D})} \le \varepsilon C\|\alpha\|_{L^2(\mathbb{D})} \le C\|\alpha\|_{L^2(\mathbb{D})} =: M.$$

Thus, G_α indeed has a fixed point, which settles the case $r = 2$ and $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$.
Next, we relax our restriction on α and assume only that it belongs to $L^2(\mathbb{D})$. Let (α_n) be a sequence in $\mathcal{D}(\mathbb{D})$ that converges to α in $L^2(\mathbb{D})$. By the first part of the proof, there is a sequence $(s_n) \subset W^{1,2}(\mathbb{D})$ such that $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s_n = 0$ and $\int_{\mathbb{T}} \operatorname{Re} \operatorname{tr}_{\mathbb{T}} s_n = 0$, satisfying $\overline{\partial} e^{s_n} = \alpha_n \overline{e^{s_n}}$ as well as (*cf.* (3.5))

$$\|s_n\|_{W^{1,2}(\mathbb{D})} \leq C\|\alpha_n\|_{L^2(\mathbb{D})} \leq C'. \tag{4.8}$$

By the Rellich–Kondrachov theorem we can find a subsequence, again denoted by (s_n), converging pointwise and in all $L^q(\mathbb{D})$, $1 \leq q < \infty$ to some function s. By dominated convergence, the functions $\bar{\partial} s_n = \alpha_n e^{-2i\operatorname{Im} s_n}$ converge to $\alpha e^{-2i\operatorname{Im} s}$ in $L^2(\mathbb{D})$. Thus, applying (2.13) with $A = s_n - s_m$, $a = \bar{\partial} s_n - \bar{\partial} s_m$, $\theta_0 = -\pi/2$, $\psi \equiv 0$, and $\lambda = 0$, we conclude that (s_n) is a Cauchy sequence in $W^{1,2}(\mathbb{D})$ which must therefore converge to s. Hence $s \in W^{1,2}(\mathbb{D})$, $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s = 0$, $\int_{\mathbb{T}} \operatorname{Re} s = 0$, and $\bar{\partial} s = \alpha e^{-2i\operatorname{Im} s}$. By Lemma 4.3, this establishes existence of s when $r = 2$. Suppose finally that $\alpha \in L^r(\mathbb{D})$ for some $r > 2$. *A fortiori* $\alpha \in L^2(\mathbb{D})$, so by what precedes there is $s \in W^{1,2}(\mathbb{D})$ such that $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s = 0$, $\int_{\mathbb{T}} \operatorname{Re} s = 0$, and $\overline{\partial} e^s = \alpha \overline{e^s}$. To see that in fact $s \in W^{1,r}(\mathbb{D})$, we apply Proposition 8.4 to get $\bar{\partial} s = \alpha e^{-2i\operatorname{Im} s} =: a \in L^r(\mathbb{D})$. Then, equation (2.13) implies that s is the unique function $A \in W^{1,r}(\mathbb{D})$ satisfying $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} A = 0$, $\int_{\mathbb{T}} \operatorname{Re} A = 0$, and $\overline{\partial} A = a$. □

4.2.2. *Uniqueness part.* In this subsection we establish uniqueness of s in the conditions of Theorem 4.1. Clearly, it is enough to consider $r = 2$. Consider two functions $w_1 = e^{s_1} F$ and $w_2 = e^{s_2} F$ meeting (3.2) on $\mathbb{D}$ with $s_j \in W^{1,2}(\mathbb{D})$, $\operatorname{tr}_{\mathbb{T}} \operatorname{Im} s_j = \psi$, and $\int_{\mathbb{T}} \operatorname{Re} s_j = \lambda$ for $j = 1, 2$. We define

$$s(z) := s_1(z) - s_2(z) \in W^{1,2}(\mathbb{D})$$

and we must prove that $s \equiv 0$. First we estimate the $\bar{\partial}$-derivative of s:

Lemma 4.5. *There is a constant $C > 0$ such that, for a.e. $z \in \mathbb{D}$, we have*

$$|\bar{\partial} s(z)| \leq C |\operatorname{Im}\, s(z)|\, |\alpha(z)|. \tag{4.9}$$

Proof. Setting $\beta := \alpha \bar{F}/F$ and using again Lemma 4.3, we find that $\bar{\partial} s_j = \beta e^{-2i \operatorname{Im} s_j}$. Hence, $\bar{\partial} s = \beta e^{-2i \operatorname{Im} s_1}\big(1 - e^{2i \operatorname{Im} s}\big)$, and (4.9) follows at once. □

Next, we extend the function s outside of $\overline{\mathbb{D}}$ by reflection:

$$s(z) := \overline{s\,(1/\bar{z})}, \qquad z \in \mathbb{C} \setminus \overline{\mathbb{D}}, \tag{4.10}$$

Observe that since s is real-valued on $\mathbb{T}$, this extension makes $s \in W^{1,2}_{loc}(\mathbb{C})$, see, for example [12, Theorem 2.54].

Lemma 4.6. *There is a constant $C > 0$ such that, for a.e. $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$,*

$$|\bar{\partial} s(z)| \leq C \frac{\big|\operatorname{Im} s(z)\big|}{|z|^2}\, |\alpha(1/\bar{z})|. \tag{4.11}$$

Proof. Putting $\zeta = 1/\bar{z} =: U(z)$ and applying the chain rule, we get (since $\partial f = 0$) that

$$\partial\left(s\left(1/\bar{z}\right)\right) = \left(\left(\partial_{\bar{\zeta}}\, s\right) \circ U\right) \partial \bar{U} = -\frac{1}{z^2} \bar{\partial} s\left(1/\bar{z}\right).$$

Thus,

$$\bar{\partial} s(z) = \overline{\partial\left(s\left(1/\bar{z}\right)\right)} = -\overline{\left(\frac{\bar{\partial} s(1/\bar{z})}{z^2}\right)}, \qquad |z| > 1,$$

and applying Lemma 4.5 gives (4.11) in view of (4.10). □

From the two previous lemmas we derive the inequality:

$$|\bar{\partial} s(z)| \le C \frac{|\mathrm{Im}\, s(z)|}{1+|z|^2} \, |\alpha(Q(z))|, \qquad a.e.\ z \in \mathbb{C}, \tag{4.12}$$

where $Q(z)$ is equal to z if $|z| \le 1$ and to $1/\bar{z}$ otherwise. Since $\alpha \in L^2(\mathbb{D})$, it follows from (4.12) and the change of variable formula that $\bar{\partial} s / s \in L^2(\mathbb{C})$. Recall now definition (2.14). We introduce two auxiliary functions ψ, ϕ on $\mathbb{C}$:

$$\psi := \mathcal{C}_2\left(\bar{\partial} s / s\right), \qquad \phi := \exp(-\psi). \tag{4.13}$$

Since $\bar{\partial} s / s \in L^2(\mathbb{C})$, we know that $\psi \in W^{1,2}_{loc}(\mathbb{C})$ with $\bar{\partial} \psi = \bar{\partial} s / s$. Consider the function $s\phi$ on $\mathbb{C}$. By Proposition 8.4, we compute from (4.13) using the Leibniz rule that $\bar{\partial}(s\phi) = 0$, hence $s\phi$ is an entire function. We claim that

$$\liminf_{R \to +\infty} \left(\frac{1}{R} \int_{\mathbb{T}_R} \log^+ |s\phi(\xi)| \, |d\xi| - \frac{1}{2} \log R \right) < 0. \tag{4.14}$$

Indeed, taking into account (4.10) and the fact that $s_{|\mathbb{D}} \in L^\ell(\mathbb{D})$ for all $1 \le \ell < \infty$ by the Sobolev embedding theorem, we get from Jensen's inequality upon choosing $\ell > 48\pi$ that

$$\begin{aligned} &\frac{1}{R^2} \int_{R<\rho<2R} \int_{0<\theta<2\pi} \log^+ |s(\rho e^{i\theta})| \, \rho \, d\rho \, d\theta \\ &\qquad \le \frac{12\pi}{\ell} \frac{4R^2}{3\pi} \int_{1/(2R)<\rho<1/R} \int_{0<\theta<2\pi} \log^+ (|s(\rho e^{i\theta})|^\ell) \, \rho \, d\rho \, d\theta \\ &\qquad \le \frac{12\pi}{\ell} \log \Big[\frac{4R^2}{3\pi} \int_{1/(2R)<\rho<1/R} \int_{0<\theta<2\pi} \max\{1, |s(\rho e^{i\theta})|^\ell\} \, \rho \, d\rho \, d\theta \Big] \\ &\qquad \le \frac{1}{\delta} \log R + C \end{aligned} \tag{4.15}$$

for some $\delta > 2$ and some $C > 0$, whenever $R \ge 1$.

In another connection, it follows from (2.15) and the Schwarz inequality that

$$\begin{aligned} \frac{1}{\pi R^2} \int_{R<\rho<2R} \int_{0<\theta<2\pi} |\psi(\rho e^{i\theta})| \, \rho \, d\rho \, d\theta &\le \frac{\|\psi\|_{L^2(\mathbb{D}_{2R})}}{\sqrt{\pi} R} \\ &= O\left((\log R)^{1/2} \right), \quad R \to +\infty. \end{aligned} \tag{4.16}$$

Since $\log^+|s\phi| \le \log^+|s| + |\psi|$, claim (4.14) easily follows from (4.15) and (4.16).
Since $\log|s\phi|$ is subharmonic on $\mathbb{C}$, for $|z|<R$ we have

$$2\pi \log|s\phi(z)| \le \frac{R+|z|}{R-|z|}\int_0^{2\pi} \log^+\left|s\phi(Re^{i\theta})\right| d\theta$$
$$\le \frac{R+|z|}{R-|z|}\,\frac{1}{R}\int_{\mathbb{T}_R} \log^+|s\phi(\xi)|\,|d\xi|$$

(see [37, Theorem 2.4.1]), so by (4.14) there is a sequence $\rho_n \to +\infty$ for which $\sup_{\mathbb{T}_{\rho_n}}|s\phi| = O(\rho_n^{1/2})$. Therefore, by an easy modification of Liouville's theorem, $s\phi$ must be a constant.
More generally, (4.12) remains valid if we replace s by $s-a$ for $a\in\mathbb{R}$, entailing that

$$\left|\frac{\bar\partial s(z)}{s(z)-a}\right| \le C\left|\frac{\alpha(Q(z))}{1+|z|^2}\right| \in L^2(\mathbb{C}), \qquad a.e.\ z\in\mathbb{C}, \quad a\in\mathbb{R}. \tag{4.17}$$

Thus, reasoning as before, we deduce that there is a complex-valued function b such that

$$(s(z)-a)\phi_a(z) \equiv b(a), \qquad a\in\mathbb{R}, \tag{4.18}$$

with

$$\psi_a := \mathcal{C}_2\left(\bar\partial s/(s-a)\right), \quad \phi_a := \exp(-\psi_a).$$

Fix $R>1$. By (4.17), Corollary 8.6, and Proposition 8.4, the sets $\{\phi_a|_{\mathbb{D}_R}\}_{a\in\mathbb{R}}$ and $\{\phi_a^{-1}|_{\mathbb{D}_R}\}_{a\in\mathbb{R}}$ are bounded in $W^{1,q}(\mathbb{D}_R)$ for $q\in[1,2)$, hence also in $L^2(\mathbb{T})$ by the trace and the Sobolev embedding theorems. Fix $A>0$ such that $\Lambda(\{\xi\in\mathbb{T} : |s(\xi)|\le A\}) = \lambda > 0$. For each $\delta>0$ we can cover the interval $[-A,A]$ by $N \le A/\delta + 1$ open intervals of length 2δ, hence there exists $a = a(\delta)\in[-A,A]$ with $\Lambda(E_a)\ge \lambda\delta/(A+\delta)$, where $E_a = \{\xi\in\mathbb{T} : |s(\xi)-a|\le\delta\}$. (We use here that s is real-valued on $\mathbb{T}$.) Moreover, we observe from (4.18) and the Schwarz inequality that

$$|b(a)|\Lambda(E_a) = \int_{E_a} |s(\xi)-a||\phi_a(\xi)|\,d\Lambda(\xi) \le \delta\|\phi_a\|_{L^2(\mathbb{T})}\Lambda(E_a)^{1/2}.$$

This lower bound on $\Lambda(E_a)$ now gives us that

$$|b(a)| \le \delta^{1/2}\sqrt{(A+\delta)/\lambda}\,\sup_{|a|\le A}\|\phi_a\|_{L^2(\mathbb{T})},$$

implying that $b(a(\delta))\to 0$ as $\delta\to 0$. By compactness, we can pick a sequence $\delta_n\to 0$ such that $a_n := a(\delta_n)\to c\in[-A,A]$. Considering the equalities $s - a_n = b(a_n)\phi_{a_n}^{-1}$ and taking into account the boundedness of $\{\phi_{a_n}^{-1}\}$ in $L^2(\mathbb{D}_R)$, we find that $s\equiv c$ on $\mathbb{D}_R$. Since R is arbitrary, s is constant on $\mathbb{C}$, and actually $s\equiv 0$ because $\int_{\mathbb{T}} s = 0$. □
A similar argument gives the following result which seems to be of independent interest.

Theorem 4.7. *If $s \in W^{1,2}_{loc}(\mathbb{C})$ satisfies*

$$|\bar{\partial}s(z)| \le |\mathrm{Im}\, s(z)|\, g(z)$$

for some non-negative function $g \in L^2(\mathbb{C})$, and if

$$\int_{\mathbb{C}\setminus\mathbb{D}} \frac{|s(\xi)|^{\ell}\, \mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}}{|\xi|^4} < \infty,$$

for some $\ell > 48\pi$, then $\mathrm{Im}\, s$ is of constant sign a.e. in $\mathbb{C}$.

The example $s(z) = i + (1+|z|)^{-\beta}$, $\beta > 0$ shows that, under these conditions, s is not necessarily a constant. On the other hand, the value 48π is not necessarily sharp.
It is interesting to compare this result to known Liouville-type theorems like [3, Proposition 3.3] and [2, Theorem 8.5.1].

Sketch of proof. For any real d, (4.18) gives us for small $\delta > 0$ that $m\{\xi \in \mathbb{D} : |\mathrm{Im}\, s(\xi)| < \delta,\ |\mathrm{Re}\, s(\xi) - d| < \delta\} \le c\delta^5/\inf_{a\in\mathbb{R}} |b(a)|^5$ with c independent of d; therefore, $m\{\xi \in \mathbb{D} : |\mathrm{Im}\, s(\xi)| < \delta,\ |\mathrm{Re}\, s(\xi)| < 1/\delta\} \le c\delta^3/\inf_{a\in\mathbb{R}} |b(a)|^5$. Since $s \in W^{1,2}_{loc}(\mathbb{C})$, by the John–Nirenberg theorem we have $m\{\xi \in \mathbb{D} : |s(\xi)| > 1/\delta\} \le c\delta^3$. Finally, if $\mathrm{Im}\, s$ changes sign in $\mathbb{D}$, then by the Hölder inequality we obtain $m\{\xi \in \mathbb{D} : |\mathrm{Im}\, s(\xi)| < \delta\} \ge c\delta^2$. As a result, passing to the limit $\delta \to 0$, we obtain that $\inf_{a\in\mathbb{R}} |b(a)| = 0$. □

4.3. **Proof of Corollary 4.2.** Uniqueness of s is established as in Theorem 4.1, except that the right hand side of (4.10) now has a minus sign because s is pure imaginary on $\mathbb{T}$. Note also that (3.5) holds by Lemma 3.1 (ii) applied with $\theta_0 = 0$.
Passing to existence of s, the argument given early in subsection 4.2.1 applies with obvious modifications to show that we may assume $\psi = 0$, $\lambda = 0$ and $F \equiv 1$. Moreover, it is enough to prove the result when $r = 2$ and $\alpha \in \mathcal{D}(\mathbb{D})$, for then the passage to $\alpha \in L^2(\mathbb{D})$ and, subsequently, to $r > 2$ is like in the theorem.
So, let us put $r = 2$, fix $\alpha \in \mathcal{D}(\mathbb{D})$, and write $s(\psi, \lambda, F)$ to emphasize the dependance on ψ, λ and F of the function $s \in W^{1,2}(\mathbb{D})$ whose existence and uniqueness is asserted by Theorem 4.1. For $u \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$, we denote by $E(u) \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$ the harmonic extension of u, *i.e.* $E(u)$ is harmonic and $\mathrm{tr}_{\mathbb{T}}\, E(u) = u$. We put $H(u) \in W^{1,2}(\mathbb{D})$ for the holomorphic function such that $\mathrm{Im}\, H(u) = E(u)$ and $\int_{\mathbb{T}} \mathrm{Re}\, H(u) = 0$. Observe that $\alpha \exp(-2iE(u))$ lies in $W^{1,2}(\mathbb{D}) \cap L^{\infty}(\mathbb{D})$; therefore, the operator $G_{\alpha e^{-2iE(u)}}$ defined by (4.6) is compact from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into itself by Lemma 4.4. In the course of the proof of Theorem 4.1, we showed that it has a unique fixed point which is none but $\mathrm{Im}\, s(0, 0, e^{H(u)}) =: \mathcal{F}(u)$. Furthermore, by (3.5), we have $\|\mathcal{F}(u)\|_{W^{1,2}(\mathbb{D})} \le C\|\alpha\|_{L^2(\mathbb{D})}$ for some absolute constant C.

Lemma 4.8. *The (nonlinear) operator $u \mapsto \mathcal{F}(u)$ is compact from $W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$ into $W^{1,2}_{\mathbb{R}}(\mathbb{D})$.*

Proof. Pick a sequence (u_n) converging to u in $W^{1/2,2}(\mathbb{T})$. By elliptic regularity, $H(u_n)$ converges to $H(u)$ in $W^{1,2}(\mathbb{D})$, and in particular $\|H(u_n)+\mathcal{F}(u_n)\|_{W^{1,2}(\mathbb{D})}$ is bounded independently of n. Besides, by (4.6) and the definition of $\mathcal{F}$, we see that

$$\mathcal{F}(u_n) = G_{\alpha e^{-2iE(u_n)}}(\mathcal{F}(u_n)) = G_\alpha(E(u_n)+\mathcal{F}(u_n)); \tag{4.19}$$

hence, Lemma 4.4 implies that $(\mathcal{F}(u_n))_{n\in\mathbb{N}}$ is relatively compact in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$. Let some subsequence, again denoted by $(\mathcal{F}(u_n))$, converges to φ in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$. Then $(E(u_n)+\mathcal{F}(u_n))$ converges to $E(u)+\varphi$ in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$, therefore by (4.19) and the continuity of G_α we obtain that $\varphi = G_{\alpha e^{-2iE(u)}}(\varphi)$. This means that $\varphi = \mathcal{F}(u)$, hence the latter is the only limit point of $(\mathcal{F}(u_n))_{n\in\mathbb{N}}$, which proves the continuity of $\mathcal{F}$.
If we assume only that $\|u_n\|_{W^{1/2,2}(\mathbb{T})}$ is bounded independently of n, then elliptic regularity still gives us that $\|E(u_n)\|_{W^{1,2}(\mathbb{D})}$ is bounded, hence $(E(u_n)+\mathcal{F}(u_n))_{n\in\mathbb{N}}$ is again bounded in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$. As before it follows that $(\mathcal{F}(u_n))_{n\in\mathbb{N}}$ is relatively compact in $W^{1,2}_{\mathbb{R}}(\mathbb{D})$, as desired. $\square$

Given $u \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$, let $\tilde{u} := -\mathrm{tr}_{\mathbb{T}}\mathrm{Re}\, H(u)$ denote the so called conjugate function of u. That is, $\tilde{u}$ is the trace of the harmonic conjugate of $E(u)$ that has zero mean on $\mathbb{T}$. Put $\mathcal{M} \subset W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$ for the subspace of functions with zero mean. By (2.13), the map $u \mapsto \tilde{u}$ is continuous from $W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$ into $\mathcal{M}$, and since $\tilde{\tilde{u}} = -u + \int_{\mathbb{T}} u$, it is a homeomorphism of $\mathcal{M}$. Pick $u \in \mathcal{M}$ and let $\varphi := \mathrm{Im}\, s(u,0,1)$. Since $s(u,0,1) - H(u) = s(0,0,e^{H(u)})$ we have $\varphi = E(u) + \mathcal{F}(u)$. Set for simplicity $R(u) := \mathcal{C}\big(\alpha \exp\{-2i(E(u)+\mathcal{F}(u))\}\big)$. Applying the trace and conjugate operators to (4.2), we see that $\mathrm{tr}_{\mathbb{T}}\mathrm{Re}\, s(u,0,1) = 0$ if and only if

$$u = tr_{\mathbb{T}}\,\mathrm{Im}\Big(R(u) - \int_{\mathbb{T}} R(u)\Big) - \overbrace{\mathrm{tr}_{\mathbb{T}}\,\mathrm{Re}\Big(R(u) - \int_{\mathbb{T}} R(u)\Big)}^{\sim}. \tag{4.20}$$

Let $B(u)$ denote the right hand side of (4.20). To complete the proof, it remains to show that the (nonlinear) operator B has a fixed point u_0 in $\mathcal{M}$. Then the function $s(u_0,0,1)$ would satisfy the conditions of the corollary.
To prove existence of a fixed point, we claim first that B is compact from $\mathcal{M}$ into itself. Indeed, by elliptic regularity, E is linear and bounded from $W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$ into $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ while $\mathcal{F}$ is compact by Lemma 4.8. A fortiori, $E+\mathcal{F}$ is bounded and continuous from $\mathcal{M}$ into $W^{1,2}_{\mathbb{R}}(\mathbb{D})$. Moreover, as $\alpha \in \mathcal{D}(\mathbb{D})$, it follows from (8.17) and the dominated convergence theorem that $h \mapsto \alpha\exp(-2ih)$ is bounded and continuous from $W^{1,2}_{\mathbb{R}}(\mathbb{D})$ into $W^{1,2}_0(\mathbb{D})$. In addition, we get from (2.12) that $\mathcal{C}$ is bounded and linear from $W^{1,2}_0(\mathbb{D})$ into $W^{2,2}(\mathbb{D})$, hence compact into $W^{1,2}(\mathbb{D})$ by the Rellich–Kondrachov theorem. Finally, by the trace theorem, $g \mapsto \mathrm{tr}_{\mathbb{T}}\mathrm{Im}\,(g - \int_{\mathbb{T}} g)$ is linear and bounded from $W^{1,2}(\mathbb{D})$ into $\mathcal{M}$. Since the conjugate operator is linear and bounded on $\mathcal{M}$ and composition with bounded continuous maps preserves compactness,

the claim follows. Appealing now to the Leray–Schauder theorem, we know that B has a fixed point if we can find a constant M such that $\|u\|_{W^{1/2,2}(\mathbb{T})} \leq M$ holds whenever $u = \varepsilon B(u)$ for some $\varepsilon \in [0,1]$. However, such a u must be equal to $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s_\varepsilon$, where $s_\varepsilon \in W^{1,2}(\mathbb{D})$ has pure imaginary trace with zero mean on $\mathbb{T}$ and e^{s_ε} satisfies (3.2) with α replaced by $\varepsilon\alpha$. Thus, from Lemma 3.1 (ii) applied with $\theta_0 = 0$, we conclude that $M = C\|\alpha\|_{L^2(\mathbb{D})}$ will do for some absolute constant C.

5. Hardy spaces on the disk

5.1. Holomorphic Hardy spaces.

For $p \in [1, \infty)$, let $H^p = H^p(\mathbb{D})$ be the Hardy space of holomorphic functions f on $\mathbb{D}$ with

$$\|f\|_{H^p} := \sup_{0<\rho<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(\rho e^{i\theta}) \right|^p d\theta \right)^{1/p} < +\infty. \tag{5.1}$$

The space H^∞ consists of bounded holomorphic functions endowed with the *sup* norm. We refer to [13, 21] for the following standard facts on holomorphic Hardy spaces.

Each $f \in H^p$ has a non-tangential limit at a.e. $\xi \in \mathbb{T}$, which is also the $L^p(\mathbb{T})$ limit of $f_\rho(\xi) := f(\rho\xi)$ as $\rho \to 1^-$ and whose norm matches the supremum in (5.1). Actually $\|f_\rho\|_{L^p(\mathbb{T})}$ is non-decreasing with ρ, hence instead of (5.1) we could as well have set[7]

$$\|f\|_{H^p} := \sup_{0<\rho<1} \left(\int_{\mathbb{T}_\rho} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty \tag{5.2}$$

where the integral is now with respect to the arclength. As usual, we keep the same notation for f and its non-tangential limit when no confusion can arise, or write sometimes $f_{|\mathbb{T}}$ to emphasize that the non-tangential limit lives on $\mathbb{T}$. Note that $f_{|\mathbb{T}}$ coincides with $\operatorname{tr} f$ when $f \in W^{1,p}(\mathbb{D})$ [5]. Each function in H^p is both the Cauchy and the Poisson integral of its non-tangential limit. As regards the non-tangential maximal function, for $1 \leq p < \infty$ and $f \in H^p$ we have

$$\|\mathcal{M}_\gamma f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}, \tag{5.3}$$

where the constant C depends only on γ and p [21, Chapter II, Theorem 3.1].

Traces of H^p-functions on $\mathbb{T}$ are exactly those functions in $L^p(\mathbb{T})$ whose Fourier coefficients of negative index do vanish. In particular, if $f \in H^p$ and $f_{|\mathbb{T}} \in L^q(\mathbb{T})$, then $f \in H^q$. It is obvious from Fubini's theorem that $H^p \subset L^p(\mathbb{D})$, but actually one can affirm more:

$$\|f\|_{L^\lambda(\mathbb{D})} \leq C\|f\|_{H^p}, \qquad p \leq \lambda < 2p, \tag{5.4}$$

[7] In fact (5.1) expresses that $|f|^p$ has a harmonic majorant whereas (5.2) bounds the L^p-norm of f on curves tending to the boundary; the first condition defines the Hardy space and the second the so-called Smirnov space. These coincide when harmonic measure and arclength are comparable on the boundary, [13, Chapter 10], [25], which is the case for smooth domains. The name "Hardy space" is then more common.

where $C = C(p, \lambda)$; for a proof see [13, Theorem 5.9]. A sequence $(z_l) \subset \mathbb{D}$ is the zero set of a nonzero H^p function, taking into account the multiplicities, if and only if it satisfies the *Blaschke condition*:

$$\sum_l (1 - |z_l|) < \infty. \tag{5.5}$$

A non-negative function $h \in L^p(\mathbb{T})$ is such that $h = |f_{\mathbb{T}}|$ for some nonzero $f \in H^p$ if and only if $\log h \in L^1(\mathbb{T})$. This entails that a nonzero H^p function cannot vanish on a subset of strictly positive Lebesgue measure on $\mathbb{T}$.
For $1 < p < \infty$ and for every $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ there exists $g \in H^p$ such that $\operatorname{Re} g = \psi$ on $\mathbb{T}$ [21, Chapter III]. Such a g is unique up to an additive pure imaginary constant, and if we normalize it so that $\int_{\mathbb{T}} \operatorname{Im} g = 0$, then $\|g\|_{H^p} \le C\|\psi\|_{L^p(\mathbb{T})}$ with $C = C(p)$. In fact $g = u + iv$ on $\mathbb{D}$, where u is the Poisson integral of ψ and v is the Poisson integral of

$$\widetilde{\psi}(e^{i\theta}) := \lim_{\varepsilon \to 0} \frac{1}{2\,\pi} \int_{\varepsilon < |\theta - t| < \pi} \frac{\psi(e^{it})}{\tan(\frac{\theta - t}{2})}\, dt \tag{5.6}$$

which is the so-called *conjugate function* of ψ. This definition carries over to $L^p(\mathbb{T})$ the conjugation operator $\psi \mapsto \widetilde{\psi}$ already introduced on $W^{1/2,2}(\mathbb{T})$ after the proof of Lemma 4.8. It is a theorem of M. Riesz that the conjugation operator maps $L^p(\mathbb{T})$ continuously into itself. By elliptic regularity, it is also continuous from $W^{1-1/p,p}(\mathbb{T})$ into itself.
When $\psi \in L^1(\mathbb{T})$, the conjugate function $\widetilde{\psi}$ is still defined pointwise almost everywhere *via* (5.6) but it does not necessarily belong to $L^1(\mathbb{T})$.
For $p \in (1, \infty)$, a non-negative function $\mathfrak{w} \in L^1(\mathbb{T})$ is said to satisfy the Muckenhoupt condition A_p if

$$\{\mathfrak{w}\}_{A_p} := \sup_I \Big(\frac{1}{\Lambda(I)} \int_I \mathfrak{w}\, d\Lambda \Big) \Big(\frac{1}{\Lambda(I)} \int_I \mathfrak{w}^{-1/(p-1)} d\Lambda \Big)^{p-1} < +\infty, \tag{5.7}$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. A theorem of Hunt, Muckenhoupt and Wheeden [21, Chapter VI, Theorem 6.2] asserts that $\mathfrak{w}$ satisfies condition A_p if and only if

$$\int_{\mathbb{T}} |\widetilde{\phi}|^p\, \mathfrak{w}\, d\Lambda \le C \int_{\mathbb{T}} |\phi|^p\, \mathfrak{w}\, d\Lambda, \qquad \phi \in L^1(\mathbb{T}), \tag{5.8}$$

where C depends only on $\{\mathfrak{w}\}_{A_p}$. In (5.8), the assumption $\phi \in L^1(\mathbb{T})$ is just a means to ensure that $\widetilde{\phi}$ is well defined.

5.2. **Pseudo-holomorphic Hardy spaces.** Given $\alpha \in L^r(\mathbb{D})$ for some $r \in [2, \infty)$ and $p \in (1, \infty)$, we define the Hardy space $G^p_\alpha(\mathbb{D})$ of those $w \in L^\gamma_{loc}(\mathbb{D})$ with $\gamma > r/(r-1)$ that satisfy (3.2), such that

$$\|w\|_{G^p_\alpha(\mathbb{D})} := \sup_{0<\rho<1} \Big(\int_{\mathbb{T}_\rho} |w(\xi)|^p\, |d\xi| \Big)^{1/p} < +\infty. \tag{5.9}$$

Denote by $\mathcal{H}^p$ the Banach space of complex measurable functions f on $\mathbb{D}$ such that $\operatorname{ess.sup}_{0<\rho<1} \rho\|f_\rho\|_{L^p(\mathbb{T})} < +\infty$. Then $G^p_\alpha(\mathbb{D})$ is identified with a real subspace of $\mathcal{H}^p$. The fact that this subspace is closed (hence a Banach space in its own right) is a part of Theorem 5.1 below. Note that if $w \in L^\gamma_{loc}(\mathbb{D})$ satisfies (3.2), then $w \in W^{1,q}_{loc}(\mathbb{D})$ for $q \in [1,2)$ by Lemma 3.1; hence the integral in (5.9) is indeed finite for *each* ρ by the trace theorem. Clearly $G^p_0(\mathbb{D}) = H^p$, but $G^p_\alpha(\mathbb{D})$ is not a complex vector space when $\alpha \not\equiv 0$. Spaces $G^1_\alpha(\mathbb{D})$ and $G^\infty_\alpha(\mathbb{D})$ could be defined similarly, but we shall not consider them.

For $r > 2$, such classes of functions were apparently introduced in [35] and subsequently considered in [27, 28, 29, 5, 14, 16, 4]. In contrast to these studies, our definition is modeled after (5.2) rather than (5.1), that is, integral means in (5.9) are with respect to arclength[8] and *not* normalized arclength. This is not important when $r > 2$, but becomes essential[9] if $r = 2$.

Below, we do consider the case $r = 2$ and stress topological connections with holomorphic Hardy spaces which are new even when $r > 2$, see Theorem 5.1 (iii).

By Lemma 3.1, each solution to (3.2) in $L^\gamma_{loc}(\mathbb{D})$, $\gamma > r/(r-1)$, factors as $w = e^s F$ where

$$\|s\|_{W^{1,r}(\mathbb{D})} \le C(r)\|\alpha\|_{L^r(\mathbb{D})} \tag{5.10}$$

and F is holomorphic in $\mathbb{D}$. Moreover, if $w \not\equiv 0$, one can impose $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \operatorname{Re} s = 0$ or $\operatorname{Re} \operatorname{tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \operatorname{Im} s = 0$ to get unique factorization. To distinguish between these two factorizations, we write $w = e^{s^{\mathfrak{r}}} F^{\mathfrak{r}}$ in the first case, and $w = e^{s^{\mathfrak{i}}} F^{\mathfrak{i}}$ in the second one; that is, $s^{\mathfrak{r}}$ is real on $\mathbb{T}$ and $s^{\mathfrak{i}}$ is pure imaginary there. If $w \equiv 0$, we put $F^{\mathfrak{r}} = F^{\mathfrak{i}} = 0$ and do not define $s^{\mathfrak{r}}$ and $s^{\mathfrak{i}}$. When $w \not\equiv 0$ (hence w is a.e. nonzero), it follows from the proof of Lemma 3.1 that if we let

$$\mathcal{R}(\beta)(z) := -\overline{\mathcal{C}(\beta)(1/\bar{z})} = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{z\bar{\beta}(\xi)}{1-\bar{\xi}z}\, d\xi \wedge \overline{d\xi}, \quad \beta \in L^r(\mathbb{D}),\ z \in \mathbb{D}, \tag{5.11}$$

then $s^{\mathfrak{r}}$ is given by

$$s^{\mathfrak{r}} = \mathcal{C}(\alpha\bar{w}/w) - \mathcal{R}(\alpha\bar{w}/w) \tag{5.12}$$

while $s^{\mathfrak{i}}$ is given by

$$s^{\mathfrak{i}} = \mathcal{C}(\alpha\bar{w}/w) + \mathcal{R}(\alpha\bar{w}/w). \tag{5.13}$$

Indeed, it is easy to check that $\mathcal{R}(\alpha\bar{w}/w)$ is a holomorphic function in $W^{1,r}(\mathbb{D})$ having zero mean on $\mathbb{T}$ and assuming conjugate values to $-\mathcal{C}(\alpha\bar{w}/w)$ there.

From (5.10) which is valid both for $s^{\mathfrak{r}}$ and $s^{\mathfrak{i}}$ we get that if $r > 2$ then

$$\|e^{\pm s^{\mathfrak{r}}}\|_{W^{1,r}(\mathbb{D})} \le C(r, \|\alpha\|_{L^r(\mathbb{D})}) \text{ and } \|e^{\pm s^{\mathfrak{i}}}\|_{W^{1,r}(\mathbb{D})} \le C(r, \|\alpha\|_{L^r(\mathbb{D})}). \tag{5.14}$$

[8] Thus, it would be more appropriate to call $G^p_\alpha(\mathbb{D})$ a pseudo-holomorphic Smirnov space.

[9] When $r = 2$, w may fail to satisfy condition (5.1) even though it meets (3.2) and (5.9). The problem lies with *small* values of r, as w needs not be locally bounded on $\mathbb{D}$.

For $r = 2$ and for $1 < q < 2$, we only deduce from (5.10) and Proposition 8.4 that

$$\|e^{\pm s^{\mathfrak{r}}}\|_{W^{1,q}(\mathbb{D})} \leq C(q, \|\alpha\|_{L^2(\Omega)}) \text{ and } \|e^{\pm s^{\mathfrak{i}}}\|_{W^{1,q}(\mathbb{D})} \leq C(q, \|\alpha\|_{L^2(\Omega)}). \quad (5.15)$$

- When $r > 2$, we conclude from (5.14) and the Sobolev embedding theorem that $e^{\pm s^{\mathfrak{r}}}$ and $e^{\pm s^{\mathfrak{i}}}$ are continuous and bounded independently of w on $\overline{\mathbb{D}}$. Hence, w belongs to $G^p_\alpha(\mathbb{D})$ if and only if $F^{\mathfrak{r}}$ or $F^{\mathfrak{i}}$ lies in H^p (in which case both do). This way $G^p_\alpha(\mathbb{D})$ inherits many properties of H^p. In particular, each $w \in G^p_\alpha(\mathbb{D})$ has a nontangential limit a.e. on $\mathbb{T}$, denoted again by w or $w_{\mathbb{T}}$ for emphasis, which is also the limit of w_ρ as $\rho \to 1^-$ in $L^p(\mathbb{T})$. Moreover, $\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}$ is a norm equivalent to (5.2) on $G^p_\alpha(\mathbb{D})$, and we might as well have used (5.1) to define the latter. Also, from Theorem 4.1, we infer that condition (5.5) characterizes the zeros of non identically vanishing functions in $G^p_\alpha(\mathbb{D})$[10].
- If $r = 2$, all we conclude *a priori* from (5.15), Lemma 8.7, and Hölder's inequality is that $F^{\mathfrak{r}}$ and $F^{\mathfrak{i}}$ belong to $\cap_{1\leq\ell<p} H^\ell$ if $w \in G^p_\alpha(\mathbb{D})$. In the other direction, $w \in \cap_{1\leq\ell<p} G^p_\alpha(\mathbb{D})$ if $F^{\mathfrak{r}}$ or $F^{\mathfrak{i}}$ lies in H^p. To clarify the matter, one should realize that factorizations $w = e^{s^{\mathfrak{r}}} F^{\mathfrak{r}}$ and $w = e^{s^{\mathfrak{i}}} F^{\mathfrak{i}}$ no longer play equivalent roles. For it may happen that $w \in G^p_\alpha(\mathbb{D})$ and $F^{\mathfrak{r}} \notin H^p$. In fact, if we let

$$w(z) := \frac{1}{\log(3/|z-1|)\,(z-1)^{1/p}}, \qquad z \in \mathbb{D}, \quad (5.16)$$

we get that

$$\Big|\frac{\bar\partial w(z)}{\overline{w(z)}}\Big| = \big(2|z-1|\log(3/|z-1|)\big)^{-1};$$

hence, $w \in G^p_\alpha(\mathbb{D})$ with $\alpha := \bar\partial w/\bar w \in L^2(\mathbb{D})$, but the factorization

$$w(z) = e^{\log\log(3/|z-1|) - a}\; e^a (z-1)^{-1/p}, \qquad a := \int_{\mathbb{T}} \log\log(3/|z-1|)\, d\Lambda(z),$$

is such that $F^{\mathfrak{r}} = e^a(z-1)^{-1/p} \notin H^p$.

On the other hand, $w \in G^p_\alpha(\mathbb{D})$ if and only if $F^{\mathfrak{i}} \in H^p$. Assume indeed that $0 \not\equiv w \in G^p_\alpha$. Since $F^{\mathfrak{i}} \in H^\ell$ for $1 \leq \ell < p$ and $e^{s^{\mathfrak{i}}_\rho}$ converges to $e^{s^{\mathfrak{i}}}$ in $W^{1,q}(\mathbb{D})$ for all $q \in [1,2)$ by Proposition 8.4, it follows from Lemma 8.7 and Hölder's inequality that $\mathrm{tr}_{\mathbb{T}} w_\rho$ converges as $\rho \to 1^-$ to $e^{\mathrm{tr}_{\mathbb{T}} s^{\mathfrak{i}}} F^{\mathfrak{i}}_{|\mathbb{T}}$ in $L^\lambda(\mathbb{T})$, for every $\lambda \in [1, p)$. Moreover, as $\mathrm{tr}_{\mathbb{T}} w_\rho$ remains bounded in $L^p(\mathbb{T})$ by (5.9), it converges weakly there to $e^{\mathrm{tr}_{\mathbb{T}} s^{\mathfrak{i}}} F^{\mathfrak{i}}_{|\mathbb{T}}$ when $\rho \to 1^-$, since this is the only weak limit possible granted the convergence of $\mathrm{tr}_{\mathbb{T}} w_\rho$ in $L^\lambda(\mathbb{T})$. In particular, $e^{\mathrm{tr}_{\mathbb{T}} s^{\mathfrak{i}}} F^{\mathfrak{i}}_{|\mathbb{T}} \in$

[10]When $r = 2$, this property has no simple analog since w is only defined $B_{1,2}$-quasi-everywhere.

$L^p(\mathbb{T})$, and since $|e^{\mathrm{tr}_{\mathbb{T}} s^{\mathfrak{i}}}| \equiv 1$ we conclude that $F^{\mathfrak{i}}_{|\mathbb{T}} \in L^p(\mathbb{T})$, and hence $F^{\mathfrak{i}} \in H^p$. Conversely, if $F^{\mathfrak{i}} \in H^p$, then w satisfies (5.9) by Corollary 8.11.

The fact that $\mathrm{tr}_{\mathbb{T}} w_\rho$ converges strongly in $L^p(\mathbb{T})$ as $\rho \to 1^-$, and not just weakly as we showed above, is a part of the next theorem, whose assertion (iii) is new even for $r > 2$.

Theorem 5.1. *Let $\alpha \in L^r(\mathbb{D})$ with $2 \le r < \infty$ and fix $p \in (1, \infty)$.*

(i) *Each $w \in G^p_\alpha(\mathbb{D})$ has a trace $w_{\mathbb{T}}$ on $\mathbb{T}$ given by*

$$w_{\mathbb{T}} := \lim_{\rho \to 1^-} tr_{\mathbb{T}} w_\rho \quad in\ L^p(\mathbb{T}). \tag{5.17}$$

When $r > 2$, the function $w_{\mathbb{T}}$ is also the non-tangential limit of w a.e. on $\mathbb{T}$.

(ii) *For some $C > 0$ depending only on $|\alpha|$ and p we have*

$$\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})} \le \|w\|_{G^p_\alpha(\mathbb{D})} \le C \|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}, \tag{5.18}$$

and $G^p_\alpha(\mathbb{D})$ is a real Banach space on which $\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}$ is a norm equivalent to (5.9).

(iii) *The map $w \mapsto F^{\mathfrak{i}}$ is a homeomorphism from $G^p_\alpha(\mathbb{D})$ onto H^p. When $r > 2$, the map $w \mapsto F^{\mathfrak{r}}$ is also such a homeomorphism.*

(iv) *If $w \in G^p_\alpha(\mathbb{D})$ and $w_{\mathbb{T}} \in L^q(\mathbb{T})$ for some $q \in (1, \infty)$, then $w \in G^q_\alpha(\mathbb{D})$. A non-negative function $h \in L^p(\mathbb{T})$ is such that $h = |w_{\mathbb{T}}|$ for some nonzero $w \in G^p_\alpha(\mathbb{D})$ if and only if $\log h \in L^1(\mathbb{T})$.*

Proof. If $r > 2$, all the properties except (iii) follow from their H^p-analogs *via* the continuity and uniform boundedness of $e^{\pm s^{\mathfrak{r}}}$ or $e^{\pm s^{\mathfrak{i}}}$ discussed earlier in this section, see also [35, 27, 5, 4].

We postpone the proof of (iii) and assume for now that $r = 2$. Take $w \in G^p_\alpha(\mathbb{D}) \setminus \{0\}$ and put $s = s^{\mathfrak{i}}$, $F = F^{\mathfrak{i}}$ to simplify notation. To prove (5.17) we need to verify that given a sequence $(\rho_n) \subset (0, 1)$ tending to 1, one can extract a subsequence (ρ_{n_k}) such that $\mathrm{tr}_{\mathbb{T}}\, w_{\rho_{n_k}}$ converges to $e^{\mathrm{tr}_{\mathbb{T}}\, s} F_{|\mathbb{T}}$ in $L^p(\mathbb{T})$.

Since s_ρ converges to s in $W^{1,2}(\mathbb{D})$, we get from Lemma 8.7 that $\mathrm{tr}_{\mathbb{T}} s_\rho$ converges to $\mathrm{tr}_{\mathbb{T}} s$ in $L^\ell(T)$, as $\rho \to 1^-$, for all $\ell \in [1, \infty)$. Moreover, as we pointed out before the theorem, $F \in H^p$, and hence $(F_\rho)_{|\mathbb{T}}$ converges to $F_{|\mathbb{T}}$ in $L^p(\mathbb{T})$. Extracting if necessary a subsequence from (ρ_n) (still denoted by (ρ_n)), we can assume that $\mathrm{tr}_{\mathbb{T}} s_{\rho_n}$ (resp. $(F_{\rho_n})_{|\mathbb{T}}$) also converges pointwise a.e. on $\mathbb{T}$ to $\mathrm{tr}_{\mathbb{T}} s$ (resp. $F_{|\mathbb{T}}$). Now, Corollary 8.11, applied with ps instead of s, implies that $\|e^{ps_\rho} F_\rho\|_{L^p(\mathbb{T})}$ is uniformly bounded as $\rho \to 1^-$. Therefore, as the weak limit coincides with the pointwise limit when both exist by Egoroff's theorem, there is a subsequence (ρ_{n_k}) such that $\left(e^{p\mathrm{Re}\, s_{\rho_{n_k}}} \left|F_{\rho_{n_k}}\right|\right)$ converges weakly to $|F|$ in $L^p(\mathbb{T})$. Letting $1/p + 1/p' = 1$, this means that for each test function $\Theta \in L^{p'}(\mathbb{T})$ we have:

$$\lim_{k\to\infty}\Big|\int_{\mathbb{T}} e^{p\mathrm{Re}\, s_{\rho_{n_k}}(z)}\big|F_{\rho_{n_k}}(z)\big|\Theta(z)\,|dz| - \int_{\mathbb{T}}|F(z)|\Theta(z)\,|dz|\Big| = 0. \tag{5.19}$$

Set $\Theta_k = \big|F_{\rho_{n_k}}\big|^{p-1} \in L^{p'}(\mathbb{T})$. Convergence of $(F_\rho)_{|\mathbb{T}}$ to $F_{|\mathbb{T}}$ in $L^p(\mathbb{T})$ implies easily that Θ_k converges to $|F|^{p-1}$ in $L^{p'}(\mathbb{T})$. In view of (5.19), this yields

$$\lim_{k\to\infty}\Big|\int_{\mathbb{T}} e^{p\mathrm{Re}\, s_{\rho_{n_k}}(z)}\big|F_{\rho_{n_k}}(z)\big|^p\,|dz| - \int_{\mathbb{T}}|F(z)|^p\,|dz|\Big| = 0.$$

Therefore, $\big\|\mathrm{tr}_{\mathbb{T}} w_{\rho_{n_k}}\big\|_{L^p(\mathbb{T})} = \big\|e^{\mathrm{Re}\, s_{\rho_{n_k}}}F_{\rho_{n_k}}\big\|_{L^p(\mathbb{T})}$ tends to $\|e^{\mathrm{tr}_{\mathbb{T}} s}F_{|\mathbb{T}}\|_{L^p(\mathbb{T})} = \big\|F_{|\mathbb{T}}\big\|_{L^p(\mathbb{T})}$ when $k\to\infty$. However, from the discussion before the theorem, we know that $\mathrm{tr}_{\mathbb{T}} w_{\rho_{n_k}}$ converges weakly to $e^{\mathrm{tr}_{\mathbb{T}} s}F_{|\mathbb{T}}$ in $L^p(\mathbb{T})$, so by uniform convexity of $L^p(\mathbb{T})$ the convergence must in fact be strong because, as we just showed, the norm of the weak limit is the limit of the norms [9, Theorem 3.32]. This proves (i).

Next, we observe by the absolute continuity of $|\alpha|^2 dm$ that for every $\varepsilon > 0$ there is $\omega(\varepsilon) > 0$ for which $\|\alpha\|_{L^2(Q_{\omega(\varepsilon)}\cap\mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a cube of sidelength $\omega(\varepsilon)$. Thus, in view of (5.13), we can apply Proposition 8.5 to $\beta := \alpha\bar{w}/w$ and obtain a strictly positive function $\widetilde{\omega}$ on $\mathbb{R}^+$, *depending only on* $|\alpha|$, such that

$$\|\partial s\|_{L^2(Q_{\widetilde{\omega}(\eta)}\cap\mathbb{D})} + \|\bar{\partial} s\|_{L^2(Q_{\widetilde{\omega}(\eta)}\cap\mathbb{D})} < \eta \tag{5.20}$$

as soon as $Q_{\widetilde{\omega}(\eta)}$ is a cube of sidelength $\widetilde{\omega}(\eta)$. A fortiori, (5.20) holds with $\mathrm{Re}\, s$ instead of s. Now, picking any $\gamma \in (0,\pi/2)$ and recalling that $|w_{\mathbb{T}}| = |F_{|\mathbb{T}}|$ because $\mathrm{Re}\, s \in W^{1,2}_{0,\mathbb{R}}(\mathbb{D})$, we deduce from (5.3) and Theorem 8.10 applied to $f = \mathrm{Re}\, s$ and $g = e^{i\mathrm{Im}\, s}F$ that the right inequality in (5.18) holds. In another connection, the left inequality is obvious from (5.17). To show that $G^p_\alpha(\mathbb{D})$ is a Banach space, consider a sequence $(w_n) \subset G^p_\alpha(\mathbb{D})$ converging in $\mathcal{H}^p$ to some function w. We must prove that $w \in G^p_\alpha$. We can assume $w \not\equiv 0$, therefore $w_n \not\equiv 0$ for n large enough. Convergence in $\mathcal{H}^p$ being stronger than in $L^p(\mathbb{D})$, *a fortiori* w_n converges to w in $\mathcal{D}'(\mathbb{D})$ and, moreover, some subsequence, again denoted by w_n, converges pointwise a.e. to w. Besides, if we write $w_n = e^{s_n}F_n$ where we mean as before that $s_n = s^{\mathrm{i}}_n$ and $F_n = F^{\mathrm{i}}_n$, we get from (5.15) that the sequence (e^{s_n}) is bounded in $W^{1,q}(\mathbb{D})$ for each $q \in (1,2)$. Therefore, by the Sobolev embedding theorem, (e^{s_n}) is bounded in $L^\ell(\mathbb{D})$ for each $\ell \in [1,\infty)$. In addition, since $|e^{\mathrm{tr}_{\mathbb{T}} s_n}| \equiv 1$, it follows from (5.18) that (F_n) is bounded in H^p, hence also in $L^\ell(\mathbb{D})$ for each $\ell \in (1,2p)$ by (5.4). Altogether, by Hölder's inequality, (w_n) is bounded in $L^\gamma(\mathbb{D})$ for some $\gamma > 2$. Consequently, some subsequence converges weakly in $L^\gamma(\mathbb{D})$, and since the weak limit coincides with the pointwise limit, if it exists, we conclude that the weak limit is w. In particular, $w \in L^\gamma(\mathbb{D})$. Moreover, by Hölder's inequality, $(\alpha\bar{w}_n)$ is bounded in $L^t(\mathbb{D})$ for some $t > 1$, and arguing as before we get that some subsequence (again denoted by $(\alpha\bar{w}_n)$) converges weakly to $\alpha\bar{w}$ there. Thus, passing to the distributional limit in the relation $\bar{\partial} w_n = \alpha \bar{w}_n$, we obtain (3.2) so that $w \in G^p_\alpha(\mathbb{D})$. This proves (ii).

We already know from Theorem 4.1 and the discussion before Theorem 5.1 that the map $w \mapsto F^{\mathfrak{i}}$ is bijective from $G^p_\alpha(\mathbb{D})$ to H^p. Since $|w_{\mathbb{T}}| = |F^{\mathfrak{i}}{}_{|\mathbb{T}}|$, it is clear from (5.18) that this map and its inverse are continuous at 0. Let now w_n converge to $w \not\equiv 0$ in $G^p_\alpha(\mathbb{D})$ and write $w_n = e^{s^{\mathfrak{i}}_n} F^{\mathfrak{i}}_n$, $w = e^{s^{\mathfrak{i}}} F^{\mathfrak{i}}$. We claim that some subsequence of $F^{\mathfrak{i}}_n$ converges to $F^{\mathfrak{i}}$ in H^p and this establishes continuity of the map at every point. As $F^{\mathfrak{i}}_{n|\mathbb{T}}$ is bounded in $L^p(\mathbb{T})$ by (5.18), some subsequence converges weakly there to $\Phi_{|\mathbb{T}}$ for some $\Phi \in H^p$. Thus, replacing w_n by a subsequence (again denoted by w_n), we may assume by the Cauchy formula that $F^{\mathfrak{i}}_n$ converges locally uniformly to Φ on $\mathbb{D}$. Note that $\Phi \not\equiv 0$ for otherwise, in view of (5.15), we would have that w_n converges to the zero distribution, contradicting that $w \not\equiv 0$. In particular, $\alpha \bar{F}^{\mathfrak{i}}_n / F^{\mathfrak{i}}_n$ converges in $L^2(\mathbb{D})$ to $\alpha \bar{\Phi}/\Phi$ by the dominated convergence theorem. Since $\bar{\partial} s^{\mathfrak{i}}_n = \alpha \bar{F}^{\mathfrak{i}}_n / F^{\mathfrak{i}}_n \exp(-2i \mathrm{Im}\, s^{\mathfrak{i}}_n)$ by Lemma 4.3 and $\|s^{\mathfrak{i}}_n\|_{W^{1,2}(\mathbb{D})}$ is uniformly bounded by (5.10), we can argue as we did after (4.8) (put $\alpha_n \equiv \alpha \bar{F}^{\mathfrak{i}}_n / F^{\mathfrak{i}}_n$ and $\theta_0 = 0$ in the discussion there) to the effect that a subsequence, again denoted by $s^{\mathfrak{i}}_n$, converges to some $\sigma \in W^{1,2}(\mathbb{D})$ such that $\mathrm{Re}\, \mathrm{tr}_{\mathbb{T}} \sigma = 0$ and $\int_{\mathbb{T}} \sigma = 0$, both a.e. and in $W^{1,2}(\mathbb{D})$. Refining the sequence if necessary, we can further assume that w_n converges a.e. to w. Taking pointwise limits we get $w = e^\sigma \Phi$, hence $\sigma = s^{\mathfrak{i}}$ and $\Phi = F^{\mathfrak{i}}$ by the uniqueness part of Corollary 4.2. Thus, $F^{\mathfrak{i}}_{|\mathbb{T}}$ is the weak limit of $F^{\mathfrak{i}}_{n|\mathbb{T}}$, and since $\|F^{\mathfrak{i}}\|_{L^p(\mathbb{T})} = \|w\|_{L^p(\mathbb{T})}$ is the limit of $\|F^{\mathfrak{i}}_n\|_{L^p(\mathbb{T})} = \|w_n\|_{L^p(\mathbb{T})}$, the convergence in fact takes place in $L^p(\mathbb{T})$, thereby proving the claim. Conversely, let $w_n = e^{s^{\mathfrak{i}}_n} F^{\mathfrak{i}}_n$ be a sequence in $G^p_\alpha(\mathbb{D})$ such that $F^{\mathfrak{i}}_n$ converges to $\Phi \not\equiv 0$ in H^p. By Corollary 4.2, $\|s^{\mathfrak{i}}_n\|_{W^{1,2}(\mathbb{D})}$ is bounded uniformly in n, and, as before, a subsequence, again denoted by $s^{\mathfrak{i}}_n$, converges in $W^{1,2}(\mathbb{D})$ to some σ such that $\mathrm{Re}\, \mathrm{tr}_{\mathbb{T}} \sigma = 0$ and $\int_{\mathbb{T}} \sigma = 0$. Refining the sequence if necessary, we can assume in view of the trace theorem that $\mathrm{tr}_{\mathbb{T}} s^{\mathfrak{i}}_n$ converges pointwise a.e. on $\mathbb{T}$ to $\mathrm{tr}_{\mathbb{T}} \sigma$. By the dominated convergence, $(w_n)_{\mathbb{T}}$ tends to $e^{\mathrm{tr}_{\mathbb{T}} \sigma} \Phi_{|\mathbb{T}}$ in $L^p(\mathbb{T})$. Using (5.18) we obtain that w_n converges in $G^p_\alpha(\mathbb{D})$ to some $w = e^{s^{\mathfrak{i}}} F^{\mathfrak{i}}$, and by the continuity proven before we conclude that $\Phi = F^{\mathfrak{i}}$. This proves (iii) when $r = 2$. That both $w \mapsto F^{\mathfrak{i}}$ and $w \mapsto F^{\mathfrak{r}}$ are homeomorphisms when $r > 2$ is similar but easier because then $s \to e^s$ is bounded and continuous from $W^{1,r}(\mathbb{D})$ into $W^{1,r}(\mathbb{D}) \subset L^\infty(\mathbb{D})$.

Finally, (iv) follows from the corresponding properties of H^p functions, the fact that $w \in G^p_\alpha$ if and only it $F^{\mathfrak{i}} \in H^p$, and the equality $|w_{\mathbb{T}}| = |F^{\mathfrak{i}}_{|\mathbb{T}}|$. $\square$

Remark 5.2. *When $r = 2$, $w_{\mathbb{T}}$ in Theorem 5.1 is not necessarily the nontangential limit of w. Indeed, if $(z_n) \subset \mathbb{D}$ is nontangentially dense on $\mathbb{T}$, then $s(z) := \sum_n 2^{-n} \log\log 2/|z - z_n|$ lies in $W^{1,2}(\mathbb{D})$ so that $e^s \in G^p_\alpha(\mathbb{D})$ for all $p \in (1,\infty)$ with $\alpha := \bar{\partial} s$ by Lemma 8.7. Yet, e^s is not even nontangentially bounded at a single $\xi \in \mathbb{T}$.*

6. The generalized conjugation operator

The M. Riesz theorem may be rephrased as follows. Given $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ with $p \in (1,\infty)$, the problem of finding a holomorphic function f in $\mathbb{D}$ such that $\mathrm{Re}\,\mathrm{tr}_{\mathbb{T}} f_\rho$ tends to ψ in $L^p(\mathbb{T})$ has a solution in H^p which is unique up to an additive imaginary constant. In fact, if we normalize it to have mean $\int_{\mathbb{T}} \psi/2\pi + ic$ on $\mathbb{T}$, then $f_{|\mathbb{T}} = \psi + i\tilde{\psi} + ic$ and we have $\|f\|_{H^p} \leq C(\|\psi\|_{L^p(\mathbb{T})} + |c|)$ for some C depending only on p.
The corresponding problem for pseudo-holomorphic functions, *i.e.* for solutions to (3.2) when $\alpha \not\equiv 0$, turns out to have a similar answer in G^p_α as long as $\alpha \in L^r(\mathbb{D})$ for some $r \geq 2$. When $r > 2$ this was essentially proven in [27], see also [5] and [4]. More precisely:

Theorem 6.1 ([27],[5],[4])**.** *Let $\alpha \in L^r(\mathbb{D})$ with $2 < r \leq \infty$ and $1 < p < \infty$. For every $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ and $c \in \mathbb{R}$ there is a unique $w \in G^p_\alpha(\mathbb{D})$ such that $Re\, w_{\mathbb{T}} = \psi$ and $\int_{\mathbb{T}} Im\, w_{\mathbb{T}} = c$. Moreover, $\|w\|_{G^p_\alpha(\mathbb{D})} \leq C(\|\psi\|_{L^p(\mathbb{T})} + |c|)$, where C depends only on p and r.*

Theorem 6.1 generalizes the M. Riesz theorem: for every $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ and $c \in \mathbb{R}$ there is a unique $\psi^\sharp_c \in L^p_{\mathbb{R}}(\mathbb{T})$ (a generalized conjugate of ψ) such that $\int_{\mathbb{T}} \psi^\sharp_c = c$ and $\psi + i\psi^\sharp_c = w_{\mathbb{T}}$ for some $w \in G^p_\alpha(\mathbb{D})$. Moreover, $\|\psi^\sharp\|_{L^p(\mathbb{T})} \leq C(\|\psi\|_{L^p(\mathbb{T})} + |c|)$. The theorem below extends this result to the case $r = 2$ where solutions to (3.2) may be locally unbounded.

Theorem 6.2. *Let $\alpha \in L^2(\mathbb{D})$ and $1 < p < \infty$. For every $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ and $c \in \mathbb{R}$ there is a unique $w \in G^p_\alpha(\mathbb{D})$ such that $Re\, w_{\mathbb{T}} = \psi$ and $\int_{\mathbb{T}} Im\, w_{\mathbb{T}} = c$. Moreover,*

$$\|w\|_{G^p_\alpha(\mathbb{D})} \leq C(\|\psi\|_{L^p(\mathbb{T})} + |c|), \tag{6.1}$$

where C depends only on p and $|\alpha|$.

Proof. We first show existence. Assume that ψ and c are not both zero; otherwise $w \equiv 0$ will do.
Let (α_n) be a sequence of functions in $L^\infty(\mathbb{D})$ converging to α in $L^2(\mathbb{D})$. By Theorem 6.1, for every n there exists $w_n \in G^p_{\alpha_n}(\mathbb{D})$ such that $\mathrm{Re}\, w_{n|\mathbb{T}} = \psi$ and $\int_{\mathbb{T}} \mathrm{Im}\, w_n = c$. Notations being as in Section 5.2, let us write $w_n = e^{s^{\mathfrak{r}}_n} F^{\mathfrak{r}}_n$ where $s^{\mathfrak{r}}_n \in W^{1,2}(\mathbb{D})$ is real with zero mean on $\mathbb{T}$ while $F^{\mathfrak{r}}_n \in H^p$. Below, we drop the superscript $\mathfrak{r}$ for simplicity.
It follows from (5.10) that $\|s_n\|_{W^{1,2}(\mathbb{D})} \leq C_0\|\alpha_n\|_{L^2(\mathbb{D})}$ for some absolute constant C_0, hence $\|s_n\|_{W^{1,2}(\mathbb{D})}$ is bounded uniformly in n. In view of the Rellich–Kondrachov theorem, we can find a subsequence, again denoted by (s_n), converging to some function s both pointwise on $\mathbb{D}$ and in $L^\ell(\mathbb{D})$ for all $\ell \in [1,\infty)$. By the trace theorem and the non integral version of the Rellich–Kondrachov theorem, we may further assume that $\mathrm{tr}_{\mathbb{T}} s_n$ converges to some function h both pointwise a.e. on $\mathbb{T}$ and in $L^\ell_{\mathbb{R}}(\mathbb{T})$. Moreover, convergence of α_n to α in $L^2(\mathbb{D})$ entails, because of (5.15), that $e^{\pm s_n}$ are bounded in $W^{1,q}(\mathbb{D})$, independently of n and ψ, for each $q \in [1,2)$. So,

invoking again the trace and the Rellich–Kondrachov theorems, we may assume upon refining s_n further that $e^{\pm \mathrm{tr}_{\mathbb{T}} s_n}$ converges to their pointwise limits $e^{\pm h}$ in $L^\ell(\mathbb{T})$, for all $\ell \in [1,\infty)$.
Thus, by Hölder's inequality, $\mathrm{Re}\,(F_n)_{|\mathbb{T}} = e^{-\mathrm{tr}_{\mathbb{T}} s_n}\psi$ converges to $e^{-h}\psi$ in $L^\lambda(\mathbb{T})$ for any $\lambda \in [1,p)$. Continuity of the conjugate operator now implies that $\widetilde{\mathrm{Re}\,(F_n)}_{|\mathbb{T}}$ in turn converges to $\widetilde{e^{-h}\psi}$ in $L^\lambda(\mathbb{T})$. Since $\int_{\mathbb{T}} \mathrm{Im}\, w_n = c$, we see by inspection that $\mathrm{Im}\,(F_n)_{|\mathbb{T}} = \widetilde{\mathrm{Re}\,(F_n)}_{|\mathbb{T}} + c_n$ where the constant c_n is such that

$$c_n \int_{\mathbb{T}} e^{\mathrm{tr}_{\mathbb{T}} s_n} + \int_{\mathbb{T}} e^{\mathrm{tr}_{\mathbb{T}} s_n} \widetilde{\mathrm{Re}\,(F_n)}_{|\mathbb{T}} = c. \tag{6.2}$$

The first integral in (6.2) converges to $\int_{\mathbb{T}} e^h > 0$, and the second integral converges to $\int_{\mathbb{T}} e^h\, \widetilde{e^{-h}\psi}$ by Hölder's inequality. Therefore, (c_n) converges to

$$c_0 := \left(c - \int_{\mathbb{T}} e^h\, \widetilde{e^{-h}\psi}\right) \Big/ \int_{\mathbb{T}} e^h, \tag{6.3}$$

and subsequently $(F_n)_{|\mathbb{T}}$ converges to

$$F_{\mathbb{T}} := e^{-h}\psi + i\, \widetilde{e^{-h}\psi} + ic_0 \tag{6.4}$$

in $L^\lambda(\mathbb{T})$, for all $\lambda \in [1,p)$. Thus, F_n converges in H^λ to F, the Poisson integral of $F_{\mathbb{T}}$. Note that F is not identically zero; otherwise, $\psi \equiv 0$ and $c = 0$, contrary to our initial assumption.
The above argument and the dominated convergence theorem give us that $\alpha_n e^{-2i\mathrm{Im}\, s_n}\bar{F}_n/F_n$ converges to $\alpha e^{-2i\mathrm{Im}\, s}\bar{F}/F$ in $L^2(\mathbb{D})$. Next, $\bar\partial s_n = \alpha_n \times$ $\times e^{-2i\mathrm{Im}\, s_n}\bar{F}_n/F_n$. By Lemma 4.3, applying (2.13) with $A = s_n - s_m$, $a = \bar\partial s_n - \bar\partial s_m$, $\theta_0 = -\pi/2$, $\psi \equiv 0$, and $\lambda = 0$, we conclude that (s_n) is a Cauchy sequence in $W^{1,2}(\mathbb{D})$ which must therefore converge to s. Hence, $s \in W^{1,2}(\mathbb{D})$ and $h = \mathrm{tr}_{\mathbb{T}} s$. Since we get in the limit that $\bar\partial s = (\alpha \bar{F}/F)e^{-2i\mathrm{Im}\, s}$, we see from Lemma 4.3 that $w := e^s F$ satisfies (3.2). Moreover, if we write $w = e^{s^{\mathfrak{r}}} F^{\mathfrak{r}}$ in the notation of Section 5.2, we find that $s^{\mathfrak{r}} = s$ and $F^{\mathfrak{r}} = F$ because s inherits from s_n the properties $\mathrm{Im}\,\mathrm{tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \mathrm{Re}\, s = 0$. As $F \in H^\lambda$ for all $\lambda \in [1,p)$, we further deduce from the discussion before (5.16) that $w \in G_\alpha^\lambda$ for all such λ. By inspection of (6.4) we get

$$\begin{aligned} w_{\mathbb{T}} = e^{\mathrm{tr}_{\mathbb{T}} s} F_{|\mathbb{T}} &= e^{\mathrm{tr}_{\mathbb{T}} s}\big(e^{-\mathrm{tr}_{\mathbb{T}} s}\psi + i\, \widetilde{e^{-\mathrm{tr}_{\mathbb{T}} s}\psi} + ic_0\big) \\ &= \psi + i\big(e^{\mathrm{tr}_{\mathbb{T}} s}\, \widetilde{e^{-\mathrm{tr}_{\mathbb{T}} s}\psi} + e^{\mathrm{tr}_{\mathbb{T}} s} c_0\big), \end{aligned} \tag{6.5}$$

where we use the fact that $h = \mathrm{tr}_{\mathbb{T}} s$ is real-valued. In particular, (6.5) entails that $\mathrm{Re}\, w_{\mathbb{T}} = \psi$.
To show that $w \in G_\alpha^p(\mathbb{D})$, we must prove in view of Theorem 5.1 that $w_{\mathbb{T}} \in L^p(\mathbb{T})$. To do this, note that $\psi \in L^p(\mathbb{T})$ by assumption and that $e^{\mathrm{tr}_{\mathbb{T}} s} c_0 \in L^p(\mathbb{T})$ by the trace and Sobolev embedding theorems. Furthermore, $p\,\mathrm{tr}_{\mathbb{T}} s \in W^{1/2,2}(\mathbb{T}) \subset VMO(\mathbb{T})$ by (8.13). By Lemma 8.2, $e^{p\mathrm{tr}_{\mathbb{T}} s}$ satisfies condition

A_p. Thus, using (5.8), we obtain

$$\|e^{\mathrm{tr}_{\mathbb{T}}s}\,\widetilde{e^{-\mathrm{tr}_{\mathbb{T}}s}\psi}\|_p^p \le C''\,\|\psi\|_p^p, \qquad C'' = C''(\{e^{p\,\mathrm{tr}_{\mathbb{T}}s}\}_{A_p}); \tag{6.6}$$

in view of (6.5) we have $w_{\mathbb{T}} \in L^p(\mathbb{T})$. This gives the existence part of Theorem 6.2.

As for uniqueness, let $w_1, w_2 \in G^p_\alpha(\mathbb{D})$ be two solutions. Set $v := w_1 - w_2 \in G^p_\alpha(\mathbb{D})$, so that $\mathrm{Re}\, v_{\mathbb{T}} = 0$, $\int_{\mathbb{T}} \mathrm{Im}\, v_{\mathbb{T}} = 0$. If we write $v = e^{\sigma^{\mathfrak{r}}}\Phi^{\mathfrak{r}}$, we observe that $\mathrm{Re}\,(\Phi^{\mathfrak{r}})_{|\mathbb{T}} \equiv 0$, and hence the H^λ function $\Phi^{\mathfrak{r}}$, $1 \le \lambda < p$, is a pure imaginary constant, say ζ. Thus, $v = \zeta e^s$ and the relations $\int_{\mathbb{T}} \mathrm{Im}\, v_{\mathbb{T}} = 0$, $\int_{\mathbb{T}} e^{\mathrm{tr}_{\mathbb{T}}\sigma} > 0$ give us $\zeta = 0$ so that $v = 0$, as desired.

Finally, we verify (6.1). By (5.18), it suffices to prove that

$$\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})} \le C(\|\psi\|_{L^p(\mathbb{T})} + |c|),$$

where C depends only on p. By (6.6), (6.5), (6.3) and Hölder's inequality, we need only establish that $\|e^{\mathrm{tr}_{\mathbb{T}}s}\|_{L^p(\mathbb{T})}$, $\{e^{p\,\mathrm{tr}_{\mathbb{T}}s}\}_{A_p}$, and $1/\int_{\mathbb{T}} e^{\mathrm{tr}_{\mathbb{T}}s}$ are bounded from above independently of ψ. We pointed out earlier in the proof that e^{s_n} are bounded in $W^{1,q}(\mathbb{D})$, independently of n and ψ, for each $q \in [1, 2)$. Since s_n tends to s in $W^{1,2}(\mathbb{D})$, boundedness of $\|e^{\mathrm{tr}_{\mathbb{T}}s}\|_{L^p(\mathbb{T})}$ follows from Proposition 8.4 and the (non-integral version of) the Sobolev embedding theorem. Next, (5.10) yields that $\|s\|_{W^{1,2}(\mathbb{D})} \le C_0\|\alpha\|_{L^2(\mathbb{D})}$ for some absolute constant C_0. Thus, using concavity of log, the Schwarz inequality, and the trace theorem, we get for some absolute constant C_1 that

$$\begin{aligned}\log\left(\frac{1}{2\pi}\int_{\mathbb{T}} e^{s(\xi)}\,|d\xi|\right) &\ge \frac{1}{2\pi}\int_{\mathbb{T}} s(\xi)\,|d\xi| \\ &\ge -\|s\|_{L^2(\mathbb{T})} \ge -C_1\|s\|_{W^{1,2}(\mathbb{D})} \ge -C_0C_1\|\alpha\|_{L^2(\mathbb{D})},\end{aligned}$$

showing that $\int_{\mathbb{T}} e^{\mathrm{tr}_{\mathbb{T}}s} \ge \exp\{-C_0C_1\|\alpha\|_{L^2(\mathbb{D})}\}$.

Finally, to majorize $\{e^{p\,\mathrm{tr}_{\mathbb{T}}s}\}_{A_p}$ independently of ψ, it suffices by Lemma 8.2 to prove that $M_{\mathrm{tr}_{\mathbb{T}}s}(J)$ (see definition (8.8)) can be made arbitrarily small as $\Lambda(J) \to 0$, uniformly with respect to ψ, as J ranges over open arcs on $\mathbb{T}$. Let ω be be a strictly positive function on $(0, +\infty)$ such that $\|\alpha\|_{L^2(Q_{\omega(\varepsilon)}\cap\mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a square of sidelength $\omega(\varepsilon)$. By (5.12) and Proposition 8.5, there is a strictly positive function $\widetilde{\omega}$ on $(0, +\infty)$, depending only on ω, such that (5.20) holds. Now, if $\Lambda(J) < 1$, it is elementary to check that $R(J, \Lambda(J))$ (*cf.* definition (8.29)) is contained in a square of sidelength $\Lambda(J)$. Therefore, if we pick $\Lambda(J) < \min\{1/2, \widetilde{\omega}(\eta)\}$, we deduce from (8.13) and Lemma 8.9 that $M_{\mathrm{tr}_{\mathbb{T}}s}(J) \le C_1\eta$, where C_1 is an absolute constant. This completes the proof of Theorem 6.2. □

7. Dirichlet problem for $\exp -W^{1,2}$ conductivity

The following connection between pseudo-holomorphic functions and conductivity equations is instrumental in [3] and was investigated in the context of pseudo-holomorphic Hardy spaces in [5, 4] when $r > 2$. We start by a 2-d isotropic conductivity equation with exp-Sobolev smooth coefficient:

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \qquad \sigma \geq 0, \quad \log \sigma \in W^{1,r}(\Omega), \quad r \in [2, \infty). \tag{7.1}$$

When $r > 2$, the assumption that $\log \sigma \in W^{1,r}(\Omega)$ simply means that $\sigma \in W^{1,r}(\Omega)$ and that $0 < c < \sigma$ (strict ellipticity). If $r = 2$, then σ lies in $W^{1,q}(\Omega)$ for all $q \in [1, 2)$ by Proposition 8.4, but it is not necessarily bounded away from zero nor infinity which makes this case particularly interesting because (7.1) may no longer be strictly elliptic.
Put $\nu := (1 - \sigma)/(1 + \sigma)$ and consider the conjugate Beltrami equation:

$$\overline{\partial} f = \nu \, \overline{\partial f} \quad \text{in } \Omega, \quad -1 \leq \nu \leq 1, \quad \operatorname{arctanh} \nu \in W^{1,r}(\Omega), \; r \in [2, \infty), \tag{7.2}$$

where the assumptions on ν correspond to those on σ given in (7.1). The fact that $\sigma \in W^{1,q}(\Omega)$ for all $q \in [1, 2)$ implies easily that the same holds for ν. If we restrict ourselves to solutions $f \in L^{\gamma}_{loc}(\Omega)$ for some $\gamma > r/(r - 1)$ and write $f = u + iv$ to separate the real and the imaginary parts, we find that (7.2) is equivalent to the generalized Cauchy–Riemann system:

$$\left\{ \begin{array}{l} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{array} \right. \tag{7.3}$$

whose compatibility condition is the conductivity equation (7.1). Hence, (7.2) is a means to rewrite (7.1) as a complex equation of the first order. Now, if we set

$$w := \frac{f - \nu \overline{f}}{\sqrt{1 - \nu^2}} = \sigma^{1/2} u + i \sigma^{-1/2} v, \qquad \alpha = \bar{\partial} \log \sigma^{1/2} \in L^r,$$

then a straightforward computation using (7.3) shows that (3.2) holds. Note that any constant c solves (7.2), the corresponding solution in (3.2) being $\sigma^{1/2} \mathrm{Re}\, c + i \sigma^{-1/2} \mathrm{Im}\, c$.
The preceding discussion makes the study of (7.2) essentially equivalent to that of (3.2), (7.1). In particular, Theorem 6.2 translates into the following result that seems to be the first to describe a class of non strictly elliptic equations with unbounded coefficients for which the Dirichlet problem is well-posed with (weighted) L^p-boundary data.

Theorem 7.1. *Let $\sigma \geq 0$ be such that $\log \sigma \in W^{1,2}(\mathbb{D})$, and fix $p \in (1, \infty)$. For every ψ such that $\psi \, tr_{\mathbb{T}} \sigma^{1/2} \in L^p(\mathbb{T})$, there exists a unique solution u to* (7.1) *such that*

$$\sup_{0<\rho<1} \left(\int_{\mathbb{T}_\rho} |u(\xi)|^p \, \sigma^{p/2}(\xi) |d\xi| \right)^{1/p} < +\infty \tag{7.4}$$

and $\lim_{\rho \to 1} tr_{\mathbb{T}}(u_\rho \sigma_\rho^{1/2}) = \psi \, tr_{\mathbb{T}} \sigma^{1/2}$ in $L^p(\mathbb{T})$. Moreover, the supremum in (7.4) *is less than $C \|\psi \sigma^{1/2}\|_{L^p(\mathbb{T})}$ for some $C = C(p, \sigma)$.*

8. Appendix

8.1. **Mean growth of Cauchy transforms.** In this subsection we prove estimate (2.15). First, we evaluate $\mathcal{C}_2(h)_{\mathbb{D}_R}$, the mean of $\mathcal{C}_2(h)$ over $\mathbb{D}_R$, when $h \in L^2(\mathbb{C})$ and $R \geq 1$. To this end, we use the following identity (see [2, Section 4.3.2]):

$$\mathcal{C}(\chi_{\mathbb{D}_R})(t) = \begin{cases} \bar{t} & \text{if} \quad |t| \leq R, \\ R^2/t & \text{if} \quad |t| > R. \end{cases} \tag{8.1}$$

If h has compact support, we deduce from (2.14), (8.1) and Fubini's theorem that

$$\begin{aligned} \mathcal{C}_2(h)_{\mathbb{D}_R} &= \frac{1}{\pi R^2} \int_{\mathbb{D}_R} \left(\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(t)}{z-t}\, dm(t) + \frac{1}{\pi} \int_{\mathbb{C}\setminus\mathbb{D}} \frac{h(t)}{t}\, dm(t) \right) dm(z) \\ &= -\frac{1}{\pi R^2} \int_{\mathbb{D}_R} h(t)\bar{t}\, dm(t) - \frac{1}{\pi} \int_{\mathbb{C}\setminus\mathbb{D}_R} \frac{h(t)}{t}\, dm(t) + \frac{1}{\pi} \int_{\mathbb{C}\setminus\mathbb{D}} \frac{h(t)}{t}\, dm(t) \\ &= -\frac{1}{\pi R^2} \int_{\mathbb{D}_R} h(t)\bar{t}\, dm(t) + \frac{1}{\pi} \int_{1\leq|t|\leq R} \frac{h(t)}{t}\, dm(t). \end{aligned} \tag{8.2}$$

By density argument, (8.2) holds for every $h \in L^2(\mathbb{C})$. Next, by (8.2) and the Schwarz inequality, we have

$$|\mathcal{C}_2(h)_{\mathbb{D}_R}| \leq \frac{\|h\|_{L^2(\mathbb{C})}}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}}\, \|h\|_{L^2(\mathbb{C})} \left(\log R\right)^{1/2}, \qquad R \geq 1. \tag{8.3}$$

In another connection, by the Poincaré inequality, we have

$$\begin{aligned} \|\mathcal{C}_2(h) - \mathcal{C}_2(h)_{\mathbb{D}_R}\|_{L^2(\mathbb{D}_R)} &\leq C_R\big(\|h\|_{L^2(\mathbb{D}_R)} + \|\mathcal{B}(h)\|_{L^2(\mathbb{D}_R)}\big) \\ &\leq 2C_R \|h\|_{L^2(\mathbb{C})}, \end{aligned} \tag{8.4}$$

where C_R is the best constant in (2.3) for $p = 2$ and $\Omega = \mathbb{D}_R$. Finally, since

$$\frac{\|\mathcal{C}_2(h)\|_{L^2(\mathbb{D}_R)}}{\sqrt{\pi}R} \leq \frac{\|\mathcal{C}_2(h) - \mathcal{C}_2(h)_{\mathbb{D}_R}\|_{L^2(\mathbb{D}_R)}}{\sqrt{\pi}R} + |\mathcal{C}_2(h)_{\mathbb{D}_R}|\,,$$

(2.15) follows from (8.3), (8.4) and the fact that $C_R = RC_1$ by homogeneity.

8.2. **Functions of vanishing mean oscillation.** The space $BMO(\mathbb{T})$ of functions with bounded mean oscillation on the unit circle consists of the functions $h \in L^1(\mathbb{T})$ such that

$$\|h\|_{BMO(\mathbb{T})} := \sup_I \frac{1}{\Lambda(I)} \int_I |h(t) - h_I|\, d\Lambda(t) < \infty, \qquad h_I := \frac{1}{\Lambda(I)} \int_I h(t)\, d\Lambda(t), \tag{8.5}$$

where Λ indicates arclength and I ranges over all subarcs of $\mathbb{T}$. Note that $\|\cdot\|_{BMO(\mathbb{T})}$ is a genuine norm modulo additive constants only. The space

$VMO(\mathbb{T})$ of functions with vanishing mean oscillation is the subspace of $BMO(\mathbb{T})$ consisting of those h for which

$$\lim_{\varepsilon\to 0}\sup_{\Lambda(I)<\varepsilon}\frac{1}{\Lambda(I)}\int_I |h(t)-h_I|\,d\Lambda(t)=0. \tag{8.6}$$

Actually, $VMO(\mathbb{T})$ is the closure in $BMO(\mathbb{T})$ of continuous functions [21, Chapter VI, Corollary 1.3 & Theorem 5.1]. The John–Nirenberg theorem asserts that there exist absolute constants C, c such that, for every $h \in BMO(\mathbb{T})$, every arc $I\subset\mathbb{T}$, and any $\lambda>0$,

$$\frac{\Lambda(\{\xi\in I\ :\ |h(\xi)-h_I|>\lambda\})}{\Lambda(I)}\le C\exp\left(\frac{-c\lambda}{\|h\|_{BMO(\mathbb{T})}}\right); \tag{8.7}$$

in fact one can take $C=e$ and $c=1/2e$, see [22, Theorem 7.1.6][11]. We also need a quantitative version of the so-called integral form of the John–Nirenberg inequality[12]. Given $h\in L^1(\mathbb{T}))$ and an arc $I\subset\mathbb{T}$, let us define

$$M_h(I):=\sup_{I'\subset I}\frac{1}{\Lambda(I')}\int_{I'}|h-h_{I'}|\,d\Lambda, \tag{8.8}$$

where the supremum is taken over all subarcs $I'\subset I$.

Lemma 8.1. *If $h\in BMO(\mathbb{T})\setminus\{0\}$ and $I\subset\mathbb{T}$ is an arc, then*

$$\int_I e^{|h|/(4eM_h(I))}d\Lambda\le(1+e)\Lambda(I)\ e^{|h_I|/(4eM_h(I))}. \tag{8.9}$$

Proof. Inspecting the standard proof of the John–Nirenberg inequality that uses recursively the Calderòn–Zygmund decomposition on dyadic subdivisions of I [21, Chapter VI, Theorem 2.1], one checks that (8.7) remains valid if we replace $\|h\|_{BMO(\mathbb{T})}$ by $M_h(I)$:

$$\frac{\Lambda\big(\{\xi\in I\ :\ |h(\xi)-h_I|>\lambda\}\big)}{\Lambda(I)}\le C\exp\left(\frac{-c\lambda}{M_h(I)}\right). \tag{8.10}$$

Pick $c'\in(0,c)$ with c as in (8.10), and set $g:=c'|h-h_I|/M_h(I)$. We compute as in [22, Corollary 7.1.7]:

$$\begin{aligned}\frac{1}{\Lambda(I)}\int_I e^g\,d\Lambda &= 1+\frac{1}{\Lambda(I)}\int_I(e^g-1)\,d\Lambda\\ &= 1+\frac{1}{\Lambda(I)}\int_0^\infty e^\lambda\Lambda(\{\xi\in I\ :\ g(\xi)>\lambda\})\,d\lambda\end{aligned}$$

[11] The argument there is given on the line but it applies mutatis mutandis to the circle.

[12] When $M_h(I)$ gets replaced by $\sup_{I'\subset I}\left(\frac{1}{\Lambda(I')}\int_{I'}|h-h_{I'}|^2\,d\Lambda\right)^{1/2}$ (a different but in fact equivalent quantity), the sharp constants in (8.9) were obtained in [39].

where the second equality follows from Fubini's theorem. Using (8.10) to estimate the distribution function of g, we find that

$$\frac{1}{\Lambda(I)}\int_I e^{c'|h-h_I|/M_h(I)}d\Lambda = \frac{1}{\Lambda(I)}\int_I e^g\,d\Lambda$$
$$\leq 1 + C\int_0^\infty e^{\lambda}e^{-c\lambda/c'}d\lambda = 1 + \frac{C}{c/c'-1}.$$

Choosing $C = e$, $c = 1/(2e)$, and $c' = 1/4e$, we obtain

$$\frac{1}{\Lambda(I)}\int_I e^{|h-h_I|/(4eM_h(I))}d\Lambda \leq 1 + e \tag{8.11}$$

from which (8.9) follows at once. □

By definition, $M_h(I)$ tends to zero uniformly with $\Lambda(I)$ if $h \in VMO(\mathbb{T})$, and Lemma 8.1 makes it clear that in this case $e^h \in L^p(\mathbb{T})$ for every $p \in [1, \infty)$. When $h \in VMO_{\mathbb{R}}(\mathbb{T})$, where subscript "$\mathbb{R}$" means "real-valued" as usual, it is well known that e^h satisfies condition A_p given in (5.7) for all $p \in (1, \infty)$. This follows for instance from (8.11) and [21, Chapter VI, Corollary 6.5]. Below, we record for later use a specific estimate for the A_p norm in terms of (8.8).

Lemma 8.2. *Let $h \in VMO_{\mathbb{R}}(\mathbb{T})$ and $p \in (1, \infty)$. Let $\eta = \eta(h, p) > 0$ be so small that $4eM_h(I)\max(1, 1/(p-1)) \leq 1$ for every arc $I \subset \mathbb{T}$ satisfying $\Lambda(I) < \eta$. Then*

$$\{e^h\}_{A_p} := \sup_I \left(\frac{1}{\Lambda(I)}\int_I e^h\,d\Lambda\right)\left(\frac{1}{\Lambda(I)}\int_I e^{-h/(p-1)}d\Lambda\right)^{p-1} \leq C \tag{8.12}$$

where C depends only on η, p, and $\|e^h\|_{L^1(\mathbb{T})}$.

Proof. If we put $p' = p/(p-1)$, then $1/(p-1) = p'-1$ and it follows easily from the definition that $\{e^h\}_{A_p} = \{e^{-h/(p-1)}\}^{(p-1)}_{A_{p'}}$. Therefore we may assume that $p \geq 2$.

Now, the left hand side of (8.12) can be rewritten as

$$\sup_I\Big(\frac{1}{\Lambda(I)}\int_I e^{h-h_I}\,d\Lambda\Big)\Big(\frac{1}{\Lambda(I)}\int_I e^{-(h-h_I)/(p-1)}d\Lambda\Big)^{p-1}.$$

If $\Lambda(I) < \eta$, then $4eM_h(I)$ and $4eM_h(I)/(p-1) < 1$, thus by (8.11) and Hölder's inequality we have

$$\frac{1}{\Lambda(I)}\int_I e^{h-h_I}\,d\Lambda \leq \Big(\frac{1}{\Lambda(I)}\int_I e^{|h-h_I|/(4eM_h(I))}\,d\Lambda\Big)^{4eM_h(I)}$$
$$\leq (1+e)^{4eM_h(I)} \leq (1+e)$$

and

$$\Big(\frac{1}{\Lambda(I)}\int_I e^{-(h-h_I)/(p-1)}\,d\Lambda\Big)^{p-1} \le \Big(\frac{1}{\Lambda(I)}\int_I e^{|h-h_I|/(4eM_h(I))}\,d\Lambda\Big)^{4eM_h(I)} \le (1+e).$$

This shows that (8.12) holds with $C=(1+e)^2$ when the supremum is restricted to those I of length less than η. In another connection, if $\Lambda(I)\ge\eta$, then obviously

$$\frac{1}{\Lambda(I)}\int_I e^h\,d\Lambda \le \eta^{-1}\|e^{|h|}\|_{L^1(\mathbb{T})}$$

and likewise, taking into account that $p\ge 2$ and using Hölder's inequality, we obtain

$$\left(\frac{1}{\Lambda(I)}\int_I e^{-h/(p-1)}\,d\Lambda\right)^{p-1} \le \frac{1}{\Lambda(I)}\int_I e^{|h|}\,d\Lambda \le \eta^{-1}\|e^{|h|}\|_{L^1(\mathbb{T})}.$$

Thus, (8.12) holds with $C=\big(\|e^{|h|}\|_{L^1(\mathbb{T})}/\eta\big)^2$ in this case. □

When Γ is a Jordan curve locally isometric to a Lipschitz graph, the definitions of $BMO(\Gamma)$, $VMO(\Gamma)$, and condition A_p on Γ which are modeled after (8.5), (8.6), and (5.7) do coincide with the standard ones [8, Section 2.5][13]. Lemma 8.1 and Lemma 8.2 carry over mechanically to this more general setting, but the significance of condition A_p with respect to the weighted L^p continuity of the conjugate operator is no longer the same if Γ is non-smooth[14]. Such considerations are not needed in this paper, but we make use at some point of the following estimate showing that $W^{1/2,2}(\Gamma)$ embeds contractively in $VMO(\Gamma)$ [10]:

$$\begin{aligned}\tfrac{1}{\Lambda(I)}\textstyle\int_I|h-h_I|\,d\Lambda &\le \tfrac{1}{(\Lambda(I))^2}\textstyle\int_{I\times I}|h(t)-h(t')|\,d\Lambda(t)d\Lambda(t')\\ &\le \tfrac{1}{\Lambda(I)}\textstyle\int_{I\times I}\frac{|h(t)-h(t')|}{\Lambda(t,t')}d\Lambda(t)d\Lambda(t')\\ &\le \Big(\textstyle\int_{I\times I}\frac{|h(t)-h(t')|^2}{(\Lambda(t,t'))^2}d\Lambda(t)d\Lambda(t')\Big)^{1/2}\\ &\le \|h\|_{W^{1/2,2}(\Gamma)},\end{aligned}\tag{8.13}$$

where the next to last step uses the Schwarz inequality. Note that if $h\in W^{1/2,2}(\Gamma)$, then

$$\|h\|_{W^{1/2,2}(I)} := \left(\int_{I\times I}\frac{|h(t)-h(t')|^2}{(\Lambda(t,t'))^2}d\Lambda(t)d\Lambda(t')\right)^{1/2}$$

tends to 0 as $\Lambda(I)\to 0$ by the absolute continuity of $\frac{|h(t)-h(t')|^2}{(\Lambda(t,t'))^2}d\Lambda(t)d\Lambda(t')$.

[13]In the standard definition, arcs $I\subset\Gamma$ are replaced by sets of type $\mathbb{D}(\xi,\rho)\cap\Gamma$ with $\xi\in\Gamma$. It is in this form that condition A_p is necessary and sufficient for weighted L^p boundedness of the singular Cauchy integral operator on Γ, see [8, Chapter 5].

[14]Even if we restrict ourselves to *constant* weights (which certainly satisfy A_p for all $p\in(1,\infty)$), the conjugate operator is generally L^p-continuous for restricted range of p only. This follows from [32, Theorem 2.1] and the fact that the Szegő projection has the same weighted L^p type as the conjugate operator on $\mathbb{T}$.

8.3. **Exp-summability of Sobolev functions at the critical exponent.** Given a bounded open set $\Omega \subset \mathbb{C}$, the Trudinger-Moser inequality [34] asserts that

$$\sup_{\substack{h \in W_0^{1,2}(\Omega) \\ \|\partial h\|^2_{L^2(\Omega)} + \|\bar\partial h\|^2_{L^2(\Omega)} \leq 1/2}} \int_\Omega e^{4\pi|h|^2} dm \leq C_{\mathrm{TM}}|\Omega| \tag{8.14}$$

for some absolute constant C_{TM}. Now, given a nonzero $f \in W_0^{1,2}(\Omega)$, put for the sake of simplicity $N_1(f) := (2\|\partial f\|^2_{L^2(\Omega)} + 2\|\bar\partial f\|^2_{L^2(\Omega)})^{1/2}$ and let further $f_1 = f/N_1(f)$. For each $\xi \in \Omega$ such that $f(\xi)$ is defined, we have either $|f(\xi)| \leq N_1^2(f)/4\pi$ or $\exp(|f(\xi)|) < \exp(4\pi|f_1(\xi)|^2)$. Thus, applying (8.14) with $h = f_1$, we obtain for $f \in W_0^{1,2}(\Omega)$ *a fortiori* that

$$\int_\Omega e^{|f|} dm \leq |\Omega| \Big(C_{\mathrm{TM}} + \exp\Big(\frac{\|\partial f\|^2_{L^2(\Omega)} + \|\bar\partial f\|^2_{L^2(\Omega)}}{2\pi} \Big) \Big). \tag{8.15}$$

Lemma 8.3. *Let $\Omega \subset \mathbb{C}$ be a bounded and Lipschitz open set. Then there exist $C_1 = C_1(\Omega)$, $C_2 = C_2(\Omega)$ such that, for every $\ell \in [1, \infty)$ and $f \in W^{1,2}(\Omega)$,*

$$\|e^{|f|}\|_{L^\ell(\Omega)} \leq C_1 \exp(C_2 \ell \|f\|^2_{W^{1,2}(\Omega)}). \tag{8.16}$$

Proof. Let $\Omega_1 \supset \overline{\Omega}$ be open and, say $|\Omega_1| \leq 2|\Omega|$. Pick $\varphi \in \mathcal{D}(\mathbb{R}^2)$ to have support in Ω_1, values in $[0,1]$, and to be identically 1 on Ω. By the extension theorem, there exists $\tilde f \in W^{1,2}(\mathbb{R}^2)$ such that $\tilde f_{|\Omega} = f$ and $\|\tilde f\|_{W^{1,2}(\mathbb{R}^2)} \leq C\|f\|_{W^{1,2}(\Omega)}$, where $C = C(\Omega)$. Then $h := \ell\varphi\tilde f$ lies in $W_0^{1,2}(\Omega_1)$ and satisfies

$$\|\partial h\|^2_{L^2(\Omega_1)} + \|\bar\partial h\|^2_{L^2(\Omega_1)} \leq \ell^2 C' \|f\|^2_{W^{1,2}(\Omega)},$$

where C' depends on C and φ. Applying (8.15) to h, we find on putting $C_2 = C'/(2\pi)$ that

$$\int_\Omega e^{\ell|f|} dm \leq \int_{\Omega_1} e^{|h|} dm \leq \left(e^{\ell^2 C_2 \|f\|^2_{W^{1,2}(\Omega)}} + C_{\mathrm{TM}} \right) |\Omega_1|,$$

that yields (8.16) upon setting $C_1 := 2(1 + C_{\mathrm{TM}})|\Omega|$. $\square$

With the help of Lemma 8.3, we now prove that e^f is fairly smooth when $f \in W^{1,2}(\Omega)$. Recall that a (possibly nonlinear) operator between Banach spaces is said to be bounded if it maps bounded sets into bounded sets.

Proposition 8.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz smooth open set. Fix $p \in (1, \infty)$ and $\ell \in [1, \min(p, 2))$. Then, the map $(g, f) \mapsto ge^f$ is continuous and bounded from $W^{1,p}(\Omega) \times W^{1,2}(\Omega)$ into $W^{1,\ell}(\Omega)$, and derivatives are computed using the Leibniz and the chain rules:*

$$\partial(ge^f) = e^f \partial g + ge^f \partial f, \qquad \bar\partial(ge^f) = e^f \bar\partial g + ge^f \bar\partial f. \tag{8.17}$$

In particular, for every $q \in [1, 2)$, the map $f \mapsto e^f$ is continuous and bounded from $W^{1,2}(\Omega)$ into $W^{1,q}(\Omega)$ and so is the map $f \mapsto e^{tr_{\partial\Omega} f}$ from $W^{1,2}(\Omega)$ into $W^{1-1/q,q}(\partial\Omega)$.

Proof. Let $g \in W^{1,p}(\Omega)$, $f \in W^{1,2}(\Omega)$, and let (f_n), (g_n) be two sequences of smooth functions on Ω converging respectively to f and g in $W^{1,2}(\Omega)$ and $W^{1,p}(\Omega)$. *We claim* that e^{f_n} converges to e^f in $L^\ell(\Omega)$ for all $\ell \in [1,\infty)$. To see this, consider first the case of real-valued functions. By the mean-value theorem and convexity of $t \mapsto e^t$, we have that

$$\begin{aligned}\int_\Omega \left|e^f - e^{f_n}\right|^\ell dm &\leq \int_\Omega |f-f_n|^\ell \left|e^f + e^{f_n}\right|^\ell dm \\ &\leq \|f-f_n\|^\ell_{L^{2\ell}(\Omega)} \|e^f + e^{f_n}\|^\ell_{L^{2\ell}(\Omega)}, \end{aligned} \tag{8.18}$$

where we use the Schwarz inequality. By the Sobolev embedding theorem, $\|f-f_n\|_{L^{2\ell}(\Omega)}$ tends to 0 as $n \to \infty$. Moreover, $\|f_n\|_{W^{1,2}(\Omega)}$ tends to $\|f\|_{W^{1,2}(\Omega)}$, hence $\|e^f + e^{f_n}\|_{L^{2\ell}(\Omega)}$ is uniformly bounded by Lemma 8.3, and the right hand side of (8.18) indeed goes to zero as $n \to \infty$. Next, if f, f_n are complex-valued, say $f = u + iv$ and $f_n = u_n + iv_n$, we write

$$\|e^f - e^{f_n}\|_{L^\ell(\Omega)} \leq \|e^u(e^{iv} - e^{iv_n})\|_{L^\ell(\Omega)} + \|e^{iv_n}(e^u - e^{u_n})\|_{L^\ell(\Omega)}.$$

By what precedes, the last term in the right hand side tends to 0 when $n \to \infty$, and so does the first since we can extract pointwise convergent subsequences from any subsequence of v_n and apply the dominated convergence theorem. This proves the claim.

Next, we observe that $g_n e^{f_n}$ is smooth on Ω and that

$$\partial(g_n e^{f_n}) = e^{f_n}\partial g_n + g_n e^{f_n} \partial f_n. \tag{8.19}$$

Assume first that $p < 2$. Then, by the Sobolev embedding theorem, (g_n) converges to g in $L^{p^*}(\Omega)$ where $p^* = 2p/(2-p) > 2$. From this and the previous claim, we deduce by Hölder's inequality that $(g_n e^{f_n})$ converges to ge^f in $L^\ell(\Omega)$ for $\ell \in [1, p^*)$, hence also in the sense of distributions. By the same token, the right hand side of (8.19) converges to $e^f \partial g + g e^f \partial f$ in $L^\ell(\Omega)$ for each $\ell \in [1,p)$. The case $p = 2$ is similar except that p^* can be taken arbitrarily large, hence the convergence in the right hand-side of (8.19) takes place in $L^\ell(\Omega)$ for all $\ell < 2$. If $p > 2$, then g is even bounded, but this does not improve the estimate. Repeating the argument for $\bar\partial(ge^f)$ proves that $(g_n e^{f_n})$ converges to ge^f in $W^{1,\ell}$ for $\ell \in [1, \min(p,2))$ and that (8.17) holds. Hence, the map $(g,f) \mapsto ge^f$ is defined from $W^{1,p}(\Omega) \times W^{1,2}(\Omega)$ into $W^{1,\ell}(\Omega)$ and (8.17) is valid. Moreover, by Lemma 8.3 and Hölder's inequality, this map is bounded. Relaxing the smoothness assumption on f_n, g_n and arguing as before shows that it is also continuous. This proves the first assertion on the Proposition. Setting $g \equiv 1$, the second assertion follows by the Sobolev embedding and the trace theorems. ☐

8.4. **Equicontinuity properties of Cauchy transforms.**

Proposition 8.5. *Let $\beta \in L^2(\mathbb{D})$ and let ω be a strictly positive function on $(0,+\infty)$ such that $\|\beta\|_{L^2(Q_{\omega(\varepsilon)}\cap\mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a square of sidelength $\omega(\varepsilon)$.*

(i) *If we set (cf.* (2.9))

$$\mathcal{C}(\beta)(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{\beta(\xi)}{\xi - z}\, d\xi \wedge \overline{d\xi}, \qquad z \in \mathbb{C},$$

then there exists a strictly positive function ω_1 *on* $(0, +\infty)$*, depending only on* ω*, such that*

$$\|\partial \mathcal{C}(\beta)\|_{L^2(Q_{\omega_1(\eta)})} + \|\bar\partial \mathcal{C}(\beta)\|_{L^2(Q_{\omega_1(\eta)})} < \eta \tag{8.20}$$

as soon as $Q_{\omega_1(\eta)}$ *is a square of sidelength* $\omega_1(\eta)$*.*

(ii) *If we set (cf.* (5.11))

$$\mathcal{R}(\beta)(z) := \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{z\bar\beta(\xi)}{1 - \bar\xi z}\, d\xi \wedge \overline{d\xi}, \qquad z \in \mathbb{D},$$

then $\mathcal{R}(\beta) \in W^{1,2}(\mathbb{D})$ *is holomorphic in* $\mathbb{D}$ *and there exists a strictly positive function* ω_2 *on* $(0, +\infty)$*, depending only on* ω*, such that*

$$\|\partial \mathcal{R}(\beta)\|_{L^2(Q_{\omega_2(\eta)} \cap \mathbb{D})} < \eta$$

as soon as $Q_{\omega_2(\eta)}$ *is a square of sidelength* $\omega_2(\eta)$*.*

Proof. Since $\beta \in L^2(\mathbb{D})$, we know that $\mathcal{C}(\beta) \in W^{1,2}_{loc}(\mathbb{C})$. Fix $\eta > 0$ and set $\delta = \min(1/3, \omega(\eta/3), \eta/(6\|\beta\|))$. For any square Q_δ, we have a nested concentric square with parallel sides $Q_{\delta^2} \subset Q_\delta$. Let $\widetilde\beta$ be the extension of β by 0 off $\mathbb{D}$. Since $\bar\partial \mathcal{C}(\beta) = \widetilde\beta$, we obtain

$$\|\bar\partial \mathcal{C}(\beta)\|_{L^2(Q_{\delta^2})} < \eta/3. \tag{8.21}$$

Next, we write

$$\partial \mathcal{C}(\beta) = \mathcal{B}(\widetilde\beta) = \mathcal{B}(\chi_{Q_\delta}\widetilde\beta) + \mathcal{B}(\chi_{\mathbb{C}\setminus Q_\delta}\widetilde\beta) \tag{8.22}$$

where $\mathcal{B}$ indicates the Beurling transform, *cf.* (2.10). As $\mathcal{B}$ is an isometry on $L^2(\mathbb{C})$, we get

$$\|\mathcal{B}(\chi_{Q_\delta}\widetilde\beta)\|_{L^2(\mathbb{C})} = \|\beta\|_{L^2(Q_\delta \cap \mathbb{D})} < \eta/3. \tag{8.23}$$

Moreover, formula (2.10) and the Cauchy-Schwarz inequality give us the pointwise estimate:

$$\mathcal{B}(\chi_{\mathbb{C}\setminus Q_\delta}\widetilde\beta)(z) \le \frac{2}{\delta}\|\beta\|_{L^2(\mathbb{D})}, \qquad z \in Q_{\delta^2}.$$

Integrating over Q_{δ^2} yields

$$\|\mathcal{B}(\chi_{\mathbb{C}\setminus Q_\delta}\widetilde\beta)\|_{L^2(Q_{\delta^2})} \le 2\delta\|\beta\|_{L^2(\mathbb{D})} < \eta/3. \tag{8.24}$$

Inequality (8.20) with $\omega_1(\eta) = \delta$ follows now from (8.21), (8.22), (8.23) and (8.24), thereby proving (i).
Consider next $\mathcal{R}(\beta)$. Clearly it is holomorphic in $\mathbb{D}$ and vanishes at 0. Furthermore, $\overline{\mathcal{R}(\beta)(z)} = -\mathcal{C}(\beta)(1/\bar z)$ and since $\mathcal{C}(\beta) \in W^{1,2}_{loc}(\mathbb{C})$ we get that $\mathcal{R}(\beta) \in W^{1,2}(\mathbb{D})$.
Once again, fix $\eta > 0$ and set $\delta = \min(\omega_1(\eta)/4, \eta/(16\|\beta\|))$. First, every square Q_δ has diameter at most $1/4$, hence is disjoint from $\mathbb{D}_{1/2}$ if it meets

$\mathcal{A}_{3/4} := \{z: \ 1 \geq |z| \geq 3/4\}$. In this case the reflection ($z \mapsto 1/\bar{z}$) of $Q_\delta \cap \mathbb{D}$ is contained in a square of sidelength $4\delta \leq \omega_1(\eta)$, and since

$$\partial\mathcal{R}(\beta)(z) = \frac{\overline{(\partial(\mathcal{C}\beta))(1/\bar{z})}}{z^2}, \qquad z \neq 0,$$

we deduce from (8.20) and the change of variable formula that $\|\partial\mathcal{R}(\beta)\|_{L^2(Q_\delta)} \leq \eta$.

Assume now that $Q_\delta \subset \mathbb{D}_{3/4}$. Differentiating under the integral sign we obtain

$$\partial\mathcal{R}(\beta) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{\bar{\beta}(\xi)}{1-\bar{\xi}z}\, d\xi \wedge \overline{d\xi} + \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{z\bar{\xi}\bar{\beta}(\xi)}{(1-\bar{\xi}z)^2}\, d\xi \wedge \overline{d\xi},$$

so that if $z \in \mathbb{D}_{3/4}$, we get by the Schwarz inequality that $|\partial\mathcal{R}(\beta)(z)| \leq 16\|\beta\|_{L^2(\mathbb{D})}$. Integrating over Q_δ yields

$$\|\partial\mathcal{R}(\beta)\|_{L^2(Q_\delta)} \leq 16\delta\|\beta\|_{L^2(\mathbb{D})} \leq \eta,$$

as desired. It remains to set $\omega_2(\eta) = \delta$. $\square$

Corollary 8.6. *Let $\beta \in L^2(\mathbb{C})$ and let ω be a strictly positive function on $(0, +\infty)$ such that $\|\beta\|_{L^2(Q_{\omega(\varepsilon)})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a square of sidelength $\omega(\varepsilon)$. If we let (cf. (2.14))*

$$\mathcal{C}_2(\beta)(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}\Big(\frac{1}{z-t} + \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}(t)}{t}\Big)\beta(t)\, dm(t), \qquad z \in \mathbb{C},$$

then there exists a strictly positive function ω_1 on $(0, +\infty)$, depending only on ω, such that

$$\|\partial\mathcal{C}_2(\beta)\|_{L^2(Q_{\omega_1(\eta)})} + \|\bar{\partial}\mathcal{C}_2(\beta)\|_{L^2(Q_{\omega_1(\eta)})} < \eta$$

as soon as $Q_{\omega_1(\eta)}$ is a square of sidelength $\omega_1(\eta)$.

Proof. This is proved in the same way as (8.20), replacing $\tilde{\beta}$ by β. $\square$

8.5. **Integral estimates on circular arcs.**

Lemma 8.7. *If $f \in W^{1,q}(\mathbb{D})$ for some $q \in (1, 2)$ and $\ell := q/(2-q)$, then*

$$\sup_{\rho\in(0,1]}\Big(\int_{\mathbb{T}_\rho} |f(\xi)|^\ell |d\xi|\Big)^{1/\ell} \leq C\|f\|_{W^{1,q}(\mathbb{D})},$$

where $C = C(q)$.

Proof. Set $f_\rho(\xi) := f(\rho\xi)$ so that

$$\Big(\int_{\mathbb{T}_\rho} |f(\xi)|^\ell |d\xi|\Big)^{1/\ell} = \rho^{1/\ell}\Big(\int_{\mathbb{T}} |f_\rho(\xi)|^\ell |d\xi|\Big)^{1/\ell}. \tag{8.25}$$

By the trace theorem and (the non-integral version of) the Sobolev embedding theorem we have

$$\Big(\int_{\mathbb{T}} |f_\rho(\xi)|^\ell |d\xi|\Big)^{1/\ell} \leq C\|f_\rho\|_{W^{1,q}(\mathbb{D})} \tag{8.26}$$

with $C = C(q)$, and from the change of variable formula we get for $\rho > 0$ that

$$\|f_\rho\|_{W^{1,q}(\mathbb{D})} = \rho^{-2/q}\|f\|_{L^q(\mathbb{D}_\rho)} + \rho^{1-2/q}\left(\|\partial f\|_{L^q(\mathbb{D}_\rho)} + \|\bar\partial f\|_{L^q(\mathbb{D}_\rho)}\right). \quad (8.27)$$

Since $1/\ell - 2/q = -1$, and in view of (8.25), (8.26), and (8.27) it remains to majorize $\rho^{-1}\|f\|_{L^q(\mathbb{D}_\rho)}$ by $C\|f\|_{W^{1,q}(\mathbb{D})}$ for some $C = C(q)$. From (2.4) we see that this is equivalent to checking the estimate:

$$\rho^{2/q-1}\left|f_{\mathbb{D}_\rho}\right| = \left|\frac{1}{\pi\rho^{3-2/q}}\int_{\mathbb{D}_\rho} f\,dm\right| \le C_1\|f\|_{W^{1,q}(\mathbb{D})}, \qquad 0 < \rho \le 1, \quad (8.28)$$

with $C_1 = C_1(q)$. Now, the Sobolev embedding theorem implies that for some $C_2 = C_2(q)$ we have $\|f\|_{L^{2q/(2-q)}(\mathbb{D})} \le C_2\|f\|_{W^{1,q}(\mathbb{D})}$, and so by Hölder's inequality,

$$\left|\int_{\mathbb{D}_\rho} f\,dm\right| \le C_2\pi^{3/2-1/q}\rho^{3-2/q}\|f\|_{W^{1,q}(\mathbb{D})}$$

which is exactly (8.28) with $C_1 = C_2\pi^{1/2-1/q}$. □

For $J \subset \mathbb{T}$ an open arc and $\delta \in (0,1)$, we denote by $R(J,\delta)$ the open curvilinear rectangle in $\mathbb{D}$ (an annulus if $J = \mathbb{T}$) defined by

$$R(J,\delta) = \{z:\ z = \rho\xi,\ \xi \in J,\ 1-\delta < \rho < 1\}. \quad (8.29)$$

Lemma 8.8. *If $f \in W_0^{1,2}(\mathbb{D})$ and $\rho \in (0,1]$, then for every arc $I \subset \mathbb{T}_\rho$ we have*

$$\left|\frac{1}{\Lambda(I)}\int_I f(\zeta)\,|d\zeta|\right| \le \frac{(1-\rho)^{1/2}}{(\Lambda(I))^{1/2}}\left(\|\partial f\|_{L^2(R(J,1-\rho)} + \|\bar\partial f\|_{L^2(R(J,1-\rho)}\right), \quad (8.30)$$

where $J \subset \mathbb{T}$ is the arc such that $\rho J = I$.

Proof. By density it suffices to prove (8.30) when $f \in \mathcal{D}(\mathbb{D})$. If we write $\zeta \in I$ as $\zeta = \rho\xi$ with $\xi \in J$, we get

$$f(\zeta) = -\int_\rho^1 \left(\partial f(t\xi)\xi + \bar\partial f(t\xi)\bar\xi\right)\,dt$$

and integrating with respect to $|d\zeta| = \rho|d\xi|$ yields

$$\left|\int_I f(\zeta)\,|d\zeta|\right| = \left|\rho\int_J\int_\rho^1 \left(\partial f(t\xi)\xi + \bar\partial f(t\xi)\right)\,dt|d\xi|\right| \le \int_{R(J,1-\rho)}\left(|\partial f(t\xi)| + \left|\bar\partial f(t\xi)\right|\right)tdt|d\xi|.$$

Since $m(R(J,1-\rho)) = \Lambda(I)(1-\rho^2)/2$, estimate (8.30) follows from the Schwarz inequality. □

Lemma 8.9. *Let J be a proper open subarc of $\mathbb{T}$ and let $\delta_0 \in (0,1)$. For every $\delta \in (0,\delta_0]$ there exists $C > 0$ depending only on δ_0 and $\Lambda(J)/\delta$ such that, for all $f \in W^{1,2}(R(J,\delta))$ (cf. definition (8.29)) we have*

$$\Big(\int_{\partial R(J,\delta)\times\partial R(J,\delta)} \frac{|f(t)-f(t')|^2}{(\Lambda(t,t'))^2} d\Lambda(t)d\Lambda(t')\Big)^{1/2} \\ \leq C\big(\|\partial f\|_{L^2(R(J,\delta))} + \|\bar\partial f\|_{L^2(R(J,\delta))}\big). \quad (8.31)$$

Proof. Pick $\delta \in (0,\delta_0]$, and write e^{ia}, e^{ib} for the endpoints of J with $a < b$ and $|a-b| < 2\pi$. The map $\varphi(\rho,\theta) := (\rho\cos\theta, \rho\sin\theta)$ is a diffeomorphism from $R := (1-\delta,1)\times(a,b)$ onto $R(J,\delta)$ satisfying $|||D\varphi||| \leq 1$ and $|||(D\varphi)^{-1}||| \leq c/(1-\delta_0)$, where $D\varphi$ indicates the derivative and $|||\cdot|||$ is the operator norm. In particular, φ^{-1} extends to a Lipschitz homeomorphism from $\partial R(J,\delta)$ onto ∂R with Lipschitz constant depending only on δ_0, and by the change of variable formula it is enough to show that if $h := f\circ\varphi$, then

$$\Big(\int_{\partial R\times\partial R} \frac{|h(t)-h(t')|^2}{(\Lambda(t,t'))^2} d\Lambda(t)d\Lambda(t')\Big)^{1/2} \leq C\big(\|\partial h\|_{L^2(R)} + \|\bar\partial h\|_{L^2(R)}\big),$$

where the constant C depends only on $\Lambda(J)/\delta = 2\pi(b-a)/\delta$. The result now follows from the fact that if $p = 2$ and Ω is a rectangle, then the constant in (2.6) depends only on the ratio of sidelengths, a fact which is obvious by homogeneity. □

8.6. **A multiplier theorem.** The next theorem is fundamental to our study of G^p_α when $\alpha \in L^2(\mathbb{D})$ but is also of independent interest. It is best stated in terms of multipliers. We use the definition (2.1) of the non-tangential maximal function $\mathcal{M}_\gamma f$. Denote by $\mathfrak{M}^{\gamma,p}$ the Banach space of complex-valued functions on $\mathbb{D}$ such that $\|\mathcal{M}_\gamma f\|_{L^p(\mathbb{T})} < \infty$. Furthermore, we use the Banach space $\mathcal{H}^p$ of functions satisfying a Hardy condition, introduced in Section 5.2.

Theorem 8.10. *Let $\gamma \in (0,\pi/2)$ and $p \in [1,\infty)$. Given $f \in W^{1,2}_{0,\mathbb{R}}(\mathbb{D})$, the multiplication by e^f is continuous from $\mathfrak{M}^{\gamma,p}$ into $\mathcal{H}^p$. More precisely, for any function g on $\mathbb{D}$, we have*

$$\sup_{0<\rho<1}\Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi|\Big)^{1/p} < C\,\|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})}, \quad (8.32)$$

where C depends on p, γ, and on $\varepsilon > 0$ so small that $\|\partial f\|_{L^2(Q_\varepsilon\cap\mathbb{D})} < C'/p$ whenever Q_ε is a square of sidelength ε, with C' depending only on γ.

Proof. First, let $\rho \in (0,\sin\gamma)$. For $\zeta \in \mathbb{T}$, $\Gamma(\zeta,\gamma)$ contains $\mathbb{T}_\rho$ and we have

$$\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi| \leq \mathcal{M}^p_\gamma g(\zeta)\int_{\mathbb{T}_\rho} e^{pf(\xi)}|d\xi|.$$

Averaging over $\zeta \in \mathbb{T}$ yields

$$\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi| \leq \frac{1}{2\pi}\Big(\int_{\mathbb{T}} \mathcal{M}_\gamma^p g(\zeta)\,|d\zeta|\Big)\Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}|d\xi|\Big). \tag{8.33}$$

By Lemma 8.7 applied to e^f in the place of f with $\ell = p$ and $q := 2p/(p+1)$, we get

$$\Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}|d\xi|\Big)^{1/p} \leq c_0\|e^f\|_{W^{1,q}(\mathbb{D})} \tag{8.34}$$

for some $c_0 = c_0(p)$. Moreover, Lemma 8.3, Proposition 8.4, Hölder's inequality, and the fact that f is real-valued imply together that for some absolute constants C_1, C_2 we have

$$\begin{aligned}\|e^f\|_{W^{1,q}(\mathbb{D})} &= \|e^f\|_{L^q(\mathbb{D})} + 2\|\partial f e^f\|_{L^q(\mathbb{D})} \leq \|e^{|f|}\|_{L^{2p}(\mathbb{D})}\big(1 + 2\|\partial f\|_{L^2(\mathbb{D})}\big)\\ &\leq C_1\big(1+\exp\big(C_2 p\|\partial f\|^2_{L^2(\mathbb{D})}\big)\big)\big(1+\|\partial f\|_{L^2(\mathbb{D})}\big).\end{aligned} \tag{8.35}$$

By (8.33), (8.34), and (8.35) we conclude that

$$\sup_{0<\rho<\sin\gamma}\Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi|\Big)^{1/p} \leq C_0\|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})} \tag{8.36}$$

for some $C_0 = C_0(p, \|\partial f\|_{L^2(\mathbb{D})})$.

Assume next that $\rho \geq \sin\gamma$. Now $\Gamma(\zeta,\gamma)$ cuts out two disjoint open arcs on $\mathbb{T}_\rho$ one of which is centered at $\xi = \rho\zeta$. Denote this arc by A_ξ. Its length $\Lambda(A_\xi)$ is independent of ζ and it is easy to check that $K_1(1-\rho) \leq \Lambda(A_\xi) \leq K_2(1-\rho)$ for strictly positive numbers K_1, K_2 depending only on γ. Take an integer N_ρ in the interval $[4\pi\rho/\Lambda(A_\xi), 4\pi\rho/\Lambda(A_\xi)+1)$, and divide $\mathbb{T}_\rho$ into N_ρ semi open arcs of equal length, say $I_{\xi_1}, \dots, I_{\xi_{N_\rho}}$, centered at equidistant points $\xi_1, \dots, \xi_{N_\rho} \in \mathbb{T}_\rho$. Put $\zeta_j = \xi_j/\rho \in \mathbb{T}$, and consider the partition of $\mathbb{T}$ into N_ρ semi open arcs $J_{\zeta_j} := I_{\xi_j}/\rho$ centered at ζ_j. By construction, if $\zeta \in J_{\zeta_j}$, then $I_{\xi_j} \subset \Gamma_{\zeta,\gamma}$. Consequently,

$$\int_{I_{\xi_j}} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi| \leq \mathcal{M}_\gamma^p g(\zeta)\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|,$$

and averaging over $\zeta \in J_{\zeta_j}$ gives us

$$\int_{I_{\xi_j}} e^{pf(\xi)}|g(\xi)|^p|d\xi| \leq \frac{1}{\Lambda(J_{\zeta_j})}\Big(\int_{J_{\zeta_j}} \mathcal{M}_\gamma^p g(\zeta)\,|d\zeta|\Big)\Big(\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|\Big).$$

Since $\Lambda(J_{\zeta_j}) = \Lambda(I_{\xi_j})/\rho$ we deduce upon summing over j that

$$\begin{aligned}&\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi|\\ &\qquad\leq \rho\Big(\int_{\mathbb{T}} \mathcal{M}_\gamma^p g(\zeta)\,|d\zeta|\Big)\sup_{1\leq j\leq N_\rho}\Big(\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|\Big).\end{aligned} \tag{8.37}$$

Let $R(J,\delta)$ be defined as in (8.29), and let C be the constant in Lemma 8.9 associated to $\delta_0 = 1-\sin\gamma$ and $\Lambda(J)/\delta = K_2/(2\sin\gamma)$; note that C depends only on γ. Since $\Lambda(J_{\zeta_j})/(1-\rho) \leq K_2/(2\sin\gamma)$, the arc $J'_{\zeta_j} \subset \mathbb{T}$ of length $(1-\rho)K_2/(2\sin\gamma)$ centered at ζ_j does contain J_{ζ_j}. Therefore, $R(J_{\zeta_j}, 1-\rho)$ is contained in $R(J'_{\zeta_j}, 1-\rho)$ and I_{ξ_j} is contained in $I'_{\xi_j} := J'_{\zeta_j}/\rho$. Hence, (8.31) *a fortiori* implies for some K depending only on γ that

$$\Big(\int_{I_{\xi_j}\times I_{\xi_j}} \frac{|f(t)-f(t')|^2}{(\Lambda(t,t'))^2} d\Lambda(t) d\Lambda(t')\Big)^{1/2} \\ \leq K\big(\|\partial f\|_{L^2(R(J'_{\zeta_j},1-\rho))} + \|\bar\partial f\|_{L^2(R(J'_{\zeta_j},1-\rho))}\big). \quad (8.38)$$

Now, it is elementary to check that $R(J'_{\zeta_j}, 1-\rho)$ is contained in a square of sidelength $K_3(1-\rho)$ (where K_3 depends only on γ), one side of which is tangent to $\mathbb{T}$ at ζ_j. So, if we let ε_1 be so small that $\|\partial \tilde f\|_{L^2(Q_{\varepsilon_1})} < 1/(8Kep)$ whenever Q_{ε_1} is a square of sidelength ε_1, we get (since f is real-valued) that

$$\|\partial f\|_{L^2(R(J'_{\zeta_j},1-\rho))} + \|\bar\partial f\|_{L^2(R(J'_{\zeta_j},1-\rho))} \leq \frac{1}{4Kep}, \\ \max(\sin\gamma, 1-\varepsilon_1/K_3) = \rho_0 \leq \rho < 1. \quad (8.39)$$

Then, from (8.13), (8.38), and (8.39), we see that for all subarcs $I \subset I_{\xi_j}$

$$\begin{aligned} \tfrac{1}{\Lambda(I)}\int_I |f-f_I|\,d\Lambda &\leq \Big(\int_{I\times I} \tfrac{|f(t)-f(t')|^2}{(\Lambda(t,t'))^2} d\Lambda(t)d\Lambda(t')\Big)^{1/2} \\ &\leq \left(\int_{I_{\xi_j}\times I_{\xi_j}} \tfrac{|f(t)-f(t')|^2}{(\Lambda(t,t'))^2} d\Lambda(t)d\Lambda(t')\right)^{1/2} \\ &\leq 1/4ep, \qquad \rho_0 \leq \rho < 1. \end{aligned} \quad (8.40)$$

If we let $h := \mathrm{tr}_{\mathbb{T}_\rho} f$, inequality (8.40) asserts that

$$M_h(I_{\xi_j}) \leq 1/4ep, \qquad \rho_0 \leq \rho < 1, \quad (8.41)$$

with $M_h(I_{\xi_j})$ defined by (8.8) where we set Γ to be $\mathbb{T}_\rho$. By Lemma 8.1, (8.41), and Hölder's inequality, for $\rho \in [\rho_0, 1)$ we have

$$\Big(\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|\Big)^{1/p} \leq (1+e)^{1/p}\exp\Big(\Big|\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} f(\zeta)\,|d\zeta|\Big|\Big). \quad (8.42)$$

In another connection, keeping in mind (8.39) and the inclusion $R(J_{\zeta_j}, 1-\rho) \subset R(J'_{\zeta_j}, 1-\rho)$, an application of (8.30) yields

$$\Big|\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} f(\zeta)\,|d\zeta|\Big| \leq \frac{(1-\rho)^{1/2}}{(\Lambda(I_{\xi_j}))^{1/2}}\,\frac{1}{4Kep}. \quad (8.43)$$

Put $\rho_1 := \max(\rho_0, K_1/(K_1+\pi))$, and assume for a while that $\rho \geq \rho_1$; in particular, $\pi\rho/(K_1(1-\rho)) > 1$, and therefore

$$\Lambda(I_{\xi_j}) \geq \frac{2\pi\rho}{1+4\pi\rho/|A_\xi|} \geq \frac{2\pi\rho}{1+4\pi\rho/(K_1(1-\rho))} \geq \frac{K_1(1-\rho)}{3}. \quad (8.44)$$

Using together (8.43) and (8.44), we obtain

$$\Big|\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} f(\zeta)\,|d\zeta|\Big| \le \frac{\sqrt{3}}{4\sqrt{K_1}Kep}, \qquad \rho_1 \le \rho < 1. \tag{8.45}$$

Plugging (8.45) in the right hand side of (8.42) and using (8.37) now gives us

$$\sup_{\rho_1\le\rho<1}\Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}\,|g(\xi)|^p\,|d\xi|\Big)^{1/p} \le (1+e)^{1/p}\exp\Big(\frac{\sqrt{3}}{4\sqrt{K_1}Kep}\Big)\|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})}.$$

To obtain (8.32), it remains to treat the case $\rho \in [\sin\gamma, \rho_1)$ when the latter interval is nonempty. First, in this range of ρ, the first two inequalities in (8.44) imply that

$$\Lambda(I_{\xi_j}) \ge c(\gamma, \rho_1). \tag{8.46}$$

On the other hand, (8.34) and (8.35) give us that

$$\Big(\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|\Big)^{1/p} \le \Big(\int_{\mathbb{T}_\rho} e^{pf(\xi)}|d\xi|\Big)^{1/p} \le C_0 \tag{8.47}$$

with C_0 as in (8.36). Therefore by (8.46) and (8.47) we have that

$$\Big(\frac{1}{\Lambda(I_{\xi_j})}\int_{I_{\xi_j}} e^{pf(\xi)}|d\xi|\Big)^{1/p} \le [c(\gamma,\rho_1)]^{-1/p}C_0, \qquad \sin\gamma \le \rho < \rho_1,$$

and using this in (8.37) completes the proof. □

Corollary 8.11. *If $w = e^sF$, where $s \in W^{1,2}(\mathbb{D})$ with $\operatorname{Re} tr_{\mathbb{T}} s \equiv 0$ and $F \in H^p$, then*

$$\sup_{0<\rho<1}\left(\frac{1}{2\pi}\int_{\mathbb{T}_\rho} |w(\xi)|^p\,|d\xi|\right)^{1/p} < +\infty.$$

Proof. This follows from (5.3) and Theorem 8.10 applied with $f = \operatorname{Re} s$ and $g = e^{i\operatorname{Im} s}F$. □

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