Existence, uniqueness of solutions and stability of nonsmooth multivalued Lur'e dynamical systems

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This paper deals with the well-posedness of a class of multivalued Lur'e systems, which consist of a nonlinear dynamical system in negative feedback interconnection with a static multivalued nonlinearity. The objective is to provide a detailed analysis of the conditions which guarantee that a certain operator, constructed from the static nonlinearity, is maximal monotone. This in turn assures the existence and the uniqueness of the solutions. Examples (nonlinear complementarity systems, nonlinear relay systems) illustrate the developments. A stability result is also given.

1. Introduction

The problem of well-posedness and stability of so-called Lur'e dynamical systems has attracted the interest of researchers in control and in applied mathematics since a long time (see *e.g.* [30] for a survey). More recently nonsmooth multivalued Lur'e systems that consist of the negative feedback interconnection of a smooth system $\dot{x}(t) = f(x(t), \lambda(t))$ with output $y = g(x, \lambda)$, with a multivalued mapping $\lambda \in \Phi(y, t)$, have been studied in [8, 10, 11, 13, 26]. This is in close connection with studies on complementarity dynamical systems [11, 17, 18, 27, 28], relay systems [21, 24, 31], projected dynamical systems [12, 27], and evolution variational inequalities [25, 29], since all these systems may be interpreted as Lur'e systems with a multivalued feedback path (see [9, 24] for surveys). These systems deserve the name of Lur'e systems because passivity constraints on both the smooth system and the multivalued mapping are often encountered in the cited works. Applications may be found in electrical circuits with ideal electronic devices [2, 3, 17], in state observers design [14], or in control of systems with friction [16].

One of the recognized difficulties in the analysis of such Lur'e systems is when the output y depends on the multiplier λ, which is common in applications (see *e.g.* Section 14.2.1 in [1]). For instance in the linear time-invariant case $y =$

$Cx + D\lambda$ for some non zero matrix D. This has been solved for dissipative linear complementarity systems (when $\Phi(\cdot)$ is the subdifferential of the indicator function of $\mathbb{R}^m_+$, $y = Cx + D\lambda$ and $f(x,\lambda) = Ax + B\lambda$) in [18]. In [13] the same problem has been tackled with a more general multivalued mapping Φ, relying either on Kato's theorem or on maximal monotone operators. In the first case a specific structure on the matrix D is imposed, while in the second case D is supposed to be positive semi-definite. In this paper the maximal monotone approach of [13] is studied in more depth, and the vector field $f(x,\lambda)$ is nonlinear with respect to x.

Notations. For $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle$ the euclidean scalar product in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, $\|x\| = \sqrt{\langle x, x\rangle}$ is the corresponding norm of x while if $M \in \mathbb{R}^{n\times n}$, then $\|M\| = \sup_{\|y\|=1}\{\|My\|\}$ is the subordinate matrix norm of M. For a multivalued application $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote respectively by $\mathcal{R}(T)$ and $\mathrm{Dom}(T)$ its range and its domain, *i.e.* $\mathcal{R}(T) = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^n : y \in Tx\}$ and $\mathrm{Dom}(T) = \{x \in \mathbb{R}^n : Tx \neq \emptyset\}$. For a matrix $M \in \mathbb{R}^{n\times n}$, we denote respectively by $\mathcal{R}(M)$ and $\ker(M)$ its range and its kernel, *i.e.* $\mathcal{R}(M) = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^n : y = Mx\}$ and $\ker(M) = \{x \in \mathbb{R}^n : Mx = 0\}$. For a nonempty set $\mathcal{C} \subset \mathbb{R}^n$, the dual cone of $\mathcal{C}$ is the nonempty closed convex cone $\mathcal{C}^\star$ defined by $\mathcal{C}^\star := \{w \in \mathbb{R}^n : \langle w, v\rangle \geq 0,\ \forall v \in \mathcal{C}\}$. The affine hull of $\mathcal{C}$ denoted by $\mathrm{aff}(\mathcal{C})$ is the intersection of all the affine sets that include C. The relative interior of $\mathcal{C}$ denoted by $\mathrm{rint}(\mathcal{C})$ is the interior of $\mathcal{C}$ relative to $\mathrm{aff}(\mathcal{C})$, *i.e.*: $\mathrm{rint}\{\mathcal{C}\} = \{x \in \mathcal{C} : \exists\ \varepsilon > 0 :\ B(x,\varepsilon) \cap \mathrm{aff}(\mathcal{C}) \subset \mathrm{aff}(\mathcal{C})\}$, where $B(x,\varepsilon) = \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\}$. The closure of a set $\mathcal{C}$ is denoted as $\overline{\mathcal{C}}$.

2. Mathematical tools in convex analysis

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi-continuous function, we denote by $\mathrm{Dom}(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ the domain of the function f. Recall that the Fenchel transform $f^\star$ of f is the proper, convex and lower semi-continuous function defined by

$$(\forall z \in \mathbb{R}^n) : \ f^\star(z) = \sup_{x\in\ \mathrm{Dom}(f)} \{\langle x, z\rangle - f(x)\}.$$

The subdifferential $\partial f(x)$ of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \{\omega \in \mathbb{R}^n : \ f(v) - f(x) \geq \langle \omega, v - x\rangle,\ \forall v \in \mathbb{R}^n\}.$$

We denote by $\mathrm{Dom}(\partial f) := \{x \in \mathbb{R}^n : \partial f(x) \neq \emptyset\}$ the domain of the subdifferential operator $\partial f : \mathbb{R}^n \to \mathbb{R}^n$. Recall that (see *e.g.* Theorem 2, Chapter 10, Section 3 in [4]):

$$\mathrm{Dom}(\partial f) \subset \mathrm{Dom}(f) \subset \overline{\mathrm{Dom}(\partial f)}. \tag{1}$$

Let x_0 be any element in the domain $\mathrm{Dom}(f)$ of f, the recession function f_∞ of f is defined by

$$(\forall x \in \mathbb{R}^n) : \ f_\infty(x) = \lim_{\lambda\to+\infty} \frac{1}{\lambda} f(x_0 + \lambda x).$$

The function $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semi-continuous function which describes the asymptotic behavior of f.

Let $K \subset \mathrm{IR}^n$ be a nonempty closed convex set. Let x_0 be any element in K. The recession cone of K is defined by

$$K_\infty = \bigcap_{\lambda>0} \frac{1}{\lambda}(K - x_0).$$

The set K_∞ is a nonempty closed convex cone that is described in terms of the directions which recede from K. The indicator function of K is denoted as Ψ_K.

Let us here recall some important properties of the recession function and recession cone (see *e.g.* Chapter 3 in [36]):

Proposition 2.1. *a) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi-continuous function. Then*

$$(\forall \alpha \geq 0,\ x \in \mathbb{R}^n) :\ f_\infty(\alpha x) = \alpha f_\infty(x), \tag{2}$$

$$(\forall\ x, v \in \mathbb{R}^n) :\ f_\infty(v) \geq f(x+v) - f(x), \tag{3}$$

and

$$(\forall x \in \mathbb{R}^n) :\ f_\infty(x) = \liminf_{t \to +\infty, v \to x} \frac{f(tv)}{t}. \tag{4}$$

b) Let $f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two proper, convex and lower semi-continuous functions. Then

$$(\forall x \in \mathbb{R}^n) :\ (f_1 + f_2)_\infty(x) \geq (f_1)_\infty(x) + (f_2)_\infty(x). \tag{5}$$

c) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function and let K be a nonempty closed convex set. Then

$$(\forall x \in \mathbb{R}^n) :\ (f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x). \tag{6}$$

d) Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex set. Then

$$(\forall x \in \mathbb{R}^n) :\ (\Psi_K)_\infty(x) = \Psi_{K_\infty}(x), \tag{7}$$

$$(\forall x \in K,\ e \in K_\infty) :\ x + e \in K. \tag{8}$$

e) If $K \subset \mathbb{R}^n$ is a nonempty closed and convex cone then $K_\infty = K$.

f) If $K \subset \mathbb{R}^n$ is a nonempty compact and convex set then $K_\infty = \{0\}$.

g) Let K_1 and K_2 be two nonempty closed convex sets. If $K_1 \cap K_2 \neq \emptyset$ then $(K_1 \cap K_2)_\infty = (K_1)_\infty \cap (K_2)_\infty$.

3. Nonsmooth nonlinear Lur'e dynamical system

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a (possibly) nonlinear operator, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times p}$ given matrices, $f \in C^0(\mathbb{R}_+; \mathbb{R})$ such that $f' \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^n)$ and $\Phi : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ a given proper convex and lower semi-continuous function. Let $x_0 \in \mathbb{R}^n$ be some initial condition, we consider the problem: Find

$x \in C^0(\mathbb{R}_+;\mathbb{R}^n)$ such that $x' \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ and x right-differentiable on $\mathbb{R}_+$, $\lambda \in C^0(\mathbb{R}_+;\mathbb{R}^p)$ and $y \in C^0(\mathbb{R}_+;\mathbb{R}^p)$ satisfying the nonsmooth Lur'e system $NSLS(A,B,C,D,f,\Phi,x_0)$:

$$\begin{cases} x(0) = x_0 \\ x'(t) = A(x(t)) + B\lambda(t) + f(t), \text{ a.e. } t \geq 0 \\ y(t) = Cx(t) + D\lambda(t), \ t \geq 0 \\ \lambda(t) \in -\partial\Phi(y(t)), \ t \geq 0. \end{cases} \tag{9}$$

We set

$$(\forall z \in \mathbb{R}^p): \ \Xi(z) := \Phi^\star(-z). \tag{10}$$

Assumption 3.1. There exists $z_0 \in \mathbb{R}^p$ at which Ξ is continuous.

Assumption 3.1 is a simple qualitative condition that is required to ensure that (see *e.g.* Proposition 2.4.5 in [33]):

$$(\forall \lambda \in \mathbb{R}^p): \ \partial\Xi(\lambda) = -\partial\Phi^\star(-\lambda).$$

The system $\lambda \in -\partial\Phi(y)$ can thus also be written as:

$$y \in -\partial\Xi(\lambda). \tag{11}$$

Using these notations, we see that the problem $NSLS(A,B,C,D,f,\Phi,x_0)$ reduces to the system:

$$\begin{cases} x(0) = x_0 \\ x'(t) = A(x(t)) + B\lambda(t) + f(t), \text{ a.e. } t \geq 0 \\ y(t) = Cx(t) + D\lambda(t), \ t \geq 0 \\ y(t) \in -\partial\Xi(\lambda(t)), \ t \geq 0. \end{cases} \tag{12}$$

Remark 3.2. Taking $\Phi(.) = \Psi_{\{0\}}$ then $\partial\Xi(.) = \{0\}$, hence our framework encapsulates equality constraints $Cx + D\lambda = 0$ which are common in applications like electrical circuits with nonsmooth multivalued electronic devices [2, 3].

It is clear from (12) that the important operator for the study of this Lur'e system is $x \mapsto (D + \partial\Xi)^{-1}(-Cx)$. As shown in [13], a way to show the well-posedness of $NSLS(A,B,C,D,f,\Phi,x_0)$ is to characterize this operator as a maximal monotone operator. The sequel of this paper is devoted to refine the characterization of the conditions under which maximal monotonicity holds.

4. Characterization of the set $\mathrm{Dom}((D + \partial\Xi)^{-1}) = \mathcal{R}(D + \partial\Xi)$

Assumption 4.1. We suppose that the matrix D is positive semi-definite, *i.e.* $(\forall x \in \mathbb{R}^p): \ \langle Dx, x\rangle \geq 0$.

It is important for applications in electrical circuits that the matrix D is allowed to be non-symmetric, with a non-zero skew-symmetric part, see [3, p. 72, p. 170]. Then (see *e.g.* [23]):

$$\ker(D) = \ker(D^T) \subset \ker(D + D^T) \tag{13}$$

Thus:
$$\mathcal{R}(D + D^T) \subset \mathcal{R}(D) = \mathcal{R}(D^T).$$
Moreover:
$$\ker(D + D^T) = \{x \in \mathbb{R}^p : \langle Dx, x\rangle = 0\}.$$
The system
$$\begin{cases} y(t) = Cx(t) + D\lambda(t) \\ y(t) \in -\partial\Xi(\lambda(t)) \end{cases}$$
may be rewritten as:
$$Cx(t) + D\lambda(t) \in -\partial\Xi(\lambda(t))$$
or also
$$-Cx(t) \in (D + \partial\Xi)(\lambda(t)). \tag{14}$$
The mapping $x \mapsto \partial\Xi(x)$ is a maximal monotone mapping with domain $\mathrm{Dom}(\partial\Xi)$ as the subdifferential of a proper convex and lower semi-continuous function (See Proposition 1, Section 2.13 in [34]). The mapping $x \mapsto Dx$ is linear monotone and continuous on $\mathbb{R}^p$ and is thus (see Proposition 1, Section 2.3 in [34]) a maximal monotone mapping with domain $\mathbb{R}^p$. Here $\mathrm{Dom}(\partial\Xi) \cap \ \mathrm{int}\ \mathrm{Dom}(D.) = \mathrm{Dom}(\partial\Xi) \neq \emptyset$ and the Sum Theorem of Rockafellar (see Theorem 32.I in [37]) ensures that the operator $D + \partial\Xi : \mathrm{Dom}(\partial\Xi) \rightrightarrows \mathcal{R}(D + \partial\Xi)$ is also maximal monotone. The inverse operator:
$$(D + \partial\Xi)^{-1} : \mathcal{R}(D + \partial\Xi) \rightrightarrows \mathrm{Dom}(\partial\Xi)$$
is thus (see Proposition 32.5 in [37]) also maximal monotone. The relation in (14) may be written as:
$$\lambda(t) \in (D + \partial\Xi)^{-1}(-Cx(t)). \tag{15}$$
Before going further, we need to provide a good characterization of the set $\mathrm{Dom}((D + \partial\Xi)^{-1}) = \mathcal{R}(D + \partial\Xi)$.

Proposition 4.2. *Let $F : \mathbb{R}^p \to I\!R \cup \{+\infty\}$ be a proper convex and lower semi-continuous function. Then*
$$\overline{\mathrm{Dom}(F^\star)} = \{q \in \mathbb{R}^p : F_\infty(v) \geq \langle q, v\rangle,\ \forall v \in \mathbb{R}^p\}$$
and
$$\mathrm{int}\{\mathrm{Dom}(F^\star)\} = \{q \in \mathbb{R}^p : F_\infty(v) > \langle q, v\rangle,\ \forall v \in \mathbb{R}^p, v \neq 0\}$$

Proof. See Corollary 13.3.4 in [35]. □

Proposition 4.3. *We have:*
$$(\forall z \in \mathbb{R}^p) : \ \Phi^\star(z) = \Xi(-z).$$
$$(\forall z \in \mathbb{R}^p) : \ \Xi^\star(z) = \Phi(-z).$$

Proof. By definition $\Xi(z) = \Phi^\star(-z)$ and thus $\Phi^\star(z) = \Xi(-z)$. Moreover, we have

$$\begin{aligned}
\Xi^\star(z) &= \sup_{x\in\mathrm{Dom}(\Xi)} \{\langle x, z\rangle - \Xi(x)\} = \sup_{x\in\mathrm{Dom}(\Xi)} \{\langle x, z\rangle - \Phi^\star(-x)\} \\
&= \sup_{-X\in\mathrm{Dom}(\Xi)} \{-\langle X, z\rangle - \Phi^\star(X)\} = \sup_{X\in\mathrm{Dom}(\Xi\circ(-id_{\mathbb{R}^p}))} \{\langle X, (-z)\rangle - \Phi^\star(X)\} \\
&= \sup_{X\in\mathrm{Dom}(\Phi^\star)} \{\langle X, (-z)\rangle - \Phi^\star(X)\} = (\Phi^\star)^\star(-z) = \Phi(-z).
\end{aligned}$$

□

Proposition 4.4. *Suppose that Assumption* 3.1 *holds. Then:*

$$\mathcal{R}(\partial\Xi) = \mathrm{Dom}(-\partial(\Phi\circ(-id_{\mathbb{R}^p}))), \tag{16}$$

and

$$\mathrm{Dom}(\partial\Xi) = \mathcal{R}(-\partial(\Phi\circ(-id_{\mathbb{R}^p}))), \tag{17}$$

Moreover it always holds that:

$$\overline{\mathcal{R}(\partial\Xi)} = \overline{\mathrm{Dom}(\Xi^\star)}. \tag{18}$$

and

$$\mathcal{R}(\partial\Xi) \subset \mathrm{Dom}(\Xi^\star), \tag{19}$$

Proof. The Fenchel correspondence

$$w \in \partial\Xi(x) \Leftrightarrow x \in \partial\Xi^\star(w) \tag{20}$$

ensures that $\mathcal{R}(\partial\Xi) = \mathrm{Dom}(\partial\Xi^\star)$ and $\mathrm{Dom}(\partial\Xi) = \mathcal{R}(\partial\Xi^\star)$ while Proposition 4.3 and Assumption 3.1 guarantee that $\partial\Xi^\star = -\partial(\Phi\circ(-id_{\mathbb{R}^p}))$. The results in (16) and (17) follow. The results in (19) and (18) are direct consequences of the property recalled in (1). □

Theorem 4.5. *Suppose that Assumptions* 3.1 *and* 4.1 *hold. Let us set*

$$\Delta(D,\Xi) = \{z \in \overline{\mathrm{Dom}(\Xi)}_\infty : Dz \in \mathrm{Dom}(\Xi_\infty)^\star\} \tag{21}$$

and

$$\Upsilon = \Xi + \Psi_{\ker(D+D^T)} + \Psi_{\Delta(D,\Xi)}. \tag{22}$$

We have

$$\mathcal{R}(D+\partial\Xi) \subset -D^T(\mathrm{Dom}(\partial\Xi)) + \overline{\mathrm{Dom}(\Upsilon^\star)}. \tag{23}$$

If $\mathrm{int}\{\mathrm{Dom}(\Upsilon^\star)\} \neq \emptyset$ *then*

$$-D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{dom}(\Upsilon^\star)\} \subset \mathcal{R}(D+\partial\Xi). \tag{24}$$

Proof. We first remark that

$$\Upsilon_\infty = \Xi_\infty + \Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)}. \tag{25}$$

Indeed, we have

$$\Upsilon_\infty = (\Xi + \Psi_{\ker(D+D^T)} + \Psi_{\Delta(D,\Xi)})_\infty = (\Xi + \Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)})_\infty.$$

The set $\ker(D+D^T)\cap\Delta(D,\Xi)$ is nonempty, closed and convex and from properties $c)$ and $d)$ of Proposition 2.1, we get:

$$\begin{aligned}(\Xi+\Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)})_\infty &= \Xi_\infty+(\Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)})_\infty\\ &= \Xi_\infty+\Psi_{(\ker(D+D^T)\cap\Delta(D,\Xi))_\infty}.\end{aligned}$$

The sets $\ker(D+D^T)$ and $\Delta(D,\Xi)$ are closed and convex and the intersection $\ker(D+D^T)\cap\Delta(D,\Xi)$ is nonempty (since $0\in\ker(D+D^T)\cap\Delta(D,\Xi)$). Thus using property $g)$ of Proposition 2.1:

$$(\ker(D+D^T)\cap\Delta(D,\Xi))_\infty=\ker(D+D^T)_\infty\cap\Delta(D,\Xi)_\infty$$

Both $\ker(D+D^T)$ (as a vector subspace of $\mathbb{R}^p$) and $\Delta(D,\Xi)$ are closed convex cones and thus from poperty $e)$ of Proposition 2.1, we obtain $\ker(D+D^T)_\infty=\ker(D+D^T)$ and $\Delta(D,\Xi)_\infty=\Delta(D,\Xi)$. The result in (25) follows.

Let us now prove the inclusion in (23). Let $q\in\mathcal{R}(D+\partial\Xi)$. There exists $x\in\mathrm{Dom}(\partial\Xi)$ such that $q\in Dx+\partial\Xi(x)$, i.e.

$$\langle Dx-q,v-x\rangle+\Xi(v)-\Xi(x)\geq 0,\quad \forall v\in\mathbb{R}^p.$$

Thus

$$\langle Dx-q,e\rangle+\Xi(x+e)-\Xi(x)\geq 0,\quad \forall e\in\ker(D+D^T),$$

where we used the fact that e may be considered to belong to any set in $\mathbb{R}^p$. For all $e\in\ker(D+D^T)$ we have $\langle Dx,e\rangle=\langle x,D^Te\rangle=-\langle x,De\rangle=-\langle D^Tx,e\rangle$ and thus

$$-\langle(D^Tx+q),e\rangle+\Xi(x+e)-\Xi(x)\geq 0,\quad \forall e\in\ker(D+D^T).$$

Using property $a)$ of Proposition 2.1, we see that:

$$\Xi_\infty(e)\geq\Xi(x+e)-\Xi(x).$$

Thus

$$-\langle(D^Tx+q),e\rangle+\Xi_\infty(e)\geq 0,\quad \forall e\in\ker(D+D^T)$$

and consequently

$$-\langle(D^Tx+q),e\rangle+\Xi_\infty(e)\geq 0,\quad \forall e\in\ker(D+D^T)\cap\Delta(D,\Xi).$$

This last inequality is clearly equivalent to

$$-\langle(D^Tx+q),e\rangle+\Xi_\infty(e)+\Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)}(e)\geq 0,\quad \forall e\in\mathbb{R}^p.$$

Using the result in (25), we get:

$$-\langle(D^Tx+q),e\rangle+\Upsilon_\infty(e)\geq 0,\quad \forall e\in\mathbb{R}^p.$$

As a consequence of Proposition 4.2, we get

$$q+D^Tx\in\overline{\mathrm{Dom}(\Upsilon^\star)}.$$

It results that

$$q \in -D^T(\mathrm{Dom}(\partial\Xi)) + \overline{\mathrm{dom}(\Upsilon^\star)},$$

which proves (23).

It remains to prove the inclusion in (24). Let $q \in -D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{Dom}(\Upsilon^\star)\}$ be given. For all $i \in \mathbb{N}, i \neq 0$, the matrix $\frac{1}{i}I + D$ is positive definite and there exists $u_i \in \mathrm{Dom}(\partial\Xi)$ such that

$$\left\langle \left(\frac{1}{i}I + D\right) u_i - q, v - u_i \right\rangle + \Xi(v) - \Xi(u_i) \geq 0, \quad \forall v \in \mathbb{R}^p. \tag{26}$$

We claim that the sequence $\{u_i\} \equiv \{u_i;\ i \in \mathbb{N}\backslash\{0\}\}$ is bounded. Suppose on the contrary that $||u_i|| \to +\infty$ as $i \to +\infty$. Then, for i large enough, $||u_i|| \neq 0$ and we may set: $z_i := \frac{u_i}{||u_i||}$. There exists a subsequence, again denoted by $\{z_i\}$, such that $\lim_{i\to+\infty} z_i = z$ with $||z|| = 1$.

Let us first remark that

$$(\forall\, i \in \mathbb{N}\backslash\{0\}): \ u_i \in \mathrm{Dom}(\partial\Xi) \subset \mathrm{Dom}(\Xi) \subset \overline{\mathrm{dom}(\Xi)}.$$

The set $\overline{\mathrm{Dom}(\Xi)}$ is nonempty, closed and convex. Let $x_0 \in \overline{\mathrm{Dom}(\Xi)}$ be any element in $\overline{\mathrm{Dom}(\Xi)}$. Let $\lambda > 0$ be given. For i large enough, $\frac{\lambda}{||u_i||} < 1$ and thus

$$\frac{\lambda}{||u_i||}u_i + \left(1 - \frac{\lambda}{||u_i||}\right) x_0 \in \overline{\mathrm{dom}(\Xi)}.$$

Taking the limit as $i \to +\infty$, we get $\lambda z + x_0 \in \overline{\mathrm{Dom}(\Xi)}$. This result holds for any $\lambda > 0$ and thus

$$z \in \bigcap_{\lambda>0} \frac{1}{\lambda}(\overline{\mathrm{Dom}(\Xi)} - x_0) = \overline{\mathrm{Dom}(\Xi)}_\infty. \tag{27}$$

Let $e \in \mathrm{Dom}(\Xi_\infty)$ be given. From part $a)$ of Proposition 2.1, we have:

$$\Xi(u_i + e) \leq \Xi(u_i) + \Xi_\infty(e) < +\infty,$$

and thus $u_i + e \in \mathrm{Dom}(\Xi)$. We may set $v = u_i + e$ in (26) to get

$$\left\langle \frac{1}{i}u_i + (Du_i - q), e \right\rangle + \Xi(u_i + e) - \Xi(u_i) \geq 0$$

and thus

$$\left\langle \frac{1}{i}u_i, e \right\rangle + \langle Du_i - q, e\rangle + \Xi_\infty(e) \geq 0.$$

Notice that $\Xi_\infty(e) < +\infty$ since $e \in \mathrm{Dom}(\Xi_\infty)$ and we may therefore divide this last relation by $||u_i||$ to get:

$$\left\langle \frac{1}{i}z_i, e \right\rangle + \left\langle Dz_i - \frac{q}{||u_i||}, e \right\rangle + \frac{1}{||u_i||}\Xi_\infty(e) \geq 0.$$

Taking the limit as $i \to +\infty$, we get $\langle Dz, e \rangle \geq 0$. This holds for any $e \in \mathrm{Dom}(\Xi_\infty)$ and thus

$$Dz \in \mathrm{Dom}(\Xi_\infty)^\star. \tag{28}$$

Let $X_0 \in \mathrm{Dom}(\Xi)$ be given and set $v = X_0$ in (26), we obtain:

$$\frac{1}{i}||u_i||^2 + \langle Du_i, u_i \rangle \leq \frac{1}{i}\langle u_i, X_0 \rangle + \langle Du_i, X_0 \rangle - \langle q, X_0 - u_i \rangle + \Xi(X_0) - \Xi(u_i). \tag{29}$$

The function Ξ is proper, convex and lower semi-continuous, and thus there exists $a \geq 0$ and $b \in \mathbb{R}$ such that:

$$\Xi(x) \geq -a||x|| + b, \quad \forall x \in \mathbb{R}^p.$$

Thus

$$\frac{1}{i}||u_i||^2 + \langle Du_i, u_i \rangle \leq a||u_i|| - b + \frac{1}{i}\langle u_i, x_0 \rangle + \langle Du_i, X_0 \rangle - \langle q, X_0 - u_i \rangle + \Xi(X_0).$$

Dividing this last relation by $||u_i||^2$, we get:

$$\begin{aligned} &\frac{1}{i}||z_i||^2 + \langle Dz_i, z_i \rangle \\ &\leq \frac{a}{||u_i||} - \frac{b}{||u_i||^2} + \left\langle Dz_i, \frac{X_0}{||u_i||} \right\rangle + \frac{1}{i}\left\langle z_i, \frac{x_0}{||u_i||} \right\rangle - \left\langle \frac{q}{||u_i||}, \frac{X_0}{||u_i||} - z_i \right\rangle + \frac{\Xi(X_0)}{||u_i||^2}. \end{aligned}$$

Taking the limit as $i \to +\infty$, we get $\langle Dz, z \rangle \leq 0$ and thus

$$z \in \ker(D + D^T)$$

since D is positive semi-definite, and using (13). Until now, we have thus proved that

$$z \in \Delta(D, \Xi) \cap \ker(D + D^T), \; z \neq 0.$$

Here $q \in -D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{dom}(\Upsilon^\star)\}$ and as a consequence of Proposition 4.2, there exists $z_0 \in \mathrm{Dom}(\partial\Xi) \subset \mathrm{Dom}(\Xi)$ such that

$$\Upsilon_\infty(v) > \langle q + D^T z_0, v \rangle, \quad \forall v \in \mathbb{R}^p.$$

Using (25), we obtain

$$\Xi_\infty(v) > \langle q + D^T z_0, v \rangle, \quad \forall v \in \ker(D + D^T) \cap \Delta(D, \Xi), \; v \neq 0,$$

and in particular

$$\Xi_\infty(z) > \langle q + D^T z_0, z \rangle. \tag{30}$$

Using now (26) with $v = z_0$, we get also:

$$\left\langle \left(\frac{1}{i}I + D\right) u_i, u_i - z_0 \right\rangle \leq -\langle q, z_0 - u_i \rangle + \Xi(z_0) - \Xi(u_i).$$

Here $\langle (\frac{1}{i}I + D)u_i, u_i \rangle > 0$ and thus

$$-\left\langle \left(\frac{1}{i}I + D\right) u_i, z_0 \right\rangle - \langle q, u_i - z_0 \rangle - \Xi(z_0) + \Xi(u_i) < 0.$$

Dividing this last relation by $||u_i||$, we get:

$$-\left\langle \left(\frac{1}{i}I + D\right) z_i, z_0 \right\rangle - \left\langle q, z_i - \frac{z_0}{||u_i||} \right\rangle - \frac{\Xi(z_0)}{||u_i||} + \frac{\Xi(||u_i||z_i)}{||u_i||} < 0.$$

Taking the limit inferior as $i \to +\infty$, we get:

$$-\langle (q + D^T z_0), z \rangle + \liminf_{i \to +\infty} \frac{\Xi(||u_i||z_i)}{||u_i||} \leq 0$$

and thus

$$-\langle (q + D^T z_0), z \rangle + \Xi_\infty(z) \leq 0,$$

which is a contradiction to (30). The sequence $\{u_i\}$ is thus bounded and we may find a subsequence, again denoted by $\{u_i\}$ such that $\lim_{i \to +\infty} u_i = u$. Let $v \in \mathbb{R}^p$ be given, we have

$$-\left\langle \left(\frac{1}{i}I + D\right) u_i - q, v - u_i \right\rangle - \Xi(v) + \Xi(u_i) \leq 0$$

and taking the limit inferior as $i \to +\infty$, we get

$$-\langle Du - q, v - u \rangle - \Xi(v) + \Xi(u) \leq 0.$$

This holds for any $v \in \mathbb{R}^p$ so that $q \in \mathcal{R}(D + \partial\Xi)$. □

Remark 4.6. i) Recalling that D is positive semi-definite, we remark that:

$$-D^T(\mathrm{Dom}(\partial\Xi)) \subset -D^T(\mathbb{R}^p) = \mathcal{R}(-D^T) = \mathcal{R}(D^T) = \mathcal{R}(D).$$

ii) If $\mathrm{Dom}(\partial\Xi) = \mathbb{R}^p$ then clearly

$$-D^T(\mathrm{Dom}(\partial\Xi)) = \mathcal{R}(D).$$

Remark 4.7. Theorem 4.5 allows us to characterize the set $\mathcal{R}(D + \partial\Xi)$. Other characterizations exist [7]. In summary, if A and B are two monotone operators that satisfy some property $(*)$ and such that $A+B$ is maximal monotone, one has $\overline{\mathcal{R}(A+B)} = \overline{\mathcal{R}(A) + \mathcal{R}(B)}$ and $\mathrm{int}[\mathcal{R}(A+B)] = \mathrm{int}[\mathcal{R}(A)+\mathcal{R}(B)]$ (see Theorems 3, 4 in [7]). The property $(*)$ is for instance satisfied by linear monotone operators A satisfying $\langle Au, u \rangle \geq \alpha |Au|^2$. When $\mathcal{R}(A) + \mathcal{R}(B)$ is closed and both operators A and B are linear, then one has $\mathcal{R}(A+B) = \mathcal{R}(A)+\mathcal{R}(B)$ (see footnote (2) page 174 in [7]).

5. An existence and uniqueness result

The problem $NSLS(A, B, C, D, f, \Phi, x_0)$ may be written as:

$$\begin{cases} x(0) = x_0 \\ x'(t) = A(x(t)) + B\lambda(t) + f(t), \text{ a.e. } t \geq 0, \\ Cx(t) + D\lambda(t) \in -\partial\Xi(\lambda(t)), \ \forall t \geq 0, \end{cases}$$

or equivalently

$$\begin{cases} x(0) = x_0 \\ x'(t) \in A(x(t)) + B(D + \partial\Xi)^{-1}(-Cx(t)) + f(t), \text{ a.e. } t \geq 0. \end{cases}$$

Assumption 5.1. There exists a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$PB = C^T.$$

Let us now denote by $R \in \mathbb{R}^{n\times n}$ a symmetric positive definite square root of P, *i.e.* $R^2 = P$. We have $R^2B = C^T$ and thus

$$RB = R^{-1}C^T = R^{-T}C^T.$$

We set:

$$(\forall t \geq 0): \; z(t) = -Rx(t).$$

The problem $NSLS(A, B, C, D, f, \Phi, x_0)$ may be thus rewritten as:

$$\begin{cases} -Rx(0) = -Rx_0 \\ -Rx'(t) \in -RA(R^{-1}Rx(t)) - RB(D + \partial\Xi)^{-1}(-CR^{-1}Rx(t)) - Rf(t), \\ \qquad\qquad \text{a.e. } t \geq 0 \end{cases}$$

or equivalently

$$\begin{cases} z(0) = -Rx_0 \\ z'(t) \in -RA(-R^{-1}z(t)) - R^{-T}C^T(D + \partial\Xi)^{-1}(CR^{-1}z(t)) - Rf(t), \; \text{a.e. } t \geq 0 \end{cases}$$

Assumption 5.2. There exists $\omega \in \mathbb{R}$ such that $RA(-R^{-1}.) + \omega I$ is maximal monotone on $\mathbb{R}^n$.

Remark 5.3. i) Assumption 5.1 was first introduced in [8]. It corresponds to an "input-output" constraint that is satisfied by observable dissipative systems [15].

ii) Assumption 5.2 holds with $\omega = 0$ if $RA(-R^{-1}.)$ is maximal monotone on $\mathbb{R}^n$.

iii) If $A(\cdot)$ is Lipschitz continuous, *i.e.* there exists $K > 0$ such that

$$(\forall x, y \in \mathbb{R}^n): \; \|A(x) - A(y)\| \leq K\|x - y|,$$

then Assumption 5.2 is satisfied with $\omega \geq K\|R\|\|R^{-1}\|$. Indeed, we have

$$\begin{aligned} &\langle RA(-R^{-1}x) - RA(-R^{-1}(y), x - y\rangle + \omega\|x - y\|^2 \\ \geq\; &-\|R\|\|A(-R^{-1}x) - A(-R^{-1}(y))\|\|x - y\| + \omega\|x - y\|^2 \\ \geq\; &-K\|R\|\|R^{-1}\|\|x - y\|^2 + \omega\|x - y\|^2 \geq 0. \end{aligned}$$

In particular if $A(\cdot)$ is piecewise-linear (hence Lipschitz continuous) $A(\cdot) + \omega I$ is piecewise-linear monotone and, so, maximal. Thus it satisfies Assumption 5.2.

Assumption 5.4. We suppose that

(a) $$\mathcal{R}(C) \cap \left(-D^T(\text{rint}\{\text{Dom}(\partial\Xi)\}) + \text{int}\{\text{Dom}(\Upsilon^\star)\}\right) \neq \emptyset,$$

(b) $$\text{Dom}(\partial\Xi) \text{ and } \text{Dom}(D + \partial\Xi)^{-1} \text{ are convex sets,}$$

(c) $$\text{aff}(-D^T(\text{Dom}(\partial\Xi)) + \text{int}(\text{Dom}(\Upsilon^\star)) = \text{aff}(\text{Dom}(D + \partial\Xi)^{-1}).$$

Proposition 5.5. *If Assumptions* 3.1, 4.1, 5.1 *and* 5.4 *hold then the operator*

$$R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}.):\mathbb{R}^n\rightrightarrows\mathbb{R}^n;\qquad x\mapsto R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}x)$$

is maximal monotone.

Proof. The theorem of maximal monotonicity under composition (see Theorem 12.43 in [36]) ensures that $R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}.)$ is maximal monotone provided that

$$\mathcal{R}(CR^{-1})\cap\operatorname{rint}\{\operatorname{Dom}((D+\partial\Xi)^{-1})\}\neq\emptyset.$$

Here $\mathcal{R}(CR^{-1})=\mathcal{R}(C)$ and from Theorem 4.5, we know that

$$-D^T(\operatorname{Dom}(\partial\Xi))+\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}\subset\mathcal{R}(D+\partial\Xi)=\operatorname{Dom}((D+\partial\Xi)^{-1})$$

and thus from Assumption 5.4 and [22, Lemma 16.2.3] one has

$$\operatorname{rint}\{-D^T(\operatorname{Dom}(\partial\Xi))+\operatorname{int}\{\operatorname{dom}(\Upsilon^\star)\}\}\subset\operatorname{rint}\{\operatorname{Dom}((D+\partial\Xi)^{-1})\}.$$

The sets $-D^T(\operatorname{Dom}(\partial\Xi))$ and $\operatorname{int}\{\operatorname{dom}(\Upsilon^\star)\}$ are convex (see [35, Theorem 3.4] and see (22)) so that (see Exercise 2.45 and Proposition 2.44 in [36]):

$$\begin{aligned}&\operatorname{rint}\{-D^T(\operatorname{Dom}(\partial\Xi))+\operatorname{int}\{\operatorname{dom}(\Upsilon^\star)\}\}\\ =\ &\operatorname{rint}\{-D^T(\operatorname{Dom}(\partial\Xi))\}+\operatorname{rint}\{\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}\}\\ =\ &-D^T(\operatorname{rint}\{\operatorname{Dom}(\partial\Xi)\})+\operatorname{rint}\{\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}\}.\end{aligned}$$

The set $K=\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}$ is open in $\mathbb{R}^p$ and thus $\operatorname{rint}\{K\}=K$. Indeed, if $x\in K$ then $x\in\operatorname{int}\{K\}=K$ so that $x\in\operatorname{rint}\{K\}$ and $K\subset\operatorname{rint}\{K\}\subset K$. Thus

$$\begin{aligned}&\operatorname{rint}\{-D^T(\operatorname{Dom}(\partial\Xi))+\operatorname{int}\{\operatorname{dom}(\Upsilon^\star)\}\}\\ =\ &-D^T(\operatorname{rint}\{\operatorname{Dom}(\partial\Xi)\})+\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}.\end{aligned}$$

It results that if $\mathcal{R}(C)\cap\big(-D^T(\operatorname{rint}\{\operatorname{Dom}(\partial\Xi)\})+\operatorname{int}\{\operatorname{Dom}(\Upsilon^\star)\}\big)\neq\emptyset$ then

$$\mathcal{R}(CR^{-1})\cap\operatorname{rint}\{\operatorname{Dom}((\mathrm{D}+\partial\Xi)^{-1})\}\neq\emptyset$$

and the result follows. □

Theorem 5.6. *Suppose that Assumptions* 3.1–5.4 *hold. Suppose also that*

$$f\in C^0([0,+\infty[;\mathbb{R}^n),\qquad f'\in L^1_{\text{loc}}(0,+\infty[;\mathbb{R}^n).$$

If

$$-Cx_0\in-D^T(\operatorname{Dom}(\partial\Xi))+\operatorname{int}\{\operatorname{dom}(\Upsilon^\star)\}$$

then there exists a unique $z\in C^0([0,+\infty[;\mathbb{R}^n)$ *such that* $z'\in L^\infty_{\text{loc}}(0,+\infty[;\mathbb{R}^n)$, z *is right-differentiable on* $[0,+\infty[$ *and satisfying the relations:*

$$\begin{cases}z(0)=-Rx_0\\ z'(t)\in-RA(-R^{-1}z(t))-R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}z(t))-Rf(t),\ \text{a.e. } t\geq0,\\ CR^{-1}z(t)\in-D^T(\operatorname{Dom}(\partial\Xi))+\overline{\operatorname{Dom}(\Upsilon^\star)},\ \forall t\geq0.\end{cases}$$

Proof. Let us set

$$T(x) = \tag{31}$$
$$\begin{cases} RA(-R^{-1}x) + R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}x) & \text{if } CR^{-1}x \in \mathrm{Dom}((D+\partial\Xi)^{-1}) \\ \emptyset & \text{if } CR^{-1}x \notin \mathrm{Dom}((D+\partial\Xi)^{-1}) \end{cases}$$

We remark that the operator $T + \omega I$ is maximal monotone. Indeed, Assumption 5.2 ensures that the operator $x \mapsto RA(-R^{-1}x) + \omega x$ is maximal monotone and Assumption 5.4 entails that the operator $x \rightrightarrows R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}x)$ is maximal monotone. We have $\mathrm{int}\{\mathrm{Dom}(RA(-R^{-1}.) + \omega I)\} = \mathbb{R}^n$ and $-Cx_0 \in -D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{dom}(\Upsilon^\star)\} \subset \mathrm{Dom}((D+\partial\Xi)^{-1})$. Thus

$$-Rx_0 \in \mathrm{int}\{\mathrm{Dom}(RA(-R^{-1}.) + \omega I)\} \cap \mathrm{Dom}(R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}.)).$$

The set $\mathrm{int}\{\mathrm{Dom}(RA(-R^{-1}.) + \omega I)\} \cap \mathrm{Dom}(R^{-T}C^T(D+\partial\Xi)^{-1}(CR^{-1}.))$ is nonempty and thus the Sum Theorem of Rockafellar (see Theorem 32.I in [37]) ensures that the operator $T + \omega I$ is also maximal monotone. Note that

$$\mathrm{Dom}(T) = \{z \in \mathbb{R}^n : \; CR^{-1}z \in \mathrm{Dom}((D+\partial\Xi)^{-1})\}.$$

Here $-Cx_0 \in \mathrm{Dom}((D+\partial\Xi)^{-1})$ and thus $-Rx_0 \in \mathrm{Dom}(T)$. Using a version of Kato's Theorem (see Corollary 2.2 in [25]), we obtain the existence of a unique $z \in C^0([0,+\infty[;\mathbb{R}^n)$ such that $z' \in L^\infty_{\mathrm{loc}}(0,+\infty[;\mathbb{R}^n)$, z is right-differentiable on $[0,+\infty[$ and satisfies the relations:

$$\begin{cases} z(0) = -Rx_0 \\ 0 \in z'(t) + T(z(t)) + Rf(t), \text{ a.e. } t \geq 0 \\ z(t) \in \mathrm{Dom}(T), \;\; \forall t \geq 0. \end{cases}$$

This gives the result since $z(t) \in \overline{\mathrm{Dom}(T)}$ entails that $CR^{-1}z(t) \in \mathrm{Dom}((D+\partial\Xi)^{-1}) \subset -D^T(\mathrm{Dom}(\partial\Xi)) + \overline{\mathrm{Dom}(\Upsilon^\star)}$. □

Corollary 5.7. *Suppose that Assumptions 3.1–5.4 hold. Suppose also that*

$$f \in C^0([0,+\infty[;\mathbb{R}^n), \qquad f' \in L^1_{\mathrm{loc}}(0,+\infty[;\mathbb{R}^n).$$

If

$$-Cx_0 \in -D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{dom}(\Upsilon^\star)\}$$

then there exists a unique $x \in C^0([0,+\infty[;\mathbb{R}^n)$ such that $x' \in L^\infty_{\mathrm{loc}}(0,+\infty[;\mathbb{R}^n)$, x is right-differentiable on $[0,+\infty[$ and satisfies the relations:

$$\begin{cases} x(0) = x_0 \\ x'(t) \in A(x(t)) + B(D+\partial\Xi)^{-1}(-Cx(t)) + f(t), \text{ a.e. } t \geq 0, \\ -Cx(t) \in -D^T(\mathrm{Dom}(\partial\Xi)) + \overline{\mathrm{Dom}(\Upsilon^\star)}, \;\; \forall t \geq 0. \end{cases} \tag{32}$$

Remark 5.8. The implicit Euler numerical method studied in [5] can be applied to the differential inclusion in (32), with order of convergence in-between $\frac{1}{2}$ and 1 depending on the multivalued mapping.

6. Examples

In this section it is shown that the above developments apply to important application cases: complementarity systems, relay systems, and mechanical systems with Coulomb's friction.

6.1. Nonlinear cone complementarity systems

Let us consider that $\Phi = \Psi_K$ where $K \subseteq \mathbb{R}^p$ is a non empty, closed convex cone. The inclusion $\lambda \in -\partial\Phi(y)$ is then equivalent to the complementarity relation $K^\star \ni \lambda \perp y \in K$, and the Lur'e system is a nonlinear cone complementarity system (NLCCS). Then $\Xi(z) = \Psi_K^\star(-z) = \Psi_{-K^\star}(-z) = \Psi_{K^\star}(z)$. Let us check Assumption 5.4. For this we first need to calculate the set $\Delta(D, \Xi)$ in (21). We have $\Xi_\infty = \Psi_{K^\star_\infty} = \Psi_{K^\star} = \Xi$, so $\mathrm{dom}(\Xi) = \mathrm{dom}(\Xi_\infty) = \mathrm{Dom}(\partial\Xi) = K^\star$, $\mathrm{dom}(\Xi_\infty)^\star = (K^\star)^\star = K$, $\overline{\mathrm{Dom}(\Xi)}_\infty = (K^\star)_\infty = K^\star$. Hence $\Delta(D, \Xi) = \{z \in K^\star :\ Dz \in K\}$. Now we have $\Upsilon = \Psi_{K^\star} + \Psi_{\ker(D+D^T)} + \Psi_{\Delta(D,\Xi)} = \Psi_{K^\star \cap \ker(D+D^T) \cap \Delta(D,\Xi)} = \Psi_{\mathcal{C}}$ where $\mathcal{C} = K^\star \cap \ker(D+D^T) \cap \Delta(D, \Xi)$. Thus $\Upsilon^\star = \Psi_{-\mathcal{C}^\star}$ and $\mathrm{int}\{\mathrm{Dom}(\Upsilon^\star)\} = \mathrm{int}\{-\mathcal{C}^\star\}$, $\mathrm{rint}\{\mathrm{Dom}(\partial\Xi)\} = \mathrm{rint}\{K^\star\}$. The conditions of Assumption 5.4 thus reduce to:

$$\begin{cases} (a) & \mathcal{R}(C) \cap \left(-D^T(\mathrm{rint}\{K^\star\}) + \mathrm{int}\{-\mathcal{C}^\star\}\right) \neq \emptyset, \\ (b) & K^\star \text{ convex, } \mathcal{R}(D + \partial\Psi_{K^\star}) \text{ convex,} \\ (c) & \mathrm{aff}(-D^T K^\star + \mathrm{int}(-\mathcal{C}^\star)) = \mathrm{aff}(\mathcal{R}(D + \partial\Psi_{K^\star})), \end{cases} \tag{33}$$

where we used that $\mathcal{R}(C) = \mathcal{R}(CR^{-1})$ in view of Assumption 5.1. Here $\partial\Psi_{K^\star} = N_{K^\star}$, the normal cone to $K^\star$. Now we have that for all $x \in K^\star$, $N_{K^\star}(x) = x^\perp \cap (-K)$, so $N_{K^\star} \subset -K$. But for $x = 0$ we have that $N_{K^\star}(0) = -K$, so finally $\mathcal{R}(N_{K^\star}) = -K$ (hence (18) holds).

• Suppose that $D = 0$. Then $\Delta(0, \Xi) = K^\star$, $\ker(D + D^T) = \mathbb{R}^p$, and $\mathcal{C} = K^\star$, $\mathcal{C}^\star = K$. Checking (33) (a) (b) (c) then boils down to verifying that:

$$\begin{cases} (a) & \mathcal{R}(C) \cap \mathrm{int}\{-K\} \neq \emptyset, \\ (b) & K^\star \text{ and } K \text{ are convex,} \\ (c) & K - K = -K - (-K), \end{cases} \tag{34}$$

where $K - K$ denotes the set $\{x - y \mid x \in K, y \in K\}$, and condition (c) is obtained using [35, Theorem 2.7]. Conditions (34) (b) and (c) trivially hold. We infer that Assumption 5.4 is satisfied provided that (34) (a) is satisfied. So provided Assumptions 5.1, 5.2 and (34) (a) are satisfied, the system is always well-posed in the sense of Theorem 5.6.

• Suppose now that D is positive definite. Then $\Delta(D, \Xi) = K^\star \cap D^{-1}K$, where $D^{-1}K = \{z \in \mathbb{R}^p \mid Dz \in K\}$, and $\ker(D + D^T) = \{0\}$. Thus $\mathcal{C} = \{0\}$, $\mathcal{C}^\star = \mathbb{R}^p$, $\Upsilon = \Psi_{K^\star} + \Psi_{\{0\}} + \Psi_{K^\star \cap D^{-1}K} = \Psi_{\{0\}}$, and $\Upsilon^\star = \Psi_{-\{0\}^\star} = \Psi_{\mathbb{R}^p} = 0$. Consequently (33) reduces to:

$$\begin{cases} (a) & \mathcal{R}(C) \cap (-D^T(\mathrm{rint}\{K^\star\}) + \mathrm{int}\{\mathbb{R}^p\}) \neq \emptyset, \\ (b) & K^\star \text{ convex, } \mathcal{R}(D + N_{K^\star}) = \mathbb{R}^p \text{ convex,} \\ (c) & \mathrm{aff}(-D^T K^\star + \mathrm{int}(\mathbb{R}^p)) = \mathrm{aff}(\mathcal{R}(D + \partial\Psi_{K^\star})). \end{cases} \tag{35}$$

Condition (35) (a) is always satisfied since $\mathcal{R}(C) \cap (-D^T(\mathrm{rint}\{K^\star\}) + \mathrm{int}\{\mathbb{R}^p\}) = \mathcal{R}(C) \cap (-D^T(\mathrm{rint}\{K^\star\}) + \mathbb{R}^p) = \mathcal{R}(C) \neq \emptyset$ and $0 \in \mathcal{R}(C)$. Hence (33) (a) is satisfied. Using (23) (24) it follows that condition (35) (b) is satisfied, and condition (35) (c) reduces to $\mathbb{R}^p = \mathbb{R}^p$. So provided Assumptions 5.1 and 5.2 are satisfied, the systems is always well-posed in the sense of Theorem 5.6.

• Suppose now that D is positive semi-definite and $K = \mathbb{R}^p_+$. In this case we get that $\Delta(D, \Xi) = \{z \in \mathbb{R}^p_+ \mid Dz \in \mathbb{R}^p_+\}$, so $\Delta(D, \Xi)$ is a $D-$invariant subspace of $\mathbb{R}^p_+$. So $\mathcal{C} = \mathbb{R}^p_+ \cap \ker(D + D^T) \cap \Delta(D, \Xi) = \ker(D + D^T) \cap \Delta(D, \Xi)$, and by [35, Corollary 16.4.2] we have that $\mathcal{C}^\star = \ker^\star(D + D^T) + \Delta^\star(D, \Xi)$. Now by [6, Example 2.2.2] it follows that $\ker^\star(D + D^T) = \ker^\perp(D + D^T) = \mathcal{R}(D + D^T)$ because $D + D^T$ is symmetric. Also it is easy to see that $\Delta^\star(D, \Xi) = \mathbb{R}^p_+$, because $\Delta(D, \Xi) \subset \mathbb{R}^p_+$ so that $\mathbb{R}^p_+ \subset \Delta^\star(D, \Xi)$ while $\Delta(D, \Xi) \subset \mathbb{R}^p_+$. So $\mathcal{C}^\star = \mathcal{R}(D + D^T) + \mathbb{R}^p_+$. Thus the conditions (33) reduce to:

$$\begin{cases} (a) & \mathcal{R}(C) \cap (-D^T\,\mathbb{R}^p_+ + \mathrm{int}(\mathcal{R}(D + D^T) + \mathbb{R}^p_+)) \neq \emptyset, \\ (b) & \mathcal{R}(D + N_{\mathbb{R}^p_+}) \text{ convex}, \\ (c) & \mathrm{aff}(-D^T\,\mathbb{R}^p_+ + \mathrm{int}(\mathcal{R}(D + D^T) + \mathbb{R}^p_+)) = \mathrm{aff}(\mathcal{R}(D + N_{\mathbb{R}^p_+})). \end{cases} \tag{36}$$

Condition (36) (a) holds since both sets contain $\{0\}$, and using (23) (24) it follows that (36) (b) holds. Thus it only remains to check (36) (c).

Remark 6.1. Existence and uniqueness of continuously differentiable solutions for any $x(0) \in \mathbb{R}^n$ have been derived for linear cone complementarity systems (LCCS) in [19, 20], with $D \geq 0$, basing on the fact that $B\lambda$ is a singleton. Our analysis and results are different. When $A(\cdot)$ is a linear operator and $K = \mathbb{R}^p_+$ the system reduces to so-called linear complementarity systems which have been studied in the field of Systems and Control [18, 28].

6.2. Nonlinear relay systems

For $x \in \mathbb{R}$ let

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0. \end{cases}$$

Let us now suppose that $\Phi(y) = |y_1| + ... + |y_p|$, so that $\partial\Phi(y) = \mathrm{Sgn}(y)$ where $\mathrm{Sgn}(y) = (\mathrm{sgn}(y_1) \;\cdots\; \mathrm{sgn}(y_p))^T$. Such systems are called *relay systems* in the Systems and Control literature. We can also consider Lipschitz single valued nonlinear terms in the relay function and incorporate them into the $A(\cdot)$ term. One has $\Xi(\lambda) = \Psi_{[-1,1]^p}(\lambda)$, hence $\mathrm{dom}(\Xi) = [-1, 1]^p$. Thus $\overline{\mathrm{Dom}(\Xi)}_\infty = \{0\}$ so that $\Delta(D, \Xi) = \{0\}$. One has $\Upsilon = \Psi_{[-1,1]^p} + \Psi_{\ker(D+D^T)} + \Psi_{\{0\}} = \Psi_{\{0\}}$, and $\Upsilon^\star = 0$. So $\mathrm{dom}(\Upsilon^\star) = \mathbb{R}^p$, and $\mathrm{Dom}(\partial\Xi) = [-1, 1]^p$. The conditions of Assumption 5.4 can be written as:

$$\begin{cases} (a) & \mathcal{R}(C) \cap (-D^T[-1, 1]^p + \mathbb{R}^p) \neq \emptyset, \\ (b) & \mathrm{Dom}(\partial\Xi) \text{ and } \mathcal{R}(D + \partial\Xi) \text{ are convex sets}, \\ (c) & \mathrm{aff}(-D^T[-1, 1]^p + \mathbb{R}^p) = \mathrm{aff}(\mathcal{R}(D + N_{[-1,1]^p})). \end{cases} \tag{37}$$

Condition (a) is satisfied since $\{0\}$ belongs to both sets. Using (23) and (24) condition (b) holds. One has $-D^T[-1,1]^p \subset \mathbb{R}^p$, and since $\mathcal{R}(N_{[-1,1]^p}) = \mathbb{R}^p$ one has $\mathcal{R}(D + N_{[-1,1]^p}) = \mathbb{R}^p$. So the condition (c) is always satisfied. Therefore provided the Assumptions 5.1 and 5.2 hold the system is well-posed in the sense of Theorem 5.6. As expected there is no constraint on the state since $\mathrm{dom}(\Upsilon^\star) = \mathbb{R}^p$.

We may state more general results when Ξ is such that $\overline{\mathrm{Dom}(\Xi)}$ is a convex compact non empty set. Then $\overline{\mathrm{Dom}(\Xi)}_\infty = \{0\}$ and $\Delta(D, \Xi) = \{0\}$. Then $\Upsilon = \Psi_{\{0\}}$, and $\Upsilon^\star = 0$. So $\mathrm{dom}(\Upsilon^\star) = \mathbb{R}^p$, and provided that $\mathrm{rint}\{\mathrm{Dom}(\partial\Xi)\} \neq \emptyset$, the condition (a) of Assumption 5.4 is satisfied. In such a case there is no constraint on the state. These results permit to extend existing well-posedness results on relay systems, see [24, §6].

Example 6.2 (system with dry friction). Let us consider a mechanical system made of two masses m_1 and m_2 moving horizontally, sliding on the ground, and linked by a spring with possible nonlinear characteristic $k(\cdot)$, see Figure 6.1. The masses are acted upon by some forces $F_{1,t}(t) + F_{1,x}(x_1, x_2, x_1', x_2')$ and $F_{2,t}(t) + F_{2,x}(x_1, x_2, x_1', x_2')$, respectively, that represent exogeneous forces and state dependent forces (like feedback control). The friction coefficients are $\mu_1 > 0$ and $\mu_2 > 0$. The positions are given by x_1 and x_2, respectively. We assume that during the sliding phases (non zero relative tangential velocity v_{tan}), the friction coefficient is varying, of the form $\mu_i(v_{tan})$. The dynamics is given by:

$$\begin{cases} m_1 x_1''(t) \in -m_1 g\mu_1 \,\mathrm{sgn}(x_1'(t)) + k(x_2, x_1) - m_1 g \mu_1(x_1'(t)) \\ \qquad\qquad +F_1(t) + F_{1,x}(x_1(t), x_2(t), x_1'(t), x_2'(t)) \\ m_2 x_2''(t) \in -m_2 g\mu_2 \,\mathrm{sgn}(x_2'(t)) - k(x_2, x_1) - m_2 g \mu_2(x_2'(t)) \\ \qquad\qquad +F_2(t) + F_{2,x}(x_1(t), x_2(t), x_1'(t), x_2'(t)) \end{cases} \tag{38}$$

The system may be written compactly as $z'(t) \in A(z(t)) - B\mathrm{Sgn}(Cz(t))$, with $z^T = (x_1\ x_1'\ x_2\ x_2')^T$. Then $B = \begin{pmatrix} 0 & 0 \\ g\mu_1 & 0 \\ 0 & 0 \\ 0 & g\mu_2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Any matrix $P = \mathrm{diag}(p_i) \in \mathbb{R}^{4\times 4}$ with $p_1 > 0$, $p_3 > 0$, $p_2 = \frac{1}{g\mu_1}$, $p_4 = \frac{1}{g\mu_2}$ satisfies $PB = C^T$. Therefore Assumption 5.1 is satisfied. Assumption 4.1 is satisfied since $D = 0$, and the nonlinear terms have to be such that Assumption 5.2 holds true.

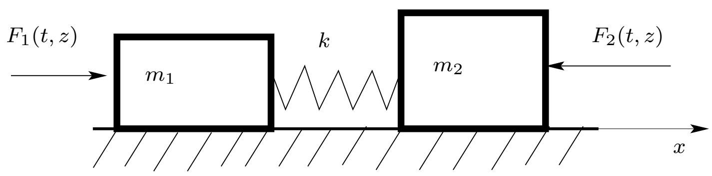

Figure 6.1: A mechanical system with Coulomb's friction.

Relay systems can also model electrical circuits with ideal Zener diodes, see for instance [3, §1.1.3, §1.1.6].

7. A stability result

In this section, we assume that Assumptions 3.1-5.4 hold and we consider the autonomous problem (*i.e.* $f \equiv 0$) that reduces to

$$\begin{cases} z(0) = -Rx_0 \\ z'(t) \in -T(z(t)), \text{ a.e. } t \geq 0, \\ CR^{-1}z(t) \in -D^T(\mathrm{Dom}(\partial\Xi)) + \overline{\mathrm{Dom}(\Upsilon^\star)}, \ \forall t \geq 0. \end{cases} \tag{39}$$

where the operator T is defined in (31). Let us set

$$\mathcal{X}_0 = \{x_0 \in \mathbb{R}^n : \ -Cx_0 \in -D^T(\mathrm{Dom}(\partial\Xi)) + \mathrm{int}\{\mathrm{Dom}(\Upsilon^\star)\}\}.$$

For $x_0 \in \mathcal{X}_0$, we denote by $t \mapsto z(t; x_0)$ the unique solution of the system in (39). Here the operator $T + \omega I$ is maximal monotone, a well-known estimation (see *e.g.* Remark 2.1 in [32]) ensures that for $x_0, y_0 \in \mathrm{Dom}(T)$, we have:

$$(\forall t \geq 0) : \ ||x(t; x_0) - x(t; y_0)|| \leq e^{\omega t}||x_0 - y_0||. \tag{40}$$

When $\omega \geq 0$ then T is said hypomonotone, when $\omega < 0$ it is said strongly monotone [36].

Assumption 7.1. We assume that

$$0 \in \mathcal{X}_0, \ \ A(0) = 0, \ \ 0 \in (D + \partial\Xi)^{-1}(0).$$

Remark 7.2. If

$$\partial\Xi(0) \neq \emptyset$$

and

$$(\forall v \in \ker(D + D^T) \cap \Delta(D, \Xi), v \neq 0) : \ \Xi_\infty(v) > 0,$$

then $0 \in \mathcal{X}_0$. Indeed, condition $\partial\Xi(0) \neq \emptyset$ entails that $0 \in -D^T(\mathrm{Dom}(\partial\Xi))$. Moreover, we have

$$(\forall v \in \mathbb{R}^p, v \neq 0) : \ \Upsilon_\infty(v) = \Xi_\infty(v) + \Psi_{\ker(D+D^T)\cap\Delta(D,\Xi)} > 0,$$

so that $0 \in \mathrm{int}\{\mathrm{Dom}(\Upsilon^\star)$.

Proposition 7.3. *Let Assumptions* 3.1, 4.1, 5.1, 5.4 *and* 7.1 *hold. Let* $\omega = 0$, *then the trivial solution of* (39) *is stable in the sense of Lyapunov. If* $\omega < 0$ *then the trivial solution is globally asymptotically stable.*

Proof. We have

$$0 \in (D + \partial\Xi)^{-1}(0) \Leftrightarrow 0 \in \partial\Xi(0) \Leftrightarrow 0 \in \partial\Phi^\star(0).$$

Asumption 7.1 ensures that

$$(\forall t \geq 0) : \ x(t; 0) = 0.$$

It results from (40), that

$$(\forall t \geq 0): \ ||x(t;x_0)|| \leq e^{\omega t}||x_0||.$$

If $\omega = 0$ then for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$(x_0 \in \mathcal{X}_0, ||x_0|| \leq \delta) \Rightarrow (\forall t \geq 0) : ||x(t;x_0)|| \leq \epsilon.$$

This relation ensures the stability of the trivial solution. If $\omega < 0$ then the trivial solution is stable and

$$(x_0 \in \mathcal{X}_0) \implies \lim_{t\to+\infty} ||x(t;x_0)|| = 0.$$

This last relation ensures the global attractivity of the trivial solution. □

Example 7.4. Consider the nonlinear relay systems of Section 6.2. Assumption 7.1 is satisfied since $0 \in (D + \partial\Xi)^{-1}(0) = \{z \in \mathbb{R}^p : \ 0 \in Dz + \partial\Xi(0) = Dz + \partial\Psi_{[-1,1]^p}(0)\}$. Therefore the stability of the trivial solution depends on the monotonicity constant of the operator T, *i.e.* of the nonlinear mapping $A(\cdot)$.

8. Conclusions

In this paper the well-posedness of multivalued Lur'e dynamical systems is studied. It relies on the maximal monotonicity of an operator that is constructed from the static multivalued part of the system. The conditions under which the maximal monotonicity holds are examined in detail. The developments are illustrated by nonlinear complementarity systems and nonlinear relay systems.

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