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Abstract

The problem of estimating an inter-event distribution on the basis of count data is addressed. A nonparametric maximum likelihood estimate of the inter-event distribution is derived utilizing the EM algorithm both in the case of an ordinary renewal process and in the case of an equilibrium renewal process. In this latter case, the iterative estimation procedure follows the basic scheme proposed by Vardi for estimating an inter-event distribution on the basis of time interval data, i.e. combines the output of the E-step corresponding to the inter-event distribution and to the length-biased distribution. A penalized likelihood approach is also investigated to provide the proposed estimation procedure with regularization capabilities. The practical estimation procedure is evaluated using simulated count data and is applied to real count data representing the elongation of coffee tree leafy axes.

1 Introduction

We consider the problem of estimating an inter-event distribution on the basis of count data collected from either ordinary or equilibrium renewal processes. In different application contexts, including plant growth follow-up (Costes *et al.* 1992), the times of occurrence of some recurrent events cannot be known or recorded but the number of events occurring between two observation dates is accessible to the observation. This type of design is encountered in medical applications for the study of the recurrence rate of a non-fatal event; see Stukel (1993) for a review. In this context, the focus is on the role of covariates which may be fixed or time-dependent while the response is simply summarized in the event rate. In the context of reliability analysis, Dattero and White (1989) proposed an estimation method whose key step is the estimation of the forward recurrence time distribution from count data collected over “short” subperiods resulting from a time sampling of the observation period (for example, for a mean time interval between consecutive events of 1, the observation period of say 10 is sectioned into fixed length subperiods of 0.1, 0.2 or 0.5). This time sampling of the observation period seems to us to be highly restrictive and may be unrealistic for many applications (particularly for plant growth follow-up discussed in this paper).

We therefore propose an estimate of the inter-event distribution which represents time intervals between consecutive events on the basis of count data. The corresponding estimation framework is strongly related to that considered by Vardi (1982b) on the basis of time interval data; see also McClean and Devine (1995) for an estimator based on a different time sampling scheme. Like Vardi (1982b) and McClean and Devine (1995), we assume that the process is discrete, and state our problem as a nonparametric maximum likelihood estimation problem.

Instead of the actual times of occurrence of events, only the number of events occurring during an observation period of known length can be observed. Hence, we are faced with an incomplete data problem and the EM algorithm (Dempster *et al.* 1977; McLachlan and Krishnan 1997) is a natural candidate to derive the nonparametric maximum likelihood estimate of the inter-event distribution. Consequently, the iterative estimation procedures we propose have all the desired properties of the EM algorithm (see McLachlan and Krishnan (1997)), particularly the monotone increasing likelihood in each iteration. It should be noted that the estimators proposed by Vardi (1982b) and McClean and Devine (1995) can be derived utilizing the EM algorithm. In the case of an equilibrium renewal process, we adopted the basic scheme of the estimation procedure proposed by Vardi (1982b) for time interval data: The output of the E-step, that is two independent pseudo-samples, one from the inter-event distribution and the other from the length-biased distribution, are combined in the M-step using the key result of Vardi (1982a).

The proposed estimation procedure is applied to the analysis of count data representing the elongation of leafy axes. The interest of renewal theory for the analysis of plant growth follow-up originated from a work of de Reffye *et al.* (1991) who discussed the pertinence of Bernoulli and Poisson processes as possible models for the elongation of leafy axes. In the context of plant growth follow-up, only count data are easily accessible to measurement because of experimental constraints. Measurement is made of the number of newly elongated leaves during a given observation period.

The design we investigate is the following. Data consist of a sample of independent count measurements of the form $\{(\tau_1, n_1), \ldots, (\tau_r, n_r)\}$ where n_1 is the number of events occurring in the observation period of length τ_1. We wish to estimate the underlying inter-event distribution on the basis of these count data. Two candidate models will be considered:

- ordinary renewal process (Section 2) where the start of the observation period coincides with the occurrence time of an event (synchronism assumption),
- equilibrium or stationary renewal process (Section 3) where the start of the observation period is independent of the process that generates the data (asynchronism assumption).

A penalized likelihood approach based on the one-step-late algorithm proposed by Green (1990) is presented in Section 4. The practical estimation procedure is then summarized in Section 5. A simulation study, whose aim was to investigate the behaviors of the proposed estimators in different contexts depending on the weight of censoring, the type of tail of the inter-event distribution and the sample size, is presented in Section 6. Section 7 is devoted to the application of this estimation procedure for analyzing the elongation of coffee tree leafy axes. Section 8 consists of concluding remarks.

2 Ordinary renewal process

In the following, the estimation problem is stated with a single count data (τ, n) in order to simplify the writing, before generalizing to the practical case of a sample of independent count data. Let $T_1, T_2, \ldots, T_n, \ldots$ be mutually independent, identically distributed discrete non-negative random variables where T_1 represents the time interval from the origin to the first event and $T_2, \ldots, T_n, \ldots$ represent the subsequent time intervals between consecutive events. Their common distribution is called the inter-event distribution. We assume that the inter-event distribution is concentrated on a finite set of time points $\{1, \ldots, M\}$. In the sequel, T is a generic random variable representing a time interval between consecutive events.

The first event occurs at time T_1, the second at time $T_1 + T_2$, the n-th at time $T_1 + \cdots + T_n$. Hence $T_1^n = T_1 + \cdots + T_n$ represents the occurrence time of the n-th event and the corresponding distribution is the n-fold convolution of the inter-event distribution. In the sequel, we will extensively use the notation $T_k^n = T_k + \cdots + T_n$ with $k < n$ (this notation transposes to the corresponding values $t_k^n = t_k + \cdots + t_n$).

The distribution of the number of events $N(\tau)$ occurring in the observation period $(0, \tau]$ (counting distribution), where τ is a strictly positive integer, is deduced from the cumulative distribution functions of the n-fold convolutions of the inter-event distribution (Cox 1962)

$$P\{N(\tau) = n\} = P(T_1^n \leq \tau) - P\left(T_1^{n+1} \leq \tau\right). \tag{1}$$

Recall that our problem consists of estimating an inter-event distribution on the basis of count data. It belongs to the class of incomplete-data problems since we only observe the number of events occurring during a given observation period $(0, \tau]$, not the time intervals between consecutive events. The incompleteness is twofold since both the date of the events occurring within the observation period and the residual time interval after the end of the observation period (right censoring) are non-observable. The estimated inter-event distribution is chosen among the set of inter-event distributions that are concentrated on $\{1, \ldots, M\}$ and hence, following Vardi (1982b), we assume that the value of M is sufficiently large (a practical solution for determining the value of M is proposed in Section 5).

The derivation of the estimator entails mainly deconvolutions under constraints given by count data. For the extraction of the contributions of the underlying time intervals between consecutive events from count data, the case of the n first complete time intervals, i.e. fully included within $(0, \tau]$, should be distinguished from the case of the final time interval which is censored on the right. If $n > 0$, we have the following convolution-type equation for the complete time intervals between consecutive events

$$\begin{aligned} P\{N(\tau) = n\} &= P\left(T_1^n \leq \tau < T_1^{n+1}\right) \\ &= \sum_{t=1}^{\min\{M, \tau-(n-1)\}} P(T_i = t)\, P\left(T_{1\backslash i}^n \leq \tau - t < T_{1\backslash i}^{n+1}\right), \qquad 1 \leq i \leq n, \end{aligned} \tag{2}$$

where $T_{1\backslash i}^n = T_1 + \cdots + T_{i-1} + T_{i+1} + \cdots + T_n$.
Using the generic random variable T, equation (2) becomes

$$P\left(T_1^n \leq \tau < T_1^{n+1}\right) = \sum_{t=1}^{\min\{M, \tau-(n-1)\}} P(T = t)\, P\left(T_1^{n-1} \leq \tau - t < T_1^n\right). \tag{3}$$

For the time interval between consecutive events originating but not terminating within $(0,\tau]$ (i.e. in which the observation date τ falls), we have

$$P\left(T_1^n \leq \tau < T_1^{n+1}\right) = \sum_{t=1}^{M} P\left(T_{n+1} = t\right) P\left(\tau - t < T_1^n \leq \tau\right). \tag{4}$$

Consider the complete-data case where each time interval between consecutive events, including the time interval in which the observation date τ falls, is observed. The complete-data likelihood is given by

$$\begin{aligned} f\left(t_1,\ldots,t_{n+1},n;\theta\right) &= P\left\{T_1 = t_1,\ldots,T_{n+1} = t_{n+1}, N\left(\tau\right) = n;\theta\right\} \\ &= \prod_{i=1}^{n+1} P\left(T_i = t_i;\theta\right) I\left(t_1^n \leq \tau < t_1^{n+1}\right) \\ &= g\left(t_1,\ldots,t_{n+1},n;\theta\right) I\left(t_1^n \leq \tau < t_1^{n+1}\right), \end{aligned}$$

where θ denotes the parameters of the inter-event distribution, i.e. its probability mass function $\{P\left(T = t\right); t = 1,\ldots,M\}$ and $I\left(\ \right)$ denotes the indicator function.

Instead of the time intervals between consecutive events, we only observe the number of events occurring during the observation period $(0,\tau]$. Hence, the objective is to find the estimate of θ which maximizes the likelihood of the observed count data

$$L\left(\theta\right) = \sum_{t_1,\ldots,t_{n+1}} f\left(t_1,\ldots,t_{n+1},n;\theta\right) = P\left\{N\left(\tau\right) = n;\theta\right\},$$

where $\sum_{t_1,\ldots,t_{n+1}}$ means sum on every possible combination $t_1,\ldots,t_{n+1}$ such that $t_1^n \leq \tau < t_1^{n+1}$.

Let $\theta^{(k)}$ denote the current value of θ at iteration k. The conditional expectation of the complete-data log-likelihood is given by

$$\begin{aligned} Q(\theta|\theta^{(k)}) &= E\left\{\log g\left(T_1,\ldots,T_{N(\tau)+1};\theta\right) | N\left(\tau\right) = n;\theta^{(k)}\right\} \\ &= \sum_{t_1,\ldots,t_{n+1}} P\left(T_1 = t_1,\ldots,T_{n+1} = t_{n+1} | T_1^n \leq \tau < T_1^{n+1};\theta^{(k)}\right) \\ &\quad \times \left\{\sum_{i=1}^{n+1} \log P\left(T_i = t_i;\theta\right)\right\} \\ &= \sum_{t=1}^{M} \left\{\sum_{i=1}^{n+1} \sum_{t_1,\ldots,t_{n+1}} P\left(T_1 = t_1,\ldots,T_{n+1} = t_{n+1} | T_1^n \leq \tau < T_1^{n+1};\theta^{(k)}\right)\right. \\ &\quad \left. \times I\left(t_i = t\right)\right\} \log P\left(T = t;\theta\right) \\ &= \sum_{t=1}^{M} \xi_t^{(k)} \log P\left(T = t;\theta\right), \end{aligned}$$

where $\xi_t^{(k)}$ is the expected multiplicity of a time interval between consecutive events of length t starting in the observation period $(0,\tau]$ (i.e. complete or right-censored) given the number of events occurring in $(0,\tau]$ and the current parameters at iteration k.

The EM algorithm maximizes $L(\theta)$ by iteratively maximizing $Q(\theta|\theta^{(k)})$ over θ. Each iteration of the EM algorithm increases $L(\theta)$ and generally the sequence of reestimated parameters $\theta^{(k)}$ converge to a local maximum of $L(\theta)$. The EM algorithm alternates two steps, the E-step which consists of computing the reestimation quantities $\xi_t^{(k)}$ from the observed count data considering the parameter $\theta^{(k)}$ at iteration k as the true parameter and the M-step which consists of choosing the next parameter value $\theta^{(k+1)}$ that maximizes $Q(\theta|\theta^{(k)})$ over θ (McLachlan and Krishnan 1997)

$$\theta^{(k+1)} = \arg\max_{\theta} \left\{ Q(\theta|\theta^{(k)}) \right\}.$$

E-step

Since all the terms involved in the derivation of $\xi_t^{(k)}$ depend on $\theta^{(k)}$, we will omit to note systematically $\theta^{(k)}$ in the sequel. If $n = 0$, the reestimation quantities $\xi_t^{(k)}$ can be written as

$$\begin{aligned} \xi_t^{(k)} &= \frac{P(T = t)\, I(t > \tau)}{P(T > \tau)} \\ &= \begin{cases} 0, & t = 1, \ldots, \tau, \\ P(T = t) / P\{N(\tau) = n\}, & t = \tau + 1, \ldots, M, \end{cases} \end{aligned} \tag{5}$$

and if $n > 0$, using the same argument as in (3) and (4), the reestimation quantities $\xi_t^{(k)}$ can be written as

$$\begin{aligned} \xi_t^{(k)} &= \frac{\sum_{i=1}^{n} P(T_i = t)\; P\left(T_{1\backslash i}^{n} \leq \tau - t < T_{1\backslash i}^{n+1}\right) + P(T_{n+1} = t)\; P(\tau - t < T_1^n \leq \tau)}{P\left(T_1^n \leq \tau < T_1^{n+1}\right)} \\ &= P(T = t)\, \frac{n\, P\left(T_1^{n-1} \leq \tau - t < T_1^n\right) + P(\tau - t < T_1^n \leq \tau)}{P\left(T_1^n \leq \tau < T_1^{n+1}\right)}, \qquad t = 1, \ldots, M. \end{aligned} \tag{6}$$

The term $P(T = t)\, n\, P\left(T_1^{n-1} \leq \tau - t < T_1^n\right)$ in (6) (which is also $P(T = t)\, n\, P\{N(\tau - t) = n - 1\}$) should be interpreted as the probability of the n equivalent ways of extracting a time interval of length t in the n complete time intervals between consecutive events within $(0, \tau]$. Note that this term is only defined for $t \leq \tau - (n-1)$ (see (3) for justification). The term $P(T = t)$ $P(\tau - t < T_1^n \leq \tau)$ in (6) corresponds to the extraction of the time interval of length t in which the observation date τ falls.

For the practical computations, (6) should be rewritten as

$$\xi_t^{(k)} = P(T = t)\, \frac{n\left\{P\left(T_1^{n-1} \leq \tau - t\right) - P(T_1^n \leq \tau - t)\right\} + P(\tau - t < T_1^n \leq \tau)}{P\{N(\tau) = n\}}. \tag{7}$$

All the terms involved in (5) and (7) are directly deduced from the computation of the counting distribution (1) which necessitates computation of the distributions of the time up to the n-th event (n-fold convolutions of the inter-event distribution) on $[n,\ \min(nM, \tau)]$ for each possible value n taken by $N(\tau)$.

M-step

In the nonparametric framework, the reestimated probabilities $p_t^{(k+1)} = P(T = t; \theta^{(k+1)})$ are directly obtained by maximization of $\sum_{t=1}^{M} \xi_t^{(k)} \log p_t$ subject to the constraint $\sum_{t=1}^{M} p_t = 1$

$$p_t^{(k+1)} = \frac{\xi_t^{(k)}}{\sum_{i=1}^{M} \xi_i^{(k)}} = \frac{\xi_t^{(k)}}{n+1}, \qquad t = 1, \ldots, M. \tag{8}$$

In practice, a sample of count measurements of the form $\{(\tau_1, n_1), \ldots, (\tau_r, n_r)\}$ is recorded. Like Vardi (1982b), we assume that we always observe at least one event in the sample. The reestimation formula (8) is directly generalizable to the maximization of the joint likelihood of r mutually independent count measurements

$$\begin{aligned} & f\left(t_{1,1}, \ldots, t_{1,n_1+1}, n_1, \ldots, t_{r,1}, \ldots, t_{r,n_r+1}, n_r; \theta\right) \\ = & \prod_{j=1}^{r} P\left\{T_{j,1} = t_{j,1}, \ldots, T_{j,n_j+1} = t_{j,n_j+1}, N_j(\tau_j) = n_j; \theta\right\}. \end{aligned}$$

The generalization of (8) simply entails accumulating the reestimation quantities corresponding to each elementary count measurement

$$p_t^{(k+1)} = \frac{\xi_t^{(k)}}{\sum_{j=1}^{r} n_j + r} \qquad \text{with} \quad \xi_t^{(k)} = \sum_{j=1}^{r} \xi_{j,t}^{(k)}.$$

3 Equilibrium renewal process

Consider now the case where the process starts a long time before the observation period whose location is independent of the process itself. The distribution of the time interval from the initial observation date to the first event is the forward recurrence time distribution (Cox 1962)

$$P(V = v) = \frac{P(T \geq v)}{\mu}, \qquad v = 1, \ldots, M, \tag{9}$$

where $\mu = \sum_{t=1}^{M} t\, P(T = t)$,
and the time interval between consecutive events in which the initial observation date falls $U + V$ has length-biased distribution

$$P(U + V = t) = \frac{t\, P(T = t)}{\mu}, \qquad t = 1, \ldots, M, \tag{10}$$

where U represents the time interval from the previous event to the initial observation date.

From the forward recurrence time distribution (9) and the length-biased distribution (10) defined above, the two following conditional distributions can be defined

$$
\begin{aligned}
P\left(U+V=t|V=v\right) &= \frac{P\left(T=t\right)}{P\left(T\geq v\right)}, \qquad t=v,\ldots,M, &(11)\\
P\left(U+V=t|V>v\right) &= \frac{\left(t-v\right)P\left(T=t\right)}{\sum_{j=v+1}^{M}\left(j-v\right)P\left(T=j\right)}, \qquad t=v+1,\ldots,M. &(12)
\end{aligned}
$$

The random variables $T_2,\ldots,T_n,\ldots$ representing the subsequent time intervals between consecutive events follow the inter-event distribution and $V,T_2,\ldots,T_n,\ldots$ are mutually independent. The distribution of the number of events occurring in the observation period $(w,w+\tau]$, where w and τ $(\tau>0)$ are fixed integers independent of the process itself, is given by

$$
P\left\{N\left(\tau\right)=n\right\}=P\left(V+T_2^n\leq\tau\right)-P\left(V+T_2^{n+1}\leq\tau\right). \tag{13}
$$

The estimation problem is now more difficult than in the ordinary case since the counting distribution (13) is written with both the inter-event distribution (and its n-fold convolutions) and the forward recurrence time distribution instead of the inter-event distribution alone (1). The step is similar to that made by Vardi (1982b) in the context of time interval data, adapting to the stationary case the Kaplan-Meier or product-limit estimator (Kaplan and Meier 1958) developed in the ordinary case.

Two choices are possible to define the complete-data problem: one can either assume that the time interval from the initial observation date to the first event is observed, or that the time interval between consecutive events in which the initial observation date falls is observed. This means basically that the inter-event distribution has to be estimated on the basis of two independent pseudo-samples resulting from the E-step, one from the inter-event distribution and another either from the forward recurrence time distribution (9) or from the length-biased distribution (10). The latter solution, adopted by Vardi (1982b) for the estimation of an equilibrium renewal process on the basis of time interval data, relies on the key result of Vardi (1982a) concerning the estimation of an inter-event distribution on the basis of two independent samples, one from the inter-event distribution and the other from the length-biased distribution. Following Vardi, we adopted this latter solution.

Consider the complete-data case where the time interval between consecutive events in which the initial observation date falls and the subsequent time intervals between consecutive events, including the time interval in which the final observation date $w+\tau$ falls, are observed. Since $U+V,T_2,\ldots,T_n,\ldots$ are mutually independent, the complete-data likelihood is given by

$$
\begin{aligned}
& f\left(u+v,t_2,\ldots,t_{n+1},n;\theta\right)\\
= \;& P\left\{U+V=u+v,T_2=t_2,\ldots,T_{n+1}=t_{n+1},N\left(\tau\right)=n;\theta\right\}\\
= \;& P\left(U+V=u+v;\theta\right)\prod_{i=2}^{n+1}P\left(T_i=t_i;\theta\right)I\left(v+t_2^n\leq\tau<v+t_2^{n+1}\right)\\
= \;& g\left(u+v,t_2,\ldots,t_{n+1},n;\theta\right)I\left(v+t_2^n\leq\tau<v+t_2^{n+1}\right).
\end{aligned}
$$

Instead of the time interval between consecutive events in which the initial observation date falls and the subsequent time intervals between consecutive events, we only observe the number of events occurring in the observation period $(w, w+\tau]$. The conditional expectation of the complete-data log-likelihood is given by

$$\begin{aligned}
Q(\theta|\theta^{(k)}) &= E\left\{\log g\left(U+V, T_2, \ldots, T_{N(\tau)+1}; \theta\right) | N(\tau) = n; \theta^{(k)}\right\} \\
&= \sum_{v,t_2,\ldots,t_{n+1}} \sum_{t_1=v}^{M} P\left(U+V=t_1, V=v, T_2=t_2, \ldots, T_{n+1}=t_{n+1}\right| \\
&\quad V+T_2^n \le \tau < V+T_2^{n+1}; \theta^{(k)}\Big) \\
&\quad \times \left\{\log P\left(U+V=t_1;\theta\right) + \sum_{i=2}^{n+1} \log P\left(T_i=t_i;\theta\right)\right\} \\
&= \sum_{t=1}^{M} \Bigg\{ \sum_{v,t_2,\ldots,t_{n+1}; v\le t} P\left(U+V=t, V=v, T_2=t_2, \ldots, T_{n+1}=t_{n+1}\right| \\
&\quad V+T_2^n \le \tau < V+T_2^{n+1}; \theta^{(k)}\Big)\Bigg\} \log \frac{tP\left(T=t;\theta\right)}{\mu} \\
&\quad + \sum_{t=1}^{M} \Bigg\{ \sum_{i=2}^{n+1} \sum_{v,t_2,\ldots,t_{n+1}} P\left(V=v, T_2=t_2, \ldots, T_{n+1}=t_{n+1}\right| \\
&\quad V+T_2^n \le \tau < V+T_2^{n+1}; \theta^{(k)}\Big) I\left(t_i=t\right)\Bigg\} \log P\left(T=t;\theta\right) \\
&= \sum_{t=1}^{M} \eta_t^{(k)} \log \frac{tP\left(T=t;\theta\right)}{\mu} + \sum_{t=1}^{M} \xi_t^{(k)} \log P\left(T=t;\theta\right), \qquad (14)
\end{aligned}$$

where $\sum_{v,t_2,\ldots,t_{n+1}}$ means sum on every possible combination $v, t_2, \ldots, t_{n+1}$ such that $v+t_2^n \le \tau$ $< v+t_2^{n+1}$ and $\eta_t^{(k)}$ is the expected multiplicity of a time interval between consecutive events of length t in which the initial observation date w falls given the number of events occurring in $(w, w+\tau]$ and the current parameters at iteration k.

E-step

In the sequel, we will omit to note systematically $\theta^{(k)}$. If $n=0$, using (12), the reestimation quantities $\eta_t^{(k)}$ in (14) can be written as

$$\begin{aligned}
\eta_t^{(k)} &= P\left(U+V=t|V>\tau\right) \\
&= \frac{(t-\tau)P\left(T=t\right)I\left(t>\tau\right)}{\sum_{j=\tau+1}^{M}(j-\tau)P\left(T=j\right)} \qquad (15) \\
&= \begin{cases} 0, & t=1,\ldots,\tau, \\ \dfrac{(t-\tau)P\left(T=t\right)}{\mu P\left\{N\left(\tau\right)=n\right\}}, & t=\tau+1,\ldots,M, \end{cases} \qquad (16)
\end{aligned}$$

where (15) is the reestimation quantity for $n=0$ obtained by Vardi (1982b) in the case of time interval data. If $n=0$, the information summarized in (τ, n) is common to both the design we study here and the design studied by Vardi on the basis of time interval data.

If $n > 0$, using (9) and (11), the reestimation quantities $\eta_t^{(k)}$ in (14) can be written as

$$\begin{aligned}\eta_t^{(k)} &= \frac{\sum_{v=1}^{\min\{t,\tau-(n-1)\}} P\left(U+V=t|V=v\right) P\left(V=v\right) P\left(T_2^n \leq \tau - v < T_2^{n+1}\right)}{P\left(V+T_2^n \leq \tau < V + T_2^{n+1}\right)} \\ &= \frac{P\left(T=t\right) \sum_{v=1}^{\min\{t,\tau-(n-1)\}} \left\{P\left(T_2^n \leq \tau - v\right) - P\left(T_2^{n+1} \leq \tau - v\right)\right\}}{\mu\, P\left\{N\left(\tau\right)=n\right\}}, \qquad t = 1,\ldots,M. \quad (17)\end{aligned}$$

The quantities $\left\{P\left(T_2^n \leq \tau - v\right) - P\left(T_2^{n+1} \leq \tau - v\right)\right\}$ can be accumulated for each possible time elapsed from the initial observation date to the next event during the computation of the reestimation quantities $\eta_t^{(k)}$ in order to save computation time.

Using the same argument as for the ordinary renewal process case, the reestimation quantities $\xi_t^{(k)}$ which are only defined for $n > 0$ can be written as

$$\begin{aligned}\xi_t^{(k)} &= P\left(T=t\right)\left\{\left(n-1\right) P\left(V+T_2^{n-1} \leq \tau - t < V + T_2^n\right)\right. \\ &\quad \left.+P\left(\tau - t < V + T_2^n \leq \tau\right)\right\} / P\left(V+T_2^n \leq \tau < V + T_2^{n+1}\right) \quad (18) \\ &= P\left(T=t\right)\left[\left(n-1\right)\left\{P\left(V+T_2^{n-1} \leq \tau - t\right) - P\left(V+T_2^n \leq \tau - t\right)\right\}\right. \\ &\quad \left.+P\left(\tau - t < V + T_2^n \leq \tau\right)\right] / P\left\{N\left(\tau\right)=n\right\}, \qquad t = 1,\ldots,M. \quad (19)\end{aligned}$$

The term $P\left(T=t\right)\left(n-1\right)P\left(V+T_2^{n-1} \leq \tau - t < V + T_2^n\right)$ in (18) should be interpreted as the probability of the $(n-1)$ equivalent ways of extracting a time interval of length t in the $(n-1)$ complete time intervals between consecutive events within $(w, w+\tau]$ (note that this term is only defined for $t \leq \tau - (n-1)$). The term $P\left(T=t\right)P\left(\tau - t < V + T_2^n \leq \tau\right)$ in (18) corresponds to the extraction of the time interval of length t in which the final observation date $w+\tau$ falls.

All the terms involved in (16) (17) and (19) are directly deduced from the computation of the counting distribution (13). Formula (13) necessitates computation of the $(n-1)$-fold convolutions of the inter-event distribution on $[n-1,\ \min\left\{\left(n-1\right)M, \tau - 1\right\}]$ and then of the distributions of the time up to the n-th event (convolution of the forward recurrence time distribution with the $(n-1)$-fold convolution of the inter-event distribution) on $[n,\ \min\left(nM, \tau\right)]$ for each possible value n taken by $N\left(\tau\right)$.

M-step

The M-step consists of maximizing

$$\sum_{t=1}^{M} \xi_t^{(k)} \log p_t + \sum_{t=1}^{M} \eta_t^{(k)} \log \frac{t\, p_t}{\sum_{i=1}^{M} i\, p_i}$$

subject to the constraint $\sum_{t=1}^{M} p_t = 1$. Applying the result of Vardi (1982a), and noting that $\sum_{t=1}^{M} \xi_t^{(k)} = n$ and $\sum_{t=1}^{M} \eta_t^{(k)} = 1$, the unique solution is

$$p_t^{(k+1)} = \frac{\xi_t^{(k)} + \eta_t^{(k)}}{n + t/\mu^{(k+1)}}, \qquad t = 1, \ldots, M, \tag{20}$$

where $\mu^{(k+1)}$ is the unique solution for μ in the equation

$$\sum_{t=1}^{M} \frac{\left(\xi_t^{(k)} + \eta_t^{(k)}\right) t}{n\mu + t} = 1. \tag{21}$$

The above equation can be solved numerically by successively bisecting the interval $[1, M]$.

It is also possible to derive a reestimation formula for the parameters of the inter-event distribution ignoring the contribution of the time interval between consecutive events in which the initial observation date falls. In this case, the reestimated probabilities $p_t^{(k+1)}$ are directly obtained by maximization of $\sum_{t=1}^{M} \xi_t^{(k)} \log p_t$ subject to the constraint $\sum_{t=1}^{M} p_t = 1$

$$p_t^{(k+1)} = \frac{\xi_t^{(k)}}{\sum_{i=1}^{M} \xi_i^{(k)}} = \frac{\xi_t^{(k)}}{n}, \qquad t = 1, \ldots, M. \tag{22}$$

The rationale behind the corresponding estimator is somewhat similar to Cox's partial likelihood idea (Cox 1975) in the sense that it is derived by maximizing part of $Q(\theta|\theta^{(k)})$; see also Denby and Vardi (1985) who used the same type of decomposition. Nevertheless, the aim underlying the decomposition of $Q(\theta|\theta^{(k)})$ is clearly different from that emphasized by Cox. For convenience, this estimator will be termed the partial likelihood (PL) estimator and will serve as a reference in simulation experiments (see Section 6) while the estimator combining the reestimation quantities $\xi_t^{(k)}$ and $\eta_t^{(k)}$ presented above will be termed the complete likelihood (CL) estimator. Since the occurrence time of the first event is a stopping time, the contribution of the time interval between consecutive events in which the initial observation date falls may safely be ignored (see Aalen and Husebye (1991) for a clear discussion of this point) but this entails a loss of information particularly in high censoring situations.

As for the ordinary renewal process case, the reestimation formulae (20) (22) are directly generalizable to the maximization of the joint likelihood of r mutually independent count measurements by accumulation of the elementary reestimation quantities corresponding to each count measurement.

4 Maximum penalized likelihood estimation using the OSL algorithm

For the regularization of the estimated inter-event distributions, a potential solution consists in incorporating a penalty term in the likelihood. In the framework of the EM algorithm, the E-step is unchanged but for the M-step, the maximization of $Q(\theta|\theta^{(k)})$ is replaced by the maximization of

$$Q(\theta|\theta^{(k)}) - \lambda J(\theta), \tag{23}$$

where λ is a tuning constant that determines the relative importance of $Q(\theta|\theta^{(k)})$ and $J(\theta)$, and $J(\theta)$ is a roughness penalty. In our case, $J(\theta)$ will be the sum of squared second differences $J(\theta) = \sum_t \{(p_{t+1} - p_t) - (p_t - p_{t-1})\}^2$.

Green (1990) demonstrated the computational economy and accelerated convergence yielded by employing the one-step-late (OSL) algorithm. The OSL algorithm solves

$$DQ(\theta|\theta^{(k)}) - \lambda DJ(\theta^{(k)}) = 0, \tag{24}$$

where D denotes the derivative operator.

The only difference between equation (24) and equating the derivatives of expression (23) to 0 is that in equation (24), the derivatives of the penalty are evaluated at the current value $\theta^{(k)}$. Both expression (23) and equation (24) have the same fixed point so the OSL algorithm converges to a maximum penalized likelihood estimate.

Let

$$\gamma_t^{(k)} = \lambda \frac{\partial J(\theta^{(k)})}{\partial p_t^{(k)}}.$$

In the case of the ordinary renewal process, the M-step (8) is replaced by

$$p_t^{(k+1)} = \frac{\xi_t^{(k)}}{\alpha^{(k+1)} + \gamma_t^{(k)}}, \qquad t = 1, \ldots, M, \tag{25}$$

where $\alpha^{(k+1)}$ is the unique solution for α in the equation

$$\sum_{t=1}^{M} \frac{\xi_t^{(k)}}{\alpha + \gamma_t^{(k)}} = 1.$$

In the case of the equilibrium renewal process, the M-step (20) is replaced by

$$p_t^{(k+1)} = \frac{\xi_t^{(k)} + \eta_t^{(k)}}{\alpha^{(k+1)} + t/\mu^{(k)} + \gamma_t^{(k)}}, \qquad t = 1, \ldots, M, \tag{26}$$

where $\alpha^{(k+1)}$ is the unique solution for α in the equation

$$\sum_{t=1}^{M} \frac{\xi_t^{(k)} + \eta_t^{(k)}}{\alpha + t/\mu^{(k)} + \gamma_t^{(k)}} = 1. \tag{27}$$

It should be noted that, in comparison with (20), $\mu^{(k+1)}$ is replaced by $\mu^{(k)}$ in (26) where $\mu^{(k)}$ is directly computed from the inter-event distribution estimated at iteration k. This modification in the spirit of the OSL algorithm enables us to keep a simple M-step.

5 Practical estimation procedure

We are now able to present an efficient practical estimation procedure based on the application of the EM algorithm. Consider that a sample of count measurements of the form

$\{(\tau_1, n_1), \ldots, (\tau_r, n_r)\}$ has been recorded. The practical estimation procedure can be summarized as follows:

- initialization with a "1-shifted" geometric inter-event distribution with parameter $p^{(0)}$ from the mean time interval between consecutive events

$$\mu = \sum_{j=1}^{r} \tau_j / \sum_{j=1}^{r} n_j .$$

  Hence $p^{(0)} = 1/\mu$.

The maximum possible time interval between consecutive events M is chosen on a quantile criterion from the initial geometric inter-event distribution (for instance M is the $(1-\epsilon)$th quantile where ϵ is a residual probability such as 10^{-5}). Hence, the initial (either ordinary or equilibrium) renewal process is always a Bernoulli process which constitutes the most neutral choice.

- computation of interval and counting distributions (detailed after this summary of the practical estimation procedure),
- computation of the log-likelihood of the count data.

do {

- accumulation of the reestimation quantities $\xi_t^{(k)}$ (5) (7) for an ordinary renewal process, or $\eta_t^{(k)}$ (16) (17) and $\xi_t^{(k)}$ (19) for an equilibrium renewal process for each count measurement,
- estimation of a nonparametric inter-event distribution ((8) or (20), or (25) or (26) for the penalized likelihood estimates),
- computation of interval and counting distributions (detailed after this summary of the practical estimation procedure),
- computation of the log-likelihood of the count data.

} while (convergence criteria on the log-likelihood of the count data)

The convergence of the estimation procedure can be monitored upon the monotone increase over iterations of the log-likelihood of the count data

$$\log L \left\{ n_1(\tau_1), \ldots, n_r(\tau_r); \theta^{(k)} \right\} = \sum_{j=1}^{r} \log P \left\{ N_j(\tau_j) = n_j; \theta^{(k)} \right\} .$$

This is a direct consequence of one of the main properties of the EM algorithm; see McLachlan and Krishnan (1997).

Computation of interval and counting distributions from the estimated inter-event distribution breaks down into the following steps:

ordinary renewal process:

- computation of the time up to the n-th event distributions (n-fold convolution of the inter-event distribution) on $[n,\ \min(nM,\tau)]$ and of the counting distribution (1),

equilibrium renewal process:

- computation of the forward recurrence time distribution, of the $(n-1)$-fold convolutions of the inter-event distribution on $[n-1,\ \min\{(n-1)M,\tau-1\}]$, of the time up to the n-th event distributions (convolution of the forward recurrence time distribution with the $(n-1)$-fold convolution of the inter-event distribution) on $[n,\ \min(nM,\tau)]$ and of the counting distribution (13).

6 Simulation study

Let us, for this simulation study, introduce the "1-shifted" negative binomial distribution with parameters r and p, NB(r,p), where r is a real number ($r>0$) and $0<p\leq 1$

$$P(T=t)=\binom{t+r-2}{r-1}p^r q^{t-1},\quad t=1,2,\ldots,\qquad \mu=1+\frac{rq}{p},\sigma^2=\frac{rq}{p^2}.$$

The performances of the different estimators evaluated in this section are related to the weight of censoring which can be quantified by the empirical distribution of the types of the underlying time intervals computed from the count data. Let f_n denote the observed frequency of $\{N(\tau)=n\}$ in the data. In the case of an ordinary renewal process, the frequencies of time interval types (censored on the right, complete i.e. fully included within the observation period) are $(\sum_n f_n, \sum_n f_n n)$ and in the case of an equilibrium renewal process, the frequencies of time interval types (censored on both ends corresponding to no-event, censored on one end i.e. censored on the left or on the right, complete) are $\left(f_0, 2\sum_{n\geq 1} f_n, \sum_{n\geq 2} f_n(n-1)\right)$.
The simulation procedure depends on the three following factors:

- the underlying inter-event distribution,
- the length of the observation period,
- the sample size.

The two following inter-event distributions were chosen for their different tail behaviors:

- NB(0.5, 0.0526): $\mu=10$, $\sigma=$ 13.08,
- NB(5, 0.357): $\mu=10$, $\sigma=$ 5.02,

These inter-event distributions may be classified by relating them to the "1-shifted" geometric distribution NB(1, 0.1) ($\mu=10$, $\sigma=$ 9.49). The first is overdispersed while the second is underdispersed.

Comparison of the PL estimator with the CL estimator. The sample size was fixed at 10000 and the number of iterations at 5000, which guarantees good convergence. The objective of this large sample experiment was to assess the bias of the PL estimator for different censoring situations. The two estimators were compared for $\tau = 10, 20, 50$ which correspond to different weights of censoring. A total of six samples were therefore generated.

In Figures 1 and 2 corresponding to the cases $\tau = 20, 50$, the graphs on the left correspond to the estimated inter-event distributions and the graphs on the right, to the counting distributions computed from the estimated inter-event distributions fitted to the count data. Note that when two distributions are completely superimposed, only the last plotted is visible. Deviances are given in Tables 1 and 2. The deviance is twice (log-likelihood of the count data for the best model minus log-likelihood of the count data for the estimated model). In our case, the best model corresponds to an exact fit and the log-likelihood of the count data for the best model is the information measure of the count data.

In high censoring situations ($\tau = 10, 20$), we observed marked differences between the two estimators (Figure 1, Table 1, Figure 2, Table 2). As expected, the counting distributions computed from the PL-estimated inter-event distributions did not give a good fit with the count data. The mean of the PL-estimated inter-event distribution is biased downwards for NB(0.5, 0.0526) (Table 1) and upwards for NB(5, 0.357) (Table 2). The direction of the bias is related to the mean of the forward recurrence time distribution (distribution of the time interval from the initial observation date to the first event) which is larger ($\mu_{\text{forward}} = 14.05$) than the mean of the inter-event distribution for NB(0.5, 0.0526) and smaller ($\mu_{\text{forward}} = 6.76$) than the mean of the inter-event distribution for NB(5, 0.357).

These results should be interpreted in the light of the distributions of the types of the underlying time intervals (censored on both ends corresponding to no-event, censored on one end, complete) computed from the count data. A detailed examination of Tables 1 and 2 shows that the PL and CL estimates are strikingly similar only if there is no time interval censored on both ends, i.e. the observed frequency of $\{N(\tau) = 0\}$ is 0 and the proportion of time intervals censored on one end is not too high. This behavior is obvious from the definition of the PL estimator (see Section 3).

A second simulation experiment was conducted with the inter-event distribution NB(5, 0.357) for $\tau = 50$ where the sample size was fixed at 100. The behavior of the estimation procedure illustrated below on the basis of a single small sample generated either by an ordinary or an equilibrium renewal process with inter-event distribution NB(5, 0.357) is typical in the sense that similar behaviors were observed with other count data samples, particularly small samples generated by either ordinary or equilibrium renewal processes with inter-event distribution NB(0.5, 0.0526).

Reestimated renewal processes along the EM iterations. The objective was to study the change in the shape of the estimated inter-event distribution, especially its smoothness/roughness along the iterative estimation procedure, for both an ordinary renewal process (Figures 3 and 4) and an equilibrium renewal process (Figures 5 and 6). The initial geometric inter-event distributions are given as references in Figures 3 and 5. The initial value of M computed on a quantile criterion from these initial geometric inter-event distributions (see Section 5) was 119 for the ordinary renewal process and 110 for the equilibrium renewal process. Both types of renewal process showed the same behavior, i.e. increasingly rough estimated inter-event distributions along the iterations (Figures 3 and 5). For both types of renewal processes, the counting distributions obtained after 10000 iterations perfectly fit the count data (Figures 4 and 6). This should be interpreted as an overfit in view of the sample size fixed at 100.

Influence of the initial inter-event distribution. The influence of the initial value of M and more generally of the initial inter-event distribution was studied using a "1-shifted" uniform distribution as the initial inter-event distribution with $M = 40, 100, 1000, 10000$ (Tables 3 and 4). The minimum value of M was chosen on the basis of the upper bound to the support of the inter-event distribution estimated by the EM algorithm initialized with a geometric distribution (see Figures 3 and 5). For both types of renewal processes, the estimated inter-event distributions after 10000 iterations are indiscernible (also with the inter-event distribution estimated by the EM algorithm initialized with a geometric distribution). Hence, for a sufficiently large value of M (and assuming that $p_t^{(0)} > 0$ for $t = 1, \ldots, M$), the proposed estimators appear to be insensitive to the initial inter-event distribution and to converge to a unique maximum.

Influence of the underlying time unit. Since the underlying time unit (both for the time interval between consecutive events and the observation period) is not part of the data -and hence should be chosen by the user- it is important to study the changes of the estimated inter-event distribution for different choices of the time unit. The influence of the time unit was evaluated by estimating an inter-event distribution on the basis of a given sample of count data generated for $\tau = 50$ but assuming for the estimation that $\tau = 100, 250$. The estimated inter-event distributions were almost identical up to a scale change (Figure 7 compared to Figure 3 for the ordinary renewal process example and Figure 8 compared to Figure 5 for the equilibrium renewal process example). The same was also true for $\tau = 250$ (estimated inter-event distributions not shown). As a consequence, the counting distributions were also almost identical (see the deviances in Tables 5 and 6). Hence, for a sufficiently large value of τ (with respect to the maximum number of events in the count data), the estimated inter-event distribution is simply scaled up or scaled down if the underlying time unit is modified.

Computation of the mean of the inter-event distribution for the estimation of an equilibrium renewal process. In the case of an equilibrium renewal process, the mean time interval between consecutive events μ can be straightforwardly estimated by

$$\widehat{\mu} = \sum_{j=1}^{r} \tau_j / \sum_{j=1}^{r} n_j. \tag{28}$$

This estimate of μ or alternatively, the value $\mu^{(k)}$ obtained at the previous iteration in the spirit of the OSL algorithm (see Section 4), can be used in place of the value $\mu^{(k+1)}$ numerically computed by the interval bisection method (21) in order to save computation time. The deviances are slightly different for 5 iterations (28.52 for $\mu^{(k+1)}$ numerically computed, 28.51 for

$\mu^{(k)}$ and 28.49 for μ estimated from count data), while they are identical thereafter (the results are those given in the column corresponding to $\tau = 50$ in Table 6). It should be noted that in the framework of the maximum penalized likelihood estimation (Section 4), the estimate of μ (28) can be used in place of the value $\mu^{(k)}$ in (26) and (27).

Sampling variability. For the two inter-event distributions NB(0.5, 0.0526) and NB(5, 0.357) and for $\tau = 20, 50$ which corresponds to different weights of censoring, we generated 1000 count data samples of size 100 both with ordinary renewal processes and with equilibrium renewal processes. This makes a total of 8 configurations. In Tables 7 to 14, adapting the design of the simulation study of Denby and Vardi (1985), we give $\widehat{F}(t)$ for three different t's. We chose the t's to be close to the 0.25, 0.5 and 0.75 quantiles of the true underlying distribution. For NB(0.5, 0.0526) (Tables 7 to 10), $t = 1, 4, 12$ and the true values are $F(1) = 0.229, F(4) = 0.476, F(12) = 0.74$. For NB(5, 0.357) (Tables 11 to 14), $t = 6, 9, 12$ and the true values are $F(6) = 0.264, F(9) = 0.521$, $F(12) = 0.731$. We chose also to give the sum of squared second differences $J(\theta)$ as a measure of the smoothness/roughness of the estimated inter-event distribution. The estimation of 1000 ordinary renewal processes on the basis of 10000 iterations of the EM algorithm takes less than 1 hour for $\tau = 20$ and about three hours for $\tau = 50$ with a Pentium III, 800 Mhz processor. For equilibrium renewal processes, the computation times are multiplied by a factor ranging from 2.8 to 4.7 if $\mu^{(k+1)}$ is numerically computed by the interval bisection method but simply by a factor ranging from 1.4 to 1.7 if $\mu^{(k)}$ is used at iteration $k+1$ in the spirit of the OSL algorithm.

As illustrated by the previous examples with a single sample (Figures 3 and 5), the smoothness/roughness measure $J(\theta)$ increases substantially after 100 iterations leading to an overfit of the count data. It should also be noted that stdv$\{\widehat{F}(t)\}$ also increases all along the iterations. The bias as given by $|avg\{\widehat{F}(t)\} - F(t)|$ is smaller for $\tau = 50$ than for $\tau = 20$. For a fixed τ, and a given type of renewal process, the bias is larger for the smallest value of t (which contributes preferentially to the highest number of events). The true probability masses are $p_1 = 0.229$, $p_4 = 0.061, p_{12} = 0.021$ for NB(0.5, 0.0526) and $p_6 = 0.08, p_9 = 0.084, p_{12} = 0.061$ for NB(5, 0.357). The large value of $p_1 = F(1)$ for NB(0.5, 0.0526) partly explains the bias observed for $t = 1$ in Tables 7 to 10.

For equilibrium renewal processes, avg(μ) tends towards the true value of μ and avg(σ) tends towards the true value of σ and the convergences are generally rapid; see Tables 8, 10, 12 and 14. For the mean of the inter-event distribution μ, this is a direct consequence of the fact that, for each estimated renewal process, the mean of the counting distribution tends in a few iterations towards the mean of the count data and $\mu = \tau / E\{N(\tau)\}$. For ordinary renewal processes, the same behavior is only observed in the case of the inter-event distribution NB(5, 0.357) for $\tau = 50$ (Table 13) which is the only case where $P(N(\tau) = 0) = 0$. For the ordinary renewal processes with inter-event distribution NB(0.5, 0.0526), avg(μ) and avg(σ) are biased downwards (Tables 7 and 9) while, for the ordinary renewal process with inter-event distribution NB(5, 0.357) for $\tau = 20$, avg(μ) and avg(σ) are biased upwards (Table 11). This is likely to be related to the overdispersion of NB(0.5, 0.0526) compared to a geometric distribution and the underdispersion of NB(5, 0.357). In Tables 7 to14, one should note that, for a given inter-event distribution, stdv(μ) is larger for $\tau = 20$ than for $\tau = 50$ and is larger for NB(0.5, 0.0526) than for NB(5, 0.357) for a fixed value of τ. Hence, stdv(μ) appears to be related both to the weight of censoring and to the dispersion of the inter-event distribution. The comparison of stdv$\{\widehat{F}(t)\}$, stdv(μ) and stdv(σ) with reference to the respective orders of magnitude of values for $\widehat{F}(t)$, μ and σ show that while $F(t)$ may be poorly estimated (because of the degree of incompleteness of the data), the estimate of μ and σ are more reliable (because of the strength of the renewal process assumption). This is particularly interesting for σ which cannot be estimated by alternative methods (while μ can be directly estimated from count data in the case of equilibrium renewal processes). The larger value of stdv(σ) in the case of the equilibrium renewal process with inter-event distribution NB(0.5, 0.0526) for $\tau = 20$ (Table 8) is likely to be related to the weight of the underlying time intervals censored on both ends.

Penalized likelihood estimation. The penalized likelihood approach was evaluated on the two count data samples used for the small sample experiment (see Figures 4 and 6). The tuning constant λ is a function of the cumulated number of events

$$\begin{aligned}\lambda &= \rho\left(\sum_{j=1}^{r} n_j + r\right) \\ &= \begin{cases} \rho\left(\sum_{t=1}^{M} \xi_t^{(k)}\right) & \text{ordinary renewal process,} \\ \rho\left(\sum_{t=1}^{M} \xi_t^{(k)} + \sum_{t=1}^{M} \eta_t^{(k)}\right) & \text{equilibrium renewal process.} \end{cases}\end{aligned}$$

The main result was that the maximum penalized likelihood estimates converged (the number of iterations was fixed at 10000) to an inter-event distribution that was similar to a reestimated inter-event distribution at a certain iteration of the EM algorithm implementing the nonparametric maximum likelihood estimator. In Figure 9 ($\rho = 0.5$) and Figure 10 ($\rho = 0.1$), the closest intermediate reestimated inter-event distribution in terms of deviance (with the corresponding number of iterations) is shown as a reference. It should be noted that the maximum penalized likelihood estimated inter-event distribution is always slightly smoother than the reference inter-event distribution (the smoothness being measured by $J(\theta)$). Since increasingly rough inter-event distributions are obtained along the iterations of the EM algorithm implementing the nonparametric maximum likelihood estimator (see Figures 3 and 5), smaller values of ρ correspond to higher numbers of iterations. In practice, for a given count data sample, different values of ρ should be tried to determine on an empirical basis the most satisfactory compromise between the smoothness of the estimated inter-event distribution and the fit of the count data. Hence, the gain of the penalized likelihood approach is limited with respect to the empirical choice of a satisfactory compromise between the smoothness of the inter-event distribution and the fit of the count data among the renewal processes reestimated along the EM iterations; see the examples in Figures 3 and 4, Table 3, and Figures 5 and 6, Table 4.

7 Application to plant growth follow-up

Elongation and branching are the two basic processes involved in plant growth. Renewal processes are mainly used for analyzing the elongation of leafy axes. In the coffee tree case, an elongated element is composed of an internode (portion of the axis between two nodes), a node, two opposite leaves and two axillary buds. The elongation process is a continuous phenomenon from microscopic early stages to the final completely elongated stage. Due to the progressive character of this process, the dates of the different stages attained by the successive elements may be difficult to observe experimentally. Hence, instead of directly performing a time interval type measurement (time interval between two consecutive elements for a given stage), a counting type measurement is performed. A reference elongation stage is precisely defined using morphological characters (color and unfolding of the leaves, for example). The number of elongated elements at successive dates is noted. The proposed estimation procedure enables estimation of an inter-event distribution which represents the time intervals between the reference stages of two consecutive elements on the basis of count data (number of newly elongated elements during an observation period).

A sample of 200 coffee tree leafy axes was observed during a period of five and a half months in the Divo experimental station (Ivory Coast). We selected three consecutive sub-periods of respective length 40, 40 and 87 days. The number of iterations was fixed at 10000 which is sufficient for a good convergence of the EM algorithm (see the deviances in Table 15). It should be noted that the estimated inter-event distributions (obtained without any explicit regularization) are smoothed while the count data are well fitted (Figures 11 and 12, Table 15). Compared to the two families of renewal processes (with inter-event distributions NB(0.5, 0.0526) and NB(5, 0.357)) studied by simulation (Section 6), these real data illustrate a more favorable context for the application of the proposed methods because of the smaller dispersions of the count data and consequently the smaller dispersions of the estimated inter-event distributions. It would be interesting to compute standard deviations for $\widehat{F}(t)$ on the basis of a bootstrap simulation. We may expect lower standard deviations for $\widehat{F}(t)$ than the standard deviations computed in the simulation experiment (see Tables 7 to 14) because of both the larger sample size and the smoother estimated inter-event distributions.

The selected period started with a rainy period followed by a drier period which is directly expressed in the estimated inter-event distributions (see Figures 11 to 13). The rainy period, which started well before the beginning of the coffee tree follow-up period, ended around day 50 (see Figure 14) while the transition between the fast growth period and the slow growth period occurred around day 80. This lag corresponds to the inertia of the plants and can be fully explained on a physiological basis. The limit between the second and the third sub-period was chosen so as to correspond to the transition between the first stage of fast growth and the second stage of slower growth (on the basis of an exploratory analysis, these two stages can be roughly considered as stationary).

The homogeneity of the sample of coffee tree leafy axes can be assessed by comparing the processes estimated over two consecutive observation periods and the process estimated over the grouped period (for which stationarity can be assumed). For the 1-80 observation period, the estimated inter-event distribution is similar to the inter-event distributions estimated over the 1-40 and 41-80 observation periods (see Figure 13) while the counting distribution is also similar to the counting distributions computed for $\tau = 80$ from the renewal processes originally estimated over the 1-40 and 41-80 observation periods (see Figure 12 and Table 16). Hence, there are no marked overdispersion (or underdisperion) phenomena and the sample of coffee tree leafy axes can be considered as homogeneous. An alternative hypothesis with some biological support would be to consider that axes which tend to grow faster during a given observation period will continue to grow faster during the subsequent observation period (and conversely axes growing slower will continue to grow slower). This would generate an overdispersion of the count data for the grouped observation period (with reference to the elementary observation periods) and the sample of coffee tree axes could not be considered as homogeneous for renewal process estimation.

The changes in environmental conditions affect mainly the event rate while the dispersion characteristics are linked to the location characteristics (see the coefficients of variation in Table 15). In the context of plant growth follow-up, renewal processes are the most simple way to analyze the dynamics of the elongation process which is otherwise studied via more clumsy experimental protocols such as destructive sampling and histological studies.

8 Concluding remarks

The large sample simulation experiment show that for moderate censoring situations, the PL and the CL estimates are close to each other while for high censoring situations, the PL-estimated inter-event distributions are biased. Whatever the weight of censoring, the CL estimates provide a valid solution in terms of count data fit. The precise shape of the estimated inter-event distribution should be considered with caution since it can be influenced by the estimation procedure, especially in high censoring situations. The reason for this is that the contributions to the estimate of the inter-event distribution for $t \leq \tau$ (see the comments in Sections 2 and 3) and $t > \tau$ (see (5) in the case of an ordinary renewal process, and (16) in the case of the CL estimator) correspond most of the time to distinct cases. Hence, if the length of the observation period τ corresponds to not too high a quantile of the inter-event distribution (say < 0.9 quantile to fix the ideas), the shape of the inter-event distribution may present a "discontinuity" at $t = \tau$ (see Figure 1 for $\tau = 20$).

The small sample simulation experiment highlights some interesting properties of the two proposed estimators:

- Under some weak conditions on the initial inter-event distribution, the reestimated inter-event distribution appears to converge to a unique maximum.
- The estimated inter-event distribution is simply scaled up or scaled down if the underlying time unit is changed. The precise validity conditions for this behavior should be studied.

This opens new avenues for future theoretical work concerning the properties of these estimators.

The behavior of the estimation procedure, i.e. increasingly rough estimated inter-event distributions along the EM iterations, can be interpreted as the result of two factors:

- The strength of the renewal process assumption that strongly constrains the estimated inter-event distributions. It should be noted that in the time interval data context studied by Vardi (1982b), the renewal process assumptions are only used for censored time intervals while time intervals fully observed are used as they are.
- The nonparametric nature of the estimated inter-event distribution that enables arbitrary shape, the only constraint being the size of the support.

Smoother estimated inter-event distributions are obtained for larger sample sizes and for smaller intrinsic dispersions of the count data (and consequently smaller dispersions of the estimated inter-event distributions). The first point is illustrated by comparing the results of the large sample simulation experiment (Figures 1 and 2) with the results of the small sample simulation experiment (Figures 3 to 6). The second point is illustrated by comparing the estimation from real data (Figures 11 and 12) with the results of the small sample simulation experiment. Hence, a sufficiently large sample size is required to apply this kind of estimation method (which is reasonable if one considers the degree of incompleteness of the data). The proposed methods are mainly useful in relatively high censoring situations since, in most applications, stationarity can only be assumed over relatively short observation periods. Moreover, interpretations may often be deduced by comparing the inter-event distributions estimated over consecutive observation periods. Hence, it is interesting to design a follow-up experiment with a sufficient number of observation dates. This also enables to assess the sample homogeneity by comparing the results obtained over consecutive observation periods and grouped observation periods as illustrated with the real data.

In comparison with methods that only consider the event rate (Stukel 1993), the proposed methods make a specific contribution by the reliable estimation of the dispersion characteristics of the inter-event distribution on the basis of count data (particularly under the equilibrium renewal process assumption).

The penalized likelihood approach can be directly transposed to estimation of an equilibrium renewal process on the basis of time interval data (Vardi 1982b). Another transposition to the context of time interval data concerns the computation of the mean of the inter-event distribution which can be avoided if, in the spirit of the OSL algorithm, $\mu^{(k+1)}$ is replaced by $\mu^{(k)}$ in the M-step of the EM algorithm proposed by Vardi (1982b).

The practical estimation procedure has been integrated into the AMAPmod software (Godin *et al.* 1997) which is freely available at http://amap.cirad.fr.

References

O. O. Aalen & E. Husebye (1991). Statistical analysis of repeated events forming renewal processes. *Statistics in Medicine*, 10, 1227-1240.

E. Costes, P. de Reffye, J. Lichou, Y. Guédon, A. Audubert & M. Jay (1992). Stochastic modelling of apricot growth units and branching. *Third International Symposium on Computer Modelling in Fruit Research and Orchard Management*, Palmerston North (New Zealand), *Acta Horticulturae*, 313, 89-98.

D. R. Cox (1962). *Renewal Theory.* Methuen, London.

D. R. Cox (1975). Partial likelihood. *Biometrika*, 62(2), 269-276.

R. Dattero & E. M. White (1989). A new estimation approach based on periodic inspection. *IEEE Transactions on Reliability*, 38(4), 436-439.

A. P. Dempster, N. M. Laird & D. B. Rubin (1977). Maximum likelihood from incomplete data via the EM algorithm (with discussion), *Journal of the Royal Statistical Society, Series B*, 39(1), 1-38.

L. Denby & Y. Vardi (1985). A short-cut method for estimation in renewal processes. *Technometrics*, 27(4), 361-373.

C. Godin, Y. Guédon, E. Costes & Y. Caraglio (1997). Measuring and analysing plants with the AMAPmod software. *Plants to Ecosystems - Advances in Computational Life Sciences*, Vol. I (M. T. Michalewicz, *ed.*), CSIRO Publishing, Collingwood, Victoria, 53-84.

P. J. Green (1990). On the use of the EM algorithm for penalized likelihood estimation. *Journal of the Royal Statistical Society, Series B*, 52(3), 443-452.

E. L. Kaplan & P. Meier (1958). Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association*, 53, 457-481.

S. McClean & C. Devine (1995). A nonparametric maximum likelihood estimator for incomplete renewal data. *Biometrika*, 82(4), 791-803.

G. J. McLachlan & T. Krishnan (1997). *The EM Algorithm and Extensions.* Wiley, New York.

P. de Reffye, E. Elguero & E. Costes (1991). Growth units construction in trees: A stochastic approach. *Acta Biotheoretica*, 39, 325-342.

T. A. Stukel (1993). Comparison of methods for the analysis of longitudinal interval count data. *Statistics in Medicine*, 12, 1339-1351.

Y. Vardi (1982a). Nonparametric estimation in the presence of length bias. *The Annals of Statistics*, 10(2), 616-620.

Y. Vardi (1982b). Nonparametric estimation in renewal processes. *The Annals of Statistics*, 10(3), 772-785.

Table 1. Equilibrium renewal process NB(0.5, 0.0526): means of the inter-event distributions and deviances.

		μ		deviance	
τ	interval type distribution	PL estimate	CL estimate	PL estimate	CL estimate
10	(0.223, 0.552, 0.225)	8.6	9.96	321.84	1.45
20	(0.078, 0.512, 0.41)	9.2	10.03	322.48	5.54
50	(0.005, 0.324, 0.671)	9.89	10.03	96.04	15.39

PL, partial likelihood; CL, complete likelihood

Table 2. Equilibrium renewal process NB(5, 0.357): means of the inter-event distributions and deviances.

		μ		deviance	
τ	interval type distribution	PL estimate	CL estimate	PL estimate	CL estimate
10	(0.096, 0.805, 0.099)	12.22	9.96	1090.66	0.06
20	(0.005, 0.66, 0.335)	10.43	10.06	354.76	0.07
50	(0, 0.333, 0.667)	9.98	9.98	4.97	4.75

PL, partial likelihood; CL, complete likelihood

Table 3. Ordinary renewal process NB(5, 0.357) for $\tau = 50$: deviances (interval type distribution: (0.18, 0.82)).

iteration	NB(1, 0.0924)	U(40)	U(100)	U(1000)	U(10000)
5	24.65	35.49	40.43	42.17	42.31
20	5.58	6.02	5.92	5.83	5.82
100	4.12	4.12	4.13	4.13	4.13
500	3.67	3.67	3.68	3.67	3.67
2000			0.35		
10000			0.04		

Table 4. Equilibrium renewal process NB(5, 0.357) for $\tau = 50$: deviances (interval type distribution: (0, 0.335, 0.665)).

iteration	NB(1, 0.0994)	U(40)	U(100)	U(1000)	U(10000)
5	28.52	33.63	52.54	42.75	36.75
20	8.88	8.32	9.75	9.63	9.57
100	3.14	3.09	3.15	3.14	3.14
500	2.37	2.38	2.38	2.37	2.37
2000			2.15		
10000			0.98		

Table 5. Ordinary renewal process NB(5, 0.357): deviances (interval type distribution: (0.18, 0.82)).

iteration	$\tau = 50$	$\tau = 100$	$\tau = 250$
5	24.65	26.69	27.93
20	5.58	5.76	5.86
100	4.12	4.13	4.14
500	3.67	3.68	3.68
2000	0.35	0.31	0.31
10000	0.04	0.03	0.03

Table 6. Equilibrium renewal process NB(5, 0.357): deviances (interval type distribution: (0, 0.335, 0.665)).

iteration	$\tau = 50$	$\tau = 100$	$\tau = 250$
5	28.52	30.49	31.66
20	8.88	9.15	9.3
100	3.14	3.13	3.12
500	2.37	2.38	2.38
2000	2.15	2.16	2.17
10000	0.98	0.89	0.9

Table 7. Ordinary renewal process NB(0.5, 0.0526) for $\tau = 20$ (theoretical interval type distribution: (0.296, 0.704)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(1)\}$	0.157	0.171	0.154	0.093	0.068	0.066
stdv$\{\widehat{F}(1)\}$	0.021	0.038	0.064	0.091	0.105	0.11
avg$\{\widehat{F}(4)\}$	0.455	0.475	0.493	0.543	0.552	0.553
stdv$\{\widehat{F}(4)\}$	0.032	0.041	0.058	0.095	0.116	0.121
avg$\{\widehat{F}(12)\}$	0.76	0.748	0.736	0.728	0.729	0.728
stdv$\{\widehat{F}(12)\}$	0.025	0.031	0.045	0.056	0.054	0.055
avg(μ)	8.8	8.96	8.96	8.97	8.97	8.97
stdv(μ)	0.667	0.688	0.688	0.688	0.687	0.688
avg(σ)	9.27	9.67	9.69	9.72	9.73	9.73
stdv(σ)	0.797	0.885	0.889	0.889	0.885	0.885
avg$\{J(\theta)\}$	0.063	0.0827	0.0802	0.2809	0.8006	1.042
avg(deviance)	7.73	6.23	5.04	3.02	1.94	1.71

Table 8. Equilibrium renewal process NB(0.5, 0.0526) for $\tau = 20$ (theoretical interval type distribution: (0.076, 0.515, 0.409)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(1)\}$	0.16	0.172	0.159	0.107	0.072	0.065
stdv$\{\widehat{F}(1)\}$	0.025	0.045	0.076	0.105	0.113	0.113
avg$\{\widehat{F}(4)\}$	0.454	0.471	0.475	0.527	0.55	0.554
stdv$\{\widehat{F}(4)\}$	0.039	0.05	0.064	0.115	0.145	0.155
avg$\{\widehat{F}(12)\}$	0.752	0.751	0.747	0.732	0.733	0.734
stdv$\{\widehat{F}(12)\}$	0.029	0.037	0.047	0.059	0.067	0.067
avg(μ)	9.77	10.05	10.06	10.05	10.04	10.04
stdv(μ)	0.882	0.932	0.933	0.932	0.931	0.931
avg(σ)	11.55	12.43	12.53	12.78	12.96	13.14
stdv(σ)	1.203	1.419	1.51	1.716	1.928	2.18
avg$\{J(\theta)\}$	0.0669	0.0863	0.0936	0.2369	0.9512	1.3574
avg(deviance)	7.23	5.82	4.99	3.69	2.88	2.68

Table 9. Ordinary renewal process NB(0.5, 0.0526) for $\tau = 50$ (theoretical interval type distribution: (0.156, 0.844)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(1)\}$	0.136	0.152	0.151	0.13	0.109	0.092
stdv$\{\widehat{F}(1)\}$	0.012	0.023	0.046	0.085	0.113	0.125
avg$\{\widehat{F}(4)\}$	0.424	0.45	0.451	0.441	0.468	0.486
stdv$\{\widehat{F}(4)\}$	0.023	0.035	0.054	0.085	0.162	0.205
avg$\{\widehat{F}(12)\}$	0.761	0.764	0.771	0.78	0.77	0.766
stdv$\{\widehat{F}(12)\}$	0.018	0.02	0.037	0.061	0.08	0.087
avg(μ)	9.63	9.83	9.84	9.85	9.85	9.85
stdv(μ)	0.555	0.587	0.588	0.59	0.59	0.59
avg(σ)	11.16	12.02	12.1	12.15	12.2	12.22
stdv(σ)	1.028	1.214	1.229	1.242	1.24	1.235
avg$\{J(\theta)\}$	0.0445	0.0583	0.0652	0.0814	0.501	1.3468
avg(deviance)	14.63	12.38	11.22	9.32	6.46	4.59

Table 10. Equilibrium renewal process NB(0.5, 0.0526) for $\tau = 50$ (theoretical interval type distribution: (0.006, 0.322, 0.672)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(1)\}$	0.135	0.153	0.159	0.155	0.166	0.143
stdv$\{\widehat{F}(1)\}$	0.013	0.026	0.051	0.095	0.14	0.159
avg$\{\widehat{F}(4)\}$	0.419	0.449	0.453	0.435	0.427	0.448
stdv$\{\widehat{F}(4)\}$	0.026	0.04	0.061	0.09	0.146	0.213
avg$\{\widehat{F}(12)\}$	0.753	0.76	0.763	0.771	0.77	0.768
stdv$\{\widehat{F}(12)\}$	0.02	0.022	0.037	0.067	0.093	0.113
avg(μ)	9.91	10.03	10.03	10.02	10.02	10.01
stdv(μ)	0.577	0.591	0.591	0.59	0.589	0.589
avg(σ)	11.86	12.82	12.91	12.98	13.03	13.06
stdv(σ)	0.985	1.171	1.192	1.238	1.271	1.288
avg$\{J(\theta)\}$	0.0443	0.0606	0.0748	0.1104	0.391	1.409
avg(deviance)	14.17	12.05	11.04	9.79	8.28	6.98

Table 11. Ordinary renewal process NB(5, 0.357) for $\tau = 20$ (theoretical interval type distribution: (0.374, 0.626)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(6)\}$	0.299	0.255	0.256	0.257	0.272	0.287
stdv$\{\widehat{F}(6)\}$	0.022	0.031	0.047	0.067	0.089	0.105
avg$\{\widehat{F}(9)\}$	0.484	0.509	0.538	0.539	0.516	0.514
stdv$\{\widehat{F}(9)\}$	0.026	0.034	0.055	0.079	0.112	0.122
avg$\{\widehat{F}(12)\}$	0.657	0.726	0.738	0.745	0.74	0.728
stdv$\{\widehat{F}(12)\}$	0.029	0.044	0.046	0.058	0.067	0.076
avg(μ)	10.86	10.34	10.31	10.29	10.3	10.38
stdv(μ)	0.497	0.494	0.494	0.498	0.503	0.57
avg(σ)	7.65	6.28	6.24	6.21	6.24	6.3
stdv(σ)	0.664	0.886	0.907	0.918	0.912	0.923
avg$\{J(\theta)\}$	0.0031	0.0009	0.0034	0.0258	0.1554	0.3319
avg(deviance)	7.13	1.54	0.63	0.39	0.22	0.13

Table 12. Equilibrium renewal process NB(5, 0.357) for $\tau = 20$ (theoretical interval type distribution: (0.005, 0.657, 0.338)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(6)\}$	0.314	0.256	0.25	0.267	0.301	0.309
stdv$\{\widehat{F}(6)\}$	0.025	0.036	0.049	0.09	0.121	0.128
avg$\{\widehat{F}(9)\}$	0.512	0.507	0.529	0.532	0.528	0.522
stdv$\{\widehat{F}(9)\}$	0.025	0.036	0.07	0.093	0.115	0.125
avg$\{\widehat{F}(12)\}$	0.683	0.726	0.736	0.726	0.714	0.725
stdv$\{\widehat{F}(12)\}$	0.024	0.035	0.042	0.052	0.072	0.093
avg(μ)	10.39	10.03	10.01	10.01	10.01	10.01
stdv(μ)	0.387	0.354	0.352	0.352	0.352	0.352
avg(σ)	6.64	5.09	5	5.01	5.03	5.07
stdv(σ)	0.348	0.499	0.53	0.546	0.562	0.581
avg$\{J(\theta)\}$	0.0021	0.0006	0.0035	0.0215	0.0898	0.4111
avg(deviance)	9.9	1.82	0.85	0.58	0.41	0.32

Table 13. Ordinary renewal process NB(5, 0.357) for $\tau = 50$ (theoretical interval type distribution: (0.176, 0.824)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(6)\}$	0.361	0.276	0.247	0.238	0.234	0.238
stdv$\{\widehat{F}(6)\}$	0.012	0.023	0.033	0.054	0.096	0.146
avg$\{\widehat{F}(9)\}$	0.528	0.492	0.5	0.512	0.532	0.527
stdv$\{\widehat{F}(9)\}$	0.013	0.018	0.038	0.089	0.125	0.156
avg$\{\widehat{F}(12)\}$	0.67	0.698	0.729	0.735	0.732	0.736
stdv$\{\widehat{F}(12)\}$	0.013	0.021	0.043	0.067	0.095	0.121
avg(μ)	10.31	10.04	10	10	10	10
stdv(μ)	0.243	0.233	0.234	0.234	0.238	0.244
avg(σ)	7.2	5.27	4.97	4.98	4.99	4.97
stdv(σ)	0.235	0.388	0.438	0.448	0.466	0.484
avg$\{J(\theta)\}$	0.0069	0.0013	0.0011	0.0079	0.0822	0.6484
avg(deviance)	23.05	5.63	3.56	2.7	1.93	1.27

Table 14. Equilibrium renewal process NB(5, 0.357) for $\tau = 50$ (theoretical interval type distribution: (0, 0.333, 0.667)): 1000 count data samples of size 100.

	iteration					
	5	20	100	500	2000	10000
avg$\{\widehat{F}(6)\}$	0.374	0.281	0.248	0.236	0.228	0.237
stdv$\{\widehat{F}(6)\}$	0.013	0.024	0.035	0.059	0.114	0.166
avg$\{\widehat{F}(9)\}$	0.544	0.5	0.502	0.513	0.532	0.539
stdv$\{\widehat{F}(9)\}$	0.013	0.019	0.037	0.087	0.14	0.191
avg$\{\widehat{F}(12)\}$	0.685	0.702	0.729	0.738	0.74	0.735
stdv$\{\widehat{F}(12)\}$	0.012	0.019	0.041	0.071	0.11	0.15
avg(μ)	10.13	10.03	10.01	10.01	10.01	10.01
stdv(μ)	0.232	0.225	0.225	0.225	0.225	0.225
avg(σ)	7.24	5.34	4.96	4.96	4.97	4.96
stdv(σ)	0.21	0.359	0.421	0.424	0.43	0.434
avg$\{J(\theta)\}$	0.0072	0.0011	0.0009	0.0076	0.0841	0.6655
avg(deviance)	24.42	6.08	3.95	3.19	2.47	1.89

Table 15. Analysis of the elongation of coffee tree leafy axes by equilibrium renewal processes.

	observation period			
	1-40	41-80	1-80	81-167
interval type dist.	(0, 0.684, 0.316)	(0.013, 0.689, 0.298)	(0, 0.423, 0.577)	(0.004, 0.796, 0.2)
μ	20.78	22.23	21.46	58.49
σ/μ	0.225	0.451	0.409	0.264
$E\{N(\tau)\}$	1.93	1.8	3.73	1.49
var$\{N(\tau)\}$	0.249	0.392	0.7	0.28
$J(\theta)$	0.0004	0.0004	0.006	$1\ 10^{-6}$
deviance	0.0003	0.0003	0.0017	$2\ 10^{-6}$

Table 16. Observation period 1-80: Comparison of the fitted counting distribution with the counting distributions computed for $\tau = 80$ from the renewal processes estimated over the 1-40 and 41-80 observation periods.

		renewal process for $\tau = 80$	
		1-40	41-80
interval type dist.	(0, 0.423, 0.577)	(0, 0.412, 0.588)	(0.001, 0.432, 0.567)
$E\{N(\tau)\}$	3.73	3.85	3.6
var$\{N(\tau)\}$	0.7	0.358	0.749

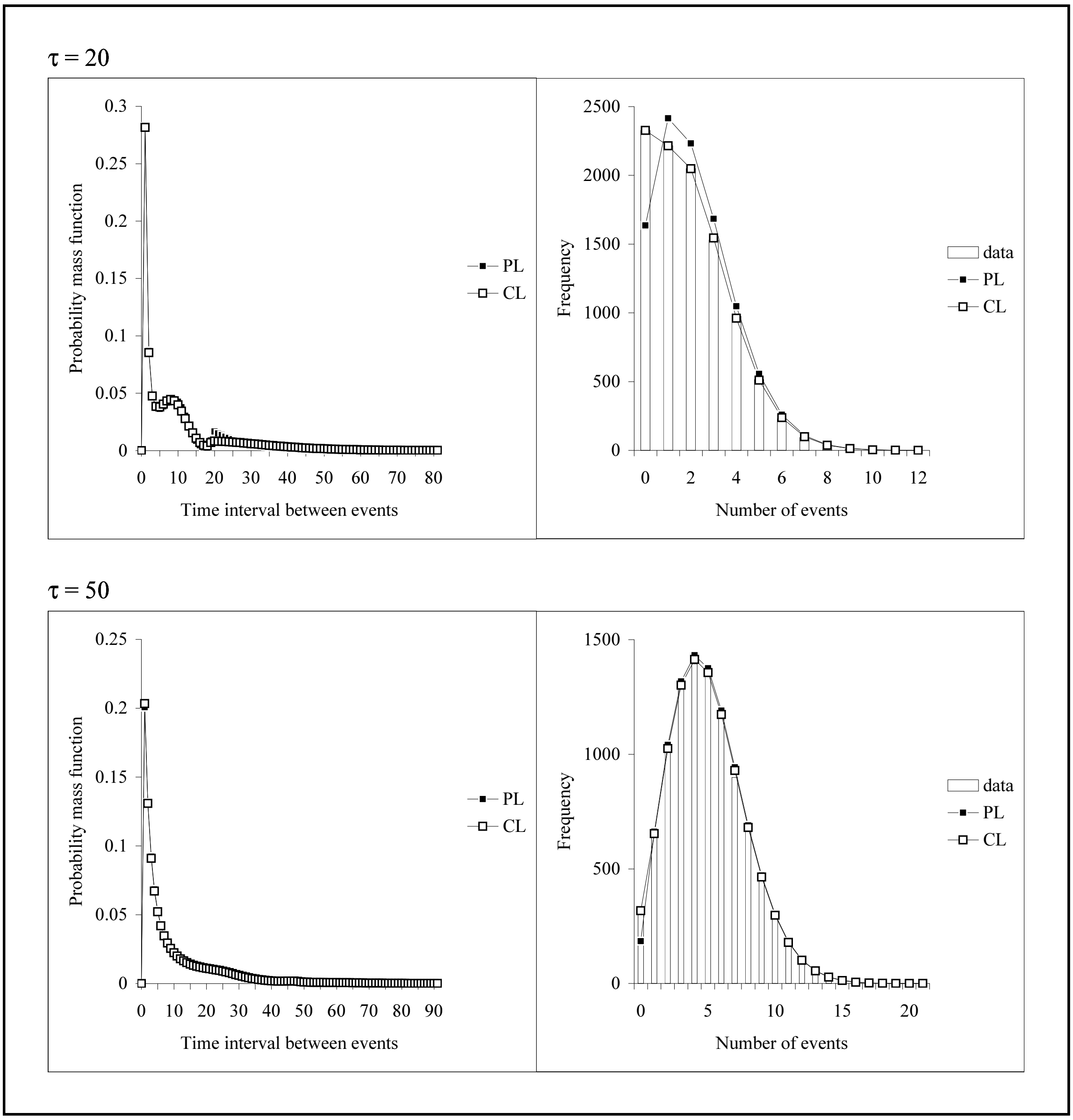

Figure 1. Equilibrium renewal process NB(0.5, 0.0526) for different observation periods τ.

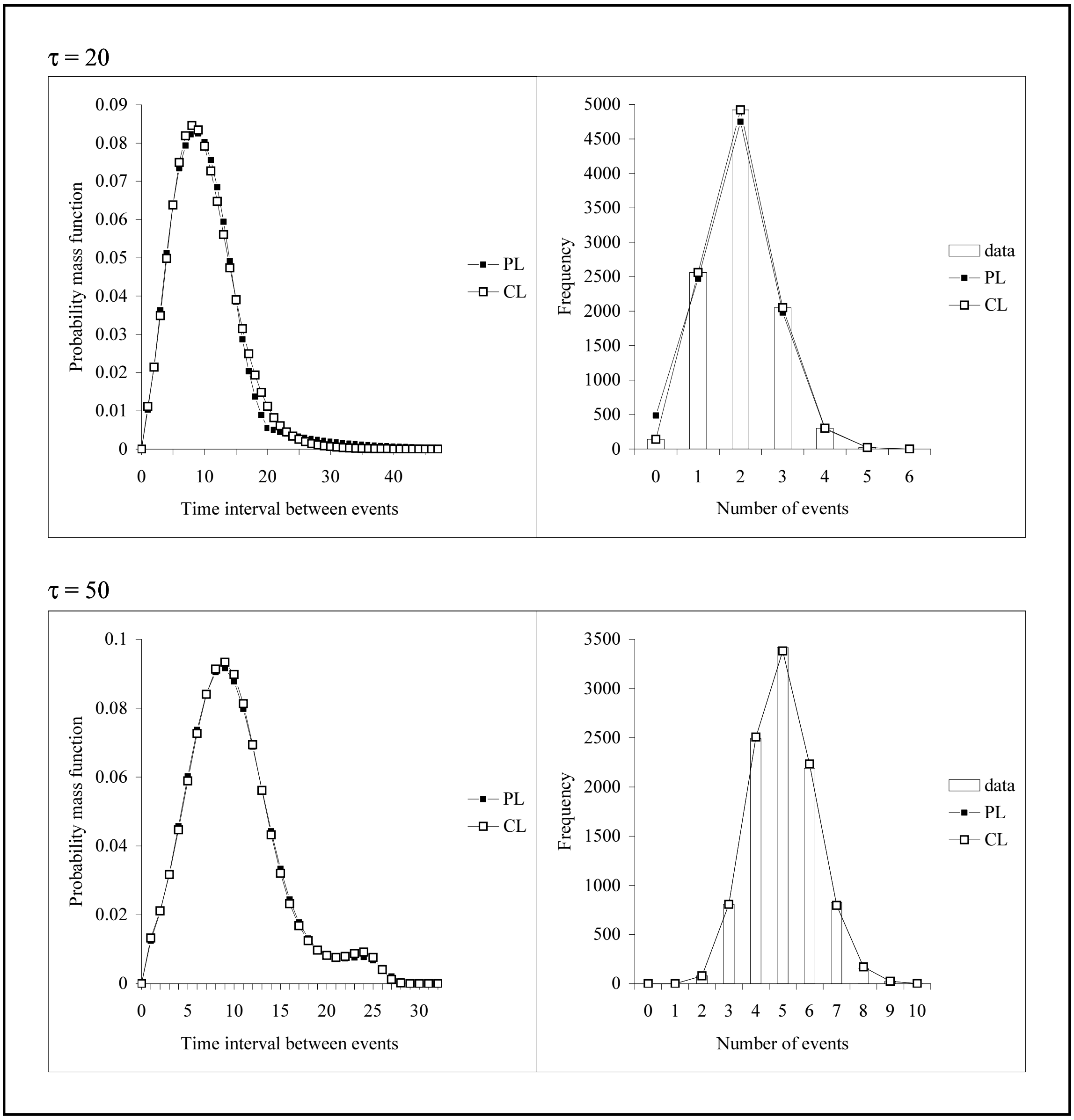

Figure 2. Equilibrium renewal process NB(5, 0.357) for different observation periods τ.

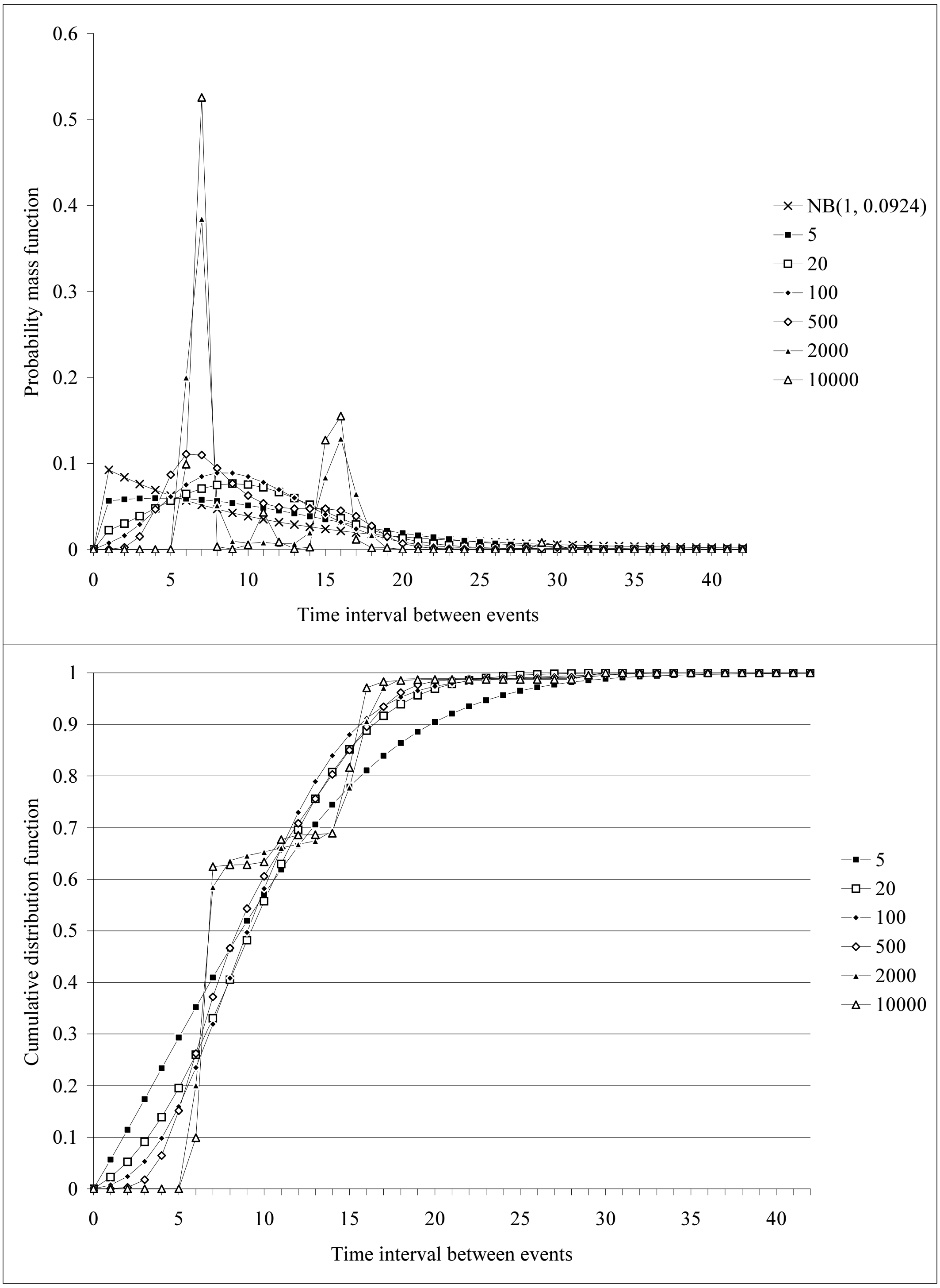

Figure 3. Ordinary renewal process NB(5, 0.357) for $\tau = 50$: reestimated inter-event distributions.

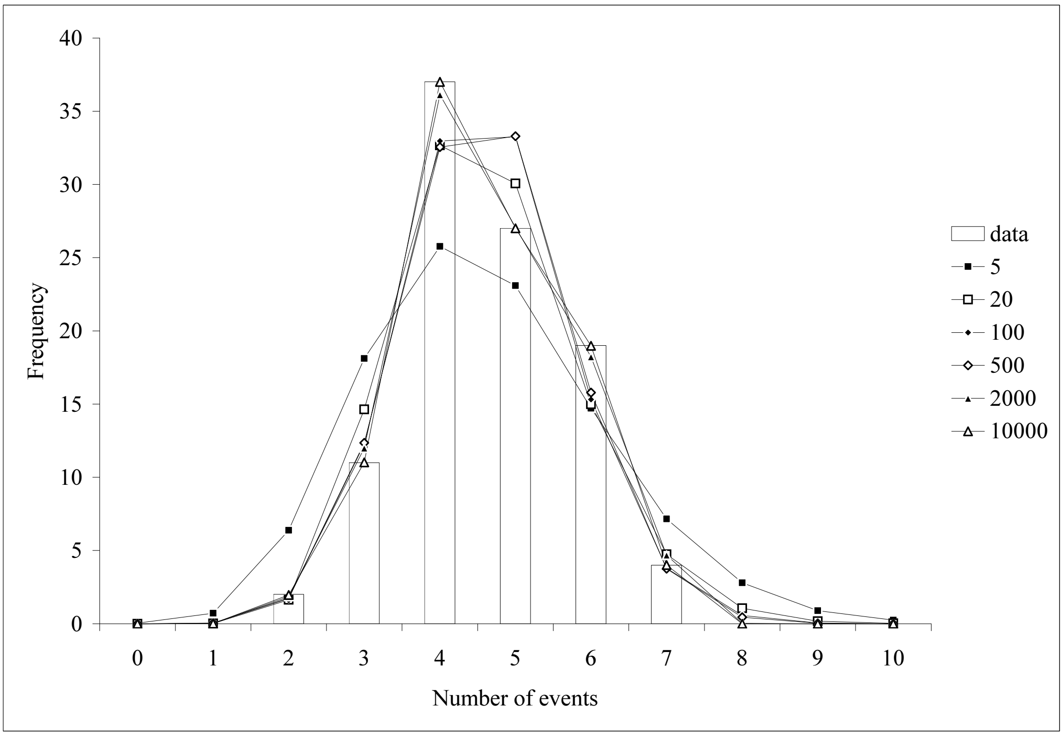

Figure 4. Ordinary renewal process NB(5, 0.357) for $\tau = 50$: counting distributions.

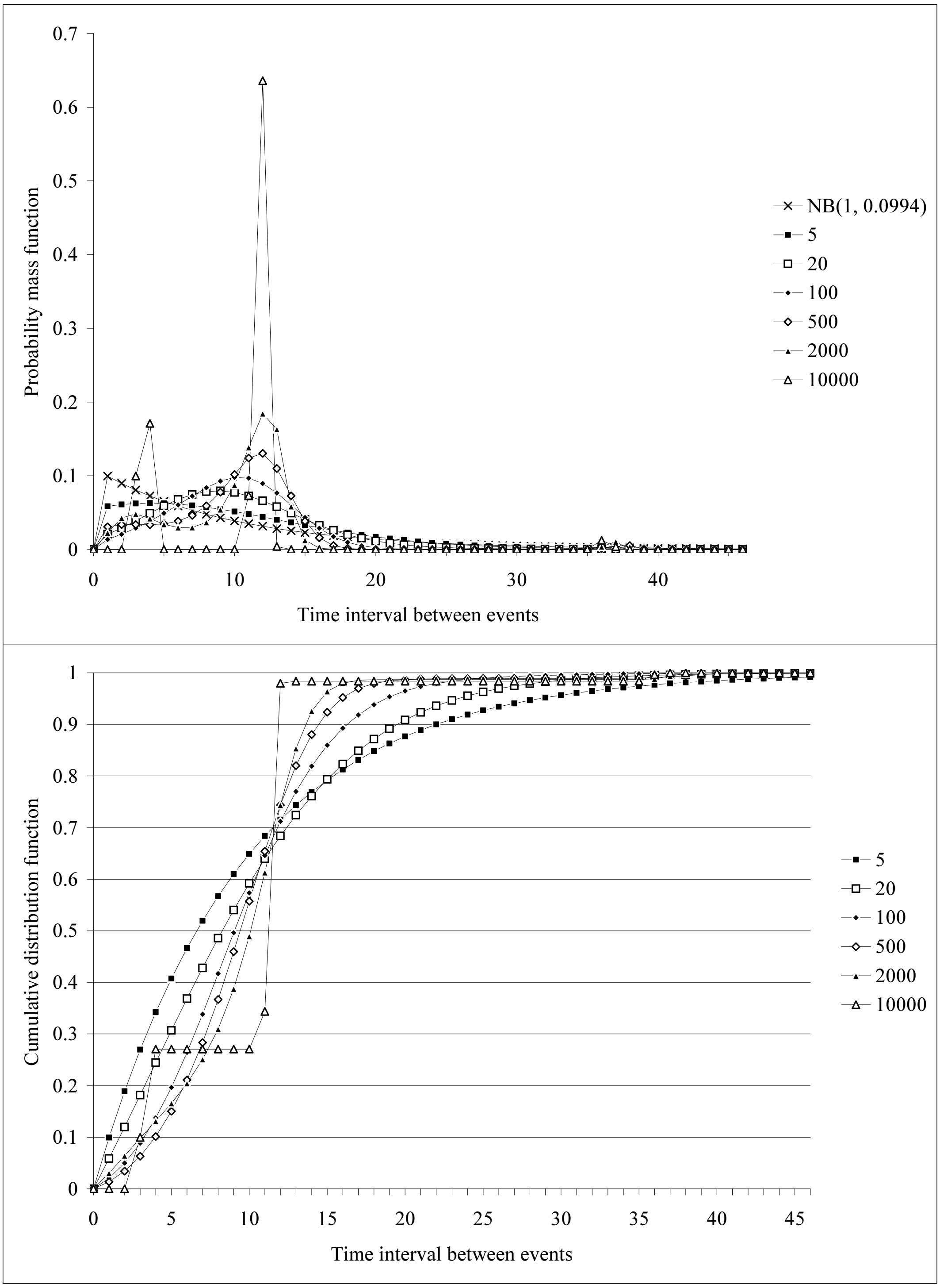

Figure 5. Equilibrium renewal process NB(5, 0.357) for $\tau = 50$: reestimated inter-event distributions.

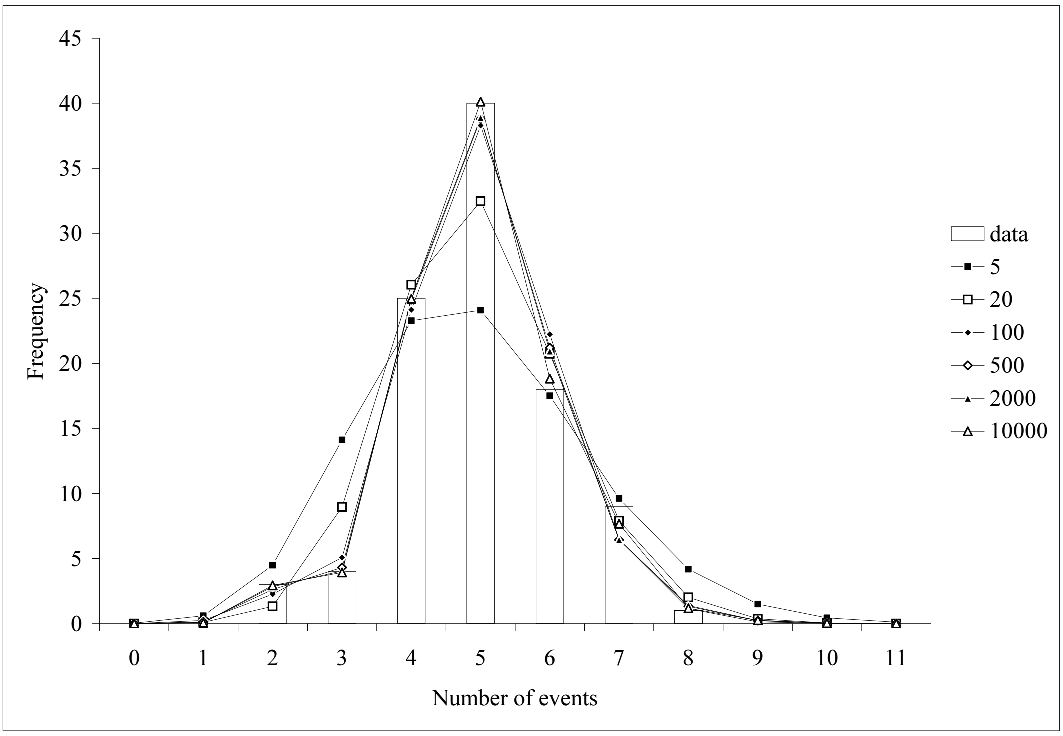

Figure 6. Equilibrium renewal process NB(5, 0.357) for $\tau = 50$: counting distributions.

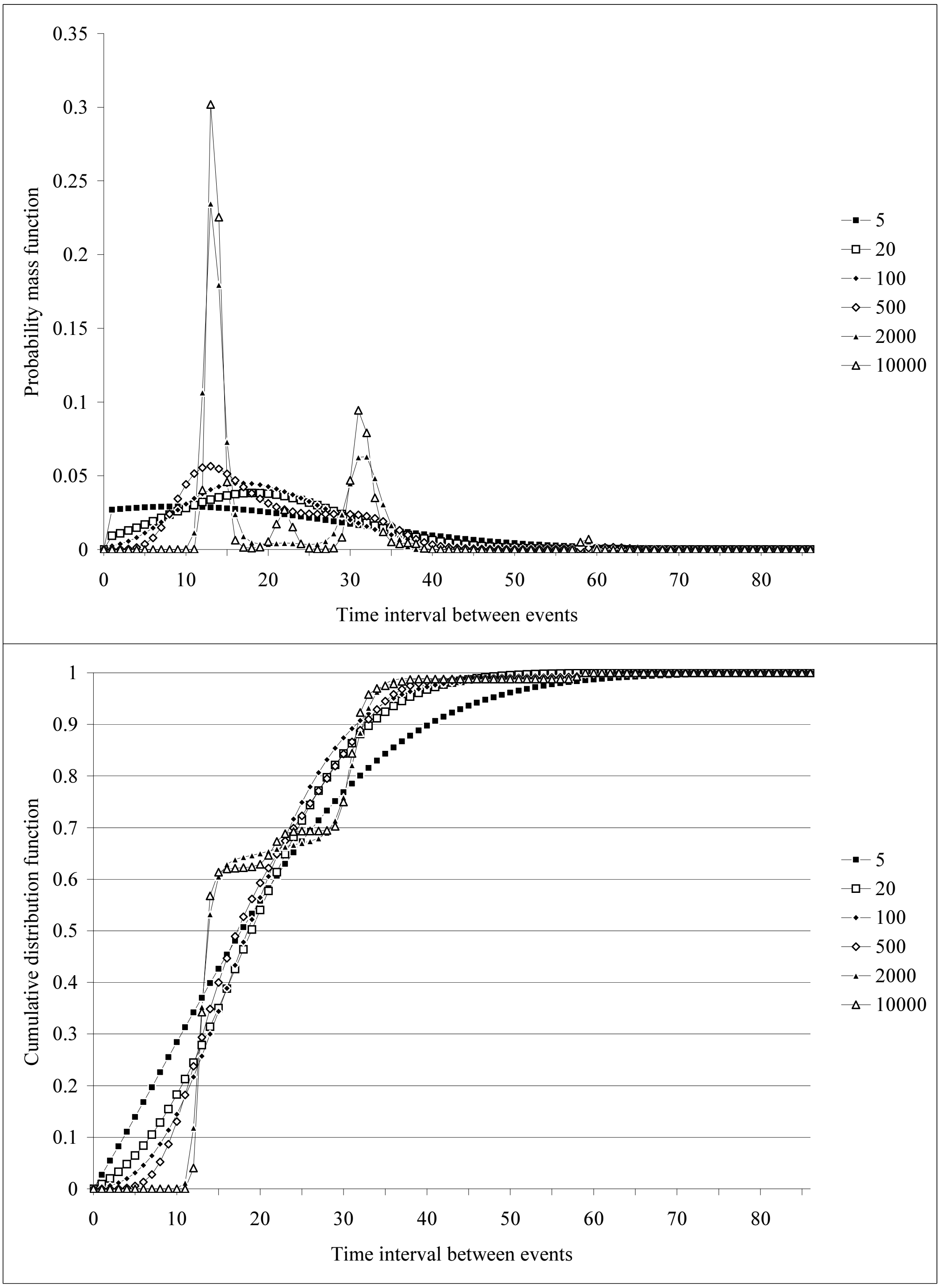

Figure 7. Ordinary renewal process NB(5, 0.357) for $\tau = 100$: reestimated inter-event distributions.

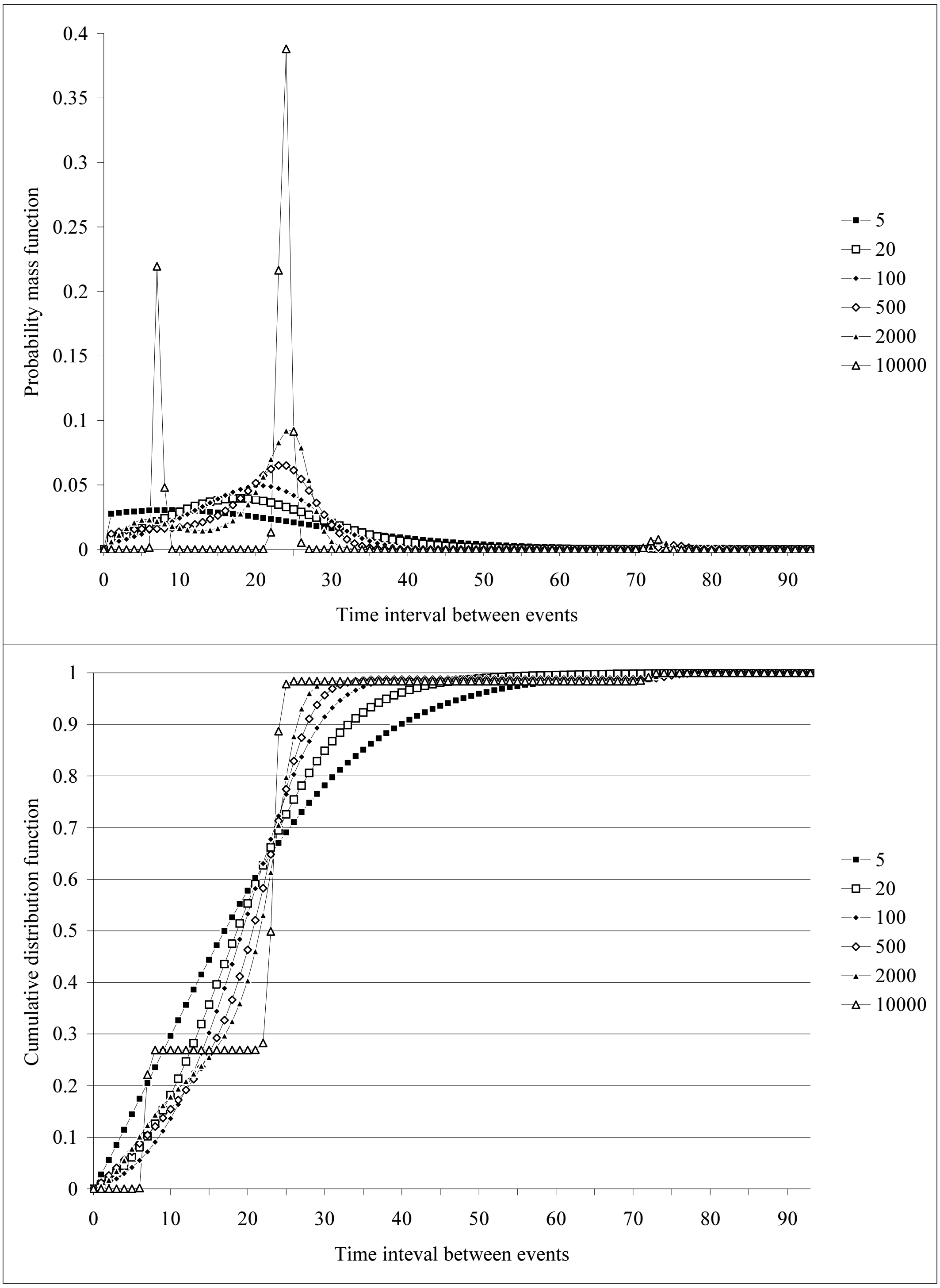

Figure 8. Equilibrium renewal process NB(5, 0.357) for $\tau = 100$: reestimated inter-event distributions.v

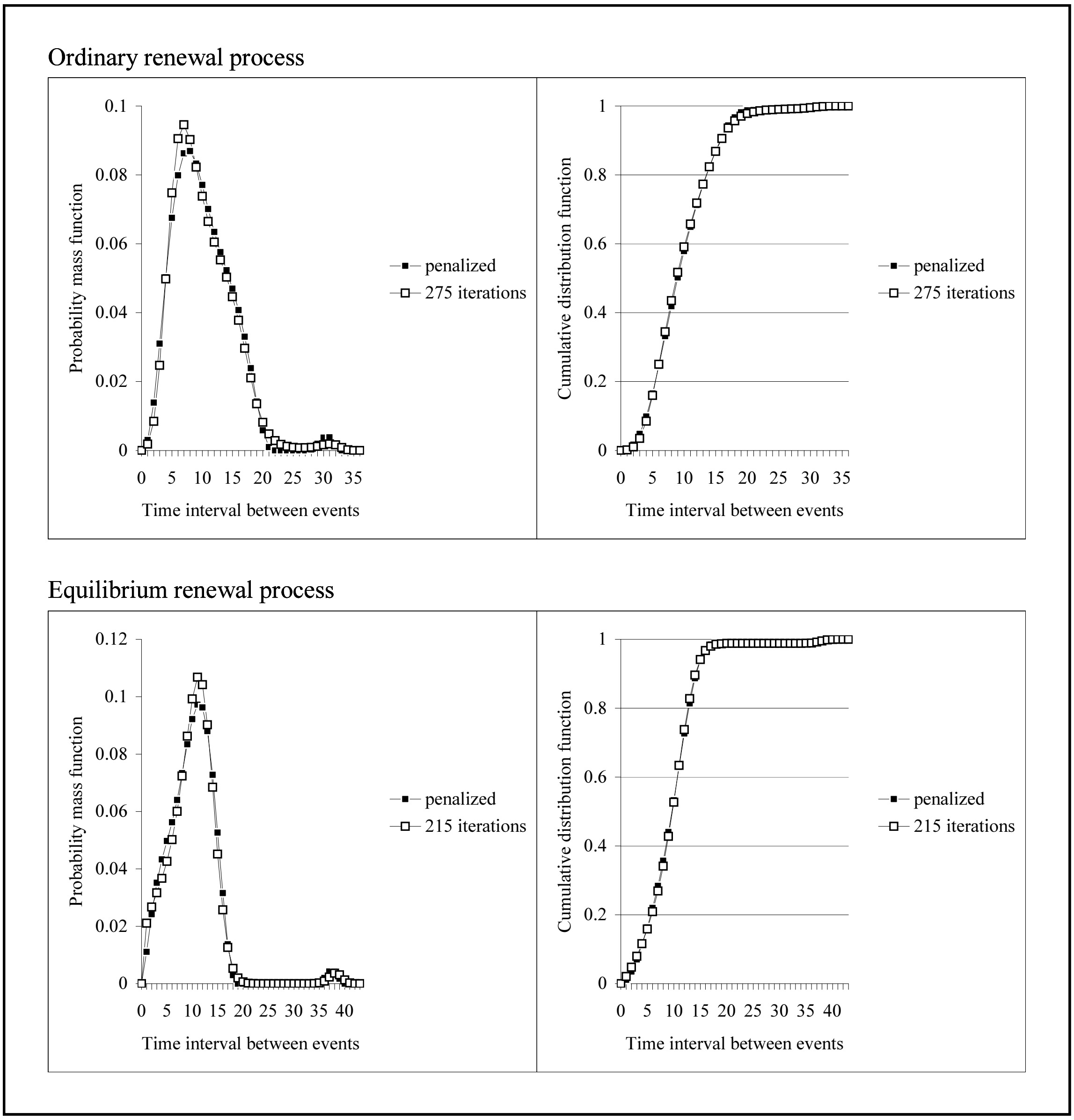

Figure 9. Renewal processes NB(5, 0.357) for $\tau = 50$: penalized likelihood estimates with $\rho = 0.5$.

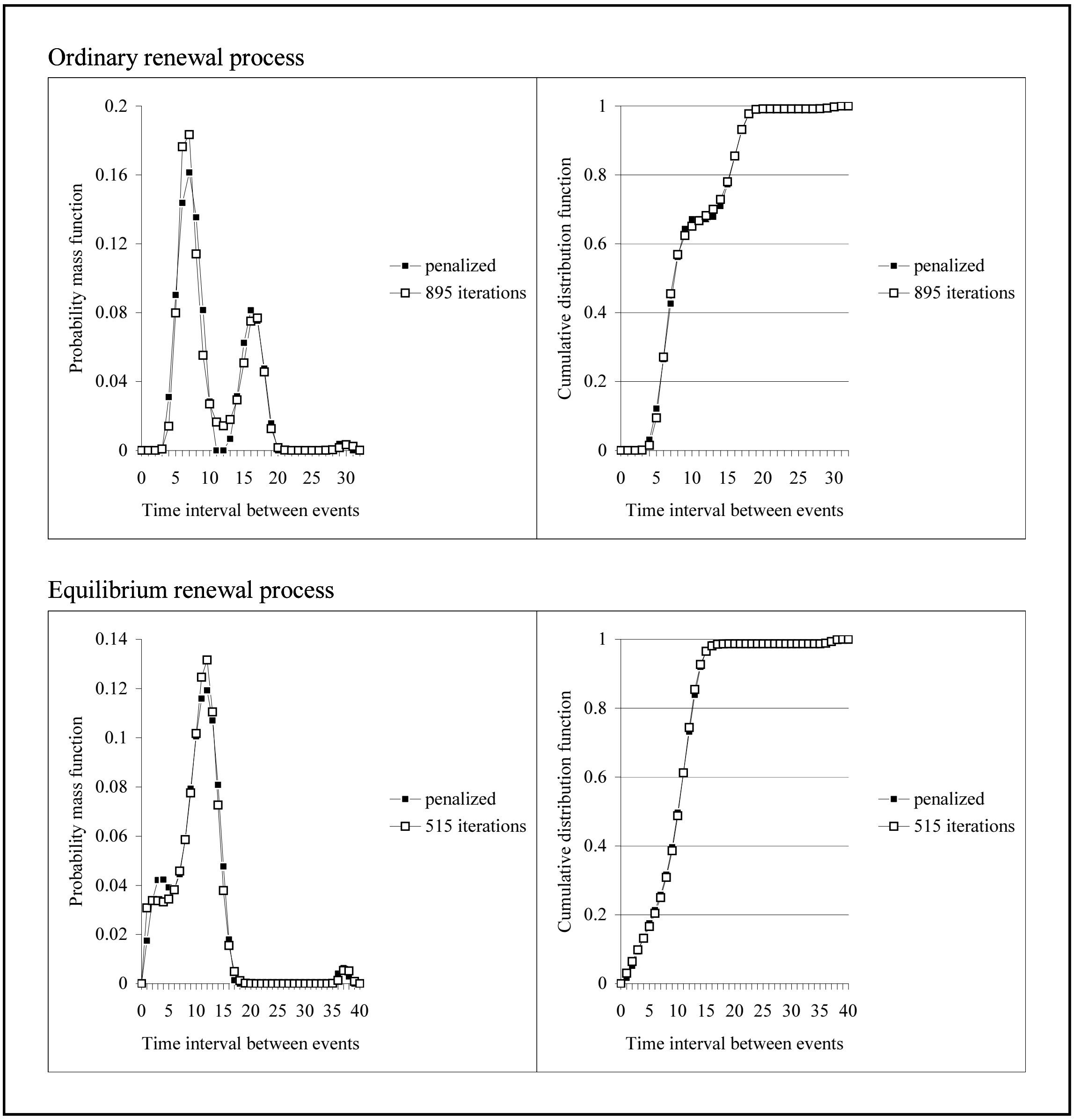

Figure 10. Renewal processes NB(5, 0.357) for $\tau = 50$: penalized likelihood estimates with $\rho = 0.1$.

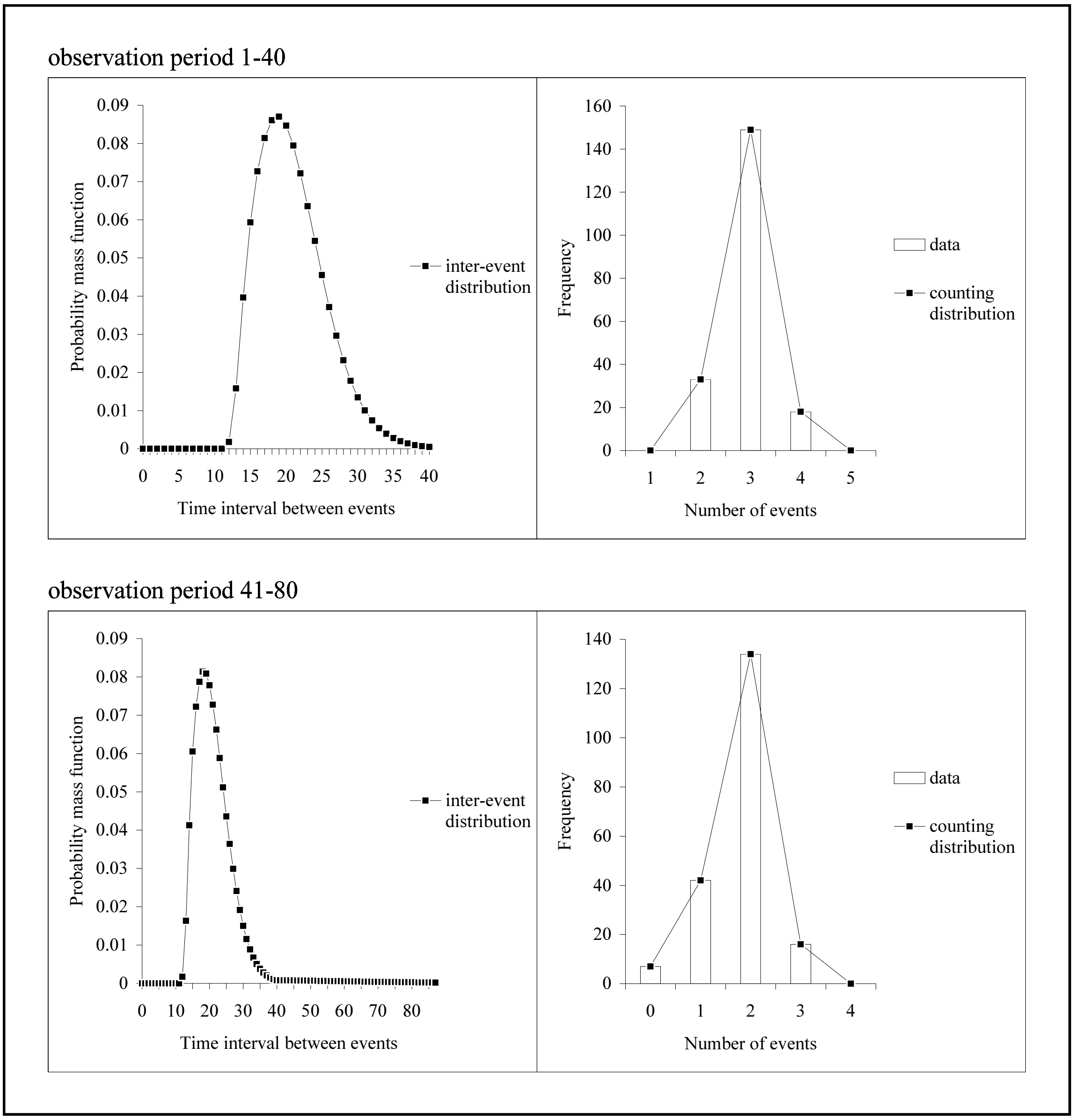

Figure 11. Analysis of the elongation of coffee tree leafy axes by equilibrium renewal processes; observation periods 1-40 and 41-80.

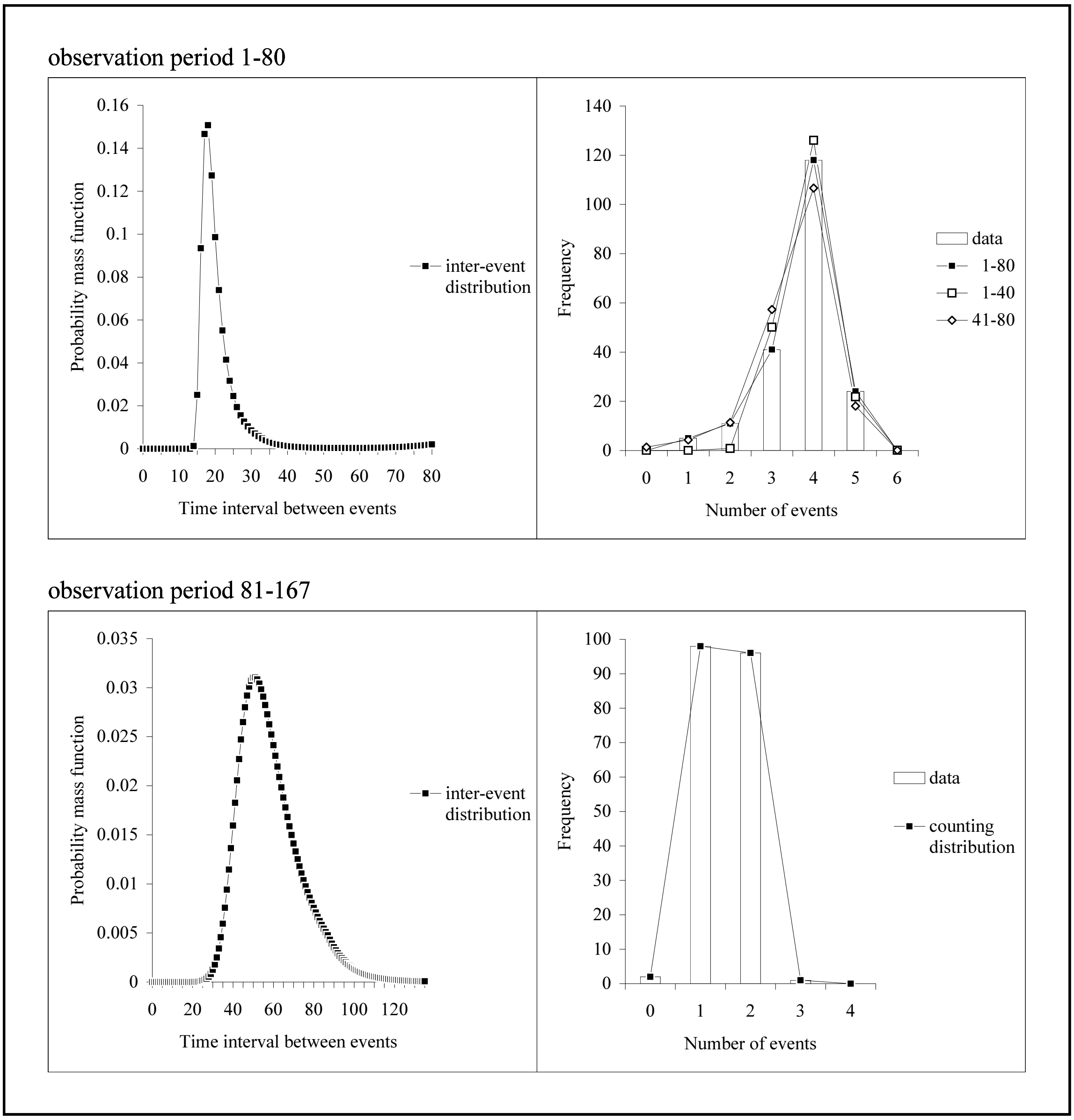

Figure 12. Analysis of the elongation of coffee tree leafy axes by equilibrium renewal processes; observation periods 1-80 and 81-167.

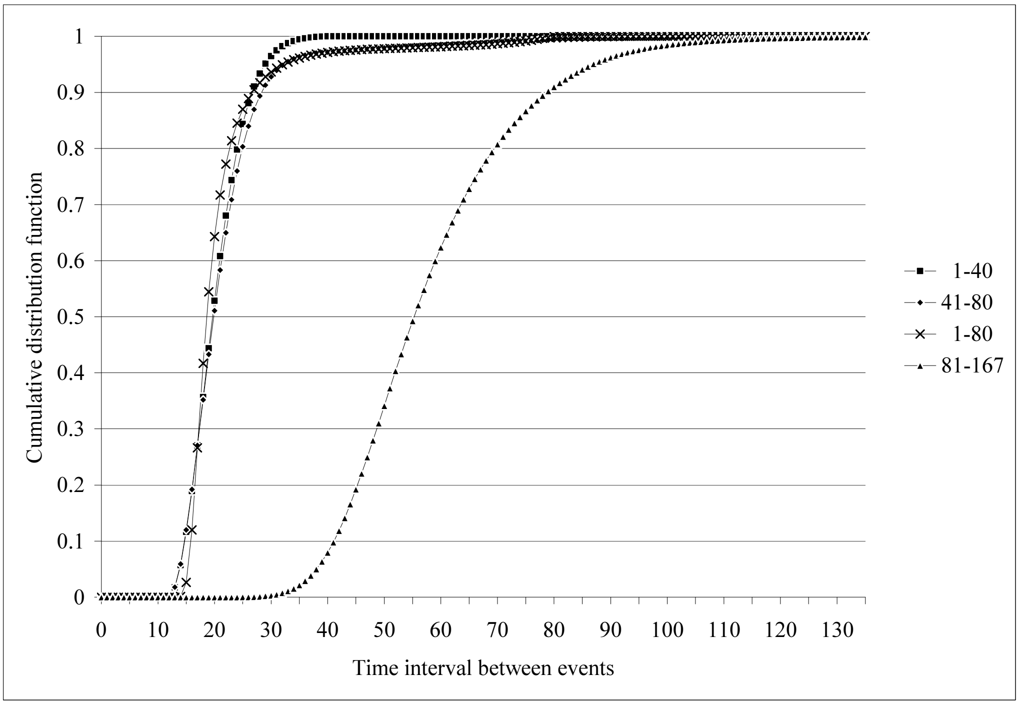

Figure 13. Elongation of coffee tree leafy axes: comparison of the inter-event distributions estimated for each observation period.

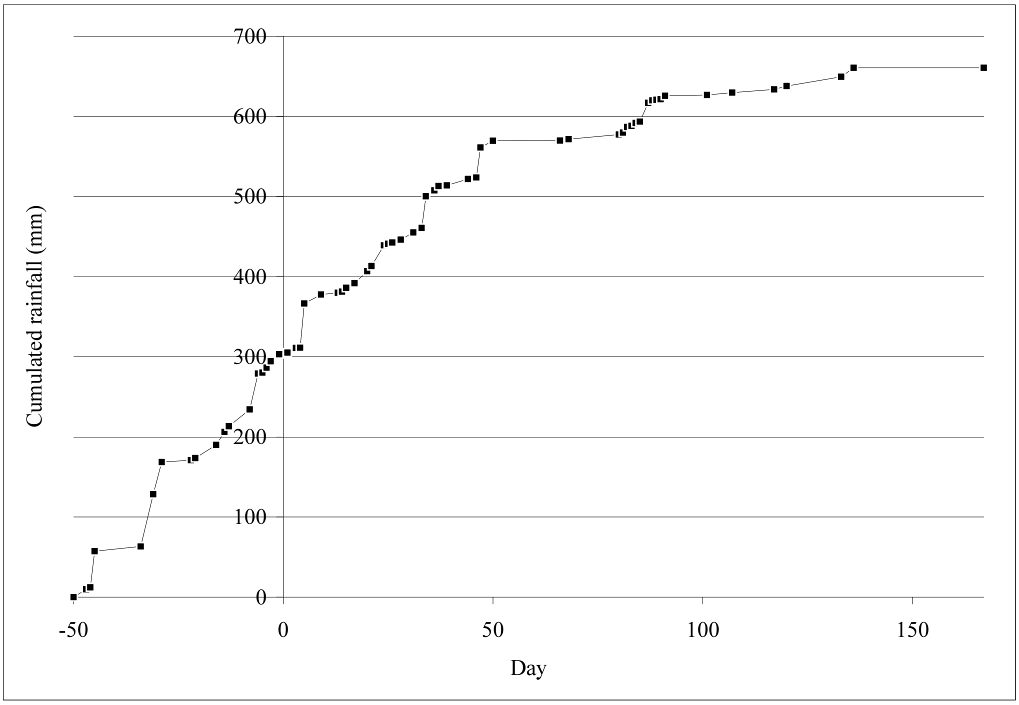

Figure 14. Rainfall during the coffee tree follow-up period.