Path Functors in Cat

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Under consideration for publication in Math. Struct. in Comp. Science

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Received 19 September 2013

We build an endofunctor in the category of small categories along with the necessary structure on it to turn it into a path object suitable for homotopy theory and modelling identity types in Martin-Löf type theory. We construct the free Grothendieck bifibration over a base category generated by an arbitrary functor to that category.

Keywords: Path object functor, Grothendieck fibration, bifibration, homotopy theory, category of fibrant objects, intensional identity type, Martin-Löf type theory.

1. Introduction

The relationship between the Martin-Löf identity type and objects of paths in various, more or less abstract models of homotopy theory is now well established (Awodey and Warren(2009); Warren(2008); Arndt and Kapulkin(2011)).

In a previous paper (Lamarche(2013)) we presented a model of the Martin-Löf intensional identity type in the category *Cat* of small categories. The dependent types were modelled by (cleft) Grothendieck bifibrations, and the identity type on a category X was an other category $\mathbf{P}(X)$ where an object was a zigzagging path of maps between two objects of X. This category was obtained by first constructing another category $\mathbf{Re}(X)$, which had the very same objects as $\mathbf{P}(X)$, and then quotienting its hom-sets to get the latter. The closest relative we know of these constructions is the path object category constructed from simplicial set in (Garner and Berg(2012)). An important theorem, that asserted that a the set of bicleavages for a bifibration Y over a category X is in bijective correspondence with the set what were called *Hurewicz actions* of $\mathbf{P}(X)$ on Y was simply left without a proof, with only the mention that it "is rather more technical and depends on a close analysis of the order enrichment $\subseteq$ on $\mathbf{Re}$ ".

The present, not really slim, paper is dedicated to just that: giving the required close analysis of the order enrichment just mentioned, and using it to prove that key theorem in (Lamarche(2013)), a paper which is perhaps better seen as an extend abstract. Our presentation is self-contained, and we think it can be read independently. An outcome of this work that goes beyond type theory or homotopy theory, and gives independent motivation to the present paper, is that we will show how to construct free (cleft) bifibrations, something which seems never to have been done before.

2. The Path Functor

We will begin by one remark on notation and one rather standard definition. The category of small categories is denoted by *Cat*. A map Y of small categories which is to have a fibration structure (a cleavage) is denoted by something like

$$Y\colon \overline{Y} \longrightarrow A\,,$$

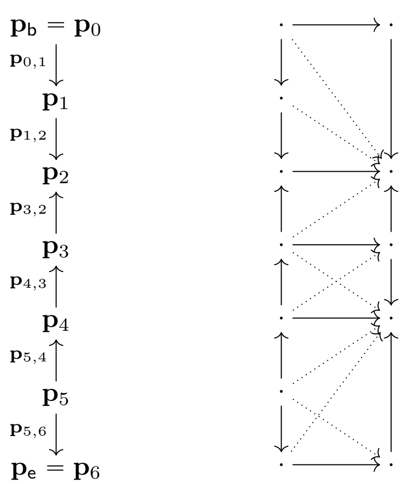
where the name of its domain is obtained by overlining the map's name, but the codomain can look arbitrary. Given the above along with an arbitrary $f\colon B \to A$ the pullback operation is denoted by

$$\begin{array}{ccc} \overline{f^*Y} & \longrightarrow & \overline{Y} \\ {\scriptstyle f^*Y}\downarrow & & \downarrow{\scriptstyle Y} \\ B & \xrightarrow[f]{} & A \end{array} \tag{1}$$

Definition 1. Let A be a set, and $\leqslant$ an order relation on it. Given $a, b \in A$ with $a \leqslant b$, we say *a is a $\leqslant$-predecessor of b* (or *b is a $\leqslant$-successor of a*) when

$$a \leqslant c \leqslant b \quad \text{implies} \quad c = a \text{ or } c = b\,.$$

Let X be a small category. Our first goal is to construct a category $\mathbf{Pe}(X)$ whose objects as we said are to be seen as paths in X. Before we get into formal definitions, let us show what we are trying to formalize. The diagram to the left below gives such an object $\mathbf{p}$ and the one to the right an example of a morphism between two such objects in a less florid notation. In the second diagram everything that can commute is assumed to do so, with the dotted diagonal arrows being obtained as composites of solid arrows. Paths will always be drawn vertically, and read from the top down. The map of path should (obviously) be read from left to right.



(2)

Definition 2. A *path* $\mathbf{p}$ in X is a quadruple

$$\mathbf{p} \;=\; \big(\mathbb{I}, \leqslant, \sqsubseteq, (\mathbf{p}_{x,x'})_{x,x'}\big)$$

where

— $(\mathbb{I}, \leqslant)$ is a nonempty finite totally ordered set.We denote its first element by b (the beginning) and its last one by e (the end of the path).
— $\sqsubseteq$ is another order structure on $\mathbb{I}$, the *diagrammatic order,* that obey the following condition, in which $\prec_{\leqslant}, \prec_{\sqsubseteq}$ mean the predecessor relation on $\leqslant, \sqsubseteq$ respectively:

$$\text{if } x \prec_{\leqslant} y \text{ then either } x \prec_{\sqsubseteq} y \text{ or } y \prec_{\sqsubseteq} x.$$

— $(\mathbf{p}_{x,x'})_{x,x' \in \mathbb{I}}$ is a diagram $(\mathbb{I}, \sqsubseteq) \to X$. That is, for every $x \in \mathbb{I}$ there is an object $\mathbf{p}_x \in X$ and for every $x \sqsubseteq x'$ there is $\mathbf{p}_{x,x'} : \mathbf{p}_x \to \mathbf{p}_{x'}$, with the usual functorial identities. In particular $\mathbf{p}_{x,x}$ is the identity on the object $\mathbf{p}_x$.

The *length* of a path $\mathbf{p}$ is $\mathrm{Card}(\mathbb{I}_{\mathbf{p}}) - 1$. Given a path $\mathbf{p} = \big(\mathbb{I}, \leqslant, \sqsubseteq, (\mathbf{p}_{x,x'})_{x,x'}\big)$ we call the structure $(\mathbb{I}_{\mathbf{p}}, \leqslant, \sqsubseteq)$ (a biposet) the *shape* of $\mathbf{p}$.

When we deal with the shape of several paths we use subscripts to distinguish whas has to be distinguished, e.g., $\mathbb{I}_{\mathbf{p}}, \sqsubseteq_{\mathbf{p}} \dots$

Thus, since $\leqslant$ is a total order, we see that $\sqsubseteq$ looks like a zigzag, whose "branches" are totally ordered and coincide with segments of $\leqslant$, each branch of $\sqsubseteq$ having the induced order from $\leqslant$ or its opposite. Notice that if we are given a poset $(\mathbb{I}, \sqsubseteq)$ with the required zigzag shape, the only additional structure which is needed to recover the order $\leqslant$ is a choice of the element b, and there can only be two possible choices of that element; the two alternative choices swap b and e. Thus Definition 2 is rather redundant, but since we will use the order $\leqslant$ a lot it is convenient to put it as part of the structure.

There are two ways we can think about the elements of the set $\mathbb{I}$. If we decide that $\mathbb{I}$ is just any set, we need to identify two paths $\mathbf{p}, \mathbf{p}'$ that differ only by the way the elements or the indexing sets $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{p}'}$ are named. Given two arbitrary paths $\mathbf{p}, \mathbf{p}'$, there is a natural definition of isomorphism between the biposets defined by the shapes $\big(\mathbb{I}_{\mathbf{p}}, \leqslant, \sqsubseteq\big)$ and $\big(\mathbb{I}_{\mathbf{p}'}, \leqslant, \sqsubseteq, \big)$: it's just a bijection between $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{p}'}$ that preserves and reflects both orders. Since $\leqslant$ is a total finite order, if an iso $\alpha \colon \mathbb{I}_{\mathbf{p}} \to \mathbb{I}_{\mathbf{p}'}$ exists it is unique, and we identify $\mathbf{p}, \mathbf{p}'$ when we have $\mathbf{p}'_{\alpha(x)} = \mathbf{p}_x$ for every $x \in \mathbb{I}_{\mathbf{p}}$.

But we can also decree that $\mathbb{I}$ is always of the form $\{0, \dots, n\}$, as is done in the left of Diagram (2). Then some operations on paths like their vertical composition forces renamings. It what follows we will try to be agnostic about these matters, but we will use numerical indexing when it is useful for us, without further apologies.

The $\leqslant$ order is read from the top down, and so a down-arrow means that $\leqslant, \sqsubseteq$ coincide and an up-arrow the opposite. A path of length zero is just an object of X; such a path is different from all the paths $\mathbf{p}$ all whose $\mathbf{p}_{x,y}$ are identities, which can have arbitrary length $n \geqslant 1$.

Definition 3. Let $(\mathbb{I}, \sqsubseteq), (\mathbb{J}, \sqsubseteq)$ be two posets. *An ordering from* $\mathbb{I}$ *to* $\mathbb{J}$ is an ordering $\sqsubseteq$ of the disjoint sum $\mathbb{I} + \mathbb{J}$ such that

— the restriction of $\sqsubseteq$ on $\mathbb{I}, \mathbb{J}$ is exactly $\sqsubseteq_{\mathbb{I}}, \sqsubseteq_{\mathbb{J}}$.
— if $x \in \mathbb{I}, y \in \mathbb{J}$ are related by the $\sqsubseteq$ order, then $x \sqsubseteq y$.

It is our intention to define a map of paths $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ as a commutative diagram

$$\begin{array}{ccc} (\mathbb{I}_{\mathbf{p}} \sqsubseteq) & & \\ \downarrow & \searrow^{\mathbf{p}} & \\ \mathrm{Sh}(\mathbf{f}) & \xrightarrow{\mathbf{f}} & X \\ \uparrow & \nearrow_{\mathbf{q}} & \\ (\mathbb{I}_{\mathbf{q}} \sqsubseteq) & & \end{array} \tag{3}$$

where $\mathrm{Sh}(\mathbf{f})$, the *shape* of $\mathbf{f}$, is an ordering from $(\mathbb{I}_{\mathbf{p}} \sqsubseteq)$ to $(\mathbb{I}_{\mathbf{q}}, \sqsubseteq)$, that satisfies a certain geometric condition. In order to enunciate this condition, we need some definitions. First

Proposition 1. Let $\mathbb{I}, \mathbb{J}$ be posets and $\sqsubseteq$ an ordering from $\mathbb{I}$ to $\mathbb{J}$. By restriction this order determines a relation $R \subseteq \mathbb{I} \times \mathbb{J}$, i.e., xRy iff $x \sqsubseteq y$. This restriction operation determines a bijection between the set of orderings from $\mathbb{I}$ to $\mathbb{J}$ and the set of relations $R \subseteq \mathbb{I} \times \mathbb{J}$ that are

— left-down-closed: $xRy, x' \sqsubseteq x$ implies $x'Ry$,
— right-up-closed: $xRy, y' \sqsupseteq y$ implies xRy'.

Proof. Very easy. □

Now it is well known that the class of posets along with left-down- and right-up-closed relations form a category, which has been dubbed by Lambek the category of posets and comparisons (Lambek(1994)) and is a very special case of the extremely general bimodule construction in enriched category theory (Lawvere(1973)), which generalises profunctors/distributors in basic category theory (Bénabou(1973)).

In other words, given comparisons $R \subseteq \mathbb{I} \times \mathbb{J}$ and $S \subseteq \mathbb{J} \times \mathbb{K}$ it is easy to check that

$$S \circ R \quad = \quad \{\, (x, z) \in \mathbb{I} \times \mathbb{K} \mid \text{there exists } y \in \mathbb{J} \text{ with } (x, y) \in R, (y, z) \in S \,\}$$

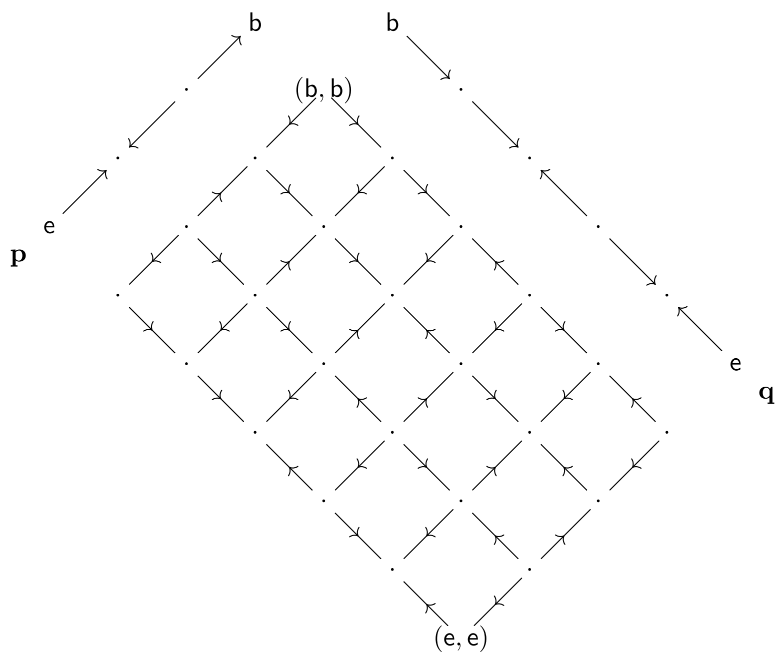
is a comparison too, and that for any poset $\mathbb{I}$ the left-down- and right-up-closure of the identity relation acts as the identity for composition (but not the identity relation itself, since it is not a comparison unless $\mathbb{I}$ is a discrete poset!).

The following construction has been used since times immemorial; for now we mention (Quillen(1973)) and promise more historically faithful citations in a future version of this paper.

Definition 4. Given an arbitrary category X we define its *category of factorizations* $\mathrm{Fac}(X)$ to have for objects the maps of X, and where a morphism $s \to r$ is a pair (m, n) of maps of appropriate source and targets such that $msn = r$.

The source and target operations provide an obvious functor $\langle \mathsf{d}_0, \mathsf{d}_1 \rangle\colon \mathrm{Fac}(X) \to X^{op} \times X$ which is a *discrete opfibration.* Quite obviously, it is the discrete opfibration associated to to the hom functor $X^{op} \times X \to \mathit{Set}$.

Given paths $\mathbf{p}, \mathbf{q}$ we denote the order $(\mathbb{I}_{\mathbf{p}}, \sqsubseteq)^{op} \times (\mathbb{I}_{\mathbf{q}}, \sqsubseteq)$ by $\trianglelefteq$. Here is a sample graphic representation of a poset of the form $(\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}, \trianglelefteq)$:



We say (b, b) is the *North Pole* and that (e, e) is the *South Pole*. The condition for a subset $R \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ to be a comparison is that it be $\trianglelefteq$-up-closed, which in this graphic representation is that it be closed downstream under the $\rightarrowtail$ arrows.

Definition 5. Given paths $\mathbf{p}, \mathbf{q}$ in X we define a *premap* $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ to be a diagram $\mathbf{f} \colon \big(\mathrm{Sh}(\mathbf{f}), \sqsubseteq_{\mathbf{f}}\big) \to X$, where $\big(\mathrm{Sh}(\mathbf{f}), \sqsubseteq_{\mathbf{f}}\big)$, the *shape* of $\mathbf{f}$ is an order from $(\mathbb{I}_{\mathbf{p}}, \sqsubseteq)$ to $(\mathbb{I}_{\mathbf{q}}, \sqsubseteq)$, that obeys the additional conditions (cf. Diagram (3))

a) $\mathbf{p}_{\mathsf{b}} \sqsubseteq \mathbf{q}_{\mathsf{b}}$ and $\mathbf{p}_{\mathsf{e}} \sqsubseteq \mathbf{p}_{\mathsf{e}}$
b) $\mathbf{f}$ restricted to $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$ is $\mathbf{p}, \mathbf{q}$.

Thus a premap can also be seen as a subset $\mathrm{At}(\mathbf{f})$ of $\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$, the *atlas* of $\mathbf{f}$, where every $(x, y) \in \mathrm{At}(\mathbf{f})$ is decorated with a map $\mathbf{f}_{x,y}$ in X, with Condition a) meaning that both the North and South pole are in $\mathrm{At}(\mathbf{f})$ and Condition b), along with the functoriality of $\mathbf{f}$, is a coherence requirement:

$$\text{Given } x, x' \in \mathbb{I}_{\mathbf{p}},\ y, y' \in \mathbb{I}_{\mathbf{q}} \text{ such that } x' \sqsubseteq x \sqsubseteq y \sqsubseteq y' \text{ then } \mathbf{q}_{y,y'} \circ \mathbf{f}_{x,y} \circ \mathbf{p}_{x',x} = \mathbf{f}_{x',y'}. \tag{4}$$

In other words, $\mathrm{At}(\mathbf{f})$ is the comparison associated to $\mathrm{Sh}(\mathbf{f})$ and the coherence requirement is equivalent to saying that the decoration $(\mathbf{f}_{x,y})_{x,y}$ is a monotone function $\mathrm{At}(\mathbf{f}) \to$

$\mathrm{Fac}(X)$ (also denoted $\mathbf{f}$) that makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{At}(\mathbf{f}) & \xrightarrow{\ \mathbf{f}\ } & \mathrm{Fac}(X) \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle \mathsf{d}_0, \mathsf{d}_1 \rangle} \\ (\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}, \trianglelefteq) & \xrightarrow[\mathbf{p}^{op} \times \mathbf{q}]{} & X^{op} \times X \end{array} \tag{5}$$

Given an arbitrary subset $R \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ let us denote by $\mathrm{Cl}(R)$ its up-$\trianglelefteq$-closure. If now $(R, \mathbf{f})$ is a pair, where R is such a subset and $\mathbf{f} \colon R \to X$ an arbitrary function $(x, y) \mapsto \mathbf{f}_{x,y}$, we are interested in knowing when $\mathbf{f}$ can be completed to a premap $\tilde{\mathbf{f}}$ where $\mathrm{At}(\tilde{\mathbf{f}}) = \mathrm{Cl}(R)$. The following is obviously a necessary condition

— Given $(x_1, y_1), (x_2, y_2) \in R$ then for every (x', y') such that $(x', y') \trianglerighteq (x_1, y_1)$ and $(x', y') \trianglerighteq (x_2, y_2)$ we have

$$\mathbf{q}_{y_1,y'} \circ \mathbf{f}_{x_1,y_1} \circ \mathbf{p}_{x',x_1} = \mathbf{q}_{y_2,y'} \circ \mathbf{f}_{x_2,y_2} \circ \mathbf{p}_{x',x_2} \ . \tag{6}$$

We say $(R, \mathbf{f})$ is a *compatible family* when this condition holds. It is also a sufficient condition, provided that $\mathrm{Cl}(R)$ contains the two poles: supposing $(R, \mathbf{f})$ is compatible, given an arbitrary $(x, y) \in \mathrm{Cl}(R)$ we can define

$$\tilde{\mathbf{f}}_{x,y} = \mathbf{f}_{x_1,y_1} \quad \text{for some } (x_1, y_1) \in R \text{ with } (x, y) \trianglerighteq (x_1, x_1)$$

since compatibility ensures that the value of $\tilde{\mathbf{f}}_{x,y}$ doesn't depend on the exact choice of (x_1, y_1).

This condition can be strengthened since $(\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}, \trianglelefteq)$ has bounded sups, meaning that if $\mathrm{Cl}\{(x_1, y_1)\} \cap \mathrm{Cl}\{(x_2, y_2)\}$ is nonempty then the least upper bound $(x_1, y_1) \vee (x_2, y_2)$ exists, and thus Equation 6 has to be checked only when $(x', y') = (x_1, y_1) \vee (x_2, y_2)$. Obviously, a premap is a compatible family, the biggest possible one that generates it. Given the finite nature of the order $\trianglelefteq$, any premap $\mathbf{f}$ is also generated by a smallest compatible family: just take the elements of $\mathrm{At}(\mathbf{f})$ that are minimal.

A premap $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ can thus be seen in two ways: as a diagram $\mathrm{Sh}(\mathbf{f}) \to X$ that extends $\mathbf{p}, \mathbf{q}$, or as a function defined on the set $\mathrm{At}(\mathbf{f})$ whose values are maps of X. Thus we can also see it as a *partial* function $(\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}, \trianglelefteq) \to X$, which allows the use of verbal shortcuts like saying "$\mathbf{f}_{x,y}$ is defined" to mean that $(x, y) \in \mathrm{At}(\mathbf{f})$.

We have started giving the latter point of view a certain "geographical" aspect, and we intend exploit that metaphor.

Observation 1. Let $t_1 = (x_1, y_1), t_2 = (x_2, y_2)$ be two elements (or points on the atlas) in $\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$. Assuming that $t_1 \neq t_2$ then there are four mutually exclusive cases that may occur:

— $x_1 = x_2$. We say that the two points are *on the same northwest-southeast diagonal.*

— $y_1 = y_2$. We say that the two points are *on the same northeast-southwest diagonal.*

— Either $x_1 < x_2$, $y_1 < y_2$ or $x_1 > x_2$, $y_1 > y_2$—i.e., t_1 and t_2 "do not cross" as viewed as arrows in a diagram like (2). Thus the two points form the north and south corner of a rectangle. In the first case we say that t_1 *is north of* t_2 *(or* t_2 *south of* t_1*)*, and in the second case that t_2 *is north of* t_1 *(or* t_1 *south of* t_2*).*

— Either $x_1 < x_2$, $y_1 > y_2$ or $x_1 > x_2$, $y_1 < y_2$—i.e., t_1 and t_2 "cross" as viewed as arrows in a diagram like (2). Thus the two points form the eastern and western corner of a rectangle. In the first case we say that t_1 *is east of* t_2 *(or* t_2 *west of* t_1*)*, and in the second case that t_2 *is east of* t_1 *(or* t_1 *west of* t_2*)*.

If $\mathbf{f}$ is a premap, the same terminology can be applied to two components $\mathbf{f}_{x_1,y_1}, \mathbf{f}_{x_2,y_2}$, by looking at $(x_1, y_1), (x_2, y_2) \in \mathrm{At}(\mathbf{f})$.

In the fourth case above there is a particular subcase worth mentioning: it could be that t_1, t_2 *are on the same parallel*; i.e., assuming that the indexing sets $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$ are of the form $\{0, \ldots, n\}$, this would be saying that $x_1 + y_1 = x_2 + y_2$, and we call this number their *latitude*. When this happens we can say that t_1 *is due east (or due west) of* t_2, whichever case is true and we can also say that they *have the same latitude*. Notice that our peculiar geography stipulates that the North Pole has latitude zero.

Definition 6. A comparison $R \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ which contains both North and South poles is said to be *contractible* is it satisfies the following two condition:

A) given $x_1, x_2, x \in \mathbb{I}_{\mathbf{p}}$ where x is $\leqslant$-between x_1, x_2, along with $(x_1, y_1), (x_2, y_2) \in R$, then tghere exists y which is $\leqslant$-between y_1, y_2 with $(x, y) \in R$.
B) given $y_1, y_2, y \in \mathbb{I}_{\mathbf{q}}$ where y is $\leqslant$-between y_1, y_2, along with $(x_1, y_1), (x_2, y_2) \in R$, then there exists x which is $\leqslant$-between x_1, x_2 with $(x, y) \in R$.

A premap $\mathbf{f}$ is said to be an *elementary map*, if $\mathrm{At}(\mathbf{f})$ is contractible.

This definition can be rephrased geographically. In the case $x_1 = x_2$ or $y_1 = y_2$, we are saying that given two points in R that are on the same diagonal (direction indifferent) then the whole segment on that diagonal between these points will be contained in R. If now the two points are at opposite corner of a rectangle, each point determines a distinct northeast-southwest diagonal and a distinct northwest-southeast one. The conctractibility conditions says that if both points are in R, then every northeast-southwest diagonal that lies between the two such diagonals determined by the points will intersect with R, and every northwest-southeast diagonal that lies between the two such diagonals determined by the points will intersect with R. Since both (b, b) and (e, e) are assumed to be in R, contractibility means in particular that *every diagonal of* $\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ *intersects with* R.

The proof of the following is rather trivial and will be left to the reader:

Proposition 2. Let $t \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ be an arbitrary subset. Then the following are equivalent:

— t is totally ordered by the induced geographical $\leqslant_{\mathbf{p}} \times \leqslant_{\mathbf{q}}$ order, and it is a maximal subset with that property,
— t is totally ordered by the induced geographical $\leqslant_{\mathbf{p}} \times \leqslant_{\mathbf{q}}$ order, and if $t_0, t_1 \ldots t_N$ is the enumeration or t induced by that total ordering, we have $t_0 = (\mathsf{b}, \mathsf{b})$, $t_N = (\mathsf{e}, \mathsf{e})$, $N = \mathrm{length}(\mathbf{p}) + \mathrm{length}(\mathbf{q})$ and for every $i < N$ we have either
  – $\pi_0(t_{i+1}) = \pi_0(t_i)$ and $\pi_1(t_{i+1}) = \pi_1(t_i) + 1$, or
  – $\pi_1(t_{i+1}) = \pi_1(t_i)$ and $\pi_0(t_{i+1}) = \pi_0(t_i) + 1$.

We call such a t a *trail*. Thus a trail is a sequence in $\mathrm{At}(\mathbf{f})$ which starts at the North pole, ends at the South pole, keeps going south, but always doing so by a minimal step, which makes the sequence maximal.

Definition 7. Let $R \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ be an arbitrary subset. We define its *western border* $\mathscr{W}(R)$ to be the subset of all elements of R that have no element in R which is due west of them (i.e. to the west and on the same parallel). Its *eastern border* $\mathscr{E}(R)$ is the subset of all elements of R that have no element in R which is due east of them.

Theorem 1. Let $R \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ be a comparison. Then the following are equivalent:

(i) R is contractible.
(ii) Every parallel intersects an element of R, and given $t_1, t_2 \in R$ with t_1 to the west of t_2, then the rectangle that t_1, t_2 define—i.e., whose sides are diagonals and whose western corner is t_1 and eastern corner is t_2—is contained in R.
(iii) Both borders $\mathscr{W}(R)$ and $\mathscr{E}(R)$ are trails and given $(x, y) \in \mathscr{W}(R)$ $(x', y') \in \mathscr{E}(R)$ that are on the same northwest-southeast diagonal, then the whole segment between them on that diagonal is contained in R.
(iv) Both borders $\mathscr{W}(R)$ and $\mathscr{E}(R)$ are trails and given $(x, y) \in \mathscr{W}(R)$ $(x', y') \in \mathscr{E}(R)$ that are on the same northeast-southwest diagonal, then the whole segment between them on that diagonal is contained in R.
(v) Both borders $\mathscr{W}(R)$ and $\mathscr{E}(R)$ are trails and given $(x, y) \in \mathscr{W}(R)$ $(x', y') \in \mathscr{E}(R)$ that are on the same parallel, then the whole segment between them on that parallel is contained in R.

Proof. The equivalence between (iii), (iv) and (v) is obvious, because all three are different ways of stating that R is a region of the discrete plane which does not contain a hole.

Now assume (i) and in order to show (ii), let (x_1, y_1) be west of (x_2, y_2), both points being in R. Thus, $y_1 < y_2$, $x_1 > x_2$, and the northern corner of the rectangle A they determine is (x_2, y_1), while the southern corner is (x_1, y_2). We claim that (x_2, y_1) is in R: since (x_1, y_1) is distinct from the North pole, it is at the southern corner of a rectangle B whose northern corner is (b, b). The nortwest-southeast diagonal determined by x_2—i. e., $(x_2, -)$—is in between B's northwest-southeast diagonals—that is, $(x_1, -)$ and $(\mathsf{b}, -)$—and thus, by contractibility, there is $(x_2, y) \in R$ inside that rectangle. That implies $y \leqslant y_1$. But then the whole northwest-southeast segment between (x_2, y) and (x_2, y_2) is in R, and since $y \leqslant y_1 < y_2$ we have (x_2, y_1) in R and that proves our claim. Naturally the same argument, dualized can be used to show that the southern corner (x_1, y_2) is in R. Given that all corners of the rectangle A are in R, trivial uses of contractibility tell us that all its sides are in R too, and thus, using contratibility for diagonals again, the whole of the rectangle, its sides and insides, is contained in R.

Let us now show that every parallel meets R. This is done by induction, starting at the North pole and going down, constructing a sequence of points $(\mathsf{b}, \mathsf{b}) = (x_0, y_0), (x_1, y_1) \dots$ that increases latitude by one at every step. So suppose that we have found $(x_i, y_i) \in R$ at latitude i. Let D be the northwest-southeast diagonal right under it, that is, $D =$

$(x_i+1,-)$. It intersects the northeast-southwest diagonal $(-,y_i)$ at (x_i+1,y_i). We know by basic contractibility that D contains at least one point, call it (x,y), in R.

If $y=y_i$, that is, if (x,y) is on the same northeast-southwest diagonal as (x_i,y_i), then we are done and take $(x_{i+1},y_{i+1})=(x,y)=(x_i+1,y_i)$. Otherwise, there are three possibilities, given that (x,y) lies on D.

— (x,y) lies on the part of D which is northwest of that intersection. Then it is west of (x_i,y_j), and assumption (i) ensures that the southern corner of the rectangle the latter forms with (x,y) is in R. We can take this southern corner, which is (x_i+1,y_i), to be (x_{i+1},y_{i+1}).
— (x,y) is right on (it *is*) the intersection, so it is (x_i+1,y_{i+1}) and we take it for (x_{i+1},y_{i+1}).
— (x,y) lies on the part of D which is southest of the intersection (x_i+1,y_{i+1}). Then these two points form the northern and southern corner of a rectangle, and by contractibility every northeast-southwest diagonal that meets that rectangle intersects with either or both its two northwest-southeast sides $(x_i,-)$ and $(x_i+1,-)$. In particular, this is true for the $(-,y_i+1)$ diagonal, and we either have that (x_i,y_i+1) or (x_i+1,y_i+1) is in R. In the first case, we are done and take $(x_{i+1},y_{i+1})=(x_i,y_i+1)$. In the second case, knowing that both (x_i,y_i) and (x_i+1,y_i+1) are in R, we use the fact that R is up-closed for the $\trianglelefteq$-order: inspection shows that whatever that order is, one of (x_i+1,y_i) or (x_i,y_i+1) (or both) is in R.

Note that we have proved that R actually contains a trail, which is stronger than stricly needed.

Now assume that (ii) holds for R. If we show that $\mathscr{W}(R)$ and $\mathscr{E}(R)$ are trails, then (iii), (iv) and (v) will follow immediately from the assumption about rectangles in (ii). So let $(x,y)\in\mathscr{W}(R)$, and (x',y') also be in $\mathscr{W}$, and on the successor latitude to (x,y). By the assumption on latitude in (ii), (x',y') is guaranteed to exist. If we show that $(x,y),(x',y')$ are on the same diagonal we are done. By definition (x',y') cannot be south or north of (x,y). But if (x',y') were west of (x,y), these two would determine a rectangle in R and the point on the northeast-southwest side of that rectangle which is immediately above (x',y') would be due west of (x,y), contradiction the assumption that $(x,y)\in\mathscr{W}(R)$, and if (x',y') were east of (x,y), they would determine a rectangle in R such that the point of the northwest-southeast side of that rectangle immediately below (x,y) would be due west of (x',y'), contradicting the assumption that $(x',y')\in\mathscr{W}(R)$. The argument for $\mathscr{E}(R)$ is the same, mutatis mutandis.

We are left to show that (i) follows from, say, (iii) and (iv). Let $(x_1,y_1),(x_2,y_2)\in R$ and x be between x_1,x_2 for the $\leqslant$ order. We can safely assume $x_1\neq x_2$, otherwise things are trivial. Then if they are on the same diagonal D (which is northeast-southwest), by assumption the intersection $D\cap R$ is a segment whose endpoints are in $\mathscr{W}(R)\cup\mathscr{E}(R)$. But then the diagonal segment determined by $(x_1,y_1),(x_2,y_2)$ is contained in D and thus in R. If $(x_1,y_1),(x_2,y_2)$ are on two distinct northeast-southwest diagonals D_1,D_2, then $D_1\cap R$ and $D_2\cap R$ are still segments, as is the case for $D\cap R$ where D is any northeast-southwest diagonal between D_1 and D_2. Thus D is guaranteed to contain a point in R and we have shown clause A) of contractibility. The same argument can be used taking a y which is $\leqslant$-between y_1, y_2, replacing northeast-southwest diagonals by northwest-southeast ones to show clause B).

This concludes the proof of the theorem. □

Corollary 1. Given a contractible comparison $R \subseteq \mathbb{I}_\mathbf{p} \times \mathbb{I}_\mathbf{q}$ and $(x, y) \in R$ then there is a trail that goes through (x, y).

Proof. For example, draw the northwest-southeast diagonal D that goes through (x, y), letting (x_1, y_1) be the westernmost point of $D \cap R$ and (x_2, y_2) be the easternmost point of $D \cap R$. Obviously $(x_1, y_1) \in \mathscr{W}(R)$ and $(x_2, y_2) \in \mathscr{E}(R)$. Then we can get such a trail by starting at (b, b), following the trail $\mathscr{W}(R)$ until (x_1, y_1), then taking D until (x_2, y_2), and finally following $\mathscr{E}(R)$ until (e, e)). □

The following is exteremly useful

Proposition 3. Let R be a comparison which is strictly contained in $\mathbb{I}_\mathbf{p} \times \mathbb{I}_\mathbf{q}$. Then there exists $(x, y) \in \mathbb{I}_\mathbf{p} \times \mathbb{I}_\mathbf{q} - R$ which is in either one of the following positions:

— $(x - 1, y)$ and $(x, y + 1)$ are in $\mathscr{W}(R)$, with $(x - 1, y) \trianglerighteq (x, y) \trianglelefteq (x, y + 1)$ or
— $(x, y - 1)$ and $(x + 1, y)$ are in $\mathscr{E}(R)$, with $(x, y - 1) \trianglerighteq (x, y) \trianglelefteq (x + 1, y)$.

Moreover, given a $S \supset R$ which is a $\subseteq$-successor, then there is a *unique* $(x, y) \in S - R$ such that $S = \mathrm{Cl}(R + \{(x, y)\})$ and it is in one of these two positions.

Proof. Because of Theorem 1 (v) there has to be a parallel P of $\mathbb{I}_\mathbf{p} \times \mathbb{I}_\mathbf{q}$ which is not all contained in R, and thus a (x', y') on that parallel which either is immediately due west of $\mathscr{W}(R) \cap P$ (with nothing between the two), or immediately due east of $\mathscr{E}(R) \cap P$. Assuming the first possibility, i.e., $(x' - 1, y' + 1) \in R$, then if both $(x' - 1, y')$ and $(x', y' + 1)$ are in $\mathscr{W}(R)$, we are done and take $(x, y) = (x', y')$. Otherwise, one of them, say $(x', y' + 1)$ is not in $\mathscr{W}(R)$. If this is the case then $(x' - 1, y' + 2) \in \mathscr{W}(R)$, because these two points are the only ways of extending the trail $\mathscr{W}(R)$ southwards from $(x' - 1, y' + 1)$. Thus we can take $(x, y) = (x', y' + 1)$ and be done if we have that $(x', y' + 2) \in \mathscr{W}(R)$. But if is not the case, we can iterate, sliding down the northwest-southeast diagonal $(x', -)$ until it reaches $\mathscr{W}(R)$, and then we will have found our (x, y). It is easy to see this is bound to happen. Then we necessarily have $(x - 1, y) \trianglerighteq (x, y) \trianglelefteq (x, y + 1)$ because this is the only possibility that allows (x, y) to be outside of R.

The other three cases are perfectly symmetrical.

Now if $S \supset R$ is a successor, there is necessarily an $(x, y) \in S - R$ in one of the positions above, otherwise S would be equal to R. If there are more than one, we can easily produce several distinct comparisons between R and S. □

Proposition 4. Let $\mathbb{I}, \mathbb{J}, \mathbb{K}$ be posets and $R \colon \mathbb{I} \to \mathbb{J}, S \colon \mathbb{J} \to \mathbb{K}$ be contractible comparisons. Then $S \circ R$ is also contractible.

Proof. The proof is trivial. □

Given elementary maps $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and $\mathbf{g} \colon \mathbf{q} \to \mathbf{r}$ we define the *horizontal composite*, written $\mathbf{gf}$ or $\mathbf{g} \circ \mathbf{f}$, as the family

$$(\mathbf{g}_{y,z} \circ \mathbf{f}_{x,y})_{x,z} \mid \text{there exists } y \text{ such that both } \mathbf{g}_{y,z} \text{ and } \mathbf{f}_{x,y} \text{ are defined.}$$

Thus $\mathrm{At}(\mathbf{gf})$ is the composite of $\mathrm{At}(\mathbf{g})$ and $\mathrm{At}(\mathbf{f})$ in the category of posets and comparisons. But we first have to show this is actually a real definition, i.e., ensure that we always have $\mathbf{g}_{y,z} \circ \mathbf{f}_{x,y} = \mathbf{g}_{y',z} \circ \mathbf{f}_{x,y'}$, whatever y, y' are. So let $x \in \mathbb{I}_{\mathbf{p}}, y, y' \in \mathbb{I}_{\mathbf{q}}, z \in \mathbb{I}_{\mathbf{r}}$ be such that the situation above happens. Let $y_0 = y, y_1, \ldots y_{n-1}, y_n = y'$ be a sequence of elements of $\mathbb{I}_{\mathbf{q}}$ that are all $\leqslant$-between y, y' and such that we always have $y_i \sqsubseteq y_{i+1} \sqsupseteq y_{i+2}$ or $y_i \sqsupseteq y_{i+1} \sqsubseteq y_{i+2}$. In other words we take all the "$\sqsubseteq$-peaks" and "$\sqsubseteq$-valleys" between y, y', and we order them using the induced $\leqslant$ suborder. If we choose an arbitrary $0 \leqslant i \leqslant n$ we know because of Condition A) in Definition 6 that $\mathbf{f}_{x,y_i}$ is guaranteed to be defined, and because of Condition B) that $\mathbf{g}_{y_i,z}$ is guaranteed to be defined. But the fact that y_i, y_{i+1} are $\sqsubseteq$-related guarantees that $\mathbf{g}_{y_i,z} \circ \mathbf{f}_{x,y_i} = \mathbf{g}_{y_{i+1},z} \circ \mathbf{f}_{x,y_{i+1}}$ and this shows $\mathbf{g}_{y,z} \circ \mathbf{f}_{x,y} = \mathbf{g}_{y',z} \circ \mathbf{f}_{x,y'}$ by induction.

Thus we get

Proposition 5. Given a small category X the set of paths and elementary maps in X, along with horizontal composition, form a category $\mathbf{Pe}(X)$.

The identity on a path $\mathbf{p}$ is the family $(\mathbf{p}_{x,x'})_{x \sqsubseteq x'}$.

Actually, it is more than just a category, but an order-enriched one.

Definition 8. Let $\mathbf{f}, \mathbf{g} \colon \mathbf{p} \to \mathbf{q}$ be maps. We write $\mathbf{f} \subseteq \mathbf{g}$ if $\mathbf{g}_{x,y}$ is defined whenever $\mathbf{f}_{x,y}$ is, and $\mathbf{g}_{x,y} = \mathbf{f}_{x,y}$.

In other words, $\mathbf{f} \subseteq \mathbf{g}$ when $\mathrm{At}(\mathbf{f}) \subseteq \mathrm{At}(\mathbf{g})$ and $\mathbf{f}$ is the restriction of $\mathbf{g}$ to $\mathrm{At}(\mathbf{f})$.

Trivially, this is an order relation, and the operation of horizontal composition is monotone in both variables.

We have obvious maps of categories $\partial_0 X, \partial_1 X \colon \mathbf{Pe}(X) \to X$ that take an elementary map of paths $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and restrict it to its endpoints, producing $\partial_0(\mathbf{f}) = \mathbf{f}_{\mathsf{b}} \colon \mathbf{p}_{\mathsf{b}} \to \mathbf{q}_{\mathsf{b}}$ and $\partial_1(\mathbf{f}) = \mathbf{f}_{\mathsf{e}} \colon \mathbf{p}_{\mathsf{e}} \to \mathbf{q}_{\mathsf{e}}$ respectively. There is also an obvious $\mathsf{r}X \colon X \to \mathbf{Pe}X$ which is a section to both ∂_0, ∂_1, that takes a map in X to the elementary map between paths of length zero that it defines.

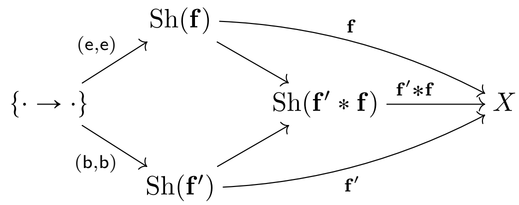
Proposition 6. The process $\mathbf{Pe}(-)$ defines an endofuntor on *Cat* and $\partial_0, \partial_1, \mathsf{r}$ are natural.

Proof. Given a map of categories $f \colon X \to Y$, since a morphism $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ in $\mathbf{Pe}(X)$ is a commutative diagram as represented in (3) we see that $f \circ \mathbf{f}$ will be a morphism in $\mathbf{Pe}(Y)$. Functoriality of the assignment $\mathbf{f} \mapsto f \circ \mathbf{f}$ is trivial to obtain, as is the naturality of $\partial_0, \partial_1, \mathsf{r}$. □

2.1. *Vertical composition*

Let $\mathbf{p}, \mathbf{p}'$ be two paths such that $\mathbf{p}_{\mathsf{e}} = \mathbf{p}'_{\mathsf{b}}$. It is quite obvious how to concatenate the two to get the vertical composite $\mathbf{p}' * \mathbf{p}$. This is just a pushout construction that involves the two orders $\leqslant, \sqsubseteq$. This also applies to maps of paths: given $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and $\mathbf{f}' \colon \mathbf{p}' \to \mathbf{q}'$ with

$\mathbf{f}_{\mathsf{e}} = \mathbf{f}'_{\mathsf{b}}$ an only slightly more involved pushout



constructs

$$\mathbf{p}' * \mathbf{p} \xrightarrow{\mathbf{f}' * \mathbf{f}} \mathbf{q}' * \mathbf{q}.$$

Notice that this is a pushout in *Cat* and not a pushout of underlying graphs. In other words, there can be situations like

$$\mathbf{p}_x \longrightarrow \mathbf{p}_{\mathsf{e}} = \mathbf{p}'_{\mathsf{b}} \longrightarrow \mathbf{q}'_{\mathsf{b}} \longrightarrow \mathbf{q}'_y \quad \text{or} \quad \mathbf{p}'_x \longrightarrow \mathbf{p}'_{\mathsf{b}} = \mathbf{p}_{\mathsf{e}} \longrightarrow \mathbf{q}_{\mathsf{e}} \longrightarrow \mathbf{q}_y$$

that create "leakage" between the original paths in the vertical composite. But

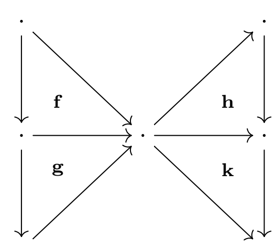
Proposition 7. We always have

$$(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f}) \;\subseteq\; (\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$$

whenever $\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}$ are four maps of path for which the above is defined.

Proof. Easy. □

This is not an equation, and here is a counterexample:



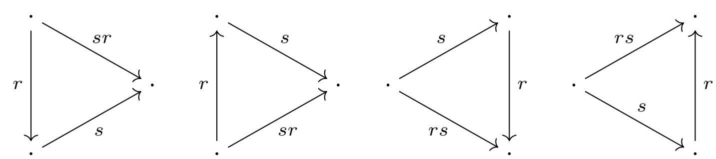
(7)

It is easy to see that in $(\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$ there is an arrow from bottom left to top right (and one from top left to bottom right) of the resulting square which is not in $(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f})$.

Thus vertical composition is not a (bi)functorial operation. It is only colax-functorial when we consider the $\subseteq$ enrichment. Nonetheless it is associative on the nose and has a real unit, and can be considered as a functorial binary operation over the category of graphs instead of *Cat*.

To simplify notation we will use the convention that the $*$ operator associates more strongly than $\circ$.

We need a notation for triangles like those we've just used. First, given a map s in a category X we write $[s]$ for the path of length 1 that has this unique map, pointing in its natural direction (downwards) and $[\![s]\!]$ for its reverse, where s points upwards. Then the four “elementary triangles” are as follows:



$[r, s\rangle$ $\quad$ $[\![r, s\rangle$ $\quad$ $\langle s, r]$ $\quad$ $\langle s, r]\!]$ (8)

This notation should be easy to memorize. Finally given an object x we write $\langle x\rangle$ for the one-object path it defines, of length zero, and given $s\colon x \to y$ in X we take $\langle s\rangle\colon \langle x\rangle \to \langle y\rangle$ to be the corresponding map of paths. Thus the operation $\langle -\rangle$ is $\mathsf{r}X$ and furnishes the identities for vertical composition.

These four triangles, along with the vertical identities, generate the whole of $\mathbf{Pe}(X)$, in a certain sense which will be made explicit later. A special and important case of this is how horizontal identities are obtained. Let $s\colon x \to y$ be a map in X. Then the identity maps of $[s]\,, [\![s]\!]$ are

$$[s, 1_y\rangle * \langle 1_x, s] \quad \text{and} \quad \langle 1_x, s]\!] * [\![s, 1_y\rangle \tag{9}$$

respectively.

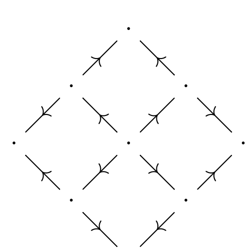
Remark 1. In what follows we will write $[s]\,, [\![s]\!]$ for these two identity maps. In other words we will use the objects-are-identity-maps definitional paradigm when dealing with horizontal composition, which simplifies notation considerably.

Definition 9. Let $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ be an elementary map of paths. We say $\mathbf{f}$ is *purely vertical* if it can be expressed as

$$\mathbf{f} = \mathbf{s}_1 * \mathbf{s}_2 * \cdots * \mathbf{s}_n \tag{10}$$

a vertical composite of a sequence of elementary triangles.

It should be obvious that here we have $n = \text{length}(\mathbf{p}) + \text{length}(\mathbf{q})$. Also obvious should be that given an elementary path $\mathbf{g}\colon \mathbf{p} \to \mathbf{q}$, that whenever we have such a composite with $\mathbf{s}_1 * \mathbf{s}_2 * \cdots * \mathbf{s}_n \subseteq \mathbf{g}$, that it defines a *trail* in $\mathbf{g}$, and vice-versa: there is a bijective correspondence between trails in $\mathbf{g}$ and maps contained in $\mathbf{g}$ that are vertical composites of elementary triangles. Moreove, a purely vertical map is one which is generated by a single trail. But such a map may be generated by several distinct trails. For instance, if an elementary map has for atlas the following diagram

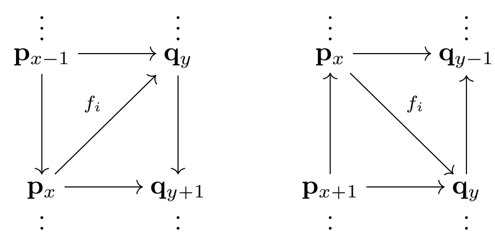


(11)

then the morphism in the middle generates the whole thing, and so will any trail that goes through it, and there are four distinct such trails. Notice also that this map has two distinct submaps, given by its western and eastern borders. These two are also purely vertical, and come from a unique trail / vertical composite of elementary triangles, and are thus $\subseteq$-minimal.

Proposition 8. Let $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ be an elementary map of paths in X. There exists a sequence $\mathbf{k}_0 \subseteq \mathbf{k}_1 \subseteq \cdots \mathbf{k}_n = \mathbf{f}$ where

— $\mathbf{k}_0$ is purely vertical,
— for every $1 \leqslant i \leqslant n$ the family $\mathbf{k}_i$ is obtained from $\mathbf{k}_{i-1}$ "by adding a single morphism f_i", in other words there is $(x, y) \in \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ such that $\mathrm{At}(\mathbf{k}_i) = \mathrm{Cl}\big(\mathrm{At}(\mathbf{k}_{i-1}) + \{(x, y)\}\big)$ and $f_i = \mathbf{f}_{x,y}$,
— every f_i can be embedded in one of the two following diagrams, where the horizontal morphisms are in $\mathbf{k}_{i-1}$



(12)

Proof. Choose an arbitrary trail in $\mathrm{At}(\mathbf{f})$ and let $\mathbf{k}_0$ be the purely vertical map obtained from it. By finiteness of $\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$ and Proposition 3, there is a sequence $\mathrm{At}(\mathbf{k}_0) = R_0 \subset R_1 \subset \cdots \subset R_n = \mathrm{At}(\mathbf{f})$ of comparisons such that R_i is obtained by R_{i-1} by adding a single pair (x, y) and taking the $\trianglelefteq$-up-closure. If we define $\mathbf{k}_i$ as the restriction of $\mathbf{f}$ to R_i, taking f_i to be $\mathbf{f}_{(x,y)}$ we have constructed our sequence. The fact that every f_i can be embedded in one of the two diagrams above is an immediate consequence of the only two positions that (x, y) can occupy with respect to R_{i-1} according to Proposition 3. □

3. Bifibrations and things that are almost bifibrations

We we will first show that the functor $\langle \partial_0, \partial_1 \rangle \colon \mathbf{Pe}(X) \to X \times X$ is very close to being a Grothendieck bifibration, i.e. a functor which is both a Grothendieck fibration and its dual (an opfibration). But we have to define some stuff to express exactly what we mean by this.

Definition 10. Let $X \colon \overline{X} \to A$ be a map of small categories. Given objects $x \in \overline{X}$ and $a \in A$ as is customary we say x *is above* a to mean $X(x) = a$, and the same goes with maps. A *system of fillers* Φ is given by the following data:

— a contravariant, "pulling back" half:
  – for every pair m, x where $m \colon b \to a$ is a map in A and x an object of $\overline{X}$ above a,

a map

$$m^*x \xrightarrow{\Phi^*_m(x)} x$$

above m along with

– for every $n\colon c \to b$ in A and every $r\colon z \to x$ which is above mn, a map

$$z \xrightarrow{\Phi^*_m(n;r)} m^*x$$

above n that makes the triangle in $\overline{X}$ commute.

— and a dual, covariant, "pushing forward" half:

– for every pair m, x where $m\colon a \to b$ is a map in A and x an object of $\overline{X}$ above a, a map

$$x \xrightarrow{\Phi^m_*(x)} m_*x$$

above m, along with

– for every $n\colon b \to c$ in A and every $r\colon x \to z$ which is above $m \circ n$, a map

$$m_*x \xrightarrow{\Phi^m_*(n;r)} z$$

above n making the triangle in $\overline{X}$ commute.

For example, if X is a bifibration, all we have to do to get a system of fillers Φ is to choose a cleavage $\Phi^*_-(-)$ and a cocleavage $\Phi^-_*(-)$. The $\Phi^*_-(-;-)$ and $\Phi^-_*(-;-)$ are obtained via the universal properties of (hyper)cartesian and (hyper)cocartesian maps, and thus are uniquely defined as soon as the cleavage and cocleavage are chosen. In a general system of fillers, the $\Phi^*_-(-;-)$ and $\Phi^-_*(-;-)$ are *choices* of maps that will make the triangles commute, and thus are not uniquely defined.

When X is a fibration, we will identify its bicleavages with systems of fillers.

Consistently with our goal, given $(x, x') \in X \times X$, we say that $\mathbf{p}$ *is above* (x, x') to mean that a path $\mathbf{p}$ is such that $\mathbf{p}_{\mathsf{b}} = x, \mathbf{p}_{\mathsf{e}} = x'$. Given $(s, s')\colon (x, x') \to (y, y')$ and $\mathbf{p}$ above (x, x') we define

$$(s, s')_*\mathbf{p} \quad = \quad \big[s'\big] * \mathbf{p} * [\![s]\!] \tag{13}$$

$$\Phi^{(s,s')}_*(\mathbf{p}) \quad = \quad [1_{x'}, s'\rangle * 1_{\mathbf{p}} * [\![1_x, s\rangle \ : \mathbf{p} \longrightarrow (s, s')_*\mathbf{p} \tag{14}$$

and given $\mathbf{q}$ above (y, y') we define

$$(s, s')^*\mathbf{q} \quad = \quad [\![s']\!] * \mathbf{q} * [s] \tag{15}$$

$$\Phi^*_{(s,s')}(\mathbf{q}) \quad = \quad \langle 1_{y'}, s']\!] * 1_{\mathbf{q}} * \langle 1_y, s] \ : (s, s')^*\mathbf{q} \longrightarrow \mathbf{q}\ . \tag{16}$$

Proposition 9. Let $\mathbf{p}$ and (s, s') be as above, and let $(r, r')\colon (y, y') \to (z, z')$ be in $X \times X$ and $\mathbf{f}\colon \mathbf{p} \to \mathbf{r}$ be above $(rs, r's')$. Then

$$[s', r'\rangle * \mathbf{f} * [\![s, r\rangle \ \circ \ \Phi^{(s,s')}_*(\mathbf{p}) \quad = \quad \mathbf{f}\ .$$

Thus we can define $\Phi_*^{(s,s')}((r,r');\mathbf{f}) = [s',r'\rangle * \mathbf{f} * [\![s,r\rangle$. Applying the same argument dually, given $(u,u')\colon (w,w') \to (x,x')$ and $\mathbf{h}\colon \mathbf{r}' \to \mathbf{q}$ above $(su, s'u')$ we define

$$\Phi^*_{(s,s')}((u,u');\mathbf{h}) = \langle u', s']\!] * \mathbf{f} * \langle u, s]$$

and we see we have constructed a system of fillers Φ.

But these filler maps $\Phi_*^-(-;-)$ and $\Phi_-^*(-;-)$ are not uniquely defined in general, and so we do not have a real bifibration. For a counterexample, suppose that (s,s') is a split mono, i.e., there is $(t,t')\colon (y,y') \to (x,x')$ with $(t,t') \circ (s,s') = (1_x, 1_{x'})$. Then

$$\big(\langle t', r'] * [s', t'\rangle * \mathbf{f} * [\![s,t\rangle * \langle t, r]\!]\big) \ \circ\ \Phi_*^{(s,s')}(\mathbf{p}) \quad = \quad \mathbf{f}\ .$$

But $\langle t', r'] * [s', t'\rangle * \mathbf{f} * [\![s,t\rangle * \langle t, r]\!]$ is not $\Phi_*^{(s,s')}((r,r');\mathbf{f}))$ because it contains two extra maps. Thus $\Phi_*^{(s,s')}(\mathbf{p})$ is not a real cocartesian in general and we do not get a standard bifibration, but something a little more general.

Definition 11. We define $\mathbf{P}(X)$ to be the quotient of $\mathbf{Pe}(X)$ under the symmetric closure of the order enrichment $\subseteq$. That is, a map $\mathbf{p} \to \mathbf{q}$ in $\mathbf{P}(X)$ is a connected component of $\mathbf{P}(X)(\mathbf{p},\mathbf{q})$. There is an obvious functor $\mathbf{Pe}(X) \to \mathbf{P}(X)$ and obvious projection $\langle \partial_0, \partial_1 \rangle\colon \mathbf{P}(X) \to X \times X$ making the obvious triangle commute. It is easy to see that $\mathbf{P}(-)$ is an endofunctor on *Cat*, with the projections and identity natural transformations as usual.

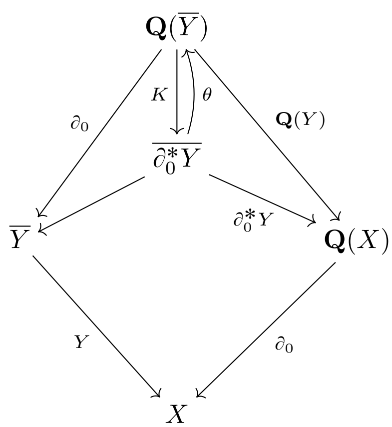
The objects of $\mathbf{P}\,(X)$ coincide with those of $\mathbf{Pe}(X)$, they are paths. The maps are equivalence classes of elementary maps of paths. These classes can be quite complex, and there is no hope of finding normal forms in general.

Notice that Inequality (7) is now an equation in the quotient, and thus vertical composition is $\mathbf{P}(X)$ is bifunctorial. Thus $\mathbf{P}(X)$ is an object of maps for an internal category whose object of objects is X. We have constructed a natural double category structure on every small category.

Remark 2. Since a map $\mathbf{p} \to \mathbf{q}$ in $\mathbf{P}(X)$ is now a class of elementary maps, and since such classes can be quite arbitrary, the concept of shape of a map of paths (and its preservation by maps of small categories) no longer appplies in general, altough the shape of objects in $\mathbf{P}(X)$ and $\mathbf{Pe}$ coincide. For instance there can be a $t\colon X \to Y$ and $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ in $\mathbf{P}(X)$ such that the class $\mathbf{P}(t)(\mathbf{f})$ in $\mathbf{P}(Y)$ contains an elementary map whose shape does not appear in the class $\mathbf{f}$. This could make life hard, but we are lucky that the class in $\mathbf{P}(X)$ of any elementary triangle is just the singleton made of that triangle. The same goes for the class of a vertical unit map $\langle s \rangle$. In addition, the reader can check that the class of a horizontal identity $[s]$ of $[\![s]\!]$ (see Equation (9)) is either a singleton or a doubleton, the latter case happening if and only if s is an iso.

This allows us to use the same notation in $\mathbf{P}(X)$ as in $\mathbf{Pe}(X)$ for calculations. For example we write $\langle s, r]$ for what would be more accurately denoted $\{\langle s, r]\}$, $\langle s \rangle$ for what should be $\{\langle s \rangle\}$, etc. Moreover a vertical composition of elementary triangles can be used to denote maps in $\mathbf{P}(X)$; it gives a representative of that map's class.

Definition 12. Let $\mathbf{Q}$ stand for one of $\mathbf{Pe}, \mathbf{P}$ and $Y : \overline{Y} \to X$ be in *Cat*. We define a *weak Hurewicz* $\mathbf{Q}$*-action on* Y to be a splitting θ of the map $K\colon \mathbf{Q}(\overline{Y}) \to \overline{\partial_0^* Y}$ determined by the universal property of pullback.



Thus a map in $\overline{\partial_0^* Y}$ is a pair $(\mathbf{f}, s)$ where $\mathbf{f}$ is a map in $\mathbf{Q}(X)$, (either an elementary map of paths in X or a set of such parallel paths) and $s \in \overline{Y}$ is above $\partial_0 X(\mathbf{f})$. Let us write $\theta(\mathbf{f}, s)$ as $\mathbf{f} \star s$. Since, according to Remark 2 we know that $\mathbf{Q}(Y)$ is guaranteed to respect the shape of triangles and vertical identities, we see that θ respects vertical identities, i.e.,

$$\langle m \rangle \star s \;=\; \langle s \rangle \,. \tag{17}$$

We say we have a *strong Hurewicz action* (or just *Hurewicz action*) if θ also respects vertical composition in the following sense: for very m in X and $s \in \overline{Y}$ above m and for every three maps $\mathbf{f}, \mathbf{g}$ in $\mathbf{Q}(X)$, s in $\overline{Y}$ such that $\mathbf{g} * \mathbf{f}$ and $\mathbf{f} \star s$ are defined, we have

$$(\mathbf{g} * \mathbf{f}) \star s \;=\; \big(\mathbf{g} \star \partial_1(\mathbf{f} \star s)\big) * (\mathbf{f} \star s) \,. \tag{18}$$

If θ is strong, we see that the composite

$$\overline{\partial_0^* Y} \xrightarrow{\;\theta\;} \mathbf{Q}(Y) \xrightarrow{\;\partial_1\;} \overline{Y}$$

is an ordinary action. That is, when $\mathbf{Q}$ is $\mathbf{P}$, the map $\partial_1 \circ \theta$ is an internal (covariant) presheaf structure on the internal category whose object of maps is $\mathbf{P}(X)$. When $\mathbf{Q}$ is $\mathbf{Pe}$, we have the equivalent to an internal presheaf, even if the category structure is not internal to *Cat*, only to the underlying graph. Let us use the term (strong) *ordinary action* for such an operation of $\mathbf{Q}(X)$ on Y, which is carried by a map $\rho\colon \overline{\partial_0^* Y} \to \overline{Y}$, and denote its associated binary operation by $*$. It obeys the usual axioms of an action by a category:

$$\partial_1(\mathbf{f}) \;=\; Y(\mathbf{f} * s) \tag{19}$$

$$\langle m \rangle * s \;=\; s \,, \tag{20}$$

$$(\mathbf{g} * \mathbf{f}) * s \;=\; \mathbf{g} * (\mathbf{f} * s), \tag{21}$$

whenever these expressions make sense. An action $\rho\colon \overline{\partial_0^* Y} \to \overline{Y}$ is said to be a *weak ordinary action* if it obeys only the first two of these equations. For example one can get a weak ordinary action by taking a weak Hurewicz action and postcomposing it with ∂_1.

Observation 2. Before the next result, notice that if we have a weak ordinary action on Y and a map s in $\overline{Y}$ with an elementary triangle $\mathbf{f}$ such that $\mathbf{f} * s$ is defined, then if $\mathbf{f}$ is a "left pointing" triangle—of the form $\langle m, n]$ or $\langle m, n]\!]$—then Equations (19), (20) ensure that the domain of $\mathbf{f} * s$ coincides with that of s, and if $\mathbf{f}$ is a "right pointing" triangle—of the form $[m, n\rangle$ or $[\![m, n\rangle$—then the codomain of $\mathbf{f} * s$ coincides with that of s.

Proposition 10. Let ρ be a weak ordinary action on the map $Y\colon \overline{Y} \to X$. Then it defines a system of fillers $\Phi\rho$ on Y. If ρ is strong, then $\Phi\rho$ is a bicleavage, guaranteeing that Y is a bifibration.

Proof. For every map m in X and x above its source (with $m\colon a \to b$), we define

$$\Phi\rho_*^m(x)\colon\ x \longrightarrow m_* x \quad \text{as} \quad \langle 1_a, m] * 1_x, \tag{22}$$

our claim about the domain of the above map being ensured by the previous Observation. If $n\colon b \to c$ is in X and $r\colon x \to z$ above nm, we define

$$\Phi\rho_*^m(n; r) \quad \text{as} \quad [m, n\rangle * r \tag{23}$$

The codomain of $\Phi\rho_*^m(n;r)$ is z by the previous Observation. And since

$$\langle a\rangle \xrightarrow{\langle 1_a, m]} [m] \xrightarrow{[m,n\rangle} \langle c\rangle$$

is a composable pair, we have that

$$x \xrightarrow{\Phi\rho_*^m(x)} m_* x \xrightarrow{\Phi\rho_*^m(n;r)} z$$

is a composable pair. Moreover

$$\begin{aligned}
\Phi\rho_*^m(n;r) \ \circ\ \Phi\rho_*^m(x) &= \big([m,n\rangle * 1_a\big) \circ \big(\langle 1_a, m] * r\big) && \text{by definition}\\
&= \big([m,n\rangle \circ \langle 1_a, m]\big) * (r \circ 1_a) && \text{by functoriality of action}\\
&= \langle nm\rangle * r && \text{trivially}\\
&= r && \text{by (20)}
\end{aligned}$$

and if we add the dual contruction we see we have obtained a system of fillers. Notice that since $[m]$ is a horizontal identity, $[m] * 1_x$ is an identity by functoriality of the action. If we now assume that the action is strong, we have

$$\begin{aligned}
[m] * 1_x &= \big([m, 1_b\rangle * \langle 1_a, m]\big) * 1_x && \text{by (9)}\\
&= [m, 1_b\rangle * \big(\langle 1_a, m]\big) * 1_x\big) && \text{by (21)}\\
&= [m, 1_b\rangle * \Phi\rho_*^m(x) && \text{by (22)}\\
&= \Phi\rho_*^m(1_b; \Phi\rho_*^m(x)) && \text{by (23)}
\end{aligned}$$

We have shown that if our system of fillers comes from a strong ordinary action, then

$$[m, 1_b\rangle * \Phi\rho_*^m(x) = \Phi\rho_*^m(1_b; \Phi\rho_*^m(x)) \quad \text{is the identity on} \quad m_* x\,. \tag{24}$$

Let now $t\colon m_* \to z$ above n be such that $t \circ \Phi\rho_*^m(x) = r$. Since the following pair of maps composes horizontally to give $[m, n\rangle$

$$[m] \xrightarrow{[m,1_b\rangle} \langle b\rangle \xrightarrow{\langle n\rangle} \langle c\rangle$$

we have

$$\begin{aligned}
\Phi\rho_*^m(n; r) &= [m, n\rangle * r && \text{by (23)}\\
&= \big(\langle n\rangle \circ [m, 1_b\rangle\big) * \big(t \circ \Phi\rho_*^m(x)\big) && \text{by assumption}\\
&= \big(\langle n\rangle * t\big) \circ \big([m, 1_b\rangle * \Phi\rho_*^m(x)\big) && \text{by functoriality}\\
&= t \circ 1_{m_* x} && \text{by (20) and (24).}
\end{aligned}$$

Thus t is forced to be $\Phi\rho_*^m(n; r)$ and fillers are uniquely defined, so we have a genuine bifibration. □

This construction is invertible.

Theorem 2. Let $Y\colon \overline{Y} \to X$ be a bifibration, Φ a bicleavage for it and let $\mathbf{Q}$ be either $\mathbf{P}$ of $\mathbf{Re}$. There exists a uniquely defined strong Hurewicz $\mathbf{Q}$-action θ such that $\Phi\rho = \Phi$, where $\rho = \partial_1 \circ \theta$ is its associated strong ordinary action.

Thus any bififration Y has

— its set of bicleavages

— the set of Hurewicz $\mathbf{Re}$-actions and the set of Hurewicz $\mathbf{P}$-actions on it

— the set of ordinary $\mathbf{Re}$-actions and the set of ordinary $\mathbf{P}$-actions on it

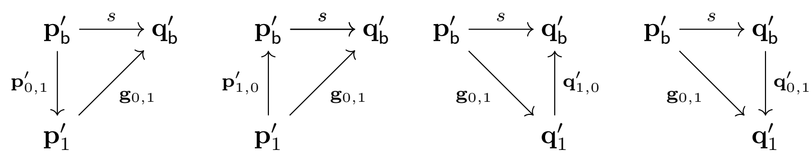
in bijective correspondence.

Let $x \in \overline{Y}$ be above a, and $m\colon a \to b$ and $n\colon c \to b$ be maps in X. Given that a Hurewicz action is functorial, it preserves horizontal identities, and $[m] \star 1_x$ and $[\![n]\!] \star 1_x$ have to be of the form $[s]$ and $[\![r]\!]$ respectively. But if we want $\Phi\rho$ to coincide with Φ we obviously need to have $[m] \star 1_x = [\Phi_*^m(x)]$ and $[\![n]\!] \star 1_x = [\![\Phi_*^n(x)]\!]$—because $[m]$ is codomain of $\langle 1_a, m]$ and $[\![n]\!]$ is domain of $\langle 1_c, n]\!]$. Let $\mathbf{p}$ be a path of length n that starts at a and look at its canonical presentation $\mathbf{p} = \mathbf{a}_1 * \cdots * \mathbf{a}_n$ as a vertical composite of n paths of length one. Since a Hurewicz action respects vertical composition, we see that the path $\mathbf{p}\star x$ decomposes as a sequence $\mathbf{b}_1 * \cdots * \mathbf{b}_n$ where every $\mathbf{b}_i$ is either obtained from a cocleavage cocartesian (when pointing down) or a cleavage cartesian (when pointing up).

Notice that the following result doesn't involve a bicleavage.

Lemma 1. Let $\mathbf{p}, \mathbf{q}$ be two paths in X, and $\mathbf{p}', \mathbf{q}'$ paths in $\overline{Y}$ such that $\mathbf{Re}(Y)(\mathbf{p}') = \mathbf{p}, \mathbf{Re}(Y)(\mathbf{q}') = \mathbf{q}$, and such that the component maps of $\mathbf{p}', \mathbf{q}'$ are cocartesian when they point down, and cartesian when they point up. Moreover let $m\colon \mathbf{p}_{\mathsf{b}} \to \mathbf{q}_{\mathsf{b}}$ in X and $s\colon \mathbf{p}'_{\mathsf{b}} \to \mathbf{q}'_{\mathsf{b}}$ be above m. Then for any elementary map $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ with $\mathbf{f}_{\mathsf{b},\mathsf{b}} = m$ there exists a *unique* elementary map $\mathbf{g}\colon \mathbf{p}' \to \mathbf{q}'$ with $\mathbf{g}_{\mathsf{b},\mathsf{b}} = s$ such that $\mathbf{Re}(Y)(\mathbf{g}) = \mathbf{f}$.

Proof. First assume $\mathbf{f}$ is an elementary triangle. That is, one of $\mathbf{p}, \mathbf{q}$ has length zero and the other has length one. Then, it suffices to inspect the four possible cases to see the result holds:



The only piece of data which is not already given in these is the diagonal map: in the first case it is uniquely determined by the cocartesianness of the vertical map, in the third casse by its cartesianness, and in the second and fourth cases by composition (we have made an arbitrary choice in the notation of the indices, where some 0s and 1s could be replaced by bs and es without change in meaning).

Then assume $\mathbf{f}$ is purely vertical, and let $\mathbf{f} = \mathbf{s}_k * \cdots * \mathbf{s}_1$ where every $\mathbf{s}_i$ is an elementary triangle. Using the previous result, we can proceed by induction, constructing $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k = \mathbf{g}$, where $\mathbf{g}_1$ is above $\mathbf{s}_1$, $\mathbf{g}_2$ is above $\mathbf{s}_2 * \mathbf{s}_1, \ldots$ and $\mathbf{g}_{k-1}$ is above $\mathbf{s}_{k-1} * \mathbf{s}_{k-2} * \cdots * \mathbf{s}_1$. Every one of the $\mathbf{g}_i$ is obviously uniquely defined.

Finally, if $\mathbf{f}$ is an arbitrary elementary map, we use Proposition 3. First we construct a sequence $\mathbf{k}_0, \mathbf{k}_1, \ldots, \mathbf{k}_n = \mathbf{f}$ where $\mathbf{k}_0$ is purely vertical and $\mathbf{k}_i$ is obtained from $\mathbf{k}_{i-1}$ by adding one significant map. There is a uniquely defined sequence $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_n = \mathbf{g}$ where $\mathbf{g}_i$ is above $\mathbf{k}_i$. One uses the previous result for $\mathbf{g}_0$, and for $i > 0$ every $\mathbf{g}_i$ is obtained from $\mathbf{g}_{i-1}$ by adding a map g_i which is in one of the two possible positions occupied by f_i in Diagrams (12). But since the vertical maps therein are either cocartesian (left diagram) or cartesian (right diagram), the map g_i—and therefore—$\mathbf{g}_i$ is uniquely defined. □

The proof of Theorem 2 is now but a formality. Let first $\mathbf{Q} = \mathbf{Pe}$. Given a path $\mathbf{p}$ in X and a x in $\overline{Y}$ above $\mathbf{p}_{\mathsf{b}}$, we already know that the bicleavage Φ forces the value of the Hurewicz action $\mathbf{p} \star x$: it has to be a composite of cartesians and cocartesians that are determined by the bicleavage. But now that this is determined, given a map of path $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and s above $\mathbf{f}_{\mathsf{b},\mathsf{b}}$ the previous Lemma forces the value of $\mathbf{f} \star s$. This action is obviously functorial, and its construction by induction shows that it respects the unit and composition law for Hurewicz actions.

Now, given $\mathbf{p}, \mathbf{q}$ as before, $m \colon \mathbf{p}_{\mathsf{b}} \to \mathbf{q}_{\mathsf{b}}$ and s above m, it is easy to see that the previous Lemma shows that the assignment $\mathbf{f} \mapsto \mathbf{f} \star s$ between sub-hom-posets (remember, the homsets are ordered by $\subseteq$)

$$\{\, \mathbf{f} \in \mathbf{Pe}(X)(\mathbf{p}, \mathbf{q}) \mid \mathbf{f}_{\mathsf{b},\mathsf{b}} = m \,\} \longrightarrow \{\, \mathbf{g} \in \mathbf{Pe}(\overline{Y})(\mathbf{p}', \mathbf{q}') \mid \mathbf{g}_{\mathsf{b},\mathsf{b}} = s \,\}$$

is an *isomorphism of posets*. If follows that it is a bijection between the connected components of these posets. But this induced mapping

$$\pi_0\{\, \mathbf{f} \in \mathbf{Pe}(X)(\mathbf{p}, \mathbf{q}) \mid \mathbf{f}_{\mathsf{b},\mathsf{b}} = m \,\} \longrightarrow \pi_0\{\, \mathbf{g} \in \mathbf{Pe}(\overline{Y})(\mathbf{p}', \mathbf{q}') \mid \mathbf{g}_{\mathsf{b},\mathsf{b}} = s \,\}$$

defines the Hurewicz $\mathbf{P}$-action. It is easy to check that it is uniquely defined and obeys the necessary axioms.

Theorem 3. Given a small category X then $\mathbf{P}(X) \to X \times X$ is a bifibration.

This will follow immediately from the following (and its dual version), which takes its notation straight from Proposition 9.

Proposition 11. Let $(s, s') \colon (x, x') \to (y, y')$ and $(r, r') \colon (y, y') \to (z, z')$ be maps in $X \times X$, $\mathbf{p}$ be above (x, x'), $\mathbf{f} \colon \mathbf{p} \to \mathbf{r}$ in $\mathbf{Pe}(Y)$ be above $(rs, r's')$ and

$$[s'] * \mathbf{p} * [\![s]\!] \xrightarrow{\ \mathbf{g}\ } \mathbf{r}$$

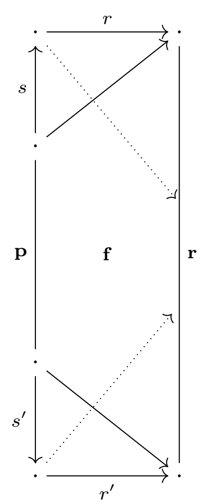
an elementary map above (r, r') such that

$$\mathbf{g} \circ \langle 1_{y'}, s']\!] * 1_{\mathbf{q}} * \langle 1_y, s] \quad = \quad \mathbf{g} \circ \Phi_*^{(s,s')}(\mathbf{p}) \quad = \quad \mathbf{f}\,. \tag{25}$$

Then $\mathbf{g} \supseteq \Phi_*^{(s,s')}((r, r'); \mathbf{f})$.

Thus all the potential fillers are identified when we quotient $\mathbf{Pe}$ to get $\mathbf{P}$ and the latter is a true bifibration.

Proof. Recall that $\Phi_*^{(s,s')}((r, r'); \mathbf{f}) = [s', r'\rangle * \mathbf{f} * [\![s, r\rangle$. Any $\mathbf{g} \colon [s'] * \mathbf{p} * [\![s]\!] \longrightarrow \mathbf{r}$ fills the outer rectangle below, whose solid lines and arrows define $\Phi_*^{(s,s')}((r, r'); \mathbf{f})$.



If $\mathbf{g}$ satisfies (25) then it has to coincide with $\mathbf{f}$ inside the quadrangle which is marked as its territory and is inside solid lines and arrows. Thus the only freedom we are allowed to construct such a $\mathbf{g}$ is to put extra maps in the position of the dotted arrows. This obviously makes $\mathbf{g} \supseteq \Phi_*^{(s,s')}((r, r'); \mathbf{f})$. □

4. Some corollaries, conclusions and hints of future work

The concept of Hurewicz action presented in this work has already been used by the author (Lamarche(2013)) to give a definition for the Martin-Löf J-operator, the elimination term for the intensional equality type (which is modelled by $\mathbf{P}(-)$). The vehicle for this is a natural map $\mathbf{P}(X) \to \mathbf{P}(\mathbf{P}(X))$ given by $\mathbf{f} \mapsto \mathbf{f} \star \langle \partial_0 \mathbf{f} \rangle$ (see also (Garner and Berg(2012)) for a very similar construction). We refer the reader to (Lamarche(2013)) for more details, where in addition a fibered version of $\mathbf{P}$ is constructed, that acts on bifibrations and not just on objects. One very desirable feature of this model is that the dependent product exists when bifibrations are used for dependent types.

One way to look at this work, which could be said to be "purely category-theoretical" (whatever pure category theory may be in the end), is to view it as the construction of free bifibrations. Given a small category X, the fact that $\big(X, \mathbf{P}(X)\big)$ is an internal category allows us to construct a very standard monad T_X on Cat/X, whose action on objects is given by

$$T_X(Y) \;=\; \partial_1 \circ \partial_0^* Y \;.$$

The objects of the category of algebras for this monad are what we have called ordinary actions, i.e., they are bicleft bifibrations, while the morphisms of algebras preserve the bicleavage. If we keep the same objects but decide that the morphisms are *pseudoalgebra* morphisms, i.e., pairs (f, α) where $f\colon Y \to Z$ is a morphism in Cat/X and α a natural iso

$$\begin{array}{ccc} T_X Y & \xrightarrow{T_X f} & T_X Z \\ \downarrow & \overset{\alpha}{\Rightarrow} & \downarrow \\ Y & \xrightarrow[f]{} & Z \end{array}$$

(obviously α is uniquely defined) then we get the more general maps that preserve cartesians and cocartesians.

Obviously the construction $X \mapsto T_X$ defines a fibered monad over the standard codomain fibration $Cat^{\to} \to Cat$.

More can be said about this internal category viewpoint. For example there is a version of Yoneda's Lemma—actually, there are at least two versions of Yoneda here, a strict one for maps that preserve the bicleavage and a "pseudo" one for maps that only preserve cartesians and cocartesians. For the interested reader who would like check this, we should say that the constant path functor $\mathbf{C}$ used in (Lamarche(2013)) is a necessary tool to construct the representable fibration associated to an object $x \in X$.

We also have all the necessary equipment to construct first and higher order homotopy groups, by quotienting path and loops objects, and thus we can define a specific notion of weak equivalence in Cat. It turns out that this is not the weak equivalences associated with the standard homotopy theory of small categories (Quillen(1973); Thomason(1980); Maltsiniotis(2005); Cisinski(2006)). In a subsequent paper we will develop this new homotopy theory for Cat, which can be presented as a *category of fibrant objects* in the sense of K. Brown (Brown(1973)), and work out its relationship with the standard homotopy theory.

Up to now the functor $\mathbf{P\!e}$ has been used as a means to construct the more internal-structure-rich $\mathbf{P}$. But it has its independent interest, one reason being that a $\mathbf{P\!e}$-homotopy—a map of the form $h\colon X \to \mathbf{P\!e}(Y)$, which doesn't actually have to be fully functorial but can be colax—induces a conventional homotopy in *Top* between the geometric realizations of $\partial_0 \circ h$ and $\partial_1 \circ h$. The problem is that we have yet to find a good notion of fibration which is compatible with $\mathbf{P\!e}$. We see in Proposition 11 that the "generalized cartesians and cocartesians" of $\mathbf{Re}(X)$ obey a 2-categorical kind of universal property. This observation is leading us to the extension of our path functor construction to the world of 2-categories.

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