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Abstract

Due to the fluctuations in their production, wind farm owners are subject to financial penalties. To limit these penalties the use of a storage device is studied. We define in this paper a large class of production/storage models in continuous time which verify the physical limits of the facility. In these models, the optimal operation of the storage device becomes an optimal stochastic control problem. We prove that this problem is equivalent to solving an Hamilton-Jacobi-Bellman PDE. Further, this PDE verifies the comparison principle and has thus a unique solution. Using a Semi-Lagrangian approach, we obtain an algorithm for this PDE. As the PDE verify the maximum principle, by using classical tools, we prove that this algorithm is convergent. Finally we present some numerical results.

1 Introduction

Due to the fossil energies limited quantity and the increasing interest on their ecological cost, the demand for renewable energy sources is growing. In particular wind energy is becoming more and more popular. Unlike other energy sources, the main drawback for renewable energy is the uncertainty and uncontrollability of the energy source. In some countries, the wind electricity producers operate on the same market as the other producers and have to follow the same rules. In particular, they have to decide at day $D-1$ the amount of power they commit to deliver for each hour of the Day D. For each of these hour, the power delivery must be constant and match their commitment. If not, the producers have to pay two financial penalties respectively proportional to overproduced and underproduced energy (see Figure 1).

In order to limit the exposure to these penalties, a first approach studied in [10], is based on financial hedging strategy. One can oppose two facts to this choice. First, the electricity market is not complete, and such strategy might not exist or might be too

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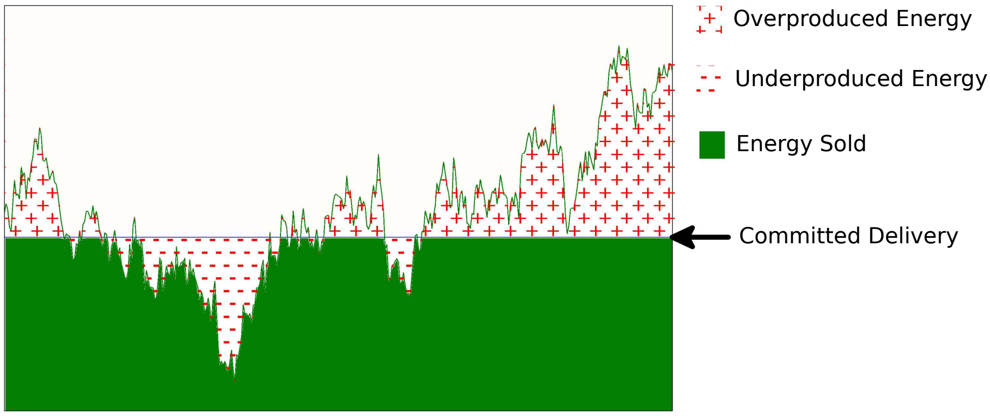

Figure 1: One hour production

expensive. Secondly, this strategy only limits the penalties but does not necessary ensure the power balance on the power grid, which could be armful (electricity shortage for some users for example).

In this paper we study another method: the use of a storage device. This device is able to convert the electricity into another form of energy (kinetic energy or compressed air for example) and vice versa. This method has been studied for example in [5, 11, 12] but only in discrete time. In these studies the authors consider mainly the hourly delivered energy. If this quantity is superior to the committed energy, the producer pays overproduction penalties, and if it is inferior he pays underproduction penalties.

It is, in fact, quite far from reality. As illustrated in Figure 1, the variations of the power production can lead to pay both underproduction and overproduction penalties for the same period. Another fact to take into account is that the using cost of the storage device is not always a linear function of the energy stored or delivered. For example, some batteries are deteriorating faster if the level of stored energy is too low. Taking into account these costs might not be possible while considering only the hourly delivered energy. To overcome these difficulties we consider a model production in continuous time.

We model the power production in continuous time by a stochastic process, solution of a stochastic differential equation (SDE). We assume that the energy prices and the penalties are known, this choice will be discussed later. At each time the producer choose how to use the storage device in order to maximize the daily gain.This leads to a stochastic optimal control problem.

Two main methods have been studied to solve this kind of problem: simulation based methods and PDE based methods. Simulation based methods (see for example [3]) are able to solve control problems in dimension greater than four. However it is known that these methods require an important number of simulations to achieve high accuracy, and thus are computationally expensive. As for PDE approaches, they are only practicable for dimensions lower than three but have in general better convergence rate. Our problem is a three dimensional problem, we choose thus to use the PDE approach.

As shown in [14], our problem is equivalent to solving an Hamilton Jacobi Bellman (HJB) PDE. Finite difference methods have been developed for solving theses equations in the general case (see for example [4]). They require to solve high dimensional non linear equations systems. Direct method, for solving these systems, if available, are extremely slow. To overcome this difficulty iterative methods have been investigated. Such methods can be found in [8], it consists in iteratively and alternatively computing a discretized solution for a given control, then replacing this control by the control which maximises the discretized Hamiltonian of the solution and so on until convergence. This method can be adapted for solving most of HJB PDE, and for particular equation it can be improved. This can be done for example if there is a finite number of admissible values for the control (or if the control admissible values can be reduced to a finite set). Then, the set of considered controls is reduced. It is thus easier to find the control which maximizes the Hamiltonian. Moreover, for some problems, there are more efficient methods which use the particularity of the associated equation.

The control part in our HJB PDE involves only a transport term, making our problem similar to the problem in [6]. In this paper a semi-Lagrangian approach is used to derive a new finite difference scheme. The resulting scheme consists in solving an optimization problem at each point of the grid. A linear system is then constructed using the result of this optimization. This linear system has a strictly dominant diagonal, thus it has always an unique solution. It is also tridiagonal and so can be efficiently solved. The solution to this system gives an approximation of the PDE solution at each point of the grid. As the policy iteration is avoided this algorithm is faster than the previous one. We adapt this method to our problem.

We start by introducing our model for the production and the stock management. In this model the production and the stock level are deterministically bounded, such that they lay in the limit of the wind farm capacities.

Next, using these bounds, we prove that our problem is equivalent to solving a modified HJB PDE. This equation is defined for all the points satisfying the physical limits of the wind farm. In particular this equation is verified on the boundary of the domain. We show that the comparison principle introduced in [7] holds for this equation. This comparison principle implies the uniqueness and the continuity of the solution in the viscosity sense.

Then, using the maximum principle, we adapt the scheme described in [6] resulting in a convergent scheme for solving our HJB PDE. This scheme avoids policy iteration and thus is faster than the algorithm developed for the general case.

Finally, we discuss the numerical results obtained in two models. The first model is similar to the models considering only the delivered energy. In this case, the strategies values can be reduced to a finite set. Using this property we improve our algorithm. The second model uses non affine costs an thus is impracticable if considering only the delivered energy.

2 The model

In this section we introduce the model, and explicit the maximization criterion.

2.1 The production

Consider the production on a period $[0, T]$. The process $W = (W_t)_{t\in[0,T]}$ describes the wind farm production over this period: at time $t \in [0, T]$ the wind farm generates W_t kW.

The production is random so we represent W as a stochastic process, solution of a particular SDE. Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and $(B_t)_{t\in[0,T]}$ a Brownian motion on this space. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ can be interpreted as the information available at each time t, and the process W is adapted with respect to this filtration. For technical reasons we also require W_0 to be of finite variance. Let $\sigma : [0,T]\times\mathbb{R} \to \mathbb{R}_+$ and $b : [0,T]\times\mathbb{R} \to \mathbb{R}_+$ be two Lipschitz functions. Given W_0, the process $(W_t)_{t\in[0,T]}$ is the unique solution of

$$W_t = W_0 + \int_0^t \sigma(s, W_s)dB_s + \int_0^t b(s, W_s)ds, \ \forall t \in [0,T]. \tag{2.1}$$

Let $\mathcal{L}$ denote the infinitesimal operator associated to the process W.

The process W is well defined, however the previous assumptions do not ensure that the trajectories of W are realistic from power production point of view. For example it might be negative which is absurd as the wind farm only produces electricity. Each wind turbine has a maximal power output, therefore the production is bounded. Let $M_W > 0$ denote the maximal capacity production. Thus the process W must take its values in $[0, M_W]$. To meet this requirement, we use the notion of inaccessible set introduced in [13] and make the following assumptions:

$$\forall t \in [0,T],\ b(t, M_W) \leq 0,\ \sigma(t, M_W) = 0 \text{ and } \mathbb{P}(W_0 \leq M_W) = 1, \tag{Prod.1}$$
$$\forall t \in [0,T],\ b(t, 0) \geq 0,\ \sigma(t, 0) = 0 \text{ and } \mathbb{P}(W_0 \geq 0) = 1. \tag{Prod.2}$$

Proposition 2.1. *1. Under the assumption* (Prod.1) *we have:*

$$\mathbb{P}(\forall s \in [0,T];\ W_s \leq M_W) = 1. \tag{2.2}$$

2. Under the assumption (Prod.2) *we have:*

$$\mathbb{P}(\forall s \in [0,T];\ W_s \geq 0) = 1. \tag{2.3}$$

Proof. We prove only the first point as the proof is similar for the second one.

Let $Y = (Y_t)_{t\in[0,T]}$ be a stochastic process such that:

$$Y_t = Y_0 + \int_0^t \tilde{\sigma}(s, Y_s)dB_s + \int_0^t \tilde{b}(s, Y_s)ds, \ \forall t \in [0,T],$$

where $\tilde{b},\ \tilde{\sigma} : [0,T]\times\mathbb{R} \to \mathbb{R}$ are Lipschitz. Using the notion of inaccessible set and Lemma 4.3 given in [9] it follows that if:

$$\forall t \in [0,T],\ \tilde{b}(t, M_W) < 0,\ \tilde{\sigma}(t, M_W) = 0 \text{ and } \mathbb{P}(Y_0 < M_W) = 1, \tag{2.4}$$

then:

$$\mathbb{P}(\forall s \in [0,T];\ Y_s < M_W) = 1.$$

For $\varepsilon > 0$ we define the process $W^\varepsilon = (W_t^\varepsilon)$ as the solution of:

$$W_t^\varepsilon = W_0 - \varepsilon + \int_0^t \sigma(s, W_s^\varepsilon)dB_s + \int_0^t (b(s, W_s^\varepsilon) - \varepsilon)ds, \ \forall s \in [0,T]. \tag{2.5}$$

If (Prod.1) holds, then W^ε verifies (2.4) and so it is bounded from above by M_W with probability one.

The SDE (2.5) is a perturbation of the SDE (2.1) and we can prove that W_t^ε converge to W_t in L^2 as ε tends to 0. Let K denote a Lipschitz constant common to σ and b. Using the Itô's Formula and some basic inequalities, we get:

$$\mathbb{E}\left[(W_t^\varepsilon - W_t)^2\right] \leq (T+1)\varepsilon^2 + (K+1)^2 \int_0^t \mathbb{E}\left[(W_s^\varepsilon - W_s)^2\right] ds, \ \forall t \in [0,T]. \tag{2.6}$$

Then Grönwall's lemma implies:

$$\mathbb{E}\left[|W_t^\varepsilon - W_t|)^2\right] \leq \varepsilon(1+T)\exp\left((K+1)^2 \frac{T}{2}\right) \underset{\varepsilon \to 0}{\to} 0, \forall t \in [0,T].$$

Therefore for all $t \in [0,T]$ there exists a positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ decreasing to 0 such that:

$$\mathbb{P}\left(\lim_{n\to+\infty} |W_t^{\varepsilon_n} - W_t| = 0\right) = 1.$$

For all $\varepsilon > 0$, the process W^ε is a.s. bounded from above by M_W, we obtain thus:

$$\mathbb{P}\left(W_t \leq M_W\right) = 1, \forall t \in [0,T].$$

Furthermore the trajectories of W are a.s. continuous, therefore (2.2) holds for the process W. □

From now on, as the process belongs to $[0, M_W]$, we consider only the functions σ and b restricted to $[0,T] \times [0, M_W]$.

For future works one can observe that this model fails to represent two characteristics of a wind farm power production. First, when the wind is too strong some wind turbines are suddenly stopped to prevent breaking. Consequently, jumps are observed in the power production. Second, their might be time periods when the wind farm does not produce power, if the wind is to strong or to slow for example. So the process should be able to stay at zero for a time. When the wind farm stops to produce, in general one cannot know when the production will resume. In our model the process W can stay at zero for a period if and only if $b(0,t) = 0$ during this period, but it will be positive as soon as $b(0,t) > 0$, and thus we know for how long the farm stopped.

We now describe the operation of the storage device.

2.2 Storage unit operation

The storage unit can store only a finite amount of energy. Denote by M_Q the maximal quantity of energy that can be stored. The chosen strategy influences the variations of the energy stored in the device. We consider that these variations depend only on the energy level, the production and the strategy. Let $f_{nrj}, f_{prod} : [0, M_W] \times [0, M_Q] \times [-1, 1] \to \mathbb{R}$, be functions such that:

$$f_{prod}(w,q,0) = 0,\ \forall (w,q) \in [0,M_W] \times [0,M_Q], \tag{Stor.1}$$
$$\forall (w,q) \in [0,M_W] \times [0,M_Q],\ \text{the function } u \to f_{nrj}(w,q,u) \text{ is increasing}, \tag{Stor.2}$$
$$\forall (w,q) \in [0,M_W] \times [0,M_Q],\ \text{the function } u \to f_{prod}(w,q,u) \text{ is decreasing}, \tag{Stor.3}$$
$$f_{nrj}(w,M_Q,u) = 0,\ \forall (w,u) \in [0,M_W] \times [0,1], \tag{Stor.4}$$
$$f_{prod}(w,0,u) = 0,\ \forall (w,u) \in [0,M_W] \times [-1,0], \tag{Stor.5}$$
$$f_{prod}(w,q,u) + w \geq 0, \forall (w,q,u) \in [0,M_W] \times [0,M_Q] \times [0,1], \tag{Stor.6}$$
$$f_{nrj}(w,q,u) \leq -f_{prod}(w,q,u),\ \forall (w,q,u) \in [0,M_W] \times [0,M_Q] \times [-1,1], \tag{Stor.7}$$
$$f_{nrj} \text{ and } f_{prod} \text{ are Lipschitz and bounded.} \tag{Stor.8}$$

At each time $t \in [0,T]$, W_t kW are produced. Let Q_t denote the energy level in the storage unit. The producer must decide how to operate the storage device by choosing $u_t \in [-1,1]$. Then $(W_t + f_{prod}(W_t,Q_t,u_t))$ kW are delivered to the grid and the variation of the stock level is $(f_{nrj}(W_t,Q_t,u_t) - f_{loss}(Q_t))\,dt$ kWh. The choice of u_t and the assumptions Stor.1-Stor.7 are interpreted as follows:

- If $u_t = 0$ the storage device is not used (Stor.1).
- If $u_t > 0$ the storage device stores energy. The more u_t is close to 1 the faster the energy is stored and the less power is delivered to the grid, (Stor.1 - Stor.3).
- If $u_t < 0$ the storage device releases energy. The more u_t is close to -1 the faster the stock is emptied and the more power is delivered to the grid, (Stor.1 - Stor.3).

The stock level must takes its values in $[0,M_Q]$, so when the stock is empty, the device cannot release (Stor.5) and cannot store anymore when full (Stor.4). Only the energy produced can be stored in the device, so the stock cannot be filled faster than energy being produced (Stor.6). Finally no energy can be created by operating the storage device (Stor.7).

The assumption (Stor.8) is made for technical reasons and may possibly be relaxed.

A strategy consists in choosing at each time t a real value $u_t \in [-1,1]$, by using the available informations $\mathcal{F}_t$ leading to Definition 2.1.

Definition 2.1.
- *Let $\mathcal{A}$ be the set of $(\mathcal{F}_t)_{t\in[0,T]}$ adapted processes taking values in $[-1,1]$. The set $\mathcal{A}$ is called the set of admissible strategies.*
- *A process $u \in \mathcal{A}$ is called an admissible strategy.*

For each admissible strategy $u \in \mathcal{A}$, and each initial stock level $Q_0 \in [0,M_Q]$, we define the process $Q^u = (Q^u_t)_{t\in[0,T]}$ representing the stock level over time, as solution of:

$$Q^u_t = Q_0 + \int_0^t f_{nrj}(W_s, Q^u_s, u_s)ds. \tag{2.7}$$

This process is well defined as stated by the following result:

Proposition 2.2. *Under the assumptions* (Prod.1),(Prod.2) *and* (Stor.1)-(Stor.8)*, for all $Q_0 \in [0,M_Q]$ and for all $u \in \mathcal{A}$:*

1. The degenerate SDE (2.7) *has a unique solution.*

2. The process Q^u, solution of SDE (2.7)*, takes its values in* $[0, M_Q]$.

Proof. We give only the main ideas of the proof as it is similar to the proof of Proposition 2.1. Under assumption (Stor.8), Cauchy-Lipschitz theorem implies existence and uniqueness of a continuous solution for at least a small (random) time. If the solution is not defined over $[0,T]$, then it has to leave $[0, M_Q]$ so it reaches 0, or M_Q before T. Let us consider for example that it reaches 0, then by the assumption Stor.5, the stock cannot be emptied any more so that Q_t stays positive. Thus Q_t does not leave $[0, M_Q]$ when reaching 0. □

Finally we introduce the gain and formulate the optimization problem.

2.3 Gain and Optimization Problem

The producer commits to deliver w_E kW during the period $[0,T]$. Afterwards, but before starting the delivery, he is informed of the energy selling price ($P > 0$ in kW^{-1}h^{-1}). So P is $\mathcal{F}_0$ measurable, and we consider it constant in the optimization problem.

We assume that the underproduction (C^- in kW^{-1}h^{-1}) and the overproduction penalties (C^+ in kW^{-1}h^{-1}) are known and constant. In practice, these costs are unknown during the production periods and are fixed afterwards. Other producers may also fail to deliver their committed power. The regulator has to balance the grid, and these corrections have a cost. Therefore the regulator fixes the penalties after the production period, in order to cover that cost. Thus the penalties should be random and not measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. However in the case of a small producer operating on a market with low wind electricity penetration we can omit the correlation between the production and these penalties. Therefore, we consider these penalty costs constant and equal to their mean value.

Using the operating device has a cost. For example, if batteries are used to store energy, the less energy is stored the faster they decay. We can also take into account the deterioration of the wind farm. We represent these costs by using a Lipschitz function $c : [0, M_W] \times [0, M_Q] \times [-1, 1] \to \mathbb{R}_+$.

For a given strategy $u \in \mathcal{A}$ the gain made over the period $[t,T]$ is:

$$\begin{aligned}\int_t^T \{ &\min(W_s + f_{prod}(W_s, Q_s^u, u_s), w_E)P - (W_s + f_{prod}(W_s, Q_s^u, u_s) - w_E)_- C^- \\ &- (W_s + f_{prod}(W_s, Q_s^u, u_s) - w_E)_+ C^+ - c(W_s, Q_s^u, u_s)\} ds \\ &= \int_t^T g(W_s, Q_s^u, u_s) ds, \end{aligned} \tag{2.8}$$

where for $x \in \mathbb{R}$, x_+ is the positive part and x_- is the negative part defined by:

$$x_+ = \max(x, 0), \qquad x_- = \max(-x, 0)$$

and the definition of g is given by the equality (2.8).

We must keep in mind that there are other periods, following this one. For these periods the level of energy stored and the power produced at time T will influence the gain. Let $h : [0, M_W] \times [0, M_Q] \to \mathbb{R}$ be Lipschitz. The value of $h(W_T, Q^u_T)$ is the potential mean gain that can be maid for the following periods given the state variable at time T.

We now define the expected gain $G : [0, T] \times [0, M_W] \times [0, M_Q] \times \mathcal{A} \to \mathbb{R}$ by:

$$G(t, w, q, u) = \mathbb{E}\left[\int_t^T g(W_s, Q^u_s, u_s)ds + h(W_T, Q^u_T)|W_t = w,\ Q^u_t = q\right]. \tag{2.9}$$

This function represents the expected gain for the period $[t, T]$ given the value of the state variable at time t and a strategy u, while taking into account the potential mean gain for the latter periods. As g and h are Lipschitz and as the production and the stock level are bounded, it follows easily that the function G is bounded.

Our aim is to maximize the expected gain by choosing the best strategy possible. To do so we define the maximal expected gain $v : [0, T] \times [0, M_W] \times [0, M_Q] \to \mathbb{R}$ by:

$$v(t, w, q) = \sup_{u\in\mathcal{A}} \{G(t, w, q, u)\}. \tag{2.10}$$

As G is bounded, v is obviously bounded.

In the sequel we do the following:

1. Approximate v.
2. Find quantitative informations about the operation of the storage device.

3 The PDE characterization

3.1 Locally bounded functions and semi-continuous functions

Under the previous hypotheses, the function v is Lipschitz (see for example [14]), but may not be smoother. However v is, in some sense, solution of a PDE which has a unique solution in the class of locally bounded functions. In order to obtain this, a notion of regularity, weaker than the continuity, is used: the semi-continuity.

Definition 3.1. *Let D be a topological space and consider a function $u : D \to \mathbb{R}$.*

- *The function u is said* upper semi-continuous *(usc) if for all $x \in D$, the following inequality holds:*
$$\limsup_{y\to x} u(y) \leq u(x).$$
- *The function u is said* lower semi-continuous *(lsc) if for all $x \in D$, the following inequality holds:*
$$\liminf_{y\to x} u(y) \geq u(x).$$

Remark. *In these definitions the value of the function u at x is included when computing the limit.*

Even if the semi-continuity is weaker than the continuity, it has similar properties, as shown in the following:

Proposition 3.1. *Let D be a topological space and $u : D \to \mathbb{R}$ a function.*

(i) If u is upper semi-continuous then $(-u)$ is lower semi-continuous.

(ii) If u is lower semi-continuous then $(-u)$ is upper semi-continuous.

(iii) Let Γ be a compact subset of D, if u is usc then u has an upper bound on Γ and attains its maximum.

(iv) Let Γ be a compact subset of D, if u is lsc then u has a lower bound on Γ and attains its minimum.

(v) The function u is continuous if and only if it is lower semi-continuous and upper semi-continuous.

The properties (i), (ii) and (v) are trivial, and the proofs of the properties (iii) and (iv) are the same as in the continuous case, using mainly the Bolzano Weierstrass theorem.

We prove that the function v is, in some sense, the unique solution of a PDE in the class of locally bounded functions. It is easy to find example of a locally bounded function which is not lower semi-continuous nor upper semi-continuous. But every locally bounded function lies between an upper semi-continuous function and a lower semi-continuous function, as stated in the following:

Definition 3.2. *Let D be a topological space and $u : D \to \mathbb{R}$ a locally bounded function.*

- *The function $u_* : D \to \mathbb{R}$ defined by:*

$$u_*(x) = \liminf_{y \to x} u(y), \tag{3.1}$$

is called the lower semi-continuous envelope *of u.*

- *The function $u^* : D \to \mathbb{R}$ defined by:*

$$u^*(x) = \limsup_{y \to x} u(y), \tag{3.2}$$

is called the upper semi-continuous envelope *of u.*

Remark. *As u is locally bounded, u^* and u_* are also locally bounded.*

Proposition 3.2. *Let D be a topological space and $u : D \to \mathbb{R}$ be a locally bonded function.*

(i) The function u^* is upper semi-continuous.

(ii) The function u_* is lower semi-continuous.

(iii) The following inequalities hold: $u_* \leq u \leq u^*$.

(iv) The function u is upper semi-continuous if and only if $u = u^*$.

(v) The function u is lower semi-continuous if and only if $u_* = u$.

(vi) The function u is continuous if and only if $u^* = u_*$.

We leave the proof of these elementary properties to the reader.

As shown by this proposition, every locally bounded function lays between two semi-continuous function. The semi-continuity being weaker than the continuity, a semi-continuous function may not be differentiable in the usual sense. We consider a notion which plays the role of the derivative in the PDE.

Definition 3.3. *Let D be a topological space and $u : D \to \mathbb{R}$ an upper semi-continuous function. For all $x \in D$, let $J^+u(x)$ denotes the elements $(p, X) \in \mathbb{R}^3 \times \mathcal{S}^3$ verifying:*

$$u(y) \leq u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \text{ as } y \to x,$$

where $\mathcal{S}^3$ is the space of real symmetric matrices of dimension 3. We call $J^+u(x)$ the second order superjet *of u at x.*

In a similar way, for lower semi-continuous functions we get:

Definition 3.4. *Let D be a topological space and $\ell : D \to \mathbb{R}$ a lower semi-continuous function. For all $x \in D$, $J^-\ell(x)$ denotes the elements $(p, X) \in \mathbb{R}^3 \times \mathcal{S}^3$ verifying:*

$$\ell(y) \geq \ell(x) + \langle p, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \text{ as } y \to x.$$

We call $J^-\ell(x)$ the second order subjet *of u at x.*

We now define the closure of these subsets.

Definition 3.5. *Let D be a topological space and $u : D \to \mathbb{R}$ an upper semi-continuous function. For all $x \in D$, $\bar{J}^+u(x)$ denotes the elements $(p, X) \in \mathbb{R}^3 \times \mathcal{S}^3$ verifying:*

$$\begin{aligned} &\exists (x_n)_{n\in\mathbb{N}} \subset D \text{ and } \exists (p_n, X_n) \in J^+u(x_n) \\ &\text{such that } x_n \to x,\ u(x_n) \to u(x) \text{ and } (p_n, X_n) \to (p, X) \text{ as } n \to +\infty. \end{aligned} \tag{3.3}$$

We call $\bar{J}^+u(x)$ the closure of the second order superjet *of u at x.*

Definition 3.6. *Let D be a topological space and $\ell : D \to \mathbb{R}$ a lower semi-continuous function. For all $x \in D$, $\bar{J}^-\ell(x)$ denotes the elements $(p, X) \in \mathbb{R}^3 \times \mathcal{S}^3$ verifying:*

$$\begin{aligned} &\exists (x_n)_{n\in\mathbb{N}} \subset D \text{ and } \exists (p_n, X_n) \in J^-\ell(x_n) \\ &\text{such that } x_n \to x,\ \ell(x_n) \to \ell(x) \text{ and } (p_n, X_n) \to (p, X) \text{ as } n \to +\infty. \end{aligned} \tag{3.4}$$

We call $\bar{J}^-\ell(x)$ the closure of the second order subjet *of u at x.*

If u is two times differentiable on an open set, any extremal point is a critical point, in which the Hessian matrix is either positive, or negative. Using the second order super/lower jet, the authors of [7] extend this result to semi-continuous functions.

Theorem 3.3. *Let D be a topological space and $u,\ \ell : D \to \mathbb{R}$ two functions respectively upper semi-continuous and lower semi-continuous. Let φ be two times continuously differentiable on an open neighbourhood of D^2. If $u(x) - \ell(y) - \varphi(x,y)$ is maximal at $(x,y) \in D^2$, then for all $\varepsilon > 0$ there exist $X,\ Y \in \mathcal{S}^3$ such that:*

$$(\partial_x\varphi(x,y), X) \in \bar{J}^+u(x),\ (-\partial_y\varphi(x,y), Y) \in \bar{J}^-\ell(x) \tag{3.5}$$

$$\text{and } -(\frac{1}{\varepsilon} + |||A|||)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2, \tag{3.6}$$

where A is the Hessian matrix of φ at (x,y), and $|||A|||$ is the matrix norm induced by the Euclidean norm of A, (more precisely $|||A|||$ is the spectral radius of A).

The inequality (3.6) is to be interpreted in the following sense. Let $M_1,\ M_2$ be elements of $\mathcal{S}$, then $M_1 \leq M_2$ means that $M_2 - M_1$ is positive.

With this generalization of the notion of differential we describe how the notion of PDE solution can be extended to non differentiable functions.

3.2 Viscosity solutions

For the sake of notational simplicity we denote by E the set $[0, M_Q] \times [0, M_W]$. We consider a PDE of the following form:

$$\begin{cases} F(\nabla^2 u(x), \nabla u(x), u(x), x) = 0,\ \forall x\ \in [0,T) \times E, \\ u(x) = h(x_1, x_2),\ \forall x \in \{T\} \times E, \end{cases} \tag{3.7}$$

where $F : \mathcal{S}^3 \times \mathbb{R}^3 \times \mathbb{R} \times [0,T) \times E$ and $h : E \to \mathbb{R}$ are continuous function. We do not require F to be linear with respect to the derivative, and thus we cannot use the definition of solution in the distribution sense. We just require F to be elliptic in the following sense:

$$F(X,p,r,x) \leq F(Y,p,s,x), \text{if } X - Y \in \mathcal{S}^3_+ \text{ and } r \leq s, \tag{3.8}$$

where $\mathcal{S}^3_+$ is the subspace of non-negative symmetric matrices of dimension 3.

We now extend the notion of solution for the PDE (3.7) to the class of locally bounded function as follows:

Definition 3.7. *Let $u : [0,T] \times E \to \mathbb{R}$ be a locally bounded function.*

1. A viscosity supersolution *of the PDE* (3.7)*, is a lower semi-continuous function u such that:*

$$F(x, u(x), p, X) \geq 0,\ \forall x \in [0,T) \times E,\ \forall (p,X) \in J^-u(x), \tag{3.9}$$

$$\text{and } u(x) \geq h(x),\ \forall x \in \{T\} \times E,\ \forall (p,X) \in J^-u(x). \tag{3.10}$$

2. A viscosity subsolution *of the PDE* (3.7)*, is an upper semi-continuous function u such that:*

$$F(x, u(x), p, X) \leq 0,\ \forall x \in [0,T) \times E,\ \forall (p,X) \in J^+u(x), \tag{3.11}$$

$$\text{and } u(x) \leq h(x),\ \forall x \in \{T\} \times E,\ \forall (p,X) \in J^+u(x). \tag{3.12}$$

3. A viscosity solution *of the PDE* (3.7)*, is a locally bounded function u, such that u^* is a viscosity subsolution of the PDE* (3.7)*, and u_* is a viscosity supersolution of the PDE* (3.7)*.*

Despite the terminology, under our hypothesis, a subsolution is not necessarily less than a subsolution. However, by making additional assumptions, this property holds and is called a comparison principle.

3.3 The HJB PDE associated to the optimization problem

Let us prove that v is the unique viscosity solution of the corresponding HJB PDE.

Proposition 3.4. *Under the assumptions* (Prod.1)-(Prod.2) *and* (Stor.1)-(Stor.8)*, the function v is continuous and is a viscosity solution of the following HJB PDE:*

$$\begin{cases} -\partial_t w(x) - \mathcal{L}(x_1, x_2) w(x) - H(x, \partial_q w(x)) = 0, \ \forall x \in [0,T) \times E, \\ w(x) = h(x), \ \forall x \in \{T\} \times E, \end{cases} \tag{3.13}$$

where $H : [0,T) \times E \times \mathbb{R} \to \mathbb{R}$ is the function defined by:

$$H(x,r) = \sup_{u \in [-1,1]} \left\{ f_{nrj}(x_2, x_3, u) r + g(x_2, x_3, u) \right\}, \ x = (x_1, x_2, x_3) \in [0,T] \times E. \tag{3.14}$$

Proof. A proof of this result is given in [14], in the case where E is open. As the process $(W_t, Q_t^u)_{t \in [0,T]}$ takes values in E, we can extend the proof to our case. □

As ∂_q^2 does not appear in (3.13) and $\sigma^2 \geq 0$ it follows that the PDE (3.13) satisfies the elliptic condition (3.8). So we can use the notion of viscosity solution, as in [7]. However, in our case F is not uniformly elliptic as defined in [7] and the PDE is required to hold on the boundary of E. Yet we show that a comparison principle holds, leading to the uniqueness of the solution

First we will prove a comparison result for the following PDE:

$$\begin{cases} -\partial_t w(x) + \lambda w(x) - \mathcal{L}(x_1, x_2) w(x) - \tilde{H}(x, \partial_q w(x)) = 0, \ \forall x \in [0,T) \times E, \\ w(x) = h(x), \ \forall x \in \{T\} \times E, \end{cases} \tag{3.15}$$

where $\lambda \in \mathbb{R}^+$, and $\tilde{H} : [0,T) \times E \times \mathbb{R} \to \mathbb{R}$ is the function defined by:

$$\tilde{H}(x,r) = \sup_{u \in [-1,1]} \left\{ f_{nrj}(x_2, x_3, u) r + \exp(\lambda(x_1 - T)) g(x_2, x_3, u) \right\}. \tag{3.16}$$

As $\lambda > 0$, and $\sigma^2 \geq 0$, the PDE (3.15) satisfies the uniform elliptic condition as stated in [7] and thus satisfies the elliptic condition (3.8).

Theorem 3.5. *Under the assumptions (Prod.1-Prod.2) and (Stor.1-Stor.8), if v_1 and v_2, are respectively subsolution and sursolution of the PDE* (3.15)*, then:*

$$v_1(x) \leq v_2(x), \forall x \in [0,T] \times E. \tag{3.17}$$

Admitting this result for now, we show that it implies a maximum principle for the PDE (3.13).

Corollary 3.6. *Under the assumptions* (Prod.1)-(Prod.2) *and* (Stor.1)-(Stor.8)*, if* v_1 *and* v_2*, are respectively subsolution and supersolution of the PDE* (3.13)*, then:*

$$v_1(x) \leq v_2(x), \forall x \in [0,T] \times E. \tag{3.18}$$

Proof. As v_1 (resp. v_2) is subsolution (resp. supersolution), straightforward computation shows that $\exp(\lambda(x_1-T))v_1$ (resp. $\exp(\lambda(x_1-T))v_2$)) is subsolution (resp. supersolution) of the PDE (3.15). Using the theorem 3.5 we get:

$$\exp(\lambda(x_1 - T))v_1(x) \leq \exp(\lambda(x_1 - T))v_2(x) \; \forall x \in [0,T] \times E.$$

Thus the comparison principle holds for the PDE (3.13). □

As stated before, the comparison principle implies the uniqueness of the solution:

Corollary 3.7. *Under the assumptions* (Prod.1)-(Prod.2) *and* (Stor.1)-(Stor.8)*,* v *is the unique viscosity solution of the PDE* (3.13)*.*

Proof. By Proposition 3.4, it follows that v is a viscosity solution of the PDE (3.7). We now prove the uniqueness. Let w be a viscosity solution of the PDE (3.13). Then w^* and v^* are viscosity subsolutions and w_* and v_* are viscosity supersolutions. As v is continuous $v_* = v = v^*$. Applying Theorem 3.6 we get the following inequalities:

$$w_* \geq v^* = v = v_* \geq w^* \text{ on } [0,T] \times E. \tag{3.19}$$

Recalling $w^* \geq w \geq w_*$, we obtain:

$$w = v \text{ on } [0,T] \times E. \tag{3.20}$$

Thus the uniqueness follows. □

For the proof of the Theorem 3.5 we use the following result:

Lemma 3.1. *Let* φ *be an upper semi-continuous function on* $([0,T] \times E)^2$ *such that:*

- $\varphi(T, w, q, T, w, q) \leq 0, \; \forall (w, q) \in E,$
- *there exists* $x \in [0,T] \times E$*, such that* $\varphi(x, x) > 0$*.*

For all $\alpha \geq 0$*, define* Φ^α *on* $([0,T] \times E)^2$ *by:*

$$\Phi^\alpha(x, y) = \varphi(x, y) - \frac{\alpha}{2}||x - y||,$$

and let M^α *denote the supremum of* Φ^α*. Then:*

1. *There exists* (x^α, y^α) *such that* $\Phi^\alpha(x^\alpha, y^\alpha) = M_\alpha$*.*
2. $\lim_{\alpha \to +\infty} ||x^\alpha - y^\alpha|| = 0.$
3. $\lim_{\alpha \to +\infty} \alpha ||x^\alpha - y^\alpha||^2 = 0.$
4. *There exists* $\alpha_0 \geq 0$ *such that for all* $\alpha \geq \alpha_0$*,* $x_1^\alpha < T$ *and* $y_1^\alpha < T$*.*

Proof. As the function φ is upper semi-continuous, the function defined on $[0,T]\times E$ by $x \to \varphi(x,x)$ is upper semi-continuous. Therefore, this function is bounded from above, and attains its maximum. Let M denotes its maximum. Due to the hypothesis M is positive.

1. For all $\alpha \geq 0$, the function Φ^α is upper semi-continuous as sum of semi-continuous functions. This function is defined on $([0,T]\times E)^2$, which is compact, so as it is upper semi-continuous it reaches its maximum and there exists (x^α, y^α) such that $\Phi^\alpha(x^\alpha, y^\alpha) = M^\alpha$.

2. Let us prove that $\lim\limits_{\alpha\to+\infty} ||x^\alpha - y^\alpha|| = 0$. If not, there exist $\delta > 0$, an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ going to infinity and a sequence $(x^{\alpha_n}, y^{\alpha_n})_{n\in\mathbb{N}}$, such that:

$$||x^{\alpha_n} - y^{\alpha_n}|| \geq \delta,\ \forall n \in \mathbb{N}.$$

The function φ is upper semi-continuous, on a compact set, so it reaches its maximum. Let M^* denote this maximum, then:

$$0 < M \leq M^{\alpha_n} = \Phi^\alpha(x^{\alpha_n}, y^{\alpha_n}) = \varphi(x^{\alpha_n}, y^{\alpha_n}) - \frac{\alpha_n}{2}||x^{\alpha_n} - y^{\alpha_n}||^2 \leq M^* - \frac{\alpha_n}{2}\delta^2.$$

Letting n go to infinity leads to a contradiction.

3. Let us prove that:

$$\lim_{\alpha\to+\infty} \alpha||x^\alpha - y^\alpha||^2 = 0. \tag{3.21}$$

If not there exist $\delta > 0$, a sequence $(\alpha_n)_{n\in\mathbb{N}}$ which goes to infinity, and a sequence $(x^{\alpha_n}, y^{\alpha_n})_{n\in\mathbb{N}}$, such that:

$$\alpha_n||x^{\alpha_n} - y^{\alpha_n}||^2 \geq \delta,\ \forall n \in \mathbb{N}.$$

Therefore:

$$0 < M \leq M^{\alpha_n} = \Phi^\alpha(x^{\alpha_n}, y^{\alpha_n}) = \varphi(x^{\alpha_n}, y^{\alpha_n}) - \frac{\alpha_n}{2}||x^{\alpha_n} - y^{\alpha_n}||^2 \leq \varphi(x^{\alpha_n}, y^{\alpha_n}) - \frac{\delta}{2}.$$

As the sequence $(x^{\alpha_n}, y^{\alpha_n})_{n\in\mathbb{N}}$ is bounded we can suppose that it is convergent. If it is not we just use a convergent subsequence. The point 2 implies that the limit is of the form $(\bar{x}, \bar{x})$, with $\bar{x} \in [0,T]\times E$. Thus, letting n go to infinity and using the upper semi-continuity of φ we obtain:

$$M \leq \varphi(\bar{x}, \bar{x}) - \frac{\delta}{2} \leq M - \frac{\delta}{2},$$

which is contradictory.

4. Finally, let us show that there exists $\alpha_0 \geq 0$ such that $x_1^\alpha < T$, $y_1^\alpha < T$ for $\alpha \geq \alpha_0$. If not, there exist a non-decreasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ going to infinity and a sequence $(x^{\alpha_n}, y^{\alpha_n})_{n\in\mathbb{N}}$, such that:

$$x_1^{\alpha_n} = T \text{ or } y_1^{\alpha_n} = T,\ \forall n \in \mathbb{N}.$$

Here again, as this sequence is bounded, we can suppose it convergent and its limit is of the form $(\bar{x}, \bar{x})$, with $\bar{x} \in \{T\}\times E$. Then:

$$0 < M \leq M_{\alpha_n} = \Phi^\alpha(x^{\alpha_n}, y^{\alpha_n}) = \varphi(x^{\alpha_n}, y^{\alpha_n}) - \frac{\alpha_n}{2}||x^{\alpha_n} - y^{\alpha_n}||^2,$$

letting n go to infinity and using the upper semi-continuity of φ leads to:

$$0 < M \leq \varphi(\bar{x}, \bar{x}) \leq 0,$$

which is contradictory. $\square$

We finally prove Theorem 3.5.

Proof of Theorem 3.5. We use the scheme given in [7]. As $v_1 - v_2$ is upper semi-continuous, it has an upper bound. Let M denote its supremum. Clearly, proving this theorem is equivalent to proving that $M \leq 0$. Let us suppose that $M > 0$. Then we define the function φ on $([0, T] \times E)^2$ by:

$$\varphi(x, y) = v_1(x) - v_2(y). \tag{3.22}$$

By the definition of viscosity subsolution and supersolution, $v_1(x) \leq h(x) \leq v_2(x)$ for all $x \in \{T\} \times E$. By upper semi-continuity $v_1 - v_2$ reaches its maximum $M > 0$, so we can apply the Lemma 3.1. Let α_0 be defined as in Lemma 3.1. Then for all $\alpha \geq \alpha_0$, there exists $(x^\alpha, y^\alpha) \in ([0, T) \times E)^2$ such that:

$$0 < M \leq M_\alpha = \varphi^\alpha(x^\alpha, y^\alpha) - \alpha||x_\alpha - y_\alpha||.$$

By Theorem 3.3, for all $\varepsilon > 0$ there exist two matrices X and Y such that:

$$(\alpha(x^\alpha - y^\alpha), X) \in \bar{J}^+ v_1(x^\alpha) \text{ and } (\alpha(x^\alpha - y^\alpha), Y) \in \bar{J}^- v_2(y^\alpha), \tag{3.23}$$

and the following inequalities holds:

$$-\left(\frac{1}{\varepsilon} + ||A||\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2, \tag{3.24}$$

where A is the Hessian matrix of $\alpha||x - y||^2$ at (x^α, y^α), and $||A||$ is the spectral radius of A. By straightforward calculation we get:

$$A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \text{ and } A^2 = 2\alpha A.$$

This last equality implies that $||A|| = 2\alpha$. With $\varepsilon = \frac{1}{\alpha}$ in (3.24):

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3A. \tag{3.25}$$

Let us define the function F by:

$$F(\nabla^2 w(x), \nabla w(x), w(x), x) = -\partial_t w - \mathcal{L}(x_1, x_2) w(x) + \lambda w(x) - \tilde{H}(x, \partial_q w(x)).$$

We prove that the following inequality holds:

$$|\tilde{H}(x, r) - \tilde{H}(y, r)| \leq C||x - y||(1 + |r|), \ \forall r \in \mathbb{R} \text{ and } \forall x, \ y \in [0, T] \times E. \tag{3.26}$$

Recall that f_{nrj} and g are Lipschitz continuous functions and let K be a Lipschitz constant for both functions. The function g is continuous and defined on a compact set therefore it is bounded. Let $\bar{M}$ denote an upper bound of $|g|$. Let $x,\ y \in [0,T] \times E$, and $r \in \mathbb{R}$, then:

$$\begin{aligned}\tilde{H}(x,r) &= \sup_{u\in[-1,1]} \{f(x_2,x_3,u)r - \exp(\lambda(x_1-T))g(x_2,x_3,u)\} \\ &= \sup_{u\in[-1,1]} \{(f(x_2,x_3,u) - f(y_2,y_3,u))r \\ &\quad - (\exp(\lambda(x_1-T)) - \exp(\lambda(y_1-T)))g(x_2,x_3,u) \\ &\quad - \exp(\lambda(y_1-T))(g(x_2,x_3,u) - g(y_2,y_3,u)) \\ &\quad + f(y_2,y_3,u)r - \exp(\lambda(y_1-T))g(y_2,y_3,u)\} \\ &\quad \text{we just artificially introduce the terms of } \tilde{H}(y,r). \\ &\leq \sup_{u\in[-1,1]} \{K||x-y|| \times |r| + e^{\lambda T}\bar{M}||x-y|| + e^{\lambda T}K||x-y|| + f(y_2,y_3,u)r \\ &\quad - \exp(\lambda(y_1-T))g(y_2,y_3,u)\} \\ &\leq \tilde{H}(y,r) + C||x-y||(1+|r|).\end{aligned}$$

The symmetry between x and y permits to conclude.

Let X and Y be two matrices such that (3.23) and (3.25) hold. Then:

$$F(x^\alpha, w_1(x^\alpha), \alpha(x^\alpha - y^\alpha), X) \leq 0 \leq F(y^\alpha, w_1(y^\alpha), \alpha(x^\alpha - y^\alpha), Y), \tag{3.27}$$

as w_1 and w_2 are respectively viscosity subsolution and viscosity supersolution of the PDE (3.15) and $x^\alpha,\ y^\alpha \in [0,T) \times E$.

By straightforward computation we obtain:

$$\begin{aligned}0 < \lambda M \leq \lambda M^\alpha &\leq \lambda(w_1(x^\alpha) - w_2(y^\alpha)) \\ &= F(x^\alpha, w_1(x^\alpha), \alpha(x^\alpha - y^\alpha), X) - F(x^\alpha, w_2(y^\alpha), \alpha(x^\alpha - y^\alpha), X) \\ &\leq F(x^\alpha, w_1(x^\alpha), \alpha(x^\alpha - y^\alpha), X) - F(y^\alpha, w_1(y^\alpha), \alpha(x^\alpha - y^\alpha), Y) \\ &\quad + F(y^\alpha, w_1(y^\alpha), \alpha(x^\alpha - y^\alpha), Y) - F(x^\alpha, w_2(y^\alpha), \alpha(x^\alpha - y^\alpha), X).\end{aligned}$$

Further, inequality (3.27), and the definition of F lead to:

$$0 < \lambda M \leq F(y^\alpha, w_1(y^\alpha), \alpha(x^\alpha - y^\alpha), Y) - F(x^\alpha, w_2(y^\alpha), \alpha(x^\alpha - y^\alpha), X) = A_1 + A_2, \tag{3.28}$$

where A_1 and A_2 are defined by:

$$\begin{aligned}A_1 &= \frac{1}{2}(\sigma^2(x_1^\alpha, x_2^\alpha)X(2,2) - \sigma^2(y_1^\alpha, y_2^\alpha)Y(2,2)), \\ A_2 &= \alpha(x_2^\alpha - y_2^\alpha)(b(x_1^\alpha, x_2^\alpha) - b(y_1^\alpha, y_2^\alpha)) + \tilde{H}(x^\alpha, \alpha(x_3^\alpha - y_3^\alpha)) - \tilde{H}(y^\alpha, \alpha(x_3^\alpha - y_3^\alpha)).\end{aligned}$$

As b is Lipschitz continuous, inequality (3.26) implies:

$$A_2 \leq \alpha K||x^\alpha - y^\alpha||^2 + C||x^\alpha - y^\alpha||(1 + \alpha||x^\alpha - y^\alpha||). \tag{3.29}$$

Let $\xi = (\xi_i)_{1\leq i\leq 6} \in \mathbb{R}^6$, be the real vector defined by: $\xi_2 = \sigma(x_1^\alpha, x_2^\alpha)$, $\xi_5 = \sigma(y_1^\alpha, y_2^\alpha)$, and all the other components equal to zero. As inequality (3.25) holds for the matrices X and Y, and as σ is Lipschitz continuous:

$$A_1 = \frac{1}{2}\xi^t \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \xi \leq \frac{3}{2}\xi^t A \xi = \frac{3\alpha}{2}(\sigma(x_1^\alpha, x_2^\alpha) - \sigma(y_1^\alpha, y_2^\alpha))^2 \leq \frac{3\alpha K^2}{2}||x^\alpha - y^\alpha||^2. \tag{3.30}$$

Finally the inequalities (3.28) - (3.30) imply that there exists a positive constant C independent of α such that:

$$0 < \lambda M < C(\alpha||x^\alpha - y^\alpha||^2 + ||x^\alpha - y^\alpha||) \to 0 \text{ when } \alpha \to +\infty \text{ by Lemma 3.1,}$$

which is absurd. Thus the assumption that M is positive leads to a contradiction. We conclude that $M \leq 0$ and the comparison principle holds. □

4 Semi-Lagrangian Discretization

A semi-Lagrangian approach is introduced in [6] for solving a HJB PDE arising in an optimal gas-storage problem. We adapt this scheme to solve the PDE (3.7).

The control part in the PDE (3.7) involves only the first order derivative in the q direction. In this sense the control part is similar to a transport problem, and semi-Lagrangian scheme have shown very good results for solving transport PDE. The main idea is to integrate the PDE (3.7) along a semi-Lagrangian trajectory, and then discretize the resulting integral.

4.1 Heuristic of the Scheme

Let us introduce some notations. We use an unequally spaced grid in the w direction for the PDE discretization represented by $[w_1 = 0, w_2, ..., w_{i_{max}} = M_W]$. Similarly, we use unequally spaced grid in the direction q denoted by $[q_1 = 0, q_2, ..., q_{j_{max}=M_Q}]$. We denote by $0 = \tau^1, ..., \tau^N = T$ the discrete time step used to discretize the PDE (3.7). Let $\Delta\tau^n$, Δw, Δq and δ be defined by:

$$\Delta\tau^n = \tau^n - \tau^{n-1},\ \Delta w = \min_i\{w_{i+1} - w_i\},\ \Delta q = \min_j\{q_{j+1} - q_j\},$$
$$\Delta\tau = \min_n\{\Delta\tau^n\},\ \delta = \min\{\Delta\tau, \Delta w, \Delta q\}.$$

The following construction of the numerical scheme is only heuristic. We suppose that v is smooth enough so that its derivatives exist. The operations performed here are to be interpreted in the formal sense. The convergence of the resulting scheme will be discussed afterwards.

Our aim is to approximate v at each grid point, for each time τ^n. Let $V^n_{i,j}$ denote an approximation of $v(\tau^n, w_i, q_j)$. The terminal condition in the PDE (3.7) gives $V^N_{i,j}$. Let $n \in \{2, ..., N\}$, be such that we have an approximation of v at each point of the grid at time τ^n.

4.1.1 Integrating along a Lagrangian Trajectory

As v is solution of the PDE (3.7), for all continuous function $Q : [\tau^{n-1}, \tau^n] \to [0, M_Q]$, $t \in [\tau^{n-1}, \tau^n]$ and $i \in \{1, ..., i_{max}\}$:

$$\partial_t v(t, w_i, Q(t)) + \mathcal{L}(t, w_i)v(t, w_i, Q(t)) + H(t, w_i, Q(t), \partial_q v(t, w_i, Q(t)) = 0,$$

then integrating over $[\tau^{n-1}, \tau^n]$:

$$\int_{\tau_{n-1}}^{\tau_n} \partial_t v(t, w_i, Q(t)) + \mathcal{L}(t, w_i)v(t, w_i, Q(t)) + H(t, w_i, Q(t), \partial_q v(t, w_i, Q(t))dt = 0.$$

Now remember the definition of H given in (3.4), and suppose that the operators sup and integral are interchangeable, then the last equation is equivalent to:

$$\sup_{u\in\mathcal{U}^*}\int_{\tau_{n-1}}^{\tau_n} \partial_t v(t,w_i,Q(t)) + \mathcal{L}(t,w_i)v(t,w_i,Q(t)) \\ + f_{nrj}(w_i,Q(t),u(t))\partial_q v(t,w_i,Q(t)) + g(w_i,Q(t),u(t))dt = 0, \tag{4.1}$$

where $\mathcal{U}^*$ is a suitable set of functions with values in $[-1,1]$. We insist on the heuristic nature of these arguments, in the general case interchanging these operators might be impossible.

The relation (4.1) holds for all continuous functions, so in particular it holds for the function Q^u solution of the following ODE:

$$Q^u(t) = q_j + \int_{\tau_{n-1}}^{t} f_{nrj}\big(w_i, Q^u(s), u(s)\big)ds, \tag{4.2}$$

which leads to:

$$\sup_{u\in\mathcal{U}^*}\int_{\tau_{n-1}}^{\tau_n} \partial_t v(t,w_i,Q^u(t)) + \partial_t Q^u(t)\partial_q v(t,w_i,Q^u(t)) \\ + \mathcal{L}(t,w_i)v(t,w_i,Q^u(t)) + g(w_i,Q^u(t),u(t))dt = 0. \tag{4.3}$$

Noting that formally:

$$\frac{d}{dt}v(t,w_i,Q^u(t)) = \partial_t v(t,w_i,Q^u(t)) + \partial_t Q^u(t)\partial_q v(t,w_i,Q^u(t)),$$

equation (4.3) becomes:

$$v(\tau^{n-1},w_i,q_j) \\ = \sup_{u\in\mathcal{U}^*}\left\{ v(t,w_i,Q^u(\tau^n)) + \int_{\tau_{n-1}}^{\tau_n} \mathcal{L}(t,w_i)v(t,w_i,Q^u(t)) + g(w_i,Q^u(t),u(t))dt \right\}. \tag{4.4}$$

4.1.2 Discretization of the Integral

Choosing $\theta\in[0,1]$, we use a $\theta-$scheme to approximate the integrals. Using this scheme in the ODE (3.7) leads to approximate $Q^u(\tau^n)$ by $\hat{q}^n_{i,j}(u_{\tau^{n-1}},u_{\tau^n})$ solution of:

$$\hat{q}^n_{i,j}(u_{\tau^{n-1}},u_{\tau^n}) = q_j + \theta\Delta\tau^n f_{nrj}(w_i,q_j,u(\tau^{n-1})) + (1-\theta)\Delta\tau^n f_{nrj}(w_i,\hat{q}^n_{i,j}(u_{\tau^{n-1}},u(\tau^n)), \tag{4.5}$$

and the integral in (4.4) is then approximated by

$$\Delta\tau^n(1-\theta)\left(\mathcal{L}(\tau^n,w_i)v\left(\tau^n,w_i,\hat{q}^n_{i,j}(u_{\tau^{n-1}},u_{\tau^n})\right) + g\left(w_i,\hat{q}^n_{i,j}(u_{\tau^{n-1}},u_{\tau^n}),u_{\tau_n}\right)\right) \\ + \Delta\tau^n\theta\left(\mathcal{L}(\tau^{n-1},w_i)v(\tau^{n-1},w_i,q_j) + g(w_i,q_j,u^{n-1}_\tau)\right). \tag{4.6}$$

As the function Q^u takes its values in $[0,M_Q]$, we require that the approximation $\hat{q}^n_{i,j}(u_{\tau^{n-1}},u_{\tau^n})$ to be also in $[0,M_Q]$. Let $U(i,j,n)$ denote the subset of $[-1,1]$ such that the equation (4.5) has a solution in $[0,M_Q]$. This is not a strong restriction as illustrated by the following result:

Proposition 4.1. *Let K_{nrj} denote the minimal Lipschitz constant for the function f_{nrj}. If $\Delta\tau^n \leq \frac{1}{K_{nrj}\theta}$, then $U(i,j,n) = [-1,1]^2$.*

Proof. Let q'_j be defined by:

$$q'_j = q_j + \theta\Delta\tau^n f_{nrj}(w_i, q_j, u(\tau^{n-1})).$$

By Assumptions (Stor.1)-(Stor.8):

$$\begin{aligned}
\Delta\tau\theta f_{nrj}(w_i, q_j, u_2) &\geq \Delta\tau\theta\left(f_{nrj}(w_i, q_j, -(u_2)_-) - f_{nrj}(w_i, 0, -(u_2)_-)\right)\\
&\geq -\Delta\tau\theta K_{nrj} q_j\\
\Delta\tau\theta f_{nrj}(w_i, q_j, u_2) &\leq \Delta\tau\theta\left(f_{nrj}(w_i, q_j, (u_2)_+) - f_{nrj}(w_i, M_Q, (u_2)_+)\right)\\
&\leq \Delta\tau\theta K_{nrj}(M - q_j).
\end{aligned}$$

So if $\Delta \leq \frac{1}{K_{nrj}\theta}$ then $q'_j \in [0, M_Q]$.

Let φ be defined by:

$$\varphi(q) = q - q'_j - (1-\theta)\Delta\tau^n f_{nrj}(w_i, q, u(\tau^n)). \tag{4.7}$$

Obviously the equation (4.5) has a solution if and only if the function φ vanishes for some $q \in [0, M_Q]$. Using the assumptions (Stor.1)-(Stor.8) we get:

$$\begin{aligned}
&\varphi(0) = -q'_j - (1-\theta)\Delta\tau^n f_{nrj}(w_i, 0, u(\tau^n)) \leq 0,\\
&\text{and } \varphi(M_Q) = M_Q - q'_j - (1-\theta)\Delta\tau^n f_{nrj}(w_i, M_Q, u(\tau^n)) \geq 0.
\end{aligned}$$

So φ vanishes for some $q \in [0, M_Q]$, and the equation (4.5) has a solution in $[0, M_Q]$. □

Then we use the approximations (4.5) and (4.6) in (4.4):

$$\begin{aligned}
v(\tau^{n-1}, w_i, q_j) = \sup_{(u_1,u_2)\in U(i,j,n)} \Big\{ & v(t, w, \hat{q}^n_{i,j}(u_1,u_2))\\
&+ \Delta\tau^n(1-\theta)\left(\mathcal{L}(\tau^n, w_i)v(\tau^n, w_i, \hat{q}^n_{i,j}(u_1,u_2)) + g\big(w_i, \hat{q}^n_{i,j}(u_1,u_2), u_2\big)\right)\\
&+ \Delta\tau^n\theta\left(\mathcal{L}(\tau^{n-1}, w_i)v(\tau^{n-1}, w_i, q_j) + g(w_i, q_j, u^{n-1}_\tau)\right)\Big\}.
\end{aligned} \tag{4.8}$$

We have now to approximate the values of v and $\mathcal{L}v$. For the grid points, we use V^n to approximate v, and the following finite differences scheme to approximate the values of $\mathcal{L}(\tau^n, w_i)v(\tau, w, q)$:

$$(LV^n)_{i,j} = \begin{cases} \left(\gamma^1_i + \beta^+_i\right)V^n_{i+1,j} - \left(\gamma^1_i + \gamma^2_i + \beta^+_i\right)V^n_{i,j} + \gamma^2_i V^n_{i-1,j} & \text{if } b(\tau^n, w_i) \geq 0,\\ \gamma^1_i V^n_{i+1,j} - \left(\gamma^1_i + \gamma^2_i - \beta^-_i\right)V^n_{i,j} + \left(\gamma^2_i - \beta^-_i\right)V^n_{i-1,j} & \text{if } b(\tau^n, w_i) < 0,\end{cases} \tag{4.9}$$

where

$$\gamma^1_i = \frac{\sigma^2(\tau^n, w_i)}{(w_{i+1} - w_i)(w_{i+1} - w_{i-1})}, \qquad \gamma^2_i = \frac{\sigma^2(\tau^n, w_i)}{(w_i - w_{i-1})(w_{i+1} - w_{i-1})},$$

$$\beta^+_i = \frac{b(\tau^n, w_i)}{x_{i+1} - x_i}, \qquad \text{and } \beta^-_i = \frac{b(\tau^n, w_i)}{x_i - x_{i-1}}.$$

Finally for the points outside the grid we use an interpolation Φ. Equation (4.8) becomes:

$$((Id - \Delta\tau^n \theta L)V^{n-1})_{i,j} = \sup_{(u_1,u_2)\in U(i,j,n)} \Big\{ (\Phi V^n)(w_i, \hat{q}^n_{i,j}(u_1,u_2)) + \Delta\tau^n \theta g(w_i, q_j, u_1)) + \Delta\tau^n(1-\theta)\big((\Phi L V^n)(w_i, \hat{q}^n_{i,j}(u_1,u_2)) + g(w_i, \hat{q}^n_{i,j}(u_1,u_2), u_2)\big)\Big\}. \tag{4.10}$$

4.1.3 The numerical scheme

Chose $\theta \in [0,1]$, for the θ scheme, and set $V^N_{i,j} = h(w_i, q_j)$. The discrete equation (4.10) can be written:

$$G^{n-1}_{i,j}(\delta, V^{n-1}_{i,j}, \{V^{n-1}_{k,j}\}_{k\neq i}, \{V^n_{i,j}\}) = 0, \forall n, i, j, \tag{4.11}$$

where $G^{n-1}_{i,j}(\delta, V^{n-1}_{i,j}, \{V^{n-1}_{k,j}\}_{k\neq i}, \{V^n_{i,j}\})$ is defined by:

$$\begin{aligned} &G^{n-1}_{i,j}(\delta, V^{n-1}_{i,j}, \{V^{n-1}_{k,j}\}_{k\neq i}, \{V^n_{i,j}\}) \\ =&((\frac{1}{\Delta\tau^n} Id - \theta L)V^{n-1})_{i,j} - \sup_{(u_1,u_2)\in U(i,j,n)} \Big\{ \frac{1}{\Delta\tau^n}(\Phi V^n)(w_i, \hat{q}^n_{i,j}(u_1,u_2)) + \theta g(w_i, q_j, u^{n-1}_\tau)) \\ &+ \Delta\tau^n(1-\theta)\Big((\Phi L V^n)\Big(w_i, \hat{q}^n_{i,j}(u_1,u_2)\Big) + g\Big(w_i, \hat{q}^n_{i,j}(u_1,u_2), u_2\Big)\Big)\Big\}. \end{aligned}$$

We refer to this system of equations, when discussing the convergence of our scheme. However when computing the scheme we use a slightly different equation in order to avoid numerical unstability. First we define:

$$\begin{aligned} \tilde{G}^{n-1}_{i,j}(\delta, V^{n-1}_{i,j}, \{V^{n-1}_{k,j}\}_{k\neq i}, \{V^n_{i,j}\}, u_1, u_2) =&(\Phi V^n)(w_i, \hat{q}^n_{i,j}(u_1,u_2)) + \Delta\tau^n \theta g(w_i, q_j, u^{n-1}_\tau) \\ &+ \Delta\tau^n(1-\theta) g(w_i, \hat{q}^n_{i,j}(u_1,u_2), u_2)) \\ &+ \Delta\tau^n(1-\theta)(\Phi L V^n)(w_i, \hat{q}^n_{i,j}(u_1,u_2)). \end{aligned}$$

The following algorithm is used:

Algorithm: Semi-Lagrangian Scheme
for $i = 1...i_{max}, j = 1...j_{max}$ **do**
  $V^N_{i,j} = h(w_i, q_j)$;
end
for $n = N-1...1$ **do**
  for $j = 1...j_{max}$ **do**
    for $i = 1...i_{max}$ **do**
      $\tilde{G}_i = \sup_{(u_1,u_2)\in U(i,j,n)} \{\tilde{G}^{n-1}_{i,j}(\delta, V^{n-1}_{i,j}, \{V^{n-1}_{k,j}\}_{k\neq i}, \{V^n_{i,j}\}, u_1, u_2)\}$;
    end
    $[V^{n-1}]_j = (Id - (1-\theta)\Delta\tau_n L(\tau_n))^{-1}\, \tilde{G}$;
  end
end

Remark. *The Algorithm is done in two steps. First we solve an optimization problem and then we solve a linear system. Therefore, the policy iteration is avoided. Using the definition of L, it follows easily that the matrix* $Id - (1-\theta)\Delta\tau_n L(\tau_n)$ *is tridiagonal, with strictly dominant diagonal. Thus, the system has a solution, which can be efficiently computed using the tridiagonal form of the matrix.*

Remark. *For* $\theta = 0$*, only the derivatives at time* τ^n *appears in the equation* (4.11)*, so* V^{n-1} *is explicitly given by the approximation at time* τ^n*. We call this scheme the explicit scheme.*
For $\theta = 1$*, only the derivatives at time* τ^{n-1} *appears in the equation* (4.11)*. We call this scheme the fully implicit scheme.*
For $\theta = 0.5$ *by analogy with usual finite differences, we call this scheme the Crank-Nicolson scheme.*

4.2 Convergence of the Semi-Lagrangian Scheme

We obtained with heuristic arguments a numerical scheme, but we have not proved its convergence. The paper [2] gives sufficient conditions for the convergence of a numerical scheme. We show that the fully implicit Semi-Lagrangian scheme verifies these conditions when Φ is the linear interpolation.

4.2.1 Stability

The solution of the PDE (3.7) is bounded, so the successive approximation V^n have to be bounded independently of the chosen grid. We call this property the stability of the scheme.

Definition 4.1. *The scheme* (4.11) *is* ℓ_∞ *stable if:*

$$||V^n||_\infty \leq C_1, \tag{4.12}$$

for $n = 1...N$*, for all* h*, where* C_1 *is a constant independent of* δ*, and* $||V^n||_\infty = \max_{i,j}|V_{i,j}|$.

The stability of the fully implicit scheme is a consequence of the following Lemma:

Lemma 4.1. *If the linear interpolation is used to approximate the values outside the grid (i.e.* Φ *is the linear interpolation), then in the case of the fully implicit scheme* $\theta = 1$*, there exists a constant* C_2 *such that:*

$$||V^n||_\infty \leq ||V^N||\infty + C_2. \tag{4.13}$$

Proof. The proof follows directly from the maximum principle applied to the discrete equation (4.11). We omit the details here, and refer to [8] for the proof of a similar result. □

4.2.2 Consistency

The quantity $G_{i,j}^n$ is in some sense a discretization of the differential operator associated to the PDE (3.7), so when δ goes to 0 it has to approximate this operator. This property is illustrated by the following definition:

Definition 4.2. *The scheme* (4.11) *is consistent if, for any smooth function φ having bounded derivatives of all orders, with $\varphi_{i,j}^n = \varphi(\tau^n, w_i, q_j)$, we have that:*

$$\lim_{\delta \to 0} |G_{i,j}^{n-1}(\delta, \varphi_{i,j}^{n-1}, \{\varphi_{k,j}^{n-1}\}_{k \neq i}, \{\varphi_{i,j}^n\} - \partial_t \varphi(\tau^n, w_i, q_j) \\ - \mathcal{L}\varphi(\tau^n, w_i, q_j) - H(\tau^n, w_i, q_j, \partial_q \varphi(\tau^n, w_i, q_j))| = 0. \tag{4.14}$$

Our scheme is not unconditionally consistent, the grid has to be well chosen. We make the following assumption:

$$\exists C_3, C_4, \text{ such that } \Delta w_{max} \leq C_3 \Delta\tau \text{ and } \Delta q_{max} \leq C_4 \Delta\tau, \tag{4.15}$$

where,

$$\Delta w_{max} = \max_i \{w_i - w_{i-1}\} \text{ and } \Delta q_{max} = \max_j \{q_j - q_{j-1}\}.$$

The following Lemma gives sufficient conditions for the convergence of the scheme.

Lemma 4.2. *Suppose that the assumption* (4.15) *is verified, and that Φ is an interpolation of order two or more (for example the linear interpolation), then the scheme* (4.11) *is consistent for all θ.*

Proof. We leave the proof to the reader as it directly follows from Taylor's expansion. □

4.2.3 Monotonicity

The authors of [2] prove the convergence for a class of monotone schemes defined as follow:

Definition 4.3. *The scheme* (4.11) *is monotone if*

$$G_{i,j}^{n-1}(\delta, X_{i,j}, \{V_{k,j}^{n-1}\}_{k \neq i}, \{V_{i,j}^n\}) \leq G_{i,j}^{n-1}(\delta, Y_{i,j}, \{V_{k,j}^{n-1}\}_{k \neq i}, \{V_{i,j}^n\}) \forall n, i, j, \ X_{i,j} \leq Y_{i,j}.$$

Lemma 4.3. *If Φ is the linear interpolation, then the fully implicit scheme* (4.11) *($\theta = 1$) is monotone.*

Proof. The proof directly follows that of monotonicity of finite difference schemes for controlled HJB equations in [1] and [8]. □

4.2.4 Convergence

Finally we show that the implicit scheme is convergent.

Definition 4.4. *The scheme* (4.11) *is convergent if:*

$$\lim_{\delta \to 0} |V_{i,j}^n - v(\tau_n, w_i, q_j)| = 0. \tag{4.16}$$

The following theorem, proved in [2], gives sufficient conditions for a scheme to be convergent.

Theorem 4.2. *If the scheme* (4.11) *is stable, consistent, monotone and if the comparison principle holds for the PDE* (3.7)*, then the scheme is convergent.*

In particular the fully implicit scheme is convergent:

Corollary 4.3. *If the linear interpolation is used, and the discretization satisfies* (4.15) *then the fully implicit scheme ($\theta = 1$) is convergent.*

5 Numerical Results

We discuss here the numerical results obtained for two models. The first is a model without strategy cost (i.e. $c \equiv 0$), and without energy loss ($f_{nrj} = f_{prod}$). As a result, the function g is piecewise affine with respect to u, and our algorithm can be improved. The second model has strategy cost without the affine property and might be impossible to model by only considering the delivered energy.

5.1 General Settings

We consider four consecutive production periods of one hour ($[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$). Let P_k (resp. w_k^E, C_k^+, C_k^-) denote the energy selling price (resp. the committed power, the overproduction, underproduction penalties) of $k-$th period, $k = 1,2,3,4$. In the fourth period we consider the final potential gain to be null ($h \equiv 0$).

We solve this problem by doing the following. We use our algorithm to solve the optimization problem for the last period. This gives the terminal condition for the third period, and we repeat the operation until the optimization problem for the first period is solved.

For the k-th period we define the functions $\sigma,\ b,\ f_{nrj},\ f_{prod}$ by:

$$\begin{aligned}
&\sigma(t,w) = (M_W - w)w,\\
&b(t,w) = w_k^E - w,\\
&f_{nrj}(w,q,u) = \begin{cases} u\min((M_q) - q, w), \text{ if } u \leq 0\\ uq \text{ if } u > 0\end{cases},\\
&f_{prod}(w,q,u) = -f_{nrj}(w,q,u).
\end{aligned}$$

It is easily seen that these functions satisfy (Prod.1), (Prod.2), (Stor.1)-(Stor.8).

We point out that these choices of functions are arbitrary. The model calibration is a tricky question and will not be discussed in this paper. We can point out two particularities for this model. First the device is a "perfect" storage device as there is no loss of energy. Second the commitments are well chosen as for each production period the process W has a mean return to w_k^E.

We use the following values:

$(W_1^E,\ W_2^E,\ W_3^E,\ W_4^E) = (2,\ 1.75,\ 0.35,\ 1)$, $(P_1,\ P_2,\ P_3,\ P_4) = (3,\ 4,\ 0.75,\ 2)$,
$(C_1^+,\ C_2^+,\ C_3^+,\ C_4^+) = (1,\ 4.8,\ 1.6,\ 1)$, $(C_1^-,\ C_2^-,\ C_3^-\ ,C_4^-) = (0.5,\ 2,\ 1,\ 4)$,
$M_W = 4,\ M_Q = 2$.

For each resolution we use uniform time step of 2 minutes ($\Delta^n \tau = 1/30$), and same step for the spatial discretization ($w_{i+1} - w_i = q_{j+1} - q_j = 1/30$). Finally we use the fully implicit scheme $\theta = 1$.

We now discuss the results obtained for two models.

5.2 Piecewise Affine Model

In this case we consider that the strategy cost is null ($c^{strat} \equiv 0$). With the chosen functions, the function g is obviously continuous and affine by part with respect to u. Therefore, for all $(w, q) \in E,\ r \in \mathbb{R}$, the function $u \to f_{nrj}(w, q, u)r + g(w, q, u)$ is affine by part. It is clear that this function reaches its maximum in a point where the slope changes or in the boundary of $[-1, 1]$. Let $U^*(w, q)$ denotes this set. Then by straightforward computation:

$$U^*(w,q) = \begin{cases} \{-1,\ 0, 1\} \text{ if } w^E \notin [w + f_{prod}(w,q,1), w + f_{prod}(w,q,-1)] \\ \{-1,\ 0, 1,\ u^*(w,q)\} \text{ if } w^E \in [w + f_{prod}(w,q,1), w + f_{prod}(w,q,-1)], \end{cases}$$

where $u^*(w, q)$ is the solution of $w + f_{prod}(w, q, u) = w^E$.

So H can be rewrite:

$$H(t, w, q, r) = \max_{u \in U^*(w,q)} \{f_{nrj}(w, q, u)r + g(w, q, u)\}.$$

Formally the search of the maximum is faster than in the general case as we just need to compare at most four values. We can easily verify that f_{nrj} is Lipschitz continuous. So Property 4.1 implies that $U(i, j, n) = [-1, 1]^2$ for all $i,\ j,\ n$ with our settings. Then using the same heuristic arguments as in the general case but this time only considering the strategy values in $u^*(w, q)$ leads to the following modified algorithm:

Algorithm: Semi-Lagrangian Scheme for affine by part model

```
for i = 1...i_max, j = 1...j_max do
|   V^N_{i,j} = h(w_i, q_j) ;
end
for n = N − 1...1 do
|   for j = 1...j_max do
|   |   for i = 1...i_max do
|   |   |   G̃_i = max_{u∈U*(w_i,q_j)} {G̃^{n−1}_{i,j}(δ, V^{n−1}_{i,j}, {V^{n−1}_{k,j}}_{k≠i}, {V^n_{i,j}}, u)} ;
|   |   end
|   |   [V^{n−1}]_j = (Id − (1 − θ)Δτ_n L(τ_n))^{−1} G̃;
|   end
end
```

With some modifications, we can show that this algorithm is monotone, stable, consistent and thus convergent.

This modified algorithm is faster as the maximum search is done by comparing at most four values. Therefore, we use this algorithm to treat this case.

The following results are obtained using `Scilab` on a computer with a 2.2GHz processor and 4Go of RAM. The run time of the algorithm is about 100 seconds. Figures 1 and 2 illustrate the approximation of the maximal expected gain at the beginning of the first and second period resulting from this modified algorithm. Figures 3 and 4 illustrate the strategy values used when executing the algorithm.

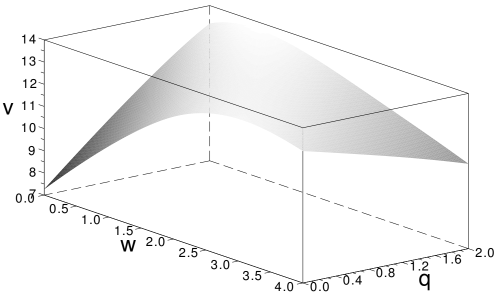

Figure 2: Maximal expected gain at time $T = 0$

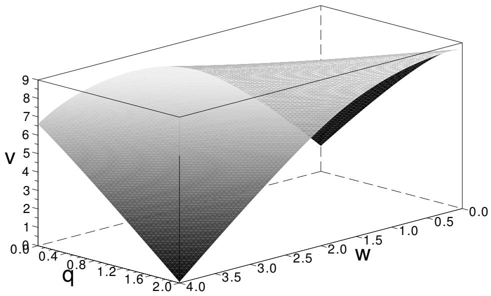

Figure 3: Maximal expected gain at time $T = 1$

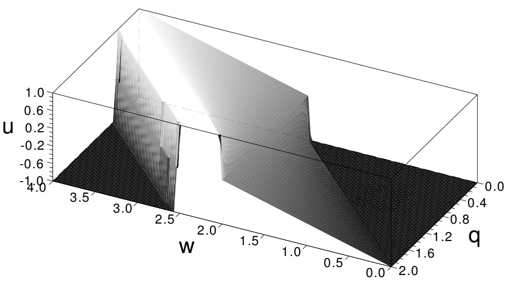

Figure 4: Strategy values in the algorithm at time $T = 0$

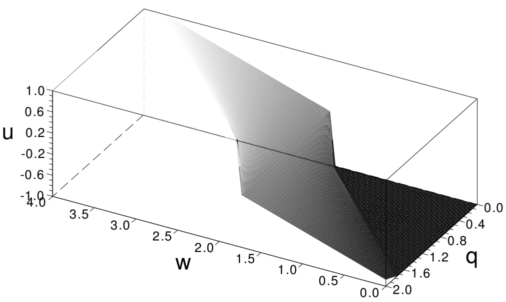

Figure 5: Strategy values in the algorithm at time $T = 1$

In Figure 4 the strategy always consists in keeping the delivery as close as possible to the committed power. This is the first strategy that comes to mind when facing this problem for the first time. However, this strategy is not used in the first period. When the production and the stock level are close to their maximum, the strategy is to empty the stock as fast as possible. Indeed, in the second period the overproduction cost is greater than in the first period. If in the first period, the production is close to its maximum, it is natural to think that it will also be the case in the second period. Therefore the producer pays cheaper overproduction penalties in the first period, and can store more overproduced energy in the second period.

We now study a case with non affine strategy cost.

5.3 Non Affine Strategy Cost

We consider the following strategy cost:

$$c^{strat}(w, q, u) = (f_{nrj}(w, q, u))^2.$$

The faster the stock level varies, the more the strategy is expensive. Because we have lost the affine property we cannot use the previous algorithm. Thus we use the general algorithm. As we use the linear interpolation, it is easy to see that the optimal strategy values is obtained when $\hat{q}$ is on the grid, or when $u \in \{-1, 1\}$. Therefore we need only to evaluate those points to find the optimal strategy values.

Iterative methods could be investigated for finding the optimal strategy value. But we want to point out that the method used here solves exactly the local optimization problem, whereas an iterative method could be biased and the resulting algorithm may not be convergent.

The following results are obtained by using `Scilab` on the same computer. The run time of the algorithm is about 120 seconds.

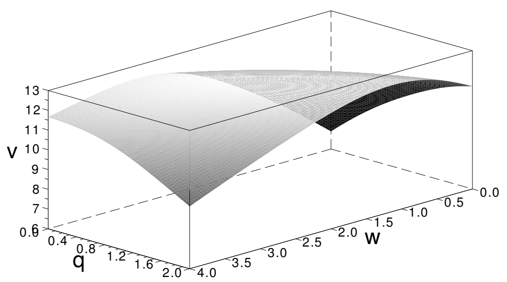

Figure 6: Maximal expected gain at time $T = 0$

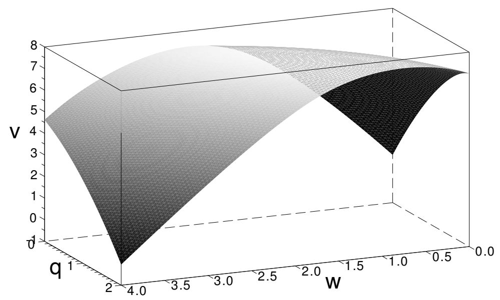

Figure 7: Maximal expected gain at time $T = 1$

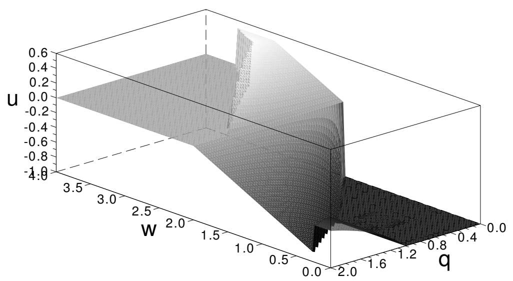

Figure 8: Strategy values in the algorithm $T = 0$

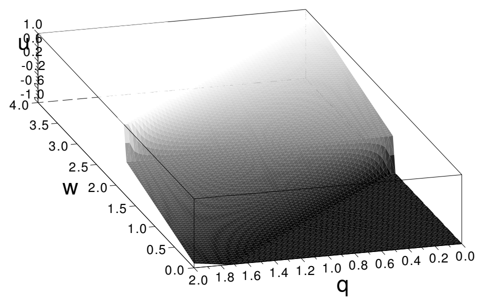

Figure 9: Strategy values in the algorithm at time $T = 1$

At the beginning of the second period the strategy values are similar to those of the previous case. The beginning of the first period is more interesting. In the first model, the strategy choice could be reduce to four: do nothing, keep the commitment, load or unload the storage device at full speed. Other strategy choices are clearly appearing in Figure 8. This illustrates the fact that the strategy values cannot be reduced to a finite set. In the first model it is already difficult, if not impossible, to guess the strategy values by using heuristic arguments. Therefore, it is harder to do so in the second model, making the quantitative informations obtained with the algorithm all the more precious.

6 Conclusion

We have defined a large class of production/storage time continuous models. In contrast to the models only considering the delivered energy, these models allow to use costs that are not piecewise affine with respect to the strategy. This class of models verifies the physical limits of the wind farm and the storage device. The model choice results in an optimal stochastic control problem.

We have proved that solving this problem is equivalent to solve a modified HJB PDE. We have established a comparison principle for this PDE which has thus an unique solution in the viscosity sense.

Next, using a semi-Lagrangian approach, we constructed a numerical scheme. This scheme is monotone, stable and consistent. As the equation verifies the maximum principle the corresponding algorithm is convergent. With this method there is no need for strategy iterations. Thus this algorithm is faster than the algorithms developed for the general case.

Finally, we have studied the results obtained for two models. The first is similar to the models which consider only the delivered energy. In this case we have shown that our algorithm may be improved to obtain a faster algorithm. The second is a model with non affine strategy cost. We used our unmodified algorithm to treat this case. Solving the PDE by this method is slower, but provides useful information about the optimal operation of the storage device.

For future work we outline some research directions:

1. Jumps are observed in the wind farm production. Therefore including a jump part in the process W makes the model more realistic.

2. The maximum search used in our algorithm is not very efficient, but is accurate. To fasten the algorithm, iterative method could be investigated. These methods are fast to provide a maximum approximation. However, if this approximation is not close enough to the maximum the algorithm may be not convergent.

3. We have used the fully implicit scheme, but the Crank-Nicolson Time-stepping schemes have shown better result for solving linear PDE. Up to now, we are not able to prove the monotonicity of the scheme for $\theta = 0.5$. Thus we cannot prove the convergence of our algorithm with the same arguments as in the fully implicit case. We also must keep in mind that with $\theta = 0.5$, the solution (4.5) of become implicit and thus may be tricky to compute.

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