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Vegas





## Silhouette of a random polytope

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Project-Teams Geometrica and Vegas

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**Abstract:** We consider random polytopes defined as the convex hull of a Poisson point process on a sphere in  $\mathbb{R}^3$  such that its average number of points is  $n$ . We show that the expectation over all such random polytopes of the maximum size of their silhouettes viewed from infinity is  $\Theta(\sqrt{n})$ .

**Key-words:** Silhouette, 3D visibility, expected complexity

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## Silhouette d'un polytope aléatoire

**Résumé :** Nous considérons des polytopes aléatoires définis comme l'enveloppe convexe d'un processus ponctuel de Poisson sur une sphère de  $\mathbb{R}^3$  dont le nombre moyen de points est  $n$ . Nous montrons que l'espérance de la taille maximale de la silhouette vu depuis l'infini d'un tel polytope est  $\Theta(\sqrt{n})$ , la taille maximale étant considérée pour l'ensemble des points de vue d'un polytope fixé, et l'espérance étant considérée sur l'ensemble des polytopes.

**Mots-clés :** Silhouette, visibilité 3D, complexité en moyenne

## 1 Introduction

The silhouette of a polytope in  $\mathbb{R}^3$  with respect to a given view point at infinity is, roughly speaking, the set of edges incident to a front and a back face. The size of the silhouette is its number of edges or equivalently vertices. Silhouettes of (non-necessarily convex) polyhedra naturally arise in computer graphics for hidden surface removal or shadow computation [DD02, EGHHZ00]. They are also important in shape recognition; Sander et al. [SGG<sup>+</sup>00] claim that the silhouette “is one of the strongest visual cues of the shape of an object”.

While the silhouette size of a polyhedron with  $n$  vertices may be linear for some view points, several experimental and theoretical studies show a sublinear behavior for a wide range of constraints. The latest result on the subject proves a bound in  $\Theta(\sqrt{n})$  on the size of the silhouette from a random view point of polyhedra of size  $n$  approximating non-convex surfaces in a reasonable way [GL08]. Prior to this result, it was widely accepted that the silhouette of a polyhedron is often of size  $\Theta(\sqrt{n})$  as, for instance, stated by Sander et al. [SGG<sup>+</sup>00]. An experimental study by Kettner and Welzl [KW97] confirmed this for a set of realistic objects, study which was extended by McGuire [McG04] to a larger database of larger objects for which the observed silhouette size was approximately  $n^{0.8}$ . In terms of theoretical results, Kettner and Welzl [KW97] first proved the  $\Theta(\sqrt{n})$  bound for the size of the silhouette, viewed from a random point at infinity, of a convex polyhedron that approximates a sphere with small Hausdorff distance. Alt et al. [AGG03] also gave conditions under which the silhouette of a convex polyhedron has sub-linear size in the worst case.

This paper addresses the size of the silhouette of a random polytope for the worst view point. We consider a Poisson point process on a sphere so the average number of points is  $n$  and we define a random polytope as the convex hull of the Poisson point process. We do not pretend that random polytopes are a good model of the objects used in graphics, but this result gives further insight explaining why silhouettes tend to be small, and it showcases interesting proof techniques. Our main result is the following.

**Theorem 1** *The expectation over all random polytopes of the maximum size of their silhouettes viewed from infinity is  $\Theta(\sqrt{n})$ .*

We first prove in Section 3 that the size of the worst-case silhouette of a random polytope viewed from infinity is in  $O(\sqrt{n \ln n})$ . We then refine this analysis in Section 4 and prove that this expected maximum size is in  $\Theta(\sqrt{n})$ .

This paper uses as a starting point the technique introduced by Devillers et al. [DGG13]. One example they consider to illustrate their technique is the expected size of the convex hull of points sampled according to a uniform Poisson process in a disk, which they bound by  $O(n^{1/3} \text{polylog}(n))$  where  $n$  is the mean of the Poisson process (this is a weaker version of a well known result, but the point is the simplicity of their proof). In the beginning of Section 3, we adapt their analysis to silhouettes viewed from a fixed direction in a straightforward manner by only modifying the density of the Poisson process. We then extend their technique in two main directions. First, we prove that our upper bounds are “reliable” in the sense that the probability of a large deviation is very small, i.e., it is very unlikely that the variables get much larger than these upper bounds on their expectation. This will allow us to bound the expectation of the maximum of a number of variables and thus to bound the *worst-case* size of the silhouette of a random polytope. Second, in Section 4, we refine the analysis in order to remove the polylogarithmic factor. The techniques introduced here are fairly generic and they can be used in other problems for removing polylogarithmic factors in expected complexities, as demonstrated by Devillers et al. (private communication which should appear in the full version of [DGG13]).

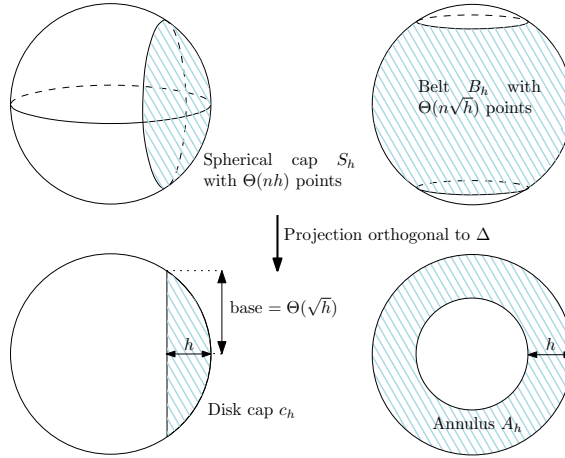


Figure 1: Left: disk and spherical caps, right: annulus and belt.

It should be stressed that the technique of Kettner and Welzl [KW97] does not easily extend to give a bound on the worst-case silhouette of random polytopes. Indeed, they compute the expected size of the silhouette as the sum of the dihedral angles of the edges and their approach is thus intrinsically tied to the average over all view points.

## 2 Preliminaries

We define a random polytope as the convex hull of a point process on the unit sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$ . For a subset  $B$  of the sphere, let  $\mathcal{A}(B)$  be its area. More formally, we consider a Poisson point process on  $\mathbb{S}^2$  with intensity  $\frac{n}{\mathcal{A}(\mathbb{S}^2)}$ , so that the mean number of vertices of the polytope is  $n$ .

For a direction  $\Delta$  that is coplanar with no face of a polytope, the silhouette of the polytope viewed from that direction is the set of its edges that are adjacent to a front face and a back face, where front and back faces are defined by the sign of the scalar product of their outer normal with  $\Delta$ , and the size of the silhouette is the number of its edges or equivalently vertices. The case where  $\Delta$  is coplanar with a face arises with probability zero but the silhouette size has to be defined since we consider the worst-case size for all viewing directions. Various definitions for the silhouette are possible (see for instance [GL08]) but, instead of detailing the possible definitions, we simply bound the size of the silhouette by the number of polytope vertices that are incident to two faces whose outer normals have scalar products with  $\Delta$  whose product is negative or null.

Next, we state some preliminary results that could be skimmed at first and referred to when needed.

**Poisson distribution.** For a subset  $B$  of the sphere, denote by  $N(B)$  the number of points of the Poisson point process that fall in  $B$ . The random variable  $N(B)$  follows a Poisson distribution of parameter  $n \frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}$  so that

- for any  $k \in \mathbb{N}$ , the probability that  $N(B) = k$ , denoted  $\mathbf{P}(N(B) = k)$ , is  $\frac{\left(n \frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}\right)^k}{k!} e^{-n \frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}}$  and
- the expectation of  $N(B)$ , denoted  $\mathbf{E}[N(B)]$ , is  $n \frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}$ .

**Spherical geometry.** In a plane orthogonal to a direction  $\Delta$ , we denote by  $D$  the projection of the sphere  $\mathbb{S}^2$  (see Figure 1). Assume that  $D$  is centered at the origin of the Euclidean plane

with a coordinate system  $(x, y)$ . We define the disk cap  $c_h$  as the set of points in the disk  $D$  with  $x \geq 1 - h$ , we call  $h$  its height and define its base as its maximum  $y$ -coordinate. We denote  $S_h$  the spherical cap of  $\mathbb{S}^2$  that projects to the disk cap  $c_h$ . Elementary geometric calculus shows that, for small  $h$ , the base is in  $\Theta(\sqrt{h})$ , the area of  $S_h$  is in  $\Theta(h)$  and thus the expected number of points in this cap is in  $\Theta(nh)$ .

In addition, we define  $A_h$  the annulus of width  $h$  as the subset of  $D$  bounded by the circles of radii  $(1 - h)$  and  $1$ . We also define the belt  $B_h$  as the part of the sphere that projects to the annulus  $A_h$ . Elementary geometric calculus shows that the width  $h$  of an annulus and the area  $\mathcal{A}(B_h)$  of the belt  $B_h$  satisfy, for small  $h$ , the asymptotic relations  $\mathcal{A}(B_h) = \Theta(\sqrt{h})$  and  $h = \Theta(\mathcal{A}(B_h)^2)$ . Hence, the expected number of points in the belt  $B_h$  is  $\mathbf{E}[N(B_h)] = \Theta(n\sqrt{h})$ .

**Large deviations.** The following lemma states a large deviation principle for a Poisson distribution. It is a classical simple result, a detailed proof can be found for instance in the appendix of [FDTT07].

**Lemma 2** *Let  $N_\lambda$  be a random variable following a Poisson distribution of parameter  $\lambda$ , then for any  $\eta > 0$ ,*

$$\mathbf{P}(N_\lambda \geq \lambda(1 + \eta)) \leq e^{-\lambda I(\eta)}$$

with  $I(\eta) = (1 + \eta) \ln(1 + \eta) - \eta$ ; note that  $I(\eta) > 0$  for  $\eta > 0$ .

A refinement of Hoeffding's inequality that allows some dependency between the variables is the following. Let  $(Y_1, \dots, Y_m)$  be random variables not necessarily independent. A family  $\{A_j\}_j$  of subsets of indices in  $\{1, \dots, m\}$  is a cover if  $\cup_j A_j = \{1, \dots, m\}$ . The cover is proper if, in each subset  $A_j$ , the random variables  $Y_i$  are independent. We denote by  $\chi$  the size of a smallest proper cover, i.e. the smallest  $k$  such that  $\{1, \dots, m\}$  is the union of  $k$  subsets of indices of independent variables.

**Lemma 3** [Jan04, Theorem 2.1] *Let  $(Y_1, \dots, Y_m)$  be random variables with range  $[a_i, b_i]$  and  $S_m = Y_1 + \dots + Y_m$ , then for any  $t > 0$*

$$\mathbf{P}(S_m - \mathbf{E}[S_m] \geq t) \leq \exp\left(-2 \frac{t^2}{\chi \sum_{i=1}^m (b_i - a_i)^2}\right).$$

### 3 Simple proof yielding a $O(\sqrt{n \ln n})$ bound

We prove in this section the following proposition which says, roughly speaking, that the size of the worst-case silhouette of a random polytope viewed from infinity is in  $O(\sqrt{n \ln n})$ .

**Proposition 4** *Consider a Poisson point process on  $\mathbb{S}^2$  with intensity  $\frac{n}{\mathcal{A}(\mathbb{S}^2)}$ , and the polytope defined as the convex hull of the points. The expectation over all such random polytopes of the maximum size of their silhouettes viewed from infinity is  $O(\sqrt{n \log n})$ .*

We consider points of view at infinity, thus the silhouette is the convex hull of the projection of the points on a plane orthogonal to the view point direction. For a view point direction  $\Delta$ , we denote by  $\text{silh}(\Delta)$  the number of vertices of the silhouette viewed from  $\Delta$ . We successively study the size of the silhouette from a fixed view point, in a neighborhood of a fixed view point, and then consider all possible view points. The idea is that the projection of the silhouette vertices are expected to be in an annulus of small width, or equivalently that the silhouette vertices are expected to be in a belt of small width on the sphere. Then, for nearby view points, we show that all possible silhouette points are expected to remain in a slightly larger belt. Finally,



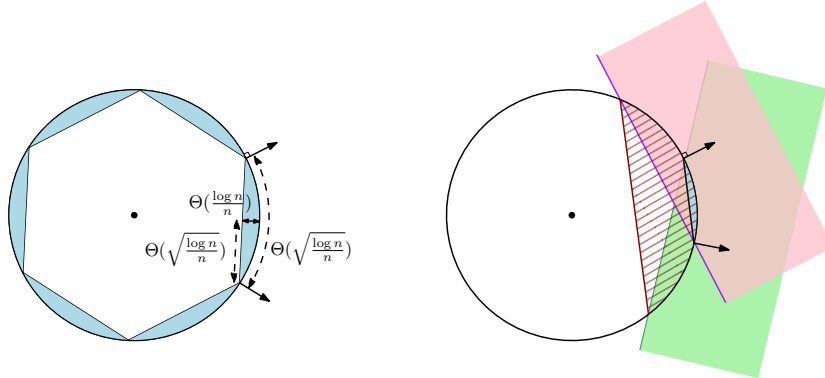


Figure 2: (a)  $\Theta(\sqrt{\frac{n}{\ln n}})$  witnesses in the projected sphere. (b) Collector (hatched) of a witness (in blue).

a covering argument of the set of view points at infinity by such neighborhoods and a large deviation principle enable us to conclude.

**Silhouette from a fixed view point.** The silhouette of a polytope viewed from a fixed direction  $\Delta_0$  is, in projection onto a plane orthogonal to  $\Delta_0$ , the convex hull of the projected polytope vertices. We are thus analyzing the size of the convex hull of points distributed in a disk according to the (non-homogeneous) point process induced by the Poisson point process of the points on the sphere.

We consider the circle  $C_0$  bounding the disk  $D_0$  that is the projection of the sphere  $\mathbb{S}^2$  in the direction orthogonal to  $\Delta_0$ . We partition  $C_0$  into arcs and consider the convex hulls of each of these arcs, which we call *witnesses*, and the spherical witnesses caps of  $\mathbb{S}^2$  that project on these witnesses (see Figure 2(a)). Associated to every witness, we consider the *range* of directions (in the plane orthogonal to  $\Delta_0$ ) defined by the rays starting at the center of disk  $D_0$  and that intersect the witness, and we define the *collector* associated to every witness as (see Figure 2(b)) the convex hull<sup>1</sup> of the union (clipped by disk  $D_0$ ) of the half-planes whose inward normals are in the corresponding range and that do not completely contain the witness.

The **key property of witnesses, ranges and collectors** is that if a witness contains a point, any point that is extreme in one of the directions of the associated range has to belong to the associated collector; furthermore the height of the collector is larger than that of its witness by at most a constant multiplicative factor (since the circular arc defining the collector is exactly three times longer than the one defining the witness). Hence, if none of the witnesses is empty, the projected vertices of the silhouette are contained in the union of the collectors. In this case, denoting  $h$  the height of the witnesses, the union of the collectors is included in an annulus of width  $\Theta(h)$ . Using the spherical geometric preliminaries, the vertices of the silhouette are thus in a belt of area  $\Theta(\sqrt{h})$ . Conversely, for the belt to contain an expected number of  $\Theta(f(n))$  points, its area should be in  $\Theta(f(n)/n)$  and the height of the witnesses should be  $h = \Theta((f(n)/n)^2)$ .

To bound the expected size of the silhouette from the view point  $\Delta_0$ , it is then sufficient to select an adequate value for the height of the witnesses such that: (a) when none of the witnesses is empty, the belt containing the silhouette has the expected number of points stated in Proposition 4,  $O(\sqrt{n \log n})$ , and (b) the probability that at least one of the witnesses is empty is small (e.g. is in  $O(\frac{1}{n})$ ), in which case the conditional expected silhouette can be crudely bounded by  $n$ .

<sup>1</sup>Considering the convex hull is not necessary but convenient since both the witness and the collector have the geometry of a disk cap as defined in the preliminaries.

Note that this setting fails to prove the upper bound of Theorem 1. Indeed, if one wants the expected number of vertices in the belt to be in  $\Theta(\sqrt{n})$ , this implies that the height of the witnesses is  $h = \Theta((\frac{\sqrt{n}}{n})^2) = \Theta(\frac{1}{n})$ . The probability that a given witness is empty is then a constant (since the law of the number of points in the witness is a Poisson distribution of parameter  $\Theta(1)$ ) and the probability that at least one witness is empty is not asymptotically small, thus property (b) is not satisfied.

To prove Proposition 4, we take a larger belt that contains an expected number of vertices in  $\Theta(\sqrt{n \ln n})$ , which implies that the height of the witnesses is  $\Theta(\frac{\ln n}{n})$  and that the expected number of vertices in each witness cap is  $\Theta(\ln n)$ . Precisely, we set the size of the ranges and witnesses so that the expected number of vertices in each witness cap is  $\alpha \ln n$  where  $\alpha$  is a constant which we define later. The base of a witness is thus in  $\Theta(\sqrt{\frac{\alpha \ln n}{n}})$ . It follows that the length of the circular arc defining a witness is also in  $\Theta(\sqrt{\frac{\alpha \ln n}{n}})$  and thus that the number of witnesses is in  $\Theta(\sqrt{\frac{n}{\alpha \ln n}})$ .<sup>2</sup> Property (b) is then satisfied since the probability that a given witness is empty is small; indeed (see Section 2), since the expected number of points in the witness cap is  $\alpha \ln n$ , the corresponding Poisson distribution has parameter  $\alpha \ln n$  and thus the probability that there is no point in the cap is  $e^{-\alpha \ln n} = n^{-\alpha}$ . The probability that (at least) one of the  $\Theta(\sqrt{\frac{n}{\alpha \ln n}})$  witnesses is empty is thus at most  $O(\sqrt{\frac{n}{\alpha \ln n}}) \cdot n^{-\alpha}$  and thus is  $O(n^{\frac{1}{2}-\alpha})$ .

We conclude by computing the expectation of the size of the silhouette conditioned by the events that all the witnesses are empty or not:

$$\begin{aligned} \mathbf{E}[\text{silh}(\Delta_0)] &= \mathbf{E}[\text{silh}(\Delta_0) | \exists i, N(W_i) = 0] \mathbf{P}(\exists i, N(W_i) = 0) \\ &\quad + \mathbf{E}[\text{silh}(\Delta_0) | \forall i, N(W_i) \neq 0] \mathbf{P}(\forall i, N(W_i) \neq 0). \end{aligned}$$

For the first term, i.e. if (at least) one witness is empty, we bound the expected size of the silhouette by the expected number of points on the sphere, which is at most  $n$  since knowing that at least one witness is empty can only decrease the expected number of points on the sphere:

$$\mathbf{E}[\text{silh}(\Delta_0) | \exists i, N(W_i) = 0] \leq \mathbf{E}[N(\mathbb{S}^2) | \exists i, N(W_i) = 0] \leq \mathbf{E}[N(\mathbb{S}^2)] = n$$

On the other hand, as noted above, if none of the witnesses is empty, the vertices of the silhouette are contained in the belt defined by the witness caps which were defined so that the belt contains an expected number of vertices in  $\Theta(\sqrt{n \ln n})$ . Denoting by  $Belt_0$  this belt and by  $\mathbf{1}$  the characteristic function, we get:<sup>3</sup>

$$\begin{aligned} \mathbf{E}[\text{silh}(\Delta_0) | \forall i, N(W_i) \neq 0] \mathbf{P}(\forall i, N(W_i) \neq 0) &= \mathbf{E}[\text{silh}(\Delta_0) \times \mathbf{1}_{\{\forall i, N(W_i) \neq 0\}}] \\ &\leq \mathbf{E}[N(Belt_0) \times \mathbf{1}_{\{\forall i, N(W_i) \neq 0\}}] \\ &\leq \mathbf{E}[N(Belt_0)] = O(\sqrt{n \ln n}). \end{aligned}$$

Summing the two terms thus gives  $\mathbf{E}[\text{silh}(\Delta_0)] \leq n\Theta(n^{\frac{1}{2}-\alpha}) + O(\sqrt{n \ln n})$  which is in  $O(\sqrt{n \ln n})$  by choosing  $\alpha \geq 1$ .

**Silhouettes from a neighborhood of a fixed view point.** We now consider the expected size of the worst-case silhouette for a set of directions in some neighborhood of  $\Delta_0$ , that is the

<sup>2</sup>The constant  $\alpha$  fixes the length of the circular arc defining the witnesses. For this length to divide the circumference of  $C_0$ ,  $\alpha$  must satisfy a constraint but this technical detail is not relevant in the proof.

<sup>3</sup>The first equality is a classical probability property which can be proved as follows with the obvious change of notation:  $\mathbf{E}[A|B] \mathbf{P}(B) = \sum_k k \mathbf{P}(A = k|B) \mathbf{P}(B) = \sum_k k \mathbf{P}(A = k \& B) = \sum_k k \mathbf{P}(A \times \mathbf{1}_B = k) = \mathbf{E}[A \times \mathbf{1}_B]$ .

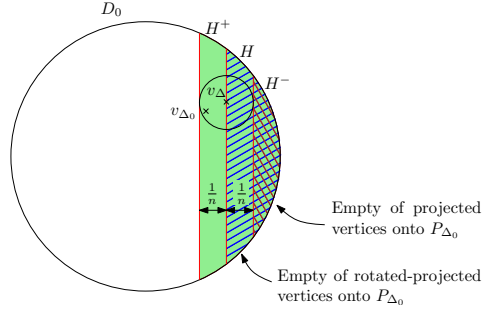


Figure 3: Empty half-planes (clipped by  $D_0$ ) associated to silhouettes from a neighborhood of a fixed view point.

set of directions that make an angle at most  $\frac{1}{n}$  with  $\Delta_0$ . We denote by  $\Delta \sim \Delta_0$  a direction in this neighborhood and we are thus interested in the variable  $\max_{\Delta \sim \Delta_0} \text{silh}(\Delta)$ . As before, we consider the witnesses and collectors in a plane  $P_{\Delta_0}$  orthogonal to  $\Delta_0$ . Also as before, if (at least) one witness is empty, we bound the expected size of the worst-case silhouette by  $n$  which is the expected total number of points. We now assume that no witness is empty in the plane  $P_{\Delta_0}$ .

The idea is to show that if a point is on the silhouette for the direction  $\Delta$ , then its projection in the direction  $\Delta_0$  (onto  $P_{\Delta_0}$ ) is, roughly speaking, in a collector enlarged by  $\frac{2}{n}$ .

Consider the silhouette of the polytope in a direction  $\Delta$ . We rotate the polytope (about the center of  $\mathbb{S}^2$ ) so that direction  $\Delta$  is mapped to  $\Delta_0$ . Then, we project the vertices of the rotated polytope onto  $P_{\Delta_0}$ . We now have two sets of points on  $P_{\Delta_0}$ : the projection of the vertices of the polytope and the projection of the vertices of the rotated polytope, and there is a trivial one-to-one mapping between the points in each set. We denote by  $v_{\Delta}$  the image of a vertex  $v$  by this rotation and projection onto  $P_{\Delta_0}$ , while we refer to  $v_{\Delta_0}$  as the projection of  $v$  onto  $P_{\Delta_0}$ . Note that for any vertex  $v$ , the distance between  $v_{\Delta_0}$  and  $v_{\Delta}$  is at most  $\frac{1}{n}$  since the vertices of the polytope move by distance at most  $\frac{1}{n}$  through the rotation and the projection onto  $P_{\Delta_0}$  decreases relative distances. In the following, we refer (in two or three dimensions) to a half-space through a point  $v$  as to an open half-space whose boundary contains  $v$ .

If a vertex  $v$  is on the silhouette for the direction  $\Delta$ , there is a half-space through  $v$  and parallel to  $\Delta$  that is empty of vertices. Through the rotation and projection onto  $P_{\Delta_0}$ , this gives a half-plane through the image  $v_{\Delta}$  of  $v$  that is empty of the rotated and projected vertices ( $v'_{\Delta}$ ) of the polytope. Denote this half-plane by  $H$  and consider, in  $P_{\Delta_0}$ , the two orthogonal translations of this half-plane by distance  $\frac{1}{n}$ ; denote them by  $H^{\pm}$  such that  $H^- \subset H \subset H^+$  and refer to Figure 3. Since the distance between two points that are in correspondence through the one-to-one mapping is at most  $\frac{1}{n}$ , the half-plane  $H^-$  contains no projection of the vertices of the (non-rotated) polytope. Thus, the part of  $H^-$  that lies in the disk  $D_0$  belongs to a collector, by definition since we assumed that no witness is empty.

As argued above, a collector has height  $\Theta(\frac{\ln n}{n})$ , thus  $H^- \cap D_0$  is included in an annulus of width  $\Theta(\frac{\ln n}{n})$  in  $D_0$ . It follows that  $H^+ \cap D_0$  is included in an annulus of width  $\Theta(\frac{\ln n}{n}) + \frac{2}{n} = \Theta(\frac{\ln n}{n})$ .<sup>4</sup> Finally, observe that the projection  $v_{\Delta_0}$  of  $v$  lies in  $H^+ \cap D_0$  since it is at distance at most  $\frac{1}{n}$  from  $v_{\Delta}$  which lies on the boundary of  $H$ . Thus all the vertices that appear on a silhouette for a direction in the neighborhood of  $\Delta_0$  lie, in projection on  $P_{\Delta_0}$ , in an annulus of

<sup>4</sup>Note that it is not correct that  $H^+ \cap D_0$  necessarily belongs to the collector enlarged by a translation of  $\frac{2}{n}$  because the boundary (in  $D_0$ ) of  $H^+$  and of the collector are not parallel. However, it is included in the enlarged annulus.

width  $\Theta(\frac{\ln n}{n})$ , and thus lie on the unit sphere in a belt with an expected number of points in  $O(\sqrt{n \log n})$ .

We conclude again by computing the conditional expectation. The expected size of the worst-case silhouette for the directions in the neighborhood of  $\Delta_0$  is at most  $n$  times the probability that (at least) a witness is empty which is in  $O(n^{\frac{1}{2}-\alpha})$ , plus the expected number of vertices in a belt which is in  $O(\sqrt{n \log n})$ . The expected size is thus  $\mathbf{E}[\max_{\Delta \sim \Delta_0} \text{silh}(\Delta)] = O(\sqrt{n \log n})$  by choosing  $\alpha \geq 1$ .

**Worst-case silhouette from any view point at infinity.** We cover the set of view points with  $O(n^2)$  disks centered on directions  $\Delta_i$ . For each such disk, the expected maximum of the size of the silhouettes is  $O(\sqrt{n \log n})$ . Using a large deviation technique, we show that there is a low probability that the silhouette from  $\Delta_0$  (or any  $\Delta_i$ ) exceeds its expectation by much.

First, by setting  $\alpha \geq 3$ , we ensure that the probability that there is a  $\Delta_i$  with an empty witness is at most  $O(n^2 n^{1/2-\alpha}) = O(1/\sqrt{n})$  and thus the contribution of this case to the expected maximum is  $O(\sqrt{n})$ .

Second, we now assume that no  $\Delta_i$  has an empty witness. Then, we consider the number of points that fall within the belt associated with the neighborhood of a direction  $\Delta_i$ . It follows a Poisson distribution of parameter  $\beta\sqrt{n \ln n}$ , thus according to Lemma 2, the probability that this number is larger than  $2\beta\sqrt{n \ln n}$  is  $O(e^{-\gamma\sqrt{n \ln n}})$  with  $\gamma > 0$ . This means that the probability that at least one of the belts associated with the  $\Delta_i$  contains more than  $2\beta\sqrt{n \ln n}$  is  $O(n^2 e^{-\gamma\sqrt{n \ln n}})$ , and thus the contribution of that case to the expected maximum is  $O(n^3 e^{-\gamma\sqrt{n \ln n}})$ .<sup>5</sup> The only remaining case is when all belts contain at most  $2\beta\sqrt{n \ln n}$  points, in which case the maximum is  $O(\sqrt{n \ln n})$ .

Summing over all cases, we obtain that the expected maximum size of the silhouette is  $O(\sqrt{n \ln n})$ .

## 4 Refined proof yielding a $\Theta(\sqrt{n})$ bound

We prove in this section Theorem 1 which states that the expected size of the worst-case silhouette of a random polytope viewed from infinity is in  $\Theta(\sqrt{n})$ . We first prove the upper bound. We follow the same proof strategy as in Section 3, except that we consider witnesses and collectors of variable sizes.

**Silhouette from a fixed view point.** For defining the witnesses, we first partition the circle in  $\lfloor \sqrt{n} \rfloor$  ranges  $R_i$  (instead of  $\Theta(\sqrt{\frac{n}{\ln n}})$  as previously), so that the associated caps of  $D_0$  then have height  $\Theta(\frac{1}{n})$ . Since many such caps will be empty with high probability, we do not use them directly as witnesses (indeed we have seen that the method of Section 3 fails in this case). Instead we define, for each range  $R_i$ , the witness  $W_i$  with height  $\frac{d_i}{n}$  where  $d_i \in \{1, \dots, 2n\}$  is the smallest integer such that  $W_i$  is non-empty (see Figure 4(a)). In the rare case where the whole sphere contains no point at all, we define the witnesses  $W_i$  as  $\mathbb{S}^2$  and set  $d_i = 2n$ . Note that the witnesses will overlap with high probability.

In the previous section, we were in some sense defining the size of all witnesses and collectors according to the largest  $d_i$ , thus building a “uniform” annulus. We are now adapting the size of each witness and collector to the local distribution of points. The resulting (topological) annulus

<sup>5</sup>The contribution of that case to the expected maximum is  $O(n^3 e^{-\gamma\sqrt{n \ln n}})$  because it is at most the product of  $O(n)$ , the expected number of points on the sphere, and  $O(n^2 e^{-\gamma\sqrt{n \ln n}})$ , the probability that at least one of the belts associated with the  $\Delta_i$  contains more than  $2\beta\sqrt{n \ln n}$ . However, note that expected total number of points on the sphere knowing that one belt contains at least  $2\beta\sqrt{n \ln n}$  is not  $n$  but it is less than  $n + 2\beta\sqrt{n \ln n}$  and thus is in  $O(n)$ .

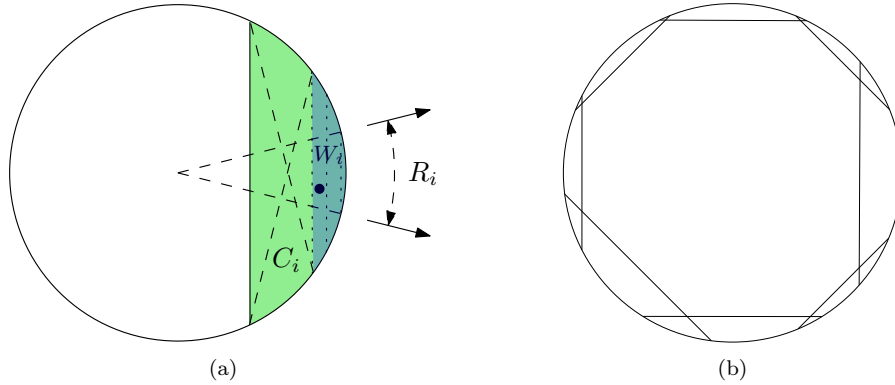


Figure 4: (a) New witness and collector. (b) Non-uniform annulus

is not uniform (see Figure 4(b)), but it is much smaller. Note that the  $d_i$  (and  $W_i$ ) are dependent random variables.

We still define the collector  $C_i$  as the convex hull of the union of the half-planes whose inward normals are in the corresponding range and that do not completely contain  $W_i$ .<sup>6</sup> Remember that the key property of collectors is that any vertex that is extreme in one of the directions of a range has to belong to the associated collector. As noticed in the previous section, the height of a collector is larger than that of its witness by at most a constant multiplicative factor. The height of collector  $C_i$  is then  $\Theta(\frac{d_i}{n})$ , and the area of the spherical cap  $\overline{C}_i$  on the sphere that projects on  $C_i$  is also  $\Theta(\frac{d_i}{n})$ . This implies that the expected number of points in the collector  $C_i$  such that  $d_i = k$  is  $\mathbf{E}[N(\overline{C}_i)|d_i = k] = O(k)$ .

We can furthermore bound the expected number of points in the collector  $C_i$  without any knowledge on  $d_i$ . For  $k \in \{1, \dots, 2n\}$ ,  $\mathbf{P}(d_i > k)$  is the probability that the candidate witness at height  $\frac{k}{n}$  is empty, that is  $O(e^{-\lambda k})$  for some  $\lambda > 0$ . The expected value of  $d_i$  is thus  $\mathbf{E}[d_i] = \sum_{0 < k \leq 2n} k \cdot \mathbf{P}(d_i = k) = \sum_{0 \leq k \leq 2n} \mathbf{P}(d_i > k) = O(\sum_{0 \leq k \leq 2n} e^{-\lambda \sqrt{k}}) = O(1)$ . This yields that  $\mathbf{E}[N(\overline{C}_i)]$  is

$$\sum_k \mathbf{E}[N(\overline{C}_i)|d_i = k] \mathbf{P}(d_i = k) \leq \sum_k O(k) \mathbf{P}(d_i = k) = O(\mathbf{E}[d_i]) = O(1).$$

We can now compute the expected size of the silhouette. First remember that every point on the silhouette is in a collector:  $\text{silh}(\Delta_0) \leq N(\cup_i \overline{C}_i) \leq \sum_i N(\overline{C}_i)$ , thus  $\mathbf{E}[\text{silh}(\Delta_0)] \leq \sum_i \mathbf{E}[N(\overline{C}_i)] = O(\sqrt{n})$ . For a fixed view point direction  $\Delta_0$ , the expected size of the silhouette is thus in  $O(\sqrt{n})$ .

**Silhouettes from a neighborhood of a fixed view point.** As in Section 3, a slight increase of the size of collectors (by a small abuse of notations, we will still denote those enlarged collectors by  $C_i$ ) shows that, for a random polytope, the expected number of points in a collector remains constant ( $\mathbf{E}[N(\overline{C}_i)|d_i = k] = O(k)$  and  $\mathbf{E}[N(\overline{C}_i)] = O(1)$ ) and thus the maximum silhouette over a small cone of view points with radius  $\frac{1}{n}$  around  $\Delta_0$  is still in  $O(\sqrt{n})$ . The details are omitted in this version for lack of space.

**Worst-case silhouette from any view point at infinity.** As in Section 3, we cover the set of view points with  $O(n^2)$  disks centered on directions  $\Delta_i$ . To compute the worst-case

<sup>6</sup>In the special case where the sphere contains no point,  $C_i = W_i = \mathbb{S}^2$  and the collectors contain all the points, i.e. none.

silhouette over all view points, we again use a large deviation technique to show that there is a low probability that the silhouette from  $\Delta_0$  (or any  $\Delta_i$ ) exceeds its expectation by much.

First, we reduce the problem of bounding the expectation of the maximum silhouette size to that of proving a large deviation result. For a positive constant  $\delta$  to be fixed later, we consider  $\delta\sqrt{n}$  as a threshold for the maximum silhouette size:

$$\mathbf{E}[\max_{\Delta} \text{silh}(\Delta)] = \mathbf{E}[\max_{\Delta} \text{silh}(\Delta) \times \mathbf{1}_{\max_{\Delta} \text{silh}(\Delta) \geq \delta\sqrt{n}}] + \mathbf{E}[\max_{\Delta} \text{silh}(\Delta) \times \mathbf{1}_{\max_{\Delta} \text{silh}(\Delta) < \delta\sqrt{n}}]. \quad (1)$$

The second term is bounded by  $\delta\sqrt{n}$ . For the first one, Cauchy-Schwartz inequality yields:<sup>7</sup>

$$\begin{aligned} \mathbf{E}[\max_{\Delta} \text{silh}(\Delta) \times \mathbf{1}_{\max_{\Delta} \text{silh}(\Delta) \geq \delta\sqrt{n}}] &\leq \mathbf{E}[\max_{\Delta} \text{silh}(\Delta)^2]^{1/2} \mathbf{P}(\max_{\Delta} \text{silh}(\Delta) \geq \delta\sqrt{n})^{1/2} \\ &\leq (n + n^2)^{1/2} \mathbf{P}(\max_{\Delta} \text{silh}(\Delta) \geq \delta\sqrt{n})^{1/2} \end{aligned}$$

since  $N(\mathbb{S}^2)$  has a Poisson distribution of parameter  $n$  and  $\mathbf{E}[\max_{\Delta} \text{silh}(\Delta)^2] \leq \mathbf{E}[N(\mathbb{S}^2)^2] = \text{var}(N(\mathbb{S}^2)) + \mathbf{E}[N(\mathbb{S}^2)]^2 = n + n^2$ .

In addition, using the covering of the view points by the  $O(n^2)$  disks and denoting  $\{\Delta \sim \Delta_0\}$  the disk of view points in the neighborhood of  $\Delta_0$ , one has  $\mathbf{P}(\max_{\Delta} \text{silh}(\Delta) \geq \delta\sqrt{n}) \leq O(n^2) \mathbf{P}(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta\sqrt{n})$ . All that remains is to prove that  $\mathbf{P}(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta\sqrt{n})$  is small.

Let  $\mathcal{Q}$  be the event that all  $d_i$  are smaller than  $n^\varepsilon$  and all  $N(\overline{C}_i)$  are smaller than  $2c_0n^\varepsilon$ , for some  $\varepsilon$  positive to be fixed later and a constant  $c_0$  such that  $\mathbf{E}[N(\overline{C}_i) | d_i = k] \leq c_0k$  for all  $k$ . We now distinguish two cases depending on  $\mathcal{Q}$ :

$$\begin{aligned} \mathbf{P}\left(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta\sqrt{n}\right) &= \mathbf{P}\left(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta\sqrt{n} \text{ and } \mathcal{Q}\right) \\ &\quad + \mathbf{P}\left(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta\sqrt{n} \text{ and not } \mathcal{Q}\right). \end{aligned} \quad (2)$$

The second term is bounded by (for any  $i$ )

$$\mathbf{P}(\text{not } \mathcal{Q}) \leq \sqrt{n} (\mathbf{P}(d_i > n^\varepsilon) + \mathbf{P}(N(\overline{C}_i) > 2c_0n^\varepsilon \mid d_i \leq n^\varepsilon)).$$

Remember that  $\mathbf{P}(d_i > n^\varepsilon) \leq e^{-\lambda n^\varepsilon}$ . Since  $\mathbf{E}[N(\overline{C}_i) \mid d_i = k] \leq c_0k$  for all  $k$ ,  $\mathbf{E}[N(\overline{C}_i) \mid d_i \leq k] \leq c_0k$  and Lemma 2 implies<sup>8</sup> that  $\mathbf{P}(N(\overline{C}_i) > 2c_0n^\varepsilon \mid d_i \leq n^\varepsilon) \leq e^{-\Omega(n^\varepsilon)}$ . Thus,  $\mathbf{P}(\text{not } \mathcal{Q})$  is exponentially small, i.e., smaller than  $\sqrt{n}e^{-\Omega(n^\varepsilon)}$ .

For the first term of (the right-hand side of) Eq. (2), we use the fact that the silhouette is included in the collectors:  $\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \leq N(\cup \overline{C}_i) \leq \sum N(\overline{C}_i)$ . In addition, we have shown that  $\mathbf{E}[N(\overline{C}_i)] = O(1)$  and this remains true conditionally to  $\mathcal{Q}$  since this event has a probability close to 1:  $\mathbf{E}[N(\overline{C}_i) | \mathcal{Q}] \leq \mathbf{E}[N(\overline{C}_i)] / \mathbf{P}(\mathcal{Q}) = O(1)$ . Denoting  $N'_i$  the random

<sup>7</sup>Indeed, considering two random variables  $A$  and  $B$  taking discrete values,  $\mathbf{E}[A \times B] = \sum_{i,j} ij \mathbf{P}(A = i \& B = j)$  and, using Cauchy-Schwartz inequality on the terms  $i\sqrt{\mathbf{P}}$  and  $j\sqrt{\mathbf{P}}$ , we get  $\mathbf{E}[A \times B]^2 \leq \sum_{i,j} i^2 \mathbf{P}(A = i \& B = j) \sum_{i,j} j^2 \mathbf{P}(A = i \& B = j)$ . Since  $\sum_j \mathbf{P}(A = i \& B = j) = \mathbf{P}(A = i)$ , the first term is equal to  $\sum_i i^2 \mathbf{P}(A = i)$ , which is equal to  $\sum_i i^2 \mathbf{P}(A^2 = i^2) = \mathbf{E}[A^2]$ . Similarly for the second term, thus  $\mathbf{E}[A \times B]^2 \leq \mathbf{E}[A^2] \mathbf{E}[B^2]$ . Now, if  $B$  is the characteristic function  $\mathbf{1}_C$ ,  $\mathbf{E}[\mathbf{1}_C^2] = \mathbf{E}[\mathbf{1}_C] = \mathbf{P}(C)$ , which proves the claim that  $\mathbf{E}[A \times B]^2 \leq \mathbf{E}[A^2] \mathbf{P}(C)$ .

<sup>8</sup>Intuitively, when we fix the depth  $h$  of the least deep point in the witness, the size of the collector is fixed and the points in the collector follow a Poisson process in the part of the collector at depth larger than  $h$ , so theorems on Poisson processes can be applied.

variable  $N(\overline{C}_i) | \mathcal{Q}$ , we have  $\mathbf{E}[\sum N'_i] = O(\sqrt{n})$  and let  $c_1$  be a positive constant such that  $\mathbf{E}[\sum N'_i] \leq c_1 \sqrt{n}$ . Note also that the variables  $N'_i$  are bounded by  $O(n^\varepsilon)$ .

$$\begin{aligned} \mathbf{P}\left(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta \sqrt{n} \text{ and } \mathcal{Q}\right) &= \mathbf{P}\left(\max_{\Delta \sim \Delta_0} \text{silh}(\Delta_0) \geq \delta \sqrt{n} \mid \mathcal{Q}\right) \mathbf{P}(\mathcal{Q}) \\ &\leq \mathbf{P}\left(\sum N'_i \geq \delta \sqrt{n}\right) \cdot 1 \\ &\leq \mathbf{P}\left(\sum N'_i - \mathbf{E}\left[\sum N'_i\right] \geq c_2 \sqrt{n}\right) \end{aligned}$$

with  $c_2 = \delta - c_1$  that can be chosen positive for  $\delta$  large enough.

The random variables  $N'_i$  are dependent, since the witnesses are not disjoint sets. On the other hand, the fact that the  $d_i$  are bounded implies that we can find a partition of the  $N'_i$  such that in each subset the  $N'_i$  are independent (i.e. the  $\overline{C}_i$  do not overlap) and such that the number of subsets is small. Then the extension of Hoeffding's inequality stated in Lemma 3 applies. Specifically, we will show that the smallest cardinal of such a partition is in  $O(n^{\varepsilon/2})$ . We postpone this proof and now apply Lemma 3 for the  $N'_i$  that are in the range  $1, \dots, O(n^\varepsilon)$ , so that  $\sum_i (\max N'_i - \min N'_i)^2 = O(n^{1/2+2\varepsilon})$  and

$$\mathbf{P}\left(\sum N'_i - \mathbf{E}\left[\sum N'_i\right] \geq c_2 \sqrt{n}\right) \leq \exp\left(\frac{-c_2^2 n}{O(n^{\varepsilon/2})O(n^{1/2+2\varepsilon})}\right) \leq e^{-\Omega(n^{(1-5\varepsilon)/2})}.$$

Hence, for  $\varepsilon < 1/5$ , the first term of Eq. (1) is exponentially small. Summing the two terms of Eq. (1) thus gives that the expected size of the maximum silhouette for all view points is in  $O(\sqrt{n})$ .

It remains to prove that, because  $\max d_i \leq n^\varepsilon$ , the smallest cardinal  $\chi$  of a partition of the  $N'_i$  in subsets of independent variables is in  $O(n^{\frac{\varepsilon}{2}})$ . An upper bound  $\chi_0$  on the number of collectors that can overlap a given collector will also be an upper bound on  $\chi$ . Indeed, the  $\chi_0$  subsets of  $N'_i$  defined for  $j \in \{1, \dots, \chi_0\}$  by  $\{N'_{j+k\chi_0}, j+k\chi_0 \leq \sqrt{n}\}$  are associated to disjoint collectors and hence independent variables  $N'_i$ . To compute  $\chi_0$ , one has to bound the number of ranges  $R_i$  contained in a cap of height  $h = O(n^\varepsilon/n)$ . The length of an arc of height  $h$  is  $2 \arccos(1-h)$  and the length of the arc associated to a range is  $2\pi/\sqrt{n}$ , hence  $\chi_0 \leq \frac{2 \arccos(1-h)}{2\pi/\sqrt{n}} = \frac{\sqrt{n}}{\pi} \arccos(1-h) \leq \frac{1}{\sqrt{2}} \sqrt{nh}$ , since for  $0 < h < 2$ ,  $\arccos(1-h) < \frac{\pi}{\sqrt{2}} \sqrt{h}$ ; with  $h = O(n^{\varepsilon-1})$ ,  $\chi_0$  and thus  $\chi$  is in  $O(n^{\frac{\varepsilon}{2}})$ .

**Lower bound.** Consider a fixed view point direction  $\Delta_0$  and the corresponding witnesses. Each witness has probability  $\Theta(1)$  of having height  $\frac{1}{n}$ ; indeed, the area of the associated spherical cap is  $\Theta(\frac{1}{n})$ , thus the expected number of vertices in it is a constant  $\xi \neq 0$ , and the probability that the spherical cap contains  $k = 0$  vertex is  $\frac{\xi^k}{k!} e^{-\xi} = e^{-\xi} < 1$ . There are  $\Theta(\sqrt{n})$  witnesses, so the expected number of witnesses of height  $\frac{1}{n}$  is  $\Theta(\sqrt{n})$  (since it can be seen as the expectation of a binomial). Every such witness contains at least one point on the convex hull and the witnesses are pairwise disjoint, thus the expected size of the silhouette in direction  $\Delta_0$  is  $\Omega(\sqrt{n})$ . It follows that the expectation of the maximum size over all directions is also  $\Omega(\sqrt{n})$ .

## 5 Conclusion

We list, in conclusion, some open problems. We proved our bound on the worst-case silhouette for polytopes defined as the convex hull of points sampled according to a Poisson distribution on the sphere. A natural extension would be to consider a Poisson distribution in the ball or points

uniformly distributed on the sphere or in the ball. A probably more difficult question is to extend our result to view points that are not necessarily at infinity. One can also consider extensions to higher dimensions. Finally, the question of concentration is also of interest. On this issue, it could be observed from our proof that the probability that the size of the worst-case silhouette is larger than  $\delta\sqrt{n}$  is exponentially small in  $n$  (with  $\delta$  defined as in the proof of Section 4). However, this does not give an estimate of the variance of the size of the worst-case silhouette. As a matter of fact, we do not even know an asymptotic equivalent for the expectation.

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