Equivalent Conditions for Elasto-Acoustics

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Abstract

We present Equivalent Conditions (ECs) for the diffraction problem of elasto-acoustic waves in a solid medium surrounded by a thin region of fluid medium. This problem is well suited for the notion of ECs : since the thickness of the layer is small with respect to the wavelength, the effect of the fluid on the solid is as a first approximation local. These conditions approximate the acoustic waves which propagate in the fluid region. We present numerical results to illustrate the accuracy of ECs.

1 Introduction

Equivalent Conditions (ECs) are usually used in the modeling of wave propagation phenomena to reduce the domain of interest. The main idea consists to replace an "exact" model inside a part of the domain by an approximate condition. This idea is pertinent when the EC can be readily handled for numerical computations.

The coupling of elastic and acoustic waves equations is essential to reproduce geophysical phenomena such as an earthquake on the Earth's surface. We can thus take into account the effects of the ocean on the propagation of seismic waves. In the context of this application, we consider that the medium consists of land areas surrounded by fluid zones whose thickness ε is very small. This raises the difficulty of applying a FEM on a mesh that combines fine cells in the fluid and much larger cells in the solid. To overcome this difficulty we use an asymptotic method to replace the fluid part by an EC. This condition is then coupled with the elastic equation and a FEM can be applied to solve the resulting boundary value problem.

We first introduce the mathematical model. Then, we present ECs up to the second order, stability and convergence results for the elastic displacement. Numerical results illustrate the accuracy of ECs.

The Mathematical Model

We consider an elasto-acoustic waves transmission problem in time-harmonic regime

$$\begin{cases} \Delta \mathsf{p}_\varepsilon + \kappa^2 \mathsf{p}_\varepsilon = 0 & \text{in} \quad \Omega_{\mathsf{f}}^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \partial_{\mathbf{n}} \mathsf{p}_\varepsilon = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathsf{p}_i & \text{on} \quad \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathsf{p}_\varepsilon \mathbf{n} - \mathsf{p}_i \mathbf{n} & \text{on} \quad \Gamma \\ \mathsf{p}_\varepsilon = 0 & \text{on} \quad \Gamma^\varepsilon , \end{cases} \tag{1}$$

set in a smooth bounded simply connected domain $\Omega^\varepsilon \subset \mathbb{R}^2$ made of a smooth connected subdomain Ω_{s} embedded in a subdomain $\Omega_{\mathsf{f}}^\varepsilon$. The domain $\Omega_{\mathsf{f}}^\varepsilon$ is a thin layer of uniform thickness ε. We denote by $\mathbf{n}$ the unit normal to Γ oriented from Ω_{s} to $\Omega_{\mathsf{f}}^\varepsilon$; $\Gamma^\varepsilon := \partial\Omega^\varepsilon$ and $\Gamma := \partial\Omega_{\mathsf{f}}^\varepsilon \cap \partial\Omega_{\mathsf{s}}$. In the system (1), the unknowns are the elastic displacement $\mathbf{u}_\varepsilon$ and the acoustic pressure p_ε. The time-harmonic wave field with angular frequency ω is characterized by using the Helmholtz equation for p_ε, and by using an anisotropic discontinuous linear elasticity system for $\mathbf{u}_\varepsilon$. The physical constants are the acoustic wave number $\kappa = \omega/c$, the speed of the sound c, the density of the solid ρ, and the density of the fluid ρ_{f}.

In the elastic equation, $\nabla\cdot$ is the divergence operator for tensors and $\underline{\underline{\sigma}}(\mathbf{u})$ is the stress tensor given by Hooke's law $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\ \underline{\underline{\varepsilon}}(\mathbf{u})$. Here $\underline{\underline{\varepsilon}}(\mathbf{u}) = (\underline{\underline{\nabla}}\mathbf{u} + \underline{\underline{\nabla}}\mathbf{u}^T)/2$ is the strain tensor, $\underline{\underline{\nabla}}$ denotes the gradient operator for tensors, and $\underline{\underline{C}} = \underline{\underline{C}}(\mathbf{x})$ is the elasticity tensor, where $\mathbf{x} \in \mathbb{R}^3$ are the cartesian coordinates. The components of $\underline{\underline{C}}$ are the elasticity moduli $C_{ijkl} \in \mathbb{R} : \underline{\underline{C}} = (C_{ijkl}(\mathbf{x}))$. The *traction operator* $\mathbf{T}$ is a surfacic differential operator defined on Γ as $\mathbf{T}(\mathbf{u}) = \sigma(\mathbf{u})\mathbf{n}$. The right-hand side p_i represents an incident wave with support on Γ.

In the framework above we address the issue of ECs for $\mathbf{u}_\varepsilon$ as $\varepsilon \to 0$. This issue is linked with the issue of ε-uniform estimates for the displacement $\mathbf{u}_\varepsilon$ and the pressure p_ε solutions of (1) since it is a main ingredient in the justification of ECs. To answer these questions, we work under usual assumptions (symmetry and positiveness) on the tensor $\underline{\underline{C}}$. Some resonant frequencies may appear in the solid domain. However, we prove uniform estimates for the elasto-acoustic field $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ as well as ECs for $\mathbf{u}_\varepsilon$ when $\varepsilon \to 0$ under a spectral assumption :

Assumption 1.1 *The angular frequency ω is not an eigenfrequency of the problem*

$$\left\{\begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \textit{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}) = 0 & \textit{on} \quad \Gamma \ . \end{array}\right.$$

2 Statement of Equivalent Conditions

We derive a hierarchy of ECs for $\mathbf{u}_\varepsilon$ set on Γ and satisfied by $\mathbf{u}_\varepsilon^k$ for all $k \in \{0,1,2\}$, i.e. $\mathbf{u}_\varepsilon^k$ solves

$$\left\{\begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k) + \omega^2 \rho \mathbf{u}_\varepsilon^k = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathsf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n})\,\mathbf{n} = \mathsf{h}_{k,\varepsilon}\mathbf{n} & \text{on} \quad \Gamma \ . \end{array}\right. \tag{2}$$

Here $\mathsf{B}_{k,\varepsilon}$ is a surfacic differential operator acting on functions defined on Γ, and $\mathsf{h}_{k,\varepsilon}$ is a data which depends on the source term p_i and ε. ECs write

$$k = 0 \ : \qquad \mathbf{T}(\mathbf{u}_0) = -\mathsf{p}_i \mathbf{n} \quad \text{on} \quad \Gamma \ , \quad (\mathbf{u}_0 = \mathbf{u}_\varepsilon^0)$$

$$k = 1 \ : \qquad \mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon \omega^2 \rho_{\mathsf{f}} \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = -\mathsf{p}_i \mathbf{n} - \varepsilon \partial_{\mathbf{n}} \mathsf{p}_i \mathbf{n} \ ,$$

$$k = 2 \ : \qquad \mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon \omega^2 \rho_{\mathsf{f}} \left(1 - \frac{\varepsilon}{2}\mathrm{c}(t)\right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = \mathsf{h}_{2,\varepsilon}\mathbf{n} \ .$$

Here, t is an *arc-length coordinate* on the curve Γ, and $\mathrm{c}(t)$ denotes the scalar *curvature* of Γ in $\mathbf{x}(t)$. These conditions show the successive corrections brought when increasing the order. For $k = 0$ the effect of the thin layer is completely neglected. The effect of the fluid part appears at the order 1 with the fluid density ρ_{f}. The influence of the geometry of Γ appears at the order 2 with its scalar curvature.

Stability and Convergence results

The validation of ECs consist to prove estimates for $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$, where $\mathbf{u}_\varepsilon^k$ is the solution of the approximate model (2), and $\mathbf{u}_\varepsilon$ solves the problem (1).

Theorem 2.1 *Under Assumption 1.1, for all $k \in \{0,1,2\}$ there exists constants $\varepsilon_k, C_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$, the problem (2) with data $\mathsf{h}_{k,\varepsilon} \in \mathrm{L}^2(\Gamma)$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathbf{H}^1(\Omega_{\mathsf{s}})$ and*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \leqslant C_k \varepsilon^{k+1} \ . \tag{3}$$

The well-posedness result for the problem (2) is proved in [2]. To estimate the difference $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$, we use a multiscale expansion for $\mathbf{u}_\varepsilon$ in power series of ε and introduce truncates series $\mathbf{u}_{k,\varepsilon}$ up to the order ε^k as intermediate quantities. The error analysis is split into two steps. We first prove uniform estimates for the difference $\mathbf{u}_\varepsilon - \mathbf{u}_{k,\varepsilon}$ [2, Thm 5.2]. Then we prove uniform estimates for the difference $\mathbf{u}_{k,\varepsilon} - \mathbf{u}_\varepsilon^k$ [2, §6.2].

3 Numerical Results.

In the numerical experiments, the computational domain for the solid Ω_{s} is an aluminum disk with a radius $R = 0.01$m embedded in water [1]. The source term is an incident wave defined as $\mathsf{p}_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1,0)$. The angular frequency is $\omega = 1.5 \times 10^6$Hz. In the domain Ω_{s}, we consider the Lamé system : $\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0$, with the coefficients $\mu \simeq 26.32 \times 10^9$ and $\lambda \simeq 51.08 \times 10^9$. The physical constants are $c = 1500$ m.s^{-1}, $\rho_{\mathsf{f}} = 1000$ kg.m^{-3}, and $\rho_{\mathsf{s}} = 2700$ kg.m^{-3}.

We use a Discontinuous Galerkin Method (IPDGM) and curved $\mathbb{P}_3$-finite elements available in the Finite Element Library Hou10ni. We compute the L^2-errors between the analytical solution of the problem (1) and each analytical solution associated with an EC of order $k \in \{0,1,2\}$. We also compute the L^2-errors for each numerical solution associated with an EC, see Fig. 1. We observe that the numerical convergence rate coincides with the theory since the L^2 error is of order ε^k.

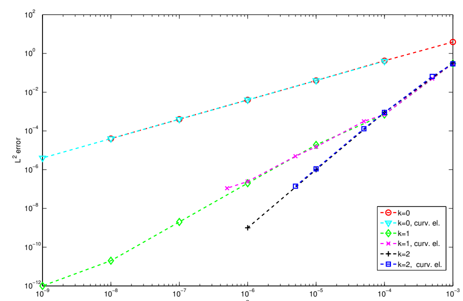

Figure 1: L^2-errors $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_{\mathsf{s}}}$ with respect to ε.

References

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