Bi-invariant means on Lie groups with Cartan-Schouten connections

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The statistical Riemannian framework was pretty well developped for finite-dimensional manifolds [28, 20, 22, 6, 7, 8, 23]. For Lie groups, left or right invariant metric provide a nice setting as the Lie group becomes a geodesically complete Riemannian manifold, thus also metrically complete. However, this Riemannian approach is fully consistent with the group operations only if a bi-invariant metric exists. Unfortunately, bi-invariant Riemannian metrics do not exist on most non compact and non-commutative Lie groups. In particular, such metrics do not exist in any dimension for *rigid-body transformations*, which form the most simple Lie group involved in biomedical image registration.

The log-Euclidean framework, initially developed for symmetric positive definite matrices [4], was proposed as an alternative for affine transformations based on the log of matrices [2] and for (some) diffeomorphisms based on Stationary Velocity Fields (SVFs) [3]. The idea is to rely on one-parameter subgroups, for which efficient algorithms exists to compute the deformation from the initial tangent vector (e.g. scaling and squaring).In [5, 24], we showed that this framework allows to define bi-invariant means on Lie groups provided that the square-root (thus the log) of the transformations was existing.

The goal of this note is to summarize the mathematical roots of these algorithms and to set the bases for comparing their properties with the left and right invariant metrics. The basis of our developments is the structure of affine connection instead of Riemannian metric. The connection defines the parallel transport, and thus a notion of geodesics (auto-parallel curves). Many local properties of Riemannian manifolds remains valid with affine connection spaces. In particular, there is still a local diffeomorphisms between the manifold and the tangent space using the exp and log maps. We explore invariant connections and show that there is a unique bi-invariant torsion-free Cartan-Schouten connection for which the geodesics are left and right translations of one-parameter subgroups. These group geodesics correspond to the ones of a left-invariant metric for the normal elements of the Lie algebra only. When a bi-invariant metric exists (we show that this is not always the case), then all elements are normal and Riemannian and group geodesics coincide. Finally we summarize the properties of the bi-invariant mean defined as the exponential barycenters of the canonical Cartan connection.

1 Cartan-Schouten connections on Lie groups

Let $\mathcal{G}$ be a Lie group, i.e. is a smooth manifold provided with an identity element Id, a smooth composition rule $(g,h) \in \mathcal{G} \times \mathcal{G} \mapsto g \circ h \in \mathcal{G}$ and a smooth inversion rule $f \mapsto f^{(-1)}$ which are both compatible with the manifold structure. We denote by $C^\infty(\mathcal{G})$ the algebra of smooth functions on the group and by $\Gamma(\mathcal{G})$ the algebra of derivations $\partial_X \phi$ of such functions (i.e. smooth vector fields on $\mathcal{G}$). The Lie bracket is $[X,Y](\phi) = \partial_X \partial_Y \phi - \partial_Y \partial_X \phi$.

The canonical automorphisms $L_g : f \mapsto g \circ f$ and $R_g : f \mapsto f \circ g$ are called the left and the right translations. The differential DL_g of the left translations maps the tangent space $T_h\mathcal{G}$ to the tangent space $T_{g.h}\mathcal{G}$ and maps maps any vector $x \in T_{\mathrm{Id}}\mathcal{G}$ to the vector $DL_g.x \in T_g\mathcal{G}$, giving rise to the left-invariant vector field $\tilde{X}|_g = DL_g.x$. The sub-algebra of left-invariant vector fields is called the Lie algebra of the group $\mathcal{G}$. It is identified to with the tangent vector space at identity provided with the additional bracket operation: $\mathfrak{g} = (T_{\mathrm{Id}}\mathcal{G}, +, ., [., .])$.

The adjoint $\mathrm{Ad}(g).x = DL_g|_{g^{(-1)}}.DR_{g^{(-1)}}|_{\mathrm{Id}}.x = DR_{g^{(-1)}}|_g.DL_g|_{\mathrm{Id}}.x$ is the automorphism of the Lie algebra obtained by differentiating the conjugation $C_g(f) = g.f.g^{(-1)}$ with respect to f. It maps each element of the group to a *linear operator* which acts on the Lie algebra: this is called the Adjoint representation. The subgroup $\mathrm{Ad}(\mathcal{G})$ of the general linear group $GL(\mathfrak{g})$ is called the *adjoint group*. The properties of this representation and the existence of bi-invariant metrics for the group $\mathcal{G}$ are linked.

1.1 One parameter subgroups

The flow $\gamma_x(t)$ of a left-invariant vector field $\tilde{X} = DL.x$ starting from Id is a one parameter subgroup, i.e. a group morphism from $(\mathcal{G}, \mathrm{Id}, .)$ to $(\mathbb{R}, 0, +)$:

$$\gamma_x(s+t) = \gamma_x(s).\gamma_x(t) = \gamma_x(t+s) = \gamma_x(t).\gamma_x(s).$$

The group exponential is defined from these one-parameter subgroups with $\mathrm{Exp}(x) = \gamma_x(1)$. It is diffeomorphic locally around 0. More precisely, since the exponential is a smooth mapping and its differential map is invertible at Id, the inverse function theorem guarantees that it is a diffeomorphism from some open neighborhood of 0 to an open neighborhood of $\mathrm{Exp}(0) = \mathrm{Id}$ [25]. This implies that one can define *without ambiguity* a logarithm in an open neighborhood of Id. In the following, we write it $x = \mathrm{Log}(g)$. The absence of an inverse function theorem in infinite dimensional Fréchet manifolds prevents the straightforward extension of this property to general groups of diffeomorphisms [13].

1.2 Affine Connection Spaces

The one parameter subgroup $\gamma_x(t)$ is a curve starting from identity with tangent vector $x \in \mathfrak{g} \simeq T_{\mathrm{Id}}\mathcal{G}$. One could wonder if this curve could be seen as a geodesic. To answer this question, we first need to define geodesics.

When one wants to compare data in the tangent space at one point of the group with data in the tangent space at another one point of the group, one needs to define a mapping between these two tangent spaces because they are not the same spaces: this is the notion of parallel transport. The (affine) connection is the infinitesimal version of this parallel transport for the tangent bundle. This is a bilinear map $(X, Y) \in \Gamma(\mathcal{G}) \times \Gamma(\mathcal{G}) \mapsto \nabla_X Y \in \Gamma(\mathcal{G})$ which is smooth and $C^\infty(\mathcal{G})$-linear in the first variable, and satisfies Leibniz rule $\nabla_X(\phi Y) = \partial_X \phi Y + \phi \nabla_X Y$ in the second variable (i.e. is a derivation).

In a local chart, a vector field $X = \sum_i x^i \partial_i$ has coordinates $x^i \in C^\infty(\mathcal{G})$. Using the above rules, we can write the connection $\nabla_X Y = \partial_X Y + \sum_{ij} x^i y^j \nabla_{\partial_i} \partial_j$, which means that the connection is completely determined by the n^3 functional coordinates (the Christoffel symbols) $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$. Thus, the connection encode how the projection from one tangent space to the neighboring one modifies the standard derivative of a vector field in a chart. On a curve, this allows us to quantifies the acceleration. Let $X|_\gamma = \dot\gamma$ be a vector field along a curve $\gamma(t)$ on $\mathcal{G}$. The covariant derivative is (in Einstein notations): $\frac{D\dot\gamma^k}{dt} = \ddot\gamma^k + \dot\gamma^i \dot\gamma^j \Gamma^k_{ij} \partial_k$.

Geodesics in an affine connection space are defined as the curves that remain parallel to themselves (auto-parallel curves), or equivalently as the curves which have no acceleration. Thus, $\gamma(t)$ is a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$. In a local coordinate system where $\dot\gamma = \sum_i \dot\gamma^i \partial_i$, the equation of the geodesics is thus: $\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0$. We retrieve the standard equation of the geodesics in Riemannian geometry without having to rely on a metric. Since geodesics are locally defined by a second order ordinary differential equation, the geodesic $\gamma_{(p,v)}(t)$ starting at any point p with any tangent vector v is defined for a sufficiently small time. Thus, the exponential map $\mathrm{Exp}_p(v) = \gamma_{(p,v)}(1)$ can be defined for a sufficiently small neighborhood and is a local diffeomorphism. Moreover, the *strong form of the Whitehead theorem* holds: each point of an affine connection space, has a *normal convex neighborhood* (NCN) in the sense that for any couple of points (p, q) in this neighborhood, there exists a unique geodesic $\gamma(t)$ joining them that is entirely contained in this neighborhood. Moreover, the geodesic $\gamma(t)$ depends smoothly on the points p and q.

1.3 Cartan-Shouten connections

Let us look at invariant connections. Left-invariant connections are completely determined by their action on the sub-algebra of left-invariant vector fields. Let $\tilde X = DL.x$ and $\tilde Y = DL.y$ be two left-invariant vector fields: the field $\nabla_{\tilde X}\tilde Y = DL(\nabla_{\tilde X}\tilde Y|_{\mathrm{Id}})$ is determined by its value at identity $\alpha(x, y) = \nabla_{\tilde X}\tilde Y|_{\mathrm{Id}} \in \mathfrak{g}$.

Among the left-invariant connections, we call *Cartan-Schouten connections* the ones for which geodesics going through identity are one parameter subgroups. This definition taken from [25, Def. 6.2 p.71] and generalizes the three classical $+$, $-$ and 0 Cartan-Schouten connections defined in [9].

Theorem 1 *Cartan-Schouten connections are characterized by the property $\alpha(x, x) = 0$ for all $x \in \mathfrak{g}$. Bi-invariant connections are characterized by the condition: $\alpha([z, x], y) + \alpha(x, [z, y]) = [z, \alpha(x, y)] \quad \forall x, y, z \in \mathfrak{g}$. The one dimensional family of connections generated by $\alpha(x, y) = \lambda[x, y]$ satisfy these two conditions.*

Moreover, there there is a unique symmetric Cartan-Schouten bi-invariant connection called the canonical Cartan connection of the Lie group *(also called* *mean or 0-connection) defined by* $\alpha(x,y) = \frac{1}{2}[x,y]$ *for all* $x,y \in \mathfrak{g}$*, i.e.* $\nabla_{\tilde{X}}\tilde{Y} = \frac{1}{2}[\tilde{X},\tilde{Y}]$ *for two left-invariant vector fields.*

It was shown by Laquer [14] that the family $\alpha(x,y) = \lambda[x,y]$ exhausts all the bi-invariant connections on compact simple Lie groups *except* for $SU(n)$ with $n > 3$ in which case there is a two-dimensional family of bi-invariant connections.

All the connections of the family $\alpha(x,y) = \lambda[x,y]$ have the same bi-invariant geodesics because they share the same symmetric part $\nabla_X Y + \nabla_Y X = \partial_X Y + \partial_Y X$. These *group geodesics* are left and right translates of one-parameter subgroups. From a computational point of view, this is particularly interesting since we can write the exponential map at any point using the group exponential:

$$\mathrm{Exp}_g(v) = \gamma_{g,v}(1) = g \circ \exp(DL_{g^{(-1)}} v) = \exp(DR_{g^{(-1)}} v) \circ g$$

Moreover, we have $g.\mathrm{Exp}(x) = \mathrm{Exp}(\mathrm{Ad}(g).x).g$ and for all g in $\mathcal{G}$, there exists an open neighborhood $\mathcal{W}_g$ of $g \in G$ (namely $\mathcal{W}_g = \mathcal{V}_e \cap g.\mathcal{V}_e.g^{(-1)}$ where $\mathcal{V}_e$ is any NCN of e) such that for all $m \in \mathcal{W}_g$ the quantities $\mathrm{Log}(m)$ and $\mathrm{Log}(g.m.g^{(-1)})$ are well-defined and are linked by the relationship $\mathrm{Log}(g.m.g^{(-1)}) = \mathrm{Ad}(g).\mathrm{Log}(m)$. Notice that in general the NCN $\mathcal{W}_g$ depends on g unless we can find a NCN $\mathcal{V}_e$ that is stable by conjugation.

Expressed in the basis of left-invariant vector fields, the torsion is $T(x,y) = 2\alpha(x,y) - [x,y]$ while the curvature is $R(x,y)z = \lambda(\lambda-1)[[x,y],z]$. For $\lambda = 0$ and $\lambda = 1$, the curvature is null (but there is torsion!). These two flat connections are called the left and right (or + and -) Cartan connections. For $\lambda = 1/2$, we get the canonical Cartan connection (also called mean or 0-connection). In fact, among the Cartan-Schouten connections, the $-$ connection is the unique one for which all the left-invariant vector fields are covariantly constant; the $+$ connection is the only one for which all the right-invariant vector fields are covariantly constant; and the 0-connection is the only one which is torsion-free (it has curvature, but its curvature tensor is covariantly constant).

2 Invariant Riemannian metrics

Let $g(X,Y)|_h = \langle\, X|_h \mid Y_h \,\rangle_h$ be a smooth bilinear symmetric form on $T\mathcal{G}$ which is positive definite everywhere (a Riemannian metric). Its Levi-Civita connection is the unique torsion free connection which is compatible with the metric: $\partial_X \langle\, Y \mid Z \,\rangle = \langle\, \nabla_X Y \mid Z \,\rangle + \langle\, Y \mid \nabla_X Z \,\rangle$. The proof of uniqueness is constructive with Khoszul formula, which uniquely defines the the scalar product of the connection with any vector field: $2\langle\, \nabla_X Y \mid Z \,\rangle = \partial_X \langle\, Y \mid Z \,\rangle + \partial_Y \langle\, X \mid Z \,\rangle - \partial_Z \langle\, X \mid Y \,\rangle + \langle\, [X,Y] \mid Z \,\rangle - \langle\, [X,Z] \mid Y \,\rangle - \langle\, [Y,Z] \mid X \,\rangle$. Let $[g^{ij}] = [g_{ij}]^{(-1)}$ be the inverse of the metric matrix. In a local coordinate system, we have $\langle\, \nabla_{\partial_i}\partial_j \mid \partial_k \,\rangle = g_{mk}\Gamma^m_{ij}$, so that Khoszul formula defines the Christoffel symbols of the Levi-Civita connection with the classical formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk} \right).$$

In the case of Lie groups, we can require the metric to be left-invariant (invariant by the left translation), or right-invariant. The left-invariance requires all left translations to be isometric. It is easy to see that such metrics are determined by the inner product at the identity. Let Q be a positive definite symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$. This metric on $T_{\mathrm{Id}}\mathcal{G}$ can be transported on each tangent space $T_g\mathcal{G}$ by left translation: $Q_g(X,Y) = Q(DL_{g^{(-1)}}X, DL_{g^{(-1)}}Y)$. Thus, it defines a left-invariant Riemannian metric on $\mathcal{G}$:

$$\langle\, X \mid Y \,\rangle_g^L = Q_g(X,Y) = Q(DL_{g^{(-1)}}X, DL_{g^{(-1)}}Y)$$

Let ad^* be the metric adjoint operator defined as the unique bilinear operator satisfying for all vector fields $X, Y, Z \in \Gamma\mathcal{G}$: $\langle\, ad^*(Y,X) \mid Z \,\rangle = \langle\, [X,Z] \mid Y \,\rangle$, and let us denote by $ad^*_{\mathfrak{g}}(x,y)$ its restriction to the Lie algebra. Rewriting Khoszul formula in the sub-algebra of left-invariant vector fields shows that the Levi-Civita connection of a left-invariant metric is characterised by:

$$\alpha(x,y) = \frac{1}{2}\left([x,y] - ad^*_{\mathfrak{g}}(x,y) - ad^*_{\mathfrak{g}}(y,x)\right)$$

Since the symmetric Cartan connection is determined by $\alpha(x,y) = \frac{1}{2}[x,y]$, we see that the geodesic $\gamma^L_{(Id,x)}(t)$ of a left-invariant Riemannian metric is a one parameter subgroups if and only if $ad^*_{\mathfrak{g}}(x,x) = 0$. Thus, the symmetric part of $ad^*_{\mathfrak{g}}$ encodes the deviation of the geodesics from one parameter subgroups. Elements $x \in \mathfrak{g}$ of the Lie algebra for which this holds are said normal.

2.1 Bi-invariant metrics

The right-invariance case is similar to the left-invariance one. In fact, *all* right-invariant metrics can be obtained from left-invariant metrics by inversion since for any two elements g, h of $\mathcal{G}$, we have $g.h = (h^{(-1)}.g^{(-1)})^{(-1)}$.

A left-invariant Riemannian metric on a Lie group is bi-invariant if and only if for all $g \in \mathcal{G}$, the adjoint operator $\mathrm{Ad}(g)$ is an isometry of the Lie algebra $\mathfrak{g}$: $\langle\, \mathrm{Ad}(g)y \mid \mathrm{Ad}(g)z \,\rangle = \langle\, y \mid z \,\rangle$, or equivalently if and only if for all elements $x, y, z \in \mathfrak{g}$:

$$\langle\, [x,y] \mid z \,\rangle + \langle\, y \mid [x,z] \,\rangle = 0 \quad \text{or} \quad ad^*_{\mathfrak{g}}(x,y) + ad^*_{\mathfrak{g}}(y,x) = 0$$

A bi-invariant metric is invariant w.r.t. inversion and has the group geodesics of $\mathcal{G}$ for geodesics. An interesting consequence is that any Lie group with a bi-invariant metric has non-negative sectional curvature $K(x,y) = \frac{1}{4}\|[x,y]\|^2$ for any two orthonormal vectors x, y of the Lie algebra. Bi-invariant Riemannian and pseudo-Riemannian metrics on Lie groups were studied in [18, 19].

If a bi-invariant metric exists for a Lie group, then $\mathrm{Ad}(g)$ is an isometry of $\mathfrak{g}$ and can thus be looked upon as an element of the orthogonal group $O(n)$ where $n = dim(\mathcal{G})$. As $O(n)$ is a *compact* group, the adjoint group $\mathrm{Ad}(\mathcal{G}) = \{\mathrm{Ad}(g)/g \in \mathcal{G}\}$ is necessarily *included in a compact set*, a situation called *relative compactness*. This notion actually provides an excellent criterion, since the theory of differential forms and their integration can be used to explicitly construct a bi-invariant metric on relatively compact subgroups [26, Theorem V.5.3.].

Theorem 2 *The Lie group $\mathcal{G}$ admits a bi-invariant metric if and only if its adjoint group $Ad(\mathcal{G})$ is relatively compact.*

For *compact* Lie groups, the adjoint group is the image of a compact set by a continuous mapping and is thus also compact. Thus, bi-invariant metrics exist in such a case. This is the case of *rotations*, for which bi-invariant Fréchet means have been extensively studied and used in practical applications. In the case of commutative Lie groups, left and right translations are identical and any left-invariant metric is trivially bi-invariant. Direct products of compact Abelian groups obviously admit bi-invariant metrics but Theorem 2 shows that in the general case, non-compact and non-commutative Lie groups which are not the direct product of such groups may fail to admit a bi-invariant metric. This is indeed the case for the semi-direct product of Euclidean motions $SE(n)$.

3 Bi-invariant means as exponential barycenters

In a manifold with Riemannian metric $\|.\|_m$ at point m, the Fréchet mean of a set of points $\{x_i\}$ are the absolute minima (the Karcher means being the local minima) of the variance $\sigma^2(m) = \frac{1}{n}\sum_i \operatorname{dist}(m, x_i)^2 = \frac{1}{n}\sum_i \| \log_m(x_i) \|_m^2$, where $\log_m$ is the Riemannian logarithmic map at the point m. When a mean points is not located on the cut locus of one of the data points, it is characterized by a null gradient of the variance, i.e $\sum_i \log_m(x_i) = 0$. The existence of Karcher means is ensured when the variance is finite at one point. The uniqueness of the Fréchet/Karcher mean was investigated in [11, 12, 16, 1, 27].

An efficient algorithm to compute a Karcher mean is the Gauss-Newton iteration

$$m_{t+1} = \exp_{m_t}\left(\tfrac{1}{n}\sum_i \log_{m_t}(x_i)\right). \tag{1}$$

This algorithm has been regularly used in the literature with varying justifications but always excellent numerical efficiency (see e.g. [22, 23] for homogeneous manifolds including $SO(3)$ and $SE(3)$, [15] for shape spaces). The study of the convergence of this specific algorithm was performed in [15, 16] in the context of the Fréchet mean, while [10] investigated more generally the convergence of algorithms of the type $m_{t+1} = \exp_{m_t}(Y(m_t))$ to the zeroes of the vector field Y on a Riemannian manifold. Very few works deal with Newton iterations on Lie groups or affine connection spaces. Notable exceptions are [21, 17] which propose Newton algorithms to optimize general functions on non compact Lie groups based on Cartan-Schouten connections.

In the particular case of Lie groups provided with a bi-invariant metric (but only in this case), the metric geodesics correspond to group geodesics and the group logarithm and Riemannian logarithm are the same. The Karcher means are automatically bi-invariant and characterized by the simpler barycentric equation:

$$\sum_i \operatorname{Log}(m^{(-1)}.x_i) = 0.$$

This equation is left-, right- and inverse-invariant, since it derives from a bi-invariant metric. The corresponding Gauss-Newton iteration can be be written as follows.

Algorithm 1 (Barycentric Fixed Point Iteration on Lie Groups.) *Initialize m_0, for example with $m_0 := x_1$. Iteratively update the estimate of the mean by: $m_{t+1} := m_t.Exp\left(\frac{1}{N}\sum_{i=1}^{N} Log({m_t}^{(-1)}.x_i)\right)$, until convergence ($\| Log(m_t^{(-1)}.m_{t+1})\|_e < \epsilon.\sigma(m_t)$).*

When a bi-invariant Riemannian metric fail to exist on a Lie group, we cannot use the notion of Fréchet or Karcher mean. However, the above barycentric iteration continues to make sense. The key idea developed in [24] is to consider Eq.(1) as an exponential barycenter of the canonical Cartan connection. This definition has all the desirable invariance properties, even when bi-invariant metrics do not exist. Moreover, we can show the existence and uniqueness of the bi-invariant mean provided the dispersion of the data is small.

Definition 1 (Bi-invariant Means) *Let $\{x_i\}$ be a finite set of data points belonging to the an open set $\mathcal{V}$ such that $Log(g^{(-1)}.x_i)$ and $Log(x_i.g^{(-1)}) = Ad(g).\, Log(g^{(-1)}.x_i)$ exists for any point $g \in \mathcal{V}$. The points $m \in \mathcal{V}_g$ which are solutions of the (group) barycentric equation $\sum_i Log(m^{(-1)}.x_i) = 0$ (if there are some) are called bi-invariant means.*

This definition is close to the Riemannian center of mass (or more specifically the Riemannian average) of [10] but uses the group logarithm instead of the Riemannian logarithm. Notice that the group geodesics generally cannot be seen as Riemannian geodesics as the canonical Cartan connection is non metric so that our definition cannot be equivalent (in general) to the Fréchet or Karcher mean of some Riemannian metric.

Theorem 3 *The bi-invariant means are left-, right- and inverse-invariant: if m is a mean of $\{x_i\}$ and $h \in G$ is any group element, then $h.m$ is a mean of $\{h.x_i\}$, $m.h$ is a mean of the points $\{x_i.h\}$ and $m^{(-1)}$ is a mean of $\{x_i^{(-1)}\}$. Moreover, if the data points belong to a sufficiently small normal convex neighborhood $\mathcal{V}$ of some point $g \in \mathcal{G}$, then there exists a unique solution of the barycentric equation in $\mathcal{V}$ and the barycentric fixed point iteration on Lie groups converges at least at a linear rate towards this unique solution, provided the initialization is close enough to g.*

References

[1] B. Afsari. Riemannian l^p center of mass: existence, uniqueness, and convexity. *Proc. of the AMS*, 180(2):655–673, Feb. 2010.

[2] V. Arsigny, O. Commowick, N. Ayache, and X. Pennec. A fast and log-Euclidean polyaffine framework for locally linear registration. *Journal of Mathematical Imaging and Vision*, 33(2):222–238, 2009.

[3] V. Arsigny, O. Commowick, X. Pennec, and N. Ayache. A log-Euclidean framework for statistics on diffeomorphisms. In *Proc. MICCAI'06*, *LNCS 4190*, p.924–931, Oct. 2006.

[4] V. Arsigny, P. Fillard, X. Pennec, and N. Ayache. Fast and simple calculus on tensors in the log-Euclidean framework. In *Proc. of MICCAI 2005*, *LNCS 3749*, p.115–122, 2005.

[5] V. Arsigny, X. Pennec, and N. Ayache. Bi-invariant means in Lie groups. application to left-invariant polyaffine transformations. Research report RR-5885, INRIA 2006.

[6] R. Bhattacharya and V. Patrangenaru. Nonparametric estimation of location and dispersion on Riemannian manifolds. *J. of Stat. Planning and Inference*, 108:23–36, 2002.

[7] R. Bhattacharya and V. Patrangenaru. Large sample theory of intrinsic and extrinsic sample means on manifolds, I. *Annals of Statistics*, 31(1):1–29, 2003.

[8] R. Bhattacharya and V. Patrangenaru. Large sample theory of intrinsic and extrinsic sample means on manifolds, II. *Annals of Statistics*, 33(3):1225–1259, 2005.

[9] E. Cartan and J. Schouten. On the geometry of the group-manifold of simple and semi-simple groups. *Proc. Akad. Wekensch, Amsterdam*, 29:803–815, 1926.

[10] D. Groisser. Newton's method, zeroes of vector fields, and the Riemannian center of mass. *Adv. in Applied Math*, 33:95–135, 2004.

[11] H. Karcher. Riemannian center of mass and mollifier smoothing. *Communications in Pure and Applied Mathematics*, 30:509–541, 1977.

[12] W. Kendall. Probability, convexity, and harmonic maps with small image I: uniqueness and fine existence. *Proc. London Math. Soc.*, 61(2):371–406, 1990.

[13] B. A. Khesin and R. Wendt. *The Geometry of Infinite Dimensional Lie groups*, volume 51 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer Verlag, 2009.

[14] H. T. Laquer. Invariant affine connections on Lie groups. *Transactions of the American Mathematical Society*, 331(2):pp. 541–551, 1992.

[15] H. Le. Locating Fréchet means with application to shape spaces. *Advances in Applied Probabilities*, 33:324–338, 2001.

[16] H. Le. Estimation of Riemannian barycenters. *LMS J. Comput. Math.*, 7:193–200, 2004.

[17] R. Mahony and R. Manton. The geometry of the Newton method on non-compact Lie groups. *Journal of Global Optimization*, 23:309–327, 2002.

[18] A. Medina. Groupes de lie munis de mtriques bi-invariantes. *Tôhoku Math. Journal*, (37):405–421, 1984.

[19] A. Medina and P. Revoy. Algèbres de lie et produit scalaire invariant. *Annales scientifiques de l'ENS, 4e série*, 18(3):553–561, 1985.

[20] J. Oller and J. Corcuera. Intrinsic analysis of statistical estimation. *Annals of Statistics*, 23(5):1562–1581, 1995.

[21] B. Owren and B. Welfert. The Newton iteration on Lie groups. *BIT Numerical Mathematics*, 40(1):121–145, 2000.

[22] X. Pennec. *L'incertitude dans les problèmes de reconnaissance et de recalage – Applications en imagerie médicale et biologie moléculaire.* Thèse de sciences (phd thesis), Ecole Polytechnique, Palaiseau (France), December 1996.

[23] X. Pennec. Intrinsic statistics on Riemannian manifolds: Basic tools for geometric measurements. *J. of Math. Imaging and Vision*, 25(1):127–154, July 2006. Preprint as INRIA RR-5093, January 2004.

[24] X. Pennec and V. Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In F. Barbaresco, A. Mishra, and F. Nielsen, editors, *Matrix Information Geometry*, pages 123–166. Springer, May 2012.

[25] M. M. Postnikov. *Geometry VI: Riemannian Geometry.* Encyclopedia of mathematical science. Springer, 2001.

[26] S. Sternberg. *Lectures on Differential Geometry.* Prentice Hall Mathematics Series. Prentice Hall Inc., 1964.

[27] L. Yang. *Medians of probability measures in Riemannian manifolds and applications to radar target detection.* PhD thesis, Poitier University, Dec. 2011.

[28] H. Ziezold. On expected figures in the plane. In A. Hübler, W. Nagel, B. Ripley, and G. Werner, editors, *Geobild '89*, *Math. Res.* (51), pages 105–110. Akademie-Verlag, Berlin, 1989.