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# Universal Point Sets for Planar Graph Drawings with Circular Arcs

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## Abstract

We prove that there exists a set  $S$  of  $n$  points in the plane such that every  $n$ -vertex planar graph  $G$  admits a plane drawing in which every vertex of  $G$  is placed on a distinct point of  $S$  and every edge of  $G$  is drawn as a circular arc.

## 1 Introduction

It is a classic result of graph theory that every planar graph has a plane *straight-line drawing*, that is, a drawing where vertices are mapped to points in the plane and edges to straight-line segments connecting the corresponding points (achieved independently by Wagner, Fáry, and Stein). Tutte [21] presented the first algorithm, the *barycentric method*, that produces such drawings. Unfortunately, the barycentric method can produce edges whose lengths are exponentially far from each other. Therefore, Rosenstiehl and Tarjan [19] asked whether every planar graph has a plane straight-line drawing where vertices lie on an integer grid of polynomial size. De Fraysseix, Pach, and Pollack [5] and, independently, Schnyder [20] answered this question in the affirmative. Their (very different) methods yield drawings of  $n$ -vertex planar graphs on a grid of size  $\Theta(n) \times \Theta(n)$ , and there are graphs (the so-called “nested triangles”) that require this grid size [10]. Later, it was apparently Mohar (according to Pach [6]) who generalized the grid question to the following problem: What is the smallest size  $f(n)$  of a *universal point set* for plane straight-line drawings of  $n$ -vertex planar graphs, that is, the smallest size (as a function of  $n$ ) of a point set  $S$  such that every  $n$ -vertex planar graph  $G$  admits a plane straight-line drawing in which the vertices of  $G$  are mapped to points in  $S$ ? The question is listed as problem #45 in the Open Problems Project [6]. Despite more than twenty years of research efforts, the best known lower bound for the value of  $f(n)$  is linear in  $n$  [4, 17, 18], while the best known upper bound is only quadratic in  $n$ , as established by de Fraysseix et al. [5] and Schnyder [20]. Universal point sets for plane straight-line drawings of planar graphs require more than  $n$  points whenever  $n \geq 15$  [3]. Recently, universal point sets with  $o(n^2)$  points have been proved to exist for straight-line planar drawings of several subclasses of planar graphs generalizing outerplanar graphs. Namely, an upper bound of  $O(n(\log n / \log \log n)^2)$  has been proven for *simply-nested planar graphs* [1] and an upper bound of  $O(n^{5/3})$  for *planar 3-trees* [14], which extends to *planar 2-trees* and hence to *series-parallel graphs*.

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Universal point sets have also been studied with respect to different drawing standards. For example, Everett et al. [13] showed that there exist sets of  $n$  points that are universal for *plane poly-line drawings with one bend per edge* of  $n$ -vertex planar graphs. On the other hand, if bend-points are required to be placed on the point-set, universal point-sets exist of size  $O(n^2/\log n)$  for drawings with one bend per edge, of size  $O(n \log n)$  for drawings with two bends per edge, and of size  $O(n)$  for drawings with three bends per edge [11].

However, smooth curves may be easier for the eye to follow and more aesthetic than poly-lines. Graph Drawing researchers have long observed that poly-lines may be made smooth by replacing each bend with a smooth curve tangent to the two adjacent line segments [7, 15]. Bekos et al. [2] formalized this observation by considering smooth curves made of line segments and circular arcs; they define the *curve complexity* of such a curve to be the number of segments and arcs it contains. A poly-line drawing with  $s$  segments per edge may be transformed into a smooth drawing with curve complexity at most  $2s - 1$ , but Bekos et al. [2] observed that in many cases the curve complexity can be made smaller than this bound. For instance, replacing poly-lines by curves in the construction of Everett et al. [13] would give rise to a drawing of curve complexity 3, but in fact every set of  $n$  collinear points is universal for smooth piecewise-circular drawings with curve complexity 2, as can be derived from the existence of topological book embeddings of planar graphs [8, 16, 2]. A *monotone topological book embedding* of a graph  $G$  is a drawing of  $G$  such that the vertices lie on a horizontal line, called *spine*, and the edges are represented by non-crossing curves, monotonically increasing in the direction of the spine. In [8, 16], it was shown that every planar graph has a monotone topological book embedding where each edge crosses the spine exactly once and consists of two semi-circles, one below and one above the spine (see Figure 2).

The difficulty of the problem of constructing a universal point set of a linear size for straight-line drawings, the aesthetical properties of smooth curves, the recent developments on drawing planar graphs with circular arcs (see, for example, [2, 12]), and the existence of universal sets of  $n$  points for drawings of planar graphs with curve complexity 2 [13] naturally give rise to the question of whether there exists a universal set of  $n$  points for drawings of planar graphs with curve complexity 1, that is, for drawings in which every edge is drawn as a single circular arc. In this paper, we answer this question in the affirmative.

We prove the existence of set  $S$  of  $n$  points on the parabolic arc  $\mathcal{P} = \{(x, y) : x \geq 0, y = -x^2\}$  such that every  $n$ -vertex planar graph  $G$  can be drawn with the vertices mapped to  $S$  and the edges mapped to non-crossing circular arcs. In the same spirit as Everett et al. [13], we draw  $G$  in two steps. In the first step, we construct a monotone topological book embedding of  $G$ . In the second step, we map the vertices of  $G$  to the points in  $S$  in the same order as they appear on the spine of the book embedding.

## 2 Circular Arcs Between Points on a Parabola

In this section, we investigate geometric properties of circular-arc drawings whose vertices lie on the parabolic arc  $\mathcal{P}$ .

In the following, when we say that a point is *to the left* of another point, we mean that the  $x$ -coordinate of the former is smaller than that of the latter. However, when we say that an arc is to the left of a point  $q$ , we mean that all the intersection points of the arc with the horizontal line through  $q$  are to the left of  $q$ . We define similarly *to the right*, *above*, and *below*, and we naturally extend these definition to non-crossing pairs of arcs. We denote by  $\mathcal{C}(p, q, r)$  the circle through three points  $p$ ,  $q$ , and  $r$ .

We start by stating a classic property of parabolas and circles.

**Lemma 1.** *For every three points  $p$ ,  $q$ , and  $r$  on  $\mathcal{P}$  with increasing  $x$ -coordinates, the circular arc from  $p$  to  $r$  and through  $q$  is below  $\mathcal{P}$  between  $p$  and  $q$  and above  $\mathcal{P}$  between  $q$  and  $r$  (see Figure 1).*

*Proof.* We first observe that a circle intersects  $\mathcal{P}$  at at most three points with positive  $x$ -coordinates (counted with multiplicity). Indeed, substituting  $y$  by  $-x^2$  in the circle equation yields a degree-4 equation in  $x$  with no monomial of degree 3. There are thus at most three changes of sign in the sequence of coefficients, and Descartes' rule of signs implies that there are at most three positive roots, counted with multiplicity.

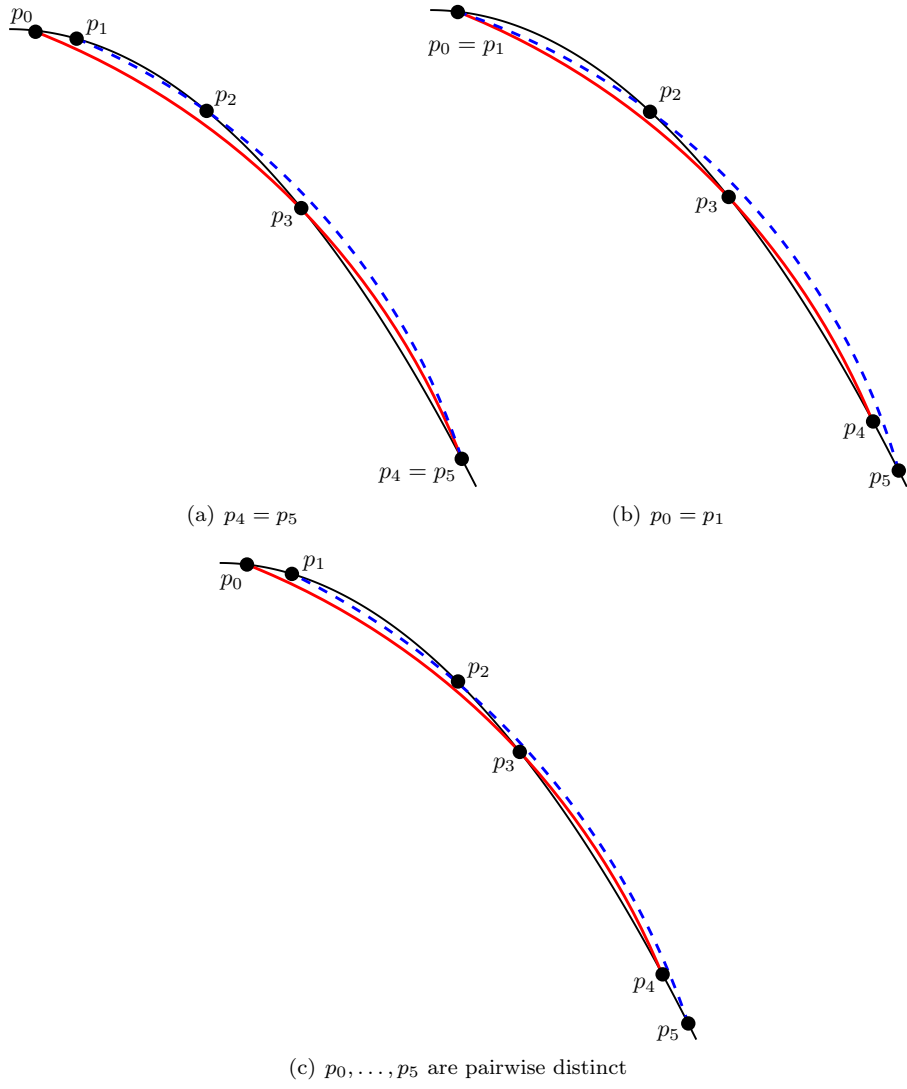


Figure 1: Three configurations of relative position of the circular arcs  $C_{0,3,4}$  (red) and  $C_{1,2,5}$  (blue dashed) defined by six points  $p_0, \dots, p_5$  lying in that order on  $\mathcal{P}$ . For readability, the figure is not to scale.

We now consider three points  $p, q,$  and  $r$  on  $\mathcal{P}$  and consider circle  $\mathcal{C}(p, q, r)$ . Since there is no other point of intersection with positive  $x$ -coordinate, and since the circle is bounded and the parabolic arc is not, the circular arc to the right of  $r$  is below the parabolic arc. The result follows since  $\mathcal{C}(p, q, r)$  crosses  $\mathcal{P}$  at  $p, q,$  and  $r$  (since, otherwise, the number of intersection points with positive  $x$ -coordinates and counted with multiplicity would be larger than three).  $\square$

Given six points  $p_0 = (x_0, y_0), \dots, p_5 = (x_5, y_5)$  in this order on  $\mathcal{P}$  (that is,  $x_0 \leq x_1 \leq \dots \leq x_5$ ), we consider two circular arcs (see Figure 1);  $C_{0,3,4}$  (red) goes through the ordered points  $p_0, p_3, p_4$  and  $C_{1,2,5}$  (blue) goes through  $p_1, p_2, p_5$ . We assume that the three points defining each arc are pairwise distinct. It should be stressed that these arcs may not be  $x$ -monotone.<sup>1</sup> The two circular arcs are, however,  $y$ -monotone—for  $C_{0,3,4}$  we argue as follows; the argument for  $C_{1,2,5}$  is similar: By Lemma 1,  $p_0$  lies on the right half-circle of  $\mathcal{C}(p_0, p_3, p_4)$ , and  $p_3$  and  $p_4$  are to the right of  $p_0$ .

We will prove, in Lemma 4, that the arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  do not intersect each other if the  $x$ -coordinate of  $p_i$  is at least twice that of  $p_{i-1}$  for  $i = 3, 4$ . For that purpose, we first consider, in the two next lemmas, the special cases where these arcs share one of their endpoints.

**Lemma 2.** *If  $p_4 = p_5$  and  $x_3 \geq x_1 + x_2$ , the two circular arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  intersect only at  $p_4 = p_5$ .*

*Proof.* Refer to Figure 1(a). We first observe that, by Lemma 1, the circular arc  $C_{0,3,4}$  is below  $\mathcal{P}$  in a neighborhood of  $p_0$ , it crosses  $\mathcal{P}$  at  $p_3$ , and it lies above  $\mathcal{P}$  in a neighborhood of  $p_4$ . Similarly  $C_{1,2,5}$  is below  $\mathcal{P}$  in a neighborhood of  $p_1$ , it crosses  $\mathcal{P}$  at  $p_2$ , and it lies above  $\mathcal{P}$  in a neighborhood of  $p_5$ .

We now argue that the two arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  intersect at a point other than  $p_4 = p_5$  if and only if the (red) arc  $C_{0,3,4}$  is to the right of the (blue) arc  $C_{1,2,5}$  in a neighborhood of  $p_4$ . Since the (red) arc  $C_{0,3,4}$  is below  $\mathcal{P}$  in a neighborhood of  $p_0$ , and  $C_{0,3,4}$  does not intersect  $\mathcal{P}$  between  $p_0$  and  $p_1$  (by Lemma 1), the (red) arc  $C_{0,3,4}$  is to the left of  $p_1$ . On the other hand, the two circular arcs intersect at most once other than at  $p_4$  (since circles intersect at most twice). Hence, if they intersect at a point  $q$  other than  $p_4$ , their horizontal ordering changes in a neighborhood of  $q$  and thus the (red) arc  $C_{0,3,4}$  is to the right of the (blue) arc  $C_{1,2,5}$  in a neighborhood of  $p_4$ .

As a consequence, we can assume without loss of generality that  $p_0$  is at the origin  $O = (0, 0)$  (that is, the topmost point of  $\mathcal{P}$ ). This can be seen as follows. First, by Lemma 1, the origin is inside  $\mathcal{C}(p_0, p_3, p_4)$ . Furthermore, since the origin is above  $p_3$  and  $p_4$ , the arc  $p_3p_4$  of  $\mathcal{C}(O, p_3, p_4)$  lies to the right of the arc  $p_3p_4$  of  $\mathcal{C}(p_0, p_3, p_4)$ . It follows that if  $C_{0,3,4}$  is to the right of  $C_{1,2,5}$  in a neighborhood of  $p_4$ , it remains to the right if  $p_0$  is placed at the origin. Hence, in the sequel, we can assume that  $x_0 = 0$ .

We now prove that if  $x_3 \geq x_1 + x_2$ , then the tangents at  $p_4 = p_5$  of the two circular arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  are distinct for any position of  $p_4 = p_5$  to the right of  $p_3$  on  $\mathcal{P}$ .

The following calculations are done in Maple. We consider the equation of  $\mathcal{C}(p_0, p_3, p_4)$ , which is the determinant

$$\begin{bmatrix} x_0 & -x_0^2 & x_0^2 + x_0^4 & 1 \\ x_3 & -x_3^2 & x_3^2 + x_3^4 & 1 \\ x_4 & -x_4^2 & x_4^2 + x_4^4 & 1 \\ x & y & x^2 + y^2 & 1 \end{bmatrix}$$

and similarly for  $\mathcal{C}(p_1, p_2, p_4 = p_5)$ . The normals to these circles at  $p_4$  are the gradient of their implicit equations evaluated at  $p_4$ . We then compute the cross product of these two vectors; more precisely, the last coordinate of the cross product, that is,  $M_x N_y - N_x M_y$ , where  $(M_x, M_y)$  and  $(N_x, N_y)$  are the normal vectors.

This expression can be factorized such that it is the product of two terms. The first is the term  $x_3 x_4 (x_3 - x_4)(x_2 - x_4)(x_1 - x_4)(x_1 - x_2)$ , which does not vanish if  $p_0, \dots, p_4$  are pairwise distinct. The second is the

<sup>1</sup> This could be seen by considering, for instance, the limit case of a circle where  $p_0$  and  $p_3$  lie at the origin and the  $x$ -coordinate of  $p_4$  is larger than one. This circle is centered at  $(0, -a)$  with  $a > 1$ . Since  $-a > -a^2$ , the rightmost point  $(a, -a)$  of the circle is above the parabola  $y = -x^2$ , thus it lies on  $C_{0,3,4}$  by Lemma 1.

following term, which we view as a polynomial in  $x_4$  whose coefficients depend on  $x_1$ ,  $x_2$ , and  $x_3$ :

$$\begin{aligned} & (x_3 - x_1 - x_2) x_4^4 \\ & + (x_1 + x_2 + x_3) (x_3 - x_1 - x_2) x_4^3 \\ & + (1 + x_1 x_2) (x_3 - x_1 - x_2) x_4^2 \\ & + (x_1 x_2 x_3^2 + x_1 x_2^2 x_3 + x_1^2 x_2 x_3 + x_3^2 - x_1^2 - x_2^2) x_4 \\ & + x_1 x_2 (1 + x_3^2) (x_1 + x_2). \end{aligned}$$

All coefficients are non-negative since  $x_3 \geq x_1 + x_2$ . Thus, the polynomial has no positive real root. In other words, the two normals are never collinear. Now, considering the limit case where  $p_4 = p_3$ , the (red) circle  $\mathcal{C}(p_0, p_3, p_4)$  is tangent to  $\mathcal{P}$  and since, by Lemma 1, the (blue) arc  $C_{1,2,5}$  is above and thus to the right of  $\mathcal{P}$  in a neighborhood of  $p_4 = p_5$  (and is not tangent to  $\mathcal{P}$  if  $p_2 \neq p_5$ ), the (blue) arc  $C_{1,2,5}$  is to the right of the (red) arc  $C_{0,3,4}$  in a neighborhood of  $p_4$ . Hence, the two arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  do not intersect except at  $p_4$ .  $\square$

**Lemma 3.** *If  $p_0 = p_1$ ,  $x_0 \geq 1$ ,  $x_3 \geq 2x_2$  and  $x_4 \geq x_0 + x_3$ , the two circular arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  intersect only at  $p_0 = p_1$ .*

*Proof.* Similarly as in the proof of Lemma 2, the two arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  intersect at a point other than  $p_0 = p_1$  if and only if the (red) arc  $C_{0,3,4}$  is to the right of the (blue) arc  $C_{1,2,5}$  in a neighborhood of  $p_0$  (see Figure 1(b)).

Furthermore, we can assume without loss of generality that  $p_5$  is at infinity, which means that  $C_{1,2,5}$  is the (straight) ray from  $p_0 = p_1$  through  $p_2$ . Indeed, for any point  $p'_5$  that lies on  $\mathcal{P}$  to the right of  $p_5$ , point  $p'_5$  lies outside the  $\mathcal{C}(p_1, p_2, p_5)$  by Lemma 1. Furthermore, since  $p'_5$  lies below  $p_1$  and  $p_2$ , the arc through  $p_1$ ,  $p_2$ , and  $p'_5$  (in order) lies to the left of  $C_{1,2,5}$  between  $p_1$  and  $p_2$ . Hence, if the (blue) arc  $C_{1,2,5}$  is to the left of the (red) arc  $C_{0,3,4}$  in a neighborhood of  $p_0$ , it remains to the left if  $p_5$  is at infinity.

Now, similarly to the proof of Lemma 2, we prove that the tangents at  $p_0 = p_1$  of  $C_{0,3,4}$  and  $C_{1,2,5}$  never coincide. With the above assumption, this is equivalent to showing that the normal to  $C_{0,3,4}$  at  $p_0$  is never orthogonal to the segment  $p_1 p_2$ . The corresponding dot product (computed in Maple) is equal to

$$\begin{aligned} & (x_4 - x_3) (x_4 - x_0) (x_3 - x_0) (x_2 - x_0) \\ & \left( (x_3 - x_2) x_4^2 + (x_3 - x_2) (x_0 + x_3) x_4 + \right. \\ & \left. ((x_0^2 - 1 - x_3 x_0 - x_3^2) x_2 + x_0^3 + x_0) \right). \end{aligned}$$

The first four terms never vanish and we want to show that the last term, seen as a polynomial in  $x_4$ , has no root  $x_4$  larger than  $x_0 + x_3$  (it can be shown that this polynomial has a positive root). For that purpose, we make the change of variable  $x_4 = t + x_0 + x_3$  which maps the interval  $(x_0 + x_3, +\infty)$  of  $x_4$  to the interval  $(0, +\infty)$  of  $t$  and maps the above degree-2 polynomial in  $x_4$  to

$$\begin{aligned} & (x_3 - x_2) t^2 + 3(x_3 - x_2) (x_0 + x_3) t - \\ & (1 + x_0^2 - 5x_0 x_3 + 3x_3^2) x_2 + \\ & x_0 + 4x_0 x_3^2 + x_0^3 + 2x_3^3 + 2x_0^2 x_3 \end{aligned}$$

whose first and second coefficients are positive and whose last coefficient is positive for any  $x_2 \in [x_0, x_3/2]$  since it is linear in  $x_2$  and takes value  $x_3(3x_0 + 2x_3)(x_3 - x_0)$  at  $x_0$  and value  $\frac{1}{2}x_3(-1 + x_3^2 + 3x_0^2 + 3x_0 x_3) + x_0 + x_0^3$  at  $x_3/2$  (which is positive since  $x_0 \geq 1$ ).<sup>2</sup> Hence, if  $x_3 \geq 2x_2$ , all coefficients of this polynomial are positive, which implies that it has no positive roots. This, in turn, means that the initial degree-2 polynomial in  $x_4$  has no root larger than  $x_0 + x_3$ .

<sup>2</sup>Note that the last coefficient is negative when  $x_2 = x_3$  which is why we consider  $x_2$  in the range  $[x_0, x_3/2]$ .

This implies that there is no position of the points  $p_0 = p_1, p_2, \dots, p_5$  such that  $x_3 \geq 2x_2$ ,  $x_4 \geq x_0 + x_3$  and such that the tangent to  $C_{0,3,4}$  is collinear with  $p_0p_2$ . Furthermore, at the limit case where  $p_2 = p_0$ , the segment  $p_0p_2$  is tangent to  $\mathcal{P}$ , and  $C_{0,3,4}$  is below and to the left of that tangent in a neighborhood of  $p_0$  (by Lemma 1). Hence, for any position of the points  $p_0 = p_1, p_2, \dots, p_5$  (as defined above) such that  $x_3 \geq 2x_2$ ,  $x_4 \geq x_0 + x_3$ , the (red) circular arc  $C_{0,3,4}$  is to the left of the segment  $p_1p_2$  in a neighborhood of  $p_0$ . Finally, as argued above when we considered  $p_5$  at infinity, this implies that for any position of the points  $p_0 = p_1, p_2, \dots, p_5$  such that  $x_3 \geq 2x_2$  and  $x_4 \geq x_0 + x_3$ , the (red) circular arc  $C_{0,3,4}$  is to the left of the (blue) circular arc  $C_{1,2,5}$  in a neighborhood of  $p_0 = p_1$ . This concludes the proof since we have proved that this is equivalent to the property that the arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  intersect only at  $p_0 = p_1$ .  $\square$

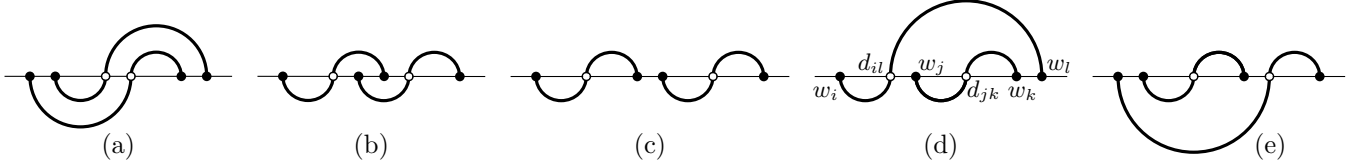


Figure 2: Relative positions of two edges in a monotone topological book embedding.

**Lemma 4.** *If  $p_0, \dots, p_5$  are pairwise disjoint and  $x_i \geq 2x_{i-1}$  for  $i = 3, 4$ , the two circular arcs  $C_{0,3,4}$  and  $C_{1,2,5}$  do not intersect.*

*Proof.* We refer to Figure 1(c) and, unless specified otherwise, an arc  $p_i p_j$  refers to the arc from  $p_i$  to  $p_j$  on the arc  $C_{0,3,4}$  or  $C_{1,2,5}$  that supports both  $p_i$  and  $p_j$ . We first prove that the arcs  $p_2 p_5$  and  $p_3 p_4$  do not intersect. For any point  $q$  on  $\mathcal{P}$  between  $p_4$  and  $p_5$ , the arc  $p_3 q$  on the circular arc through  $p_0, p_3, q$  lies above the concatenation of the arcs  $p_3 p_4$  of  $C_{0,3,4}$  and  $p_4 q$  of  $\mathcal{P}$  (since the circular arcs  $p_3 q$  and  $p_3 p_4$  lie above  $\mathcal{P}$ , by Lemma 1, and  $\mathcal{C}(p_0, p_3, p_4)$  and  $\mathcal{C}(p_0, p_3, q)$  intersect only at  $p_0$  and  $p_3$ ). It follows that if arc  $p_3 p_4$  intersects arc  $p_2 p_5$ , then arc  $p_3 q$  also intersects arc  $p_2 p_5$  for any position of  $q$  between  $p_4$  and  $p_5$  on  $\mathcal{P}$ . This implies that, for the limit case where  $q = p_5$ , arc  $C_{1,2,5}$  and the circular arc through  $p_0, p_3$ , and  $q = p_5$  intersect in some point other than  $q = p_5$ , which is not the case by Lemma 2.

We now prove, similarly, that the arcs  $p_0 p_3$  and  $p_1 p_2$  do not intersect. For any point  $q$  on  $\mathcal{P}$  between  $p_0$  and  $p_1$ , the arc  $q p_2$  on the circular arc through  $q, p_2, p_5$  lies below the concatenation of the arcs  $q p_1$  of  $\mathcal{P}$  and  $p_1 p_2$  of  $C_{1,2,5}$ . It follows that if arc  $p_1 p_2$  intersects arc  $p_0 p_3$ , then arc  $q p_2$  also intersects arc  $p_0 p_3$  for any position of  $q$  between  $p_0$  and  $p_1$  on  $\mathcal{P}$ . This implies that, for the limit case where  $q = p_0$ , arc  $C_{0,3,4}$  and the circular arc through  $q = p_0, p_2$ , and  $p_5$  intersect in some point other than  $q = p_0$ , which is not the case by Lemma 3.

Finally, arcs  $p_1 p_2$  of  $C_{1,2,5}$  and  $p_3 p_4$  of  $C_{0,3,4}$  do not intersect because they lie on different sides of  $\mathcal{P}$  and similarly for arcs  $p_0 p_3$  of  $C_{0,3,4}$  and  $p_2 p_5$  of  $C_{1,2,5}$ . Hence, the two arcs  $C_{0,3,4}$  or  $C_{1,2,5}$  do not intersect.  $\square$

### 3 Universal Point Set for Circular Arc Drawings

In this section, we construct a set of  $n$  points on  $\mathcal{P}$  and, by using the lemmata of the previous section, we prove that it is universal for plane circular arc drawings of  $n$ -vertex planar graphs.

Consider  $n^2$  points  $q_0, \dots, q_{n^2-1}$  on the parabolic arc  $\mathcal{P}$  such that  $x_0 \geq 1$  and  $x_i \geq 2x_{i-1}$  for  $i = 1, \dots, n^2 - 1$ . For our universal point set, we take the  $n$  points  $p_i = q_{ni}$  for  $i = 0, \dots, n - 1$ . We call the points in  $q_0, \dots, q_{n^2-1}$  that are not in the universal point set *helper points*.

**Theorem 5.** *Every  $n$ -vertex planar graph can be drawn with the vertices on  $p_0, \dots, p_{n-1}$  and circular edges that do not intersect except at common endpoints.*

*Proof.* Consider any planar graph  $G$ . Construct a monotone topological book embedding  $\Gamma$  of  $G$  in which all edges are drawn with a spine crossing [8, 16]. Denote by  $w_0, \dots, w_{n-1}$  the order of the vertices of  $G$  on

the spine in  $\Gamma$ . We substitute every spine crossing with a *dummy* vertex. The relative position of any two edges in  $\Gamma$  is as depicted in Figure 2 (in which two edges may share their endpoints). For  $0 \leq i \leq n - 1$ , we map vertex  $w_i$  to point  $p_i$ . Furthermore, for each  $0 \leq i \leq n - 2$ , we map the dummy vertices that lie in between  $w_i$  and  $w_{i+1}$  on the spine in  $\Gamma$  to distinct helper points in between  $p_i$  and  $p_{i+1}$ , so that the order of the dummy vertices on  $\mathcal{P}$  is the same as on the spine in  $\Gamma$ . (We postpone the proof that there are enough points  $q_i$  to map the dummy vertices.) We finally draw every edge  $(w_i, w_j)$  of  $G$  containing a dummy vertex  $d_l$  as a circular arc passing through  $p_i$ , through  $p_j$ , and through the helper point to which vertex  $d_l$  has been mapped to. We prove that the resulting drawing is plane.

By Lemmata 2, 3, and 4, two edges whose relative positions in  $\Gamma$  are as depicted in Figure 2(a) do not intersect except possibly at a common endpoint.

For the pairs of edges whose relative positions in  $\Gamma$  are as depicted in Figures 2(b) and 2(c), it is straightforward to check that they do not intersect either because they are separated by  $\mathcal{P}$ , or because they are  $y$ -monotone and hence they are separated by a horizontal line.

Consider two edges  $(w_i, w_l)$  and  $(w_j, w_k)$  whose relative position in  $\Gamma$  is as depicted in Figure 2(d) (the argument for pairs of edges as in Figure 2(e) is analogous). Let  $d_{il}$  and  $d_{jk}$  be the dummy vertices of  $(w_i, w_l)$  and  $(w_j, w_k)$ , respectively. Let  $q_{il}$  and  $q_{jk}$  be the points on  $\mathcal{P}$  to which  $d_{il}$  and  $d_{jk}$  are mapped. Arcs  $p_i q_{il}$  and  $p_j p_k$  do not intersect because they are both  $y$ -monotone and their endpoints are separated by a horizontal line. Arcs  $q_{il} p_l$  and  $p_j q_{jk}$  do not intersect because they are separated by  $\mathcal{P}$ . Hence, it suffices to prove that arcs  $q_{jk} p_k$  and  $q_{il} p_l$  do not intersect. These two arcs are above and to the right of  $\mathcal{P}$  (by Lemma 1) and  $q_{il}, q_{jk}, p_k, p_l$  are ordered from top to bottom. It is thus sufficient to prove that there exists a curve from  $q_{jk}$  to  $p_k$  that is to the right of  $q_{jk} p_k$  and that does not intersect  $q_{il} p_l$ . Consider the ( $y$ -monotone) arc from  $q_{jk}$  to  $p_k$  of the circle  $\mathcal{C}(p_i, q_{jk}, p_k)$ . It is indeed to the right of the arc  $q_{jk} p_k$  (of  $\mathcal{C}(p_j, q_{jk}, p_k)$ ) because  $p_i$  is inside  $\mathcal{C}(p_j, q_{jk}, p_k)$  (by Lemma 1) and  $p_i, q_{jk}$ , and  $p_k$  are ordered on the parabola. Furthermore, this new arc does not intersect  $q_{il} p_l$  because in the case where  $w_i = w_j$ ,  $w_k$  and  $w_l$  are in this order on the spine—that's the situation depicted in Figure 2(a)—we know that the corresponding circular arcs do not intersect.

It remains to show that there are enough helper points to map the dummy vertices. There are  $n - 1$  helper points  $q_{ni+1}, \dots, q_{n(i+1)-1}$  between each pair of points  $p_i = q_{ni}$  and  $p_{i+1} = q_{n(i+1)}$ . It thus suffices to prove that there are at most  $n - 1$  dummy vertices in between  $w_i$  and  $w_{i+1}$  along the spine in  $\Gamma$ .

Let  $(u_1, v_1), \dots, (u_k, v_k)$  be  $k$  edges in the book embedding that define consecutive dummy vertices on the spine. If no vertex  $w_i$  lies in between these dummy vertices on the spine in  $\Gamma$ , the  $k$  edges are such that  $u_1, \dots, u_k, v_1, \dots, v_k$  are ordered from left to right on the spine in  $\Gamma$ ; see Figure 3(a). Now, consider the graph that consists of these edges plus the edges  $(u_i, u_{i+1}), (v_i, v_{i+1})$ , for  $i = 1, \dots, k - 1$ ; see Figure 3(b). This graph is outerplanar. It has at most  $n$  vertices and, thus, at most  $n - 3$  chords. On the other hand, it has exactly  $k - 2$  chords:  $(u_2, v_2), \dots, (u_{k-1}, v_{k-1})$ . This implies that  $k - 2 \leq n - 3$  and  $k \leq n - 1$ , which concludes the proof.  $\square$

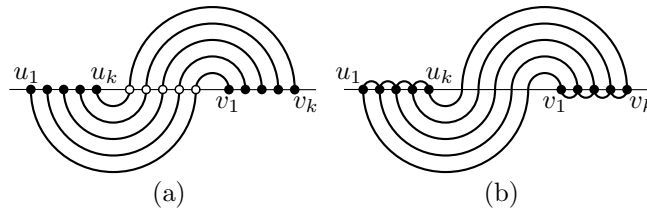


Figure 3: (a)  $k$  edges of a monotone topological book embedding that defines  $k$  consecutive dummy vertices (spine crossings). (b) Augmented outerplanar graph.



## 4 Conclusions

We proved the existence of a universal point set with  $n$  points for plane circular arc drawings of planar graphs. The universal point set we constructed has an area of  $2^{O(n^2)}$ . It would be interesting, also for practical visualization purposes, to construct a universal point set with  $n$  points for plane circular arc drawings of planar graphs within polynomial area. We remark that (relaxing the requirement that the set have exactly  $n$  points) a universal point set with  $O(n)$  points and within  $2^{O(n)}$  area for plane circular arc drawings of planar graphs is  $Q = \{q_0, \dots, q_{4n-7}\}$ , where the helper points are defined as in Section 3. To construct a plane circular-arc drawing of a planar graph  $G$  on  $Q$ , it suffices to map vertices and dummy vertices of a monotone topological book embedding of  $G$  to the points of  $Q$  in the order they appear in the book embedding. The geometric lemmata of Section 2 ensure that the resulting drawing is plane.

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