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# On a nonlocal, nonlinear Schrödinger equation occuring in Plasma Physics

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ABSTRACT

In this work, our goal is to study the Cauchy problem for a nonlocal, nonlinear Schrödinger equation occuring in Plasma Physics. We shall construct solutions in the energy space; we give some sufficient conditions on the initial data which ensure that the solutions are global and we show that in some cases, finite time blow up occurs. We prove the existence of standing waves, solution to these equations and provide some stability results.

## 1 Introduction

The goal of this work is to study the following system :

$$\left\{ \begin{array}{c} i\phi_t + \Delta\phi = -div(|\nabla\psi|^\sigma \nabla\psi), \\ \Delta\psi = \phi, \\ \phi(x,0) = \phi_0(x), \end{array} \right. \tag{1}$$

for $t \geq 0$ and $x \in R^3$.
This system is equivalent to :

$$\left\{ \begin{array}{c} i(\nabla\psi)_t + \Delta(\nabla\psi) = \nabla(-\Delta)^{-1} div(|\nabla\psi|^\sigma \nabla\psi), \\ \nabla\psi(x,0) = \nabla\psi_0(x). \end{array} \right. \tag{2}$$

The physical case occurs for $\sigma = 2$ and was introduced by Musher, Rubenchick and Zakharov[5] as a model for a nonlinear Plasma.
The plan of the work is the following:

2 Existence and finite time blow-up results.
3 Standing waves.
4 Radial standing waves.
5 Return to the finite time blow-up.

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6 Stability for $\sigma < 4/3$.
7 Instability for $\sigma > 4/3$.
We will not give complete proofs, we refer the reader to Colin[2] for more details.

## 2 Existence and finite time blow-up results.

If $\phi$ is a solution to Eq. 1, one can show that the following quantities are invariants of the motion:

$$m(t) = \int_{R^3} |\nabla\psi|^2 dx = m(0), \tag{3}$$

$$E(t) = \int_{R^3} \frac{1}{2}|\Delta\psi|^2 - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2} = E(0). \tag{4}$$

Therefore, we define the energy space $H$ as the completion of the set of $\mathcal{C}^\infty$ fonctions $\psi$ with compact support endowed with the norm $|\nabla\psi|_{H^1}$ ($H^1$ denoting the standard Sobolev space on $R^3$). Because of the Sobolev imbedding, $H$ turns out to be equal to $\{\psi \in L^6,\ \nabla\psi \in H^1\}$ normed by $|\nabla\psi|_{H^1}$.
With these notations, we have:

**Theorem 1** *Let $0 < \sigma < 4$ and $\psi_0 \in H$, then there exists an unique maximal solution to Eq. 2 on $[0, T(\psi_0)[$ in $\mathcal{C}([0, T(\psi_0)[, H)$. The equalities Eq. 3 and Eq. 4 hold. Moreover, if $T(\psi_0) < \infty$ then*

$$\lim_{t \to T(\psi_0)} |\nabla\psi(t)|_{H^1} = +\infty.$$

*If $0 < \sigma < 4/3$, then $T(\psi_0) = \infty$.*
*If $4/3 \le \sigma < 4$ and if $|\psi_0|_H$ is sufficiently small, then $T(\psi_0) = \infty$.*

**Proof:** For the proof of this Theorem, we remark that the operator $\nabla(\Delta)^{-1}div$ is a Calderón-Zygmund operator, hence it maps $L^p$ into $L^p$ for all $1 < p < \infty$. Therefore, we can use the standard methods for Schrödinger type equations (see Kato[3] for example). We first prove that Eq. 2 is equivalent to the following integral equation:

$$\nabla\psi = \mathcal{T}(\psi), \tag{5}$$

where

$$\mathcal{T}(\psi) \equiv S(t)\nabla\psi_0 - i\int_0^t S(t-s)\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)(s)ds, \tag{6}$$

$S(t)$ denoting the free propagator of the equation

$$i\nabla\psi_t + \Delta\nabla\psi = 0.$$

Using the $L^p - L^q$ estimates (see Kato[3]), one can prove that $\mathcal{T}$ is a contraction in the convenient spaces, hence there exists a local solution to Eq. 2.

If $\sigma < 4/3$, using Eq. 3, 4 and a Gagliardo-Nirenberg inequality, we obtain a bound of $\psi$ in the space $H$, and hence the solution is global.
If $\sigma \geq 4/3$ and if $|\nabla\psi_0|_{H^1}$ is small, one can show, using a Gagliardo-Nirenberg inequality, that $E(t)$ controls the $L^2$ norm of $\Delta\psi$, therefore the solution is global in time, see Colin[2] for details. ■

For the case where $|\nabla\psi_0|_H$ is not small, we have the following result:

**Theorem 2** *i) Let $\psi_0 \in H$ be such that*

$$\int |x|^2|\nabla\psi_0|^2 < \infty,$$

*then the solution $\psi$ to Eq. 2 satisfies:*

$$|x||\nabla\psi| \in L^\infty(0,t,L^2),$$

*for all $t < T(\psi_0)$.*
*ii)If $\sigma \geq 4/3$, there exists some radial initial data such that the corresponding solutions blow up in finite time.*

**Sketch of proof:**
i) In order to prove that the quantity $|\,|x||\nabla\psi|\,|_{L^2}$ persists, we work on the integral equation Eq. 5. We have to estimate the commutator of $x$ and $\nabla(-\Delta)^{-1}div$. This can be done using the Fourier transform and some estimates on the Riesz potential (see Stein[7]). Next, we show that if $|x||\nabla\psi_0| \in L^2(R^3)$, then there exists $T_0 > 0$ such that $|x||\nabla\psi| \in L^\infty(0,t,L^2)$ for all $t < T_0$. Then it is sufficient to prove than $T(\psi_0)$ and $T_0$ are equal. This is done by supposing the contrary and working on Eq. 5.

ii) To prove the second part of Theorem 2, we make two elementary remarks:
1)$P = \nabla(-\Delta)^{-1}div$ is the projector on the set of the gradients in $(L^2(R^3))^3$.
2)Every radial vector of $(L^2(R^3))^3$ is a gradient.

These remarks prove that the restriction of Eq. 2 to the radial functions is:

$$i(\nabla\psi)_t + \Delta\nabla\psi = |\nabla\psi|^\sigma\nabla\psi.$$

This equation satisfies the Virial identity:

$$\frac{d^2}{dt^2}\int |x|^2|\nabla\psi|^2dx = 16E(\psi_0) + \frac{16}{\sigma+2}(1-\frac{3\sigma}{4})\int |\nabla\psi|^{\sigma+2}, \tag{7}$$

where

$$E(\psi_0) = \int \frac{1}{2}|\Delta\psi_0|^2 - \frac{1}{\sigma+2}|\nabla\psi_0|^{\sigma+2}.$$

Hence, in the case where $\sigma \geq \frac{4}{3}$ and $E(\psi_0) < 0$, the solution has to blow up in finite time. ■

## 3 Standing waves.

The goal of this section is to find $\psi(x)$ and $\omega \in R$ such that $e^{i\omega t}\psi(x)$ is a solution to Eq. 2 .

**Theorem 3** *For all* $0 < \sigma < 4$ *and all* $\omega > 0$*, there exists a function* $\psi_\omega \in H$ *satisfying:*

$$\begin{cases} -\omega\phi_\omega + \Delta\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega), \\ \Delta\psi_\omega = \phi_\omega, \end{cases}$$

*i.e.* $e^{i\omega t}\psi_\omega(x)$ *is a solution to Eq. 2.*

**Proof:** We solve the following minimization problem:

$$inf\{-\int |\nabla\psi|^{\sigma+2}, \psi \in H, \int |\nabla\psi|^2 = \lambda, \int |\Delta\psi|^2 = \mu\},$$

where $\lambda$ and $\mu$ are convenient parameters. In order to solve this problem we take a minimizing sequence and thanks to the concentration-compactness method of P.L. Lions[4], we prove that this sequence is compact up to translations, providing a minimum $\psi$. Writing the Euler-Lagrange equation satisfied by $\phi$, there exists two Lagrange multipliers $\alpha$ and $\beta$ such that:

$$\alpha\Delta\psi + \beta\Delta^2\psi = -div(|\nabla\psi|^\sigma \nabla\psi). \tag{8}$$

We now make a scaling on $\psi$, and using the so-called Pohozahev identity (i.e. multiplying Eq. 8 by $x.\nabla\psi$ and integrating over all $R^3$), we find that $\omega$ has to be positive. Hence $e^{i\omega t}\psi(x)$ is a standing wave solution to Eq. 1. ■

## 4 Radial Standing waves.

The goal of this section is to prove that there exists some radial standing waves solution to the system 1. Our result in this direction reads as follows.

**Theorem 4** *For all* $\omega > 0$ *and* $0 < \sigma < 4$*, there exists radial functions* $\psi_\omega$ *satisfying*

$$\begin{cases} -\omega\phi_\omega + \Delta\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega), \\ \Delta\psi_\omega = \phi_\omega. \end{cases}$$

**Method:**
The system 1 in the radial case is equivalent to the following ordinary differential equation:

$$f'' + \frac{2}{r}f' - (\omega + \frac{2}{r^2})f = -|f|^\sigma f,$$

where $f = \psi_r$.
We find a solution to this equation by considering the following minimization problem:

$$inf\{-\int f^{\sigma+2} r^2 dr, f \in H^1_r(R^3) \text{ such that } \int f'^2 r^2 dr + 2\int f^2 dr = \mu, \ \int f^2 dr = \lambda\},$$

where $H^1_r(R^3)$ denotes the set of all radial functions which are in the Sobolev space $H^1(R^3)$. Using the compactness of $H^1_r(R^3) \subset L^{\sigma+2}$, due to Strauss[8], we are able to prove that the minimizing sequences are compact in $H^1(R^3)$, providing therefore a minimum. We conclude as in the previous section. ■

There is an open question: are the two familly of standing waves the same?

## 5 Return to the finite time blow up.

Thanks to the previous section, we prove:

**Proposition 1** *Let $\sigma = 4/3$, then for all $t_0 > 0$, there exists a radial function $\psi_0$ such that the corresponding solution to Eq. 2 blows up exactly at $t = t_0$.*

**Proof:** The restriction of Eq. 2 to the radial functions is:

$$iu_t + \Delta u - \frac{2}{|x|^2} u = -|u|^{4/3} u, \tag{9}$$

where $u = \psi_r$.
This equation satisfies the pseudo-conformal transformation law, *i.e.* if $u$ is a solution to Eq. 9 then

$$v(x,t) = \frac{1}{t^{\frac{3}{2}}} e^{\frac{i|x|^2}{4t}} \bar{u}(\frac{x}{t}, \frac{1}{t})$$

is also a solution to Eq. 9.

We now apply this transformation with $u =$ any radial standing wave, and the proposition is proved. ■

## 6 Stability for $\sigma < 4/3$.

**Theorem 5** *For $\sigma < 4/3$, the set $S_\omega$ of standing waves defined by the minimization problem of section 3 is stable:*
*for all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\forall \psi_0 \in H, \ \inf_{\psi_\omega \in S_\omega} ||\psi_0 - \psi_\omega||_H < \delta$$

$$\Rightarrow \inf_{\psi_\omega \in S_\omega} ||\psi(.,t) - \psi_\omega||_H \leq \epsilon.$$

**Sketch of proof:**
We first proved that the problems

$$(P1) \quad inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H,\ \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2,$$

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\}$$

and

$$(P2) \quad inf\{\frac{1}{2}\int |\Delta\psi|^2 - \frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2,$$

$$\psi \in H,\ \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2\},$$

where $\psi_\omega$ is any standing wave that we found in section 3, are equivalent.
In order to do that, we use the concentration-compactness lemma to prove that $(P2)$ has a solution, and a careful scaling analysis shows that this solution is also a solution to $(P1)$. This method gives us the compactness of the minimizing sequences, which implies the stability according to the method of Cazenave and Lions[1]. ■

# 7 Instability for $\sigma > 4/3$.

In this case, we prove:

**Theorem 6** *For $\sigma > 4/3$, the orbit $e^{i\theta}\psi_\omega$ is unstable, i.e. there exists $\epsilon_0 > 0$ and $\psi_n^0 \to \psi_\omega$ in $H$ such that*

$$\sup_{t>0} \inf_{\theta\in R} ||\psi_n(.,t) - e^{i\theta}\psi_\omega||_H \geq \epsilon_0.$$

**Sketch of proof:**
The problems

$$(P1) \quad inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H,\ \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2,$$

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\}$$

and

$$(P3) \quad inf\{\frac{1}{2}\int |\Delta\psi|^2 - \frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2,$$

$$\psi \in H,\ \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\},$$

where $\psi_\omega$ is any standing wave, are equivalent (we apply the same method than in the previous section).

We then define

$$d(\omega) = inf\{\frac{1}{2}\int |\Delta\psi|^2 - \frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2,$$
$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\}.$$

A simple scaling leads to

$$d(\omega) = \frac{\omega^{\frac{2}{\sigma}-\frac{1}{2}}}{3}\int |\Delta\psi_1|^2,$$

so that $d$ is convex.
With these notations and results we are in the framework of Shatah and Strauss[6], which leads to Theorem 6. ■

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