On the standing waves solutions to a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics

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Abstract

In this work, our goal is to study the standing waves solutions to some generalization of the following system.

$$\left\{ \begin{array}{c} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{array} \right.$$

We shall prove the existence of standing waves, solution to this equations and provide some stability results.

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Contents

1 Introduction and statement of the results.

1.1 Introduction.

In a previous work ([4]), we study the Cauchy problem for the following familly of system:

$$\left\{ \begin{array}{c} i\phi_t + \sum_{k,l=1}^{3} a_{kl} \frac{\partial^2 \phi}{\partial x_k \partial x_l} = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x), \end{array} \right. \tag{1}$$

where $a_{kl} = a_{lk}$ are real constants, the matrix (a_{kl}) being nonsingular and $\sigma > 0$.
These systems are generalizations of the following ones which occures in Plasma Physics (see [6]).

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases} \tag{2}$$

If ϕ is a solution of (1), one can show that the following quantities are invariants of the motion:

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0), \tag{3}$$

$$E(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx = E(0), \tag{4}$$

where q is the following hermitian form:

$$q(\mathbf{u}) = \sum_{k,l=1}^{3} a_{kl}u_k\bar{u}_l.$$

Indeed, multiplying the first equation of (1) by $\bar{\psi}$ leads, after integration, to:

$$-i\int \nabla\psi_t.\nabla\bar{\psi} + \int a_{kl}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i} = \int |\nabla\psi|^{\sigma+2}. \tag{5}$$

But $\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$ is real, indeed an integration by parts gives:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l} = \int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_i}.$$

Hence, taking the imaginary part of (5) leads to:

$$Re\int \nabla\psi_t.\nabla\bar{\psi} = 0$$

and we obtain (3). On the other hand, multiplying the first equation of (1) by $\bar{\psi}_t$ and using the same method, we arrive at (with the summation convention):

$$-i\int \nabla\psi_t.\nabla\bar{\psi}_t + \int a_{kl}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i} = \int |\nabla\psi|^{\sigma}\nabla\psi.\nabla\bar{\psi}_t. \tag{6}$$

But:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}=\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_i},$$

therefore,

$$Re(\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l})=\frac{1}{2}\int\frac{d}{dt}a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$$

$$=\frac{1}{2}\frac{d}{dt}\int\sum_i q(\nabla\frac{\partial\psi}{\partial x_i}).$$

Since

$$Re(\int|\nabla\psi|^\sigma\nabla\psi.\nabla\bar{\psi}_t)=(\sigma+2)\frac{d}{dt}(\int|\nabla\psi|^{\sigma+2}),$$

we now obtain (4) using (6).

Let us recall the results that we have obtained concerning the Cauchy problem in [4].
We introduce

$$H=\{\psi\in L^6\cap C_0(R^3),\nabla\psi\in H^1\},$$

endowed with the norm $||\nabla\psi||_{H^1}$.

Theorem 1. *Let* $0<\sigma<4$.

$*$*Let* $\psi_0\in H$, *then there exists an unique maximal solution on* $[0,T(\psi_0)[$ $\psi\in C([0,T(\psi_0)[,H)$ *to:*

$$\begin{cases} i(\nabla\psi)_t+L(\nabla\psi)=\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)\\ \psi(x,0)=\psi_0(x)\end{cases}$$

$*$*Moreover,* $\phi=\Delta\psi\in L^r(0,t,L^{\sigma+2})$ *with* $\frac{2}{r}+\frac{3}{\sigma+2}=\frac{3}{2}$ *for all* $t<T$.

$*$*The function* ψ *is a solution to:*

$$\begin{cases} i\phi_t+L\phi=-div(|\nabla\psi|^\sigma\nabla\psi)\\ \nabla\psi=\phi\\ \phi(x,0)=\phi_0(x)\end{cases}$$

and if $0<\sigma<3$, *it is the only solution in* $C([0,T[,H)$.

$*$*The solution* ψ *depends continuously on* ψ_0 *in* $C([0,T[,H)$ *in the following sense: if* $\psi_0^n\to\psi_0$ *in* H *then for all* $T<T(\psi_0)$, *if* n *is sufficiently large, the corresponding solutions exist on a common interval* $[0,T]$ *and* $\psi_n\to\psi$ *in* $C([0,T],H)$.

We have a result of regularity:

Theorem 2. *If $\psi_0 \in H$ with $\nabla\psi_0 \in H^2$, then the solution given by Theorem 1 satisfies*

$$\nabla\psi \in C([0, T(\psi_0)[, H^2).$$

In some cases, we are able to prove that the solution exists globally for all $t > 0$.

Theorem 3. *a)Let $\psi_0 \in H$, then the solution of (1) satisfies:*

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0),$$

$$E(t) = \int_{R^3} \left(\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2}\right)dx = E(0),$$

where q is the following hermitian form:

$$q(u) = \sum_{k,l=1}^{3} a_{kl} u_k \bar{u}_l.$$

b)If the matrix (a_{kl}) is negative, then $||\psi||_H$ remains bounded and the solution is global in time.

c)If the matrix (a_{kl}) is positive, then if $\sigma < \frac{4}{3}$, $||\psi||_H$ remains bounded and the solution is global in time. If $\sigma \geq \frac{4}{3}$, then if $||\psi_0||_H$ is sufficiently small, $||\psi||_H$ remains bounded and the solution is global in time.

d)If ψ is a solution of (2), if $\nabla\psi_0 \in H^{m+1} \cap W^{m+1,\frac{6}{5}}$ for $m \geq 4$, then there exists $\delta > 0$ such that, if

$$||\nabla\psi_0||_{H^{m+1}} + ||\nabla\psi_0||_{W^{m+1,\frac{6}{5}}} < \delta,$$

then the solution is global in time and there exists $\bar{\phi}$ satisfying

$$\begin{cases} i\bar{\phi}_t + \Delta\bar{\phi} = 0 \\ \Delta\bar{\psi} = \bar{\phi}, \end{cases}$$

such that:

$$\begin{cases} ||\nabla\psi||_{W^{m-2,6}} \leq \frac{C}{1+t} \\ ||\nabla\psi||_{H^{m+1}} \leq C \\ ||\nabla\psi - \nabla\bar{\psi}||_{H^{m+1}} \leq \frac{C}{1+t}. \end{cases}$$

We have the following finite-time blow-up result:

Theorem 4. *Let $\psi_0 \in H$ be such that $\int |x|^2 |\nabla\psi|^2 < \infty$, then the solution of (1) with $L = \Delta$ satisfies*

$$|x|\ |\nabla\psi| \in L^\infty(0,t,L^2) \cap L^r(0,t,L^{\sigma+2}),$$

for all $t < T(\psi_0)$.

There exists some radial initial values such that the corresponding solutions blow up in finite time.

1.2 Statement of the results.

The main results of this work are:

Theorem 5. *Let us suppose that the matrix a_{ij} is positive, and let $0 < \sigma < 4$. For every $\omega > 0$, there exists a function $\psi_\omega \in H$ satisfying:*

$$\begin{cases} -\omega\phi_\omega + L\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega \end{cases}$$

i.e. $e^{i\omega t}\psi_\omega(x)$ satisfies (1).

Moreover, ψ_ω is a solution to the following minimization problem:

$$(P1)\ inf\{-\int |\nabla\psi|^{\sigma+2}, \psi \in H, \int |\nabla\psi|^2 = \lambda, \int \sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) = \mu\}$$

for convenient λ, μ and $\nabla\psi \in \mathcal{C}^2(R^3)$.

Moreover

$$\int |\nabla\psi_\omega|^{\sigma+2}, \int \sum_{i=1}^{3} q(\nabla\frac{\partial\psi_\omega}{\partial x_i})\ and\ \int |\nabla\psi_\omega|^2$$

are independent of the solution ψ_ω of $(P1)$ that we consider.

We will prove this theorem in next section, adapting to our functionnal setting the concentration-compactness arguments of P.L. Lions [5].
About standing waves, let us mention the following result.

Theorem 6. *For all $\omega > 0$ and $0 < \sigma < 4$, there exists radial functions ψ_ω such that $\frac{\partial}{\partial r}\psi_\omega \in \mathcal{C}^2(R^3)$ satisfying:*

$$\left\{ \begin{array}{c} -\omega\phi_\omega + \Delta\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{array} \right.$$

We prove this theorem using compactness properties of spaces of radial functions (see Strauss [8]).
Moreover, these radial standing waves are related to the finite time blow up by the following proposition.

Proposition 1 *Let $\sigma = 4/3$, then for all $t_0 > 0$, there exists a radial function ψ_0 so that the corresponding solution to:*

$$\left\{ \begin{array}{c} i\phi_t + \Delta\phi = -div(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi = \phi \end{array} \right.$$

blows up exactly at $t = t_0$.

We can now state some stability results for these standing waves:

Theorem 7. $*$*If $0 < \sigma < \frac{4}{3}$, we consider*

$$(P2)\ inf\{\frac{1}{2}\int\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2, \int|\nabla\psi|^2 = \int|\nabla\psi_\omega|^2\}$$

where ψ_ω is a solution of (P1). Then (P1) and (P2) are equivalent.
$*$*The set S_ω of standing waves (solutions of (P1) or (P2)) is stable,ie for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall\psi_0 \in H,\ inf_{\psi_\omega\in S_\omega}||\psi_0 - \psi_\omega||_H < \delta \to inf_{\psi_\omega\in S_\omega}||\psi(.,t) - \psi_\omega||_H \le \epsilon.$$

The first part of this theorem will be proved using concentration-compactness methods, the second part adapting the method of Cazenave-Lions [2].

Theorem 8. $*$*For $0 < \sigma < 4$ we consider:*

$$(P3)\ inf\{\frac{1}{2}\int\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,$$

$$\int \sum_{i=1}^{3} q(\nabla \frac{\partial \psi}{\partial x_i}) = \int \sum_{i=1}^{3} q(\nabla \frac{\partial \psi_\omega}{\partial x_i})\}$$

where ψ_ω is a solution of (P1). Then (P1) and (P3) are equivalent.

$*$If $\sigma > \frac{4}{3}$, the orbite $e^{i\theta}\psi_\omega$ is unstable, i.e. there exists $\epsilon_0 > 0$ and $\psi_n^0 \to \psi_\omega$ in H such that:

$$sup_{t>0} inf_{\theta \in R} ||\psi_n(.,t) - e^{i\theta}\psi_\omega||_H \geq \epsilon_0.$$

The first part of this theorem will be proved using concentration-compactness methods. For the second part, we will prove that our equations enter in the framework of Shatah-Strauss [7].
The results of this paper were announced in [3].

2 Standing waves solutions

We want to find solutions to

$$\begin{cases} i\phi_t + L\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

of the form $e^{i\omega t}u(x)$, where L is elliptic. Therefore if we make a scaling, we can restrict ourself to the case where $L = \Delta$; i.e. we want to solve:

$$\begin{cases} -\omega u + \Delta u = -div(|\nabla v|^\sigma \nabla v) \\ \Delta v = u \end{cases} \tag{7}$$

In order to solve (7), we introduce the following minimization problem:

$$inf\{-\int |\nabla\psi|^{\sigma+2},\ \phi \in H,\ \int |\nabla\psi|^2 = 1,\ \int |\Delta\psi|^2 = 1\},$$

where $H = \{\psi \in L^6,\ \nabla\psi \in H^1\}$.

If ψ is a solution of the minimization problem, then there exists two numbers α and β (Lagrange muliplier) such that:

$$\beta\Delta^2\psi - \alpha\Delta\psi = -div(|\nabla\psi|^\sigma \nabla\psi).$$

A suitable scaling will give a solution of (7).

2.1 Concentration-compactness lemma

A few notations are in order.
We set

$$I(\lambda,\mu) = inf(-\int |\nabla\psi|^{\sigma+2},\ \psi \in S_{\lambda,\mu}),$$

for $\lambda, \mu > 0$, and

$$S_{\lambda,\mu} = \{\psi \in H,\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}.$$

We now have:

Lemma 1.

$$I(\lambda,\mu) = I(1,1)\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and $-\infty < I(1,1) < 0$.

Proof: The Lemma follows obviously from the Gagliardo-Nirenberg's inequality:

$$|\nabla\psi|^{\sigma+2}_{L^{\sigma+2}} \leq C|\Delta\psi|^{3\sigma/2}_{L^2}|\nabla\psi|^{2-\sigma/2}_{L^2}.$$

■

We will use the concentration-compactness lemma of P.L. Lions (see [5]). The aim of this section is to prove the following version of this lemma:

Lemma 2. *1)Let* ρ_n *a sequence in* $L^1(R^3)$ *satisfying* $\rho_n \geq 0$ *in* R^3 *and* $\int \rho_n dx = \lambda$ *for a* $\lambda > 0$ *fixed. Then there exists a subsequence* ρ_{n_k} *satisfying one of the three following alternatives:*

i)(Compactness) : there exists $y_{n_k} \in R^3$ *such that* $\forall\epsilon > 0,\ \exists R < \infty$ *such that:*

$$\int_{y_{n_k}+B_R} \rho_{n_k}(x)dx \geq \lambda - \epsilon.$$

ii)(Vanishing) : $\forall R < \infty$

$$sup_{y\in R^3} \int_{y+B_R} \rho_{n_k} \to 0$$

when $k \to \infty$.

iii)(Dichotomy) : there exists $\alpha \in]0, \lambda[$ *such that* $\forall \epsilon > 0$*, there exists* $\rho^1_{n_k}, \rho^2_{n_k} \in L^1_+(R^3)$ *such that if* $k \geq k_0$ *:*

$$|\int \rho^1_{n_k} dx - \alpha| < \epsilon, \;\; |\rho_{n_k} - (\rho^1_{n_k} + \rho^2_{n_k})|_{L^1} \leq \epsilon,$$

$\rho^1_{n_k}$ *has compact support and* $dist(supp\rho^1_{n_k}, supp\rho^2_{n_k}) \to \infty$ *as* $k \to \infty$.

2)Moreover, if $\rho_n = |\nabla u_n|^2$*, with* $u_n \in H$*, if* u_n *is bounded in* H *and if iii) occures, then one can take* $\rho^i_{n_k} = (|\nabla u^i_{n_k}|^2)^2$ *for* $i = 1, 2$ *and* $u^i_{n_k}$ *are bounded in* H *and satisfy*

$$\int |\Delta u_{n_k}|^2 \geq \int (|\Delta u^1_{n_k}|^2 + |\Delta u^2_{n_k}|^2) - \epsilon$$

and

$$|\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|_{L^p} \leq \delta_p(\epsilon),$$

with $\delta_p(\epsilon) \to 0$ *when* $\epsilon \to 0$ *for* $2 \leq p < 6$.

Proof: The 1) is lemma I1 of P.L. Lions [5] partI. The 2) is an adaptation of lemma III1 in [5] partI.

1) Let us recall briefly how to prove the first part, (see [5] for details). We introduce:

$$Q_n(t) = sup_{y \in R^3} \int_{y+B_t} \rho_n(x) dx$$

The Q_n are nondecreasing, nonnegative functions, uniformly bounded on R^+. We can extract a subsequence $Q_{n_k}(t) \to Q(t)$ for every $t \geq 0$, where Q is a nondecreasing, nonnegative function. Let $\alpha = lim_{t \to \infty} Q(t) \in [0, \lambda]$.

If $\alpha = 0$ then $Q \equiv 0$ and vanishing occures.

If $\alpha = \lambda$, no mass disappears : compactness occures (see [5] for details).

If $0 < \alpha < \lambda$: some mass disappears, and dichotomy occures. Let us explain why: let $\epsilon > 0$, choose R such that $Q(R) > \alpha - \epsilon$. If k is sufficiently large we have: $\alpha - \epsilon < Q_{n_k}(R) < \alpha + \epsilon$. Moreover, one can find $R_k \to \infty$ such that $Q_{n_k}(R_k) \leq \alpha + \epsilon$. Finally, there exists $y_k \in R^3$ such that $\int_{y_k + B_R} \rho_{n_k}(x) dx \in]\alpha - \epsilon, \alpha + \epsilon[$. Let $\rho^1_{n_k} = \rho_{n_k} 1_{y_k + B_R}$ and $\rho^2_{n_k} = \rho_{n_k} 1_{R^3 - (y_k + B_{R_k})}$ then

$$\int \{\rho_{n_k} - \rho^1_{n_k} - \rho^2_{n_k}\} dx = \int_{R \leq |x - y_k| \leq R_k} \rho_{n_k} dx \leq (Q_{n_k}(R_k) + \epsilon) - (Q_{n_k}(R) - \epsilon) \leq 4\epsilon$$

The condition on the supports is satisfied and this implies that $\rho^1_{n_k} \rho^2_{n_k} = 0$ a.e.

2)To prove 2), we make a truncation of $\rho_{n_k} = |\nabla u_{n_k}|^2$ with regular functions: let $\epsilon > 0$, $\xi \in \mathcal{D}(R^3)$, $0 \leq \xi \leq 1$, with $\xi \equiv 1$ if $|x| \leq 1$ and $\xi \equiv 0$ if $|x| \geq 2$. Let $\phi = 1 - \xi$, $\xi_\mu = \xi(\frac{\cdot}{\mu})$ and $\phi_\mu = \phi(\frac{\cdot}{\mu})$. We take R_0 such that $Q(R) \leq \alpha - \epsilon$ for $R \geq R_0$. Let $v \in H$ such that $|v|_H \leq M$ where $M \geq sup_n |u_n|_H$.
We then have:

Lemma 3. *There exists a constant $C(M)$ such that:*

$$|\int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx| \leq \frac{C(M)}{R} \tag{8}$$

and

$$|\int |\nabla(\xi_R v)|^2 dx - \int \xi_R^2 |\nabla v|^2 dx| \leq \frac{C(M)}{R} \tag{9}$$

Proof: Let us prove the first one for example. We have:

$$|\int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx| = \int \Delta\xi_R^2 v^2 + 4\int (\nabla\xi_R.\nabla v)^2$$

$$+4\int \Delta\xi_R v \nabla\xi_R.\nabla v + 4\int \nabla\xi_R.\nabla v \xi_R \Delta v + 2\int \Delta\xi_R v \xi_R \Delta v,$$

using Hölder's and Cauchy-Schwartz inequalities we obtain that this last quantity is lower or equal to:

$$(\int (\Delta\xi_R)^3)^{1/3}(\int v^6)^{1/6} + 4\int \nabla\xi_R^2 \nabla v^2 + 4(\int \Delta\xi_R^3)^{1/3}(\int v^6)^{1/6}(\int \nabla\xi_R^2 \nabla v)^{1/2}$$

$$+4(\int \nabla\xi_R^2 \nabla v^2)^{1/2}(\int \xi_R^2 \Delta v^2)^{1/2} + 2(\int \Delta\xi_R^3)^{1/3}(\int v^6)^{1/6}(\int \xi_R^2 \Delta v^2)^{1/2}.$$

But $\nabla\xi_R(x) = \frac{1}{R}\nabla\xi(\frac{x}{R})$ and $\Delta\xi_R(x) = \frac{1}{R^2}\Delta\xi(\frac{x}{R})$. Moreover $\xi, \nabla\xi, \Delta\xi$ are bounded and $\int |\nabla v|^2, \int |\Delta v|^2 \leq M^2$. Since $(\int v^6)^{1/6} \leq C(\int |\nabla v|^2)^{1/2}$ for all $v \in H$, the above expression is bounded by $\frac{C(M)}{R}$ as claimed. ∎

Choose $R_1 \geq R_0$ such that $\frac{C(M)}{R_1} \leq \epsilon$, thus

$$Q(R_1) \geq \alpha - \epsilon. \tag{10}$$

If k is sufficiently large, then

$$Q_{n_k}(R_1) \leq \int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 + \epsilon. \tag{11}$$

We define $u^1_{n_k} = \xi_{R_1}(. - y_k)u_{n_k} \in H$ and:

$$|\int |\nabla u^1_{n_k}|^2 dx - \alpha| \leq |\int |\nabla u^1_{n_k}|^2 dx - \int \xi^2_{R_1}(. - y_k)|\nabla u_{n_k}|^2 dx|$$

$$+|\int \xi^2_{R_1}(. - y_k)|\nabla u_{n_k}|^2 dx - \alpha|.$$

(9) implies that this last quantity is smaller than:

$$\epsilon + |\int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 - \alpha| \leq 3\epsilon,$$

by (10) and (11).
We may find $R_k \to \infty$ such that

$$Q_{n_k}(2R_k) \leq \alpha + 2\epsilon, \tag{12}$$

take $\phi_k = \phi_{R_k}(. - y_k)$ and $u^2_{n_k} = \phi_k u_{n_k} \in H$. We have the condition on the supports and

$$\int |\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|^2 dx = \int |\nabla(\{1 - \xi_{R_1}(x - y_k) - \phi_k\}u_{n_k})|^2 dx$$

$$\leq \frac{2C(M)}{R_1} + \int \{1 - \xi_{R_1}(. - y_k) - \phi_k\}^2 |\nabla u_{n_k}|^2 dx$$

$$\leq 2\epsilon + \int_{R_1 \leq |x - y_k| \leq 2R_k} |\nabla u_{n_k}|^2 dx,$$

using (11):

$$\leq 2\epsilon + Q_{n_k}(2R_k) - Q_{n_k}(R_1) + \epsilon,$$

by (12):

$$\leq 3\epsilon + \alpha + 2\epsilon - Q_{n_k}(R_1).$$

But if $k \geq k_0$, $|Q_{n_k}(R_1) - \alpha| \leq \epsilon$, this implies that

$$\int |\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|^2 dx \leq 6\epsilon$$

Now interpolating L^p between L^2 and L^6 for $2 \leq p < 6$ leads to

$$|\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|_{L^p} \leq \delta_p(\epsilon).$$

Moreover

$$|\int |\Delta u^1_{n_k}|^2 dx - \int \xi^2_{R_1}(x - y_k)|\Delta u_{n_k}|^2 dx| \leq \epsilon,$$

and lemma 2 is proved. ■

2.2 Application of the concentration-compactness lemma.

The aim of this section is to prove:

Proposition 2. *For all $\lambda, \mu > 0$, the following minimization problem:*

$$inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H,\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}$$

has a solution.
Moreover, every minimizing sequence is relatively compact in H up to translations.

Proof: We follow [5]. Take $\psi_n \in H$ satisfying $-|\nabla\psi_n|^{\sigma+2}_{L^{\sigma+2}} \to I(\lambda,\mu)$ with $\int |\nabla\psi_n|^2 = \lambda$ and $\int |\Delta\psi_n|^2 = \mu$. Let us apply lemma 2 with $\rho_n = |\nabla\psi_n|^2$.

Suppose that iii) occures: We apply 2) of lemma 2: there exists $\psi^1_{n_k}, \psi^2_{n_k}$ bounded in H such that for $k \geq k_0$

$$|\nabla\psi_{n_k} - (\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k})|_{L^{\sigma+2}} \leq \delta_{\sigma+2} \to 0.$$

On the other hand, since $\nabla\psi^i_{n_k}$ in bounded in H^1 and hence in $L^{\sigma+2}$, we see that

$$\int |\nabla\psi_{n_k}|^{\sigma+2} \leq K\delta_{\sigma+2}(\epsilon) + \int |\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k}|^{\sigma+2}.$$

Moreover $dist(supp\nabla\psi^1_{n_k}, supp\nabla\psi^2_{n_k}) \to \infty$ so that $(\nabla\psi^1_{n_k})^2(\nabla\psi^2_{n_k})^2 = 0$ a.e. This leads to

$$\int (\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k})^{\sigma+2} = \int |\nabla\psi^1_{n_k}|^{\sigma+2} + \int |\nabla\psi^2_{n_k}|^{\sigma+2}.$$

Finally we obtain

$$-\int |\nabla\psi_{n_k}|^{\sigma+2} \geq -K\delta_{\sigma+2}(\epsilon) - \int |\nabla\psi^1_{n_k}|^{\sigma+2} - \int |\nabla\psi^2_{n_k}|^{\sigma+2}.$$

Then:

$$\begin{aligned} -\textstyle\int |\nabla\psi_{n_k}|^{\sigma+2} \geq & -K\delta_{\sigma+2}(\epsilon) + I(\textstyle\int |\nabla\psi^1_{n_k}|^2, \int |\Delta\psi^1_{n_k}|^2) \\ & +I(\textstyle\int |\nabla\psi^2_{n_k}|^2, \int |\Delta\psi^2_{n_k}|^2), \end{aligned} \tag{13}$$

with

$$|\int |\nabla\psi^1_{n_k}|^2 - \alpha| \leq \epsilon,$$

$$|\int |\nabla\psi^2_{n_k}|^2 - (\lambda-\alpha)| \le \epsilon$$

and

$$\mu = \int |\Delta\psi_{n_k}|^2 \ge \int |\Delta\psi^1_{n_k}|^2 + \int |\Delta\psi^2_{n_k}|^2 - \epsilon.$$

Hence possibly extracting subsequences:

$$\begin{cases} \int |\nabla\psi^1_{n_k}|^2 \to_{k\to\infty} \bar{\alpha}_\epsilon \\ \int |\nabla\psi^2_{n_k}|^2 \to_{k\to\infty} \bar{\beta}_\epsilon \\ \int |\Delta\psi^1_{n_k}|^2 \to_{k\to\infty} \mu^\epsilon_1 \\ \int |\Delta\psi^2_{n_k}|^2 \to_{k\to\infty} \mu^\epsilon_2, \end{cases}$$

with

$$|\bar{\alpha}_\epsilon - \alpha| \le \epsilon$$

$$|\bar{\beta}_\epsilon - (\beta-\alpha)| \le \epsilon$$

$$\mu \ge \mu^\epsilon_1 + \mu^\epsilon_2 - \epsilon.$$

We let $k\to\infty$ in (13) and we obtain:

$$I(\lambda,\mu) \ge -K\delta_{\sigma+2}(\epsilon) + I(\bar{\alpha}_\epsilon, \mu^\epsilon_1) + I(\bar{\beta}_\epsilon, \mu^\epsilon_2).$$

Since $\mu^\epsilon_1 + \mu^\epsilon_2 - \epsilon \le \mu$, possibly extracting subsequences: $\mu^\epsilon_1 \to \mu_1$, $\mu^\epsilon_2 \to \mu_2$, with $0 \le \mu_1 + \mu_2 \le \mu$. This leads to

$$I(\lambda,\mu) \ge I(\alpha,\mu_1) + I(\lambda-\alpha,\mu_2).$$

Using the explicit value of $I(\lambda,\mu)$ given by lemma 1, we get that

$$\mu^{3\sigma/4} \le (\frac{\alpha}{\lambda})^{1-\sigma/4}\mu_1^{3\sigma/4} + (1-\frac{\alpha}{\lambda})^{1-\sigma/4}\mu_2^{3\sigma/4} < \mu_1^{3\sigma/4} + \mu_2^{3\sigma/4}$$

which is a contradiction since $\mu \ge \mu_1 + \mu_2$. So dichotomy does not occur.

Suppose that vanishing occurs: We use lemma I1 part II of P.L. Lions [5]:

Lemma 4. *Let $1 < p \le \infty$, $1 \le q < \infty$ with $q \ne \frac{Np}{N-p}$ if $p < N$. Suppose that u_n is bounded in $L^q(R^N)$, and ∇u_n is bounded in $L^p(R^n)$ and*

$$sup_{y\in R^N} \int_{y+B_R} |u_n|^q dx \to_{n\to\infty} 0$$

for one $R > 0$.
Then $u_n \to 0$ as $n\to\infty$ in $L^\alpha(R^N)$ for all $\alpha \in]q, \frac{Np}{N-p}[$

For the proof see [5].
In our case $p = 2$, $N = 3$, $q = 2$. Then $\nabla u_n \to 0$ in $L^{\sigma+2}$, this is a contradiction since $I(\lambda,\mu) < 0$, and vanishing does not occur.

Hence i) occurs:
$*$We define $\tilde{\psi}_n = \psi_n(. + y_n)$, possibly extracting a subsequence we have $\nabla\tilde{\psi}_n \rightharpoonup \nabla\tilde{\psi}$ in H^1 and $L^{\sigma+2}$ weak. Moreover $\lambda \geq \int_{B_R} |\nabla\tilde{\psi}_n|^2 dx \geq \lambda - \epsilon$; then since $\nabla\tilde{\psi}_n$ is bounded in H^1_{loc}, $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^2_{loc} strong and this leads to:

$$\lambda \geq \int_{B_R} |\nabla\tilde{\psi}|^2 \geq \lambda - \epsilon.$$

Letting $R \to \infty$ and $\epsilon \to 0$ gives:

$$\int |\nabla\tilde{\psi}|^2 = \lambda = lim_n \int |\nabla\tilde{\psi}_n|^2.$$

This implies that $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^2 strong.
$*$But $\nabla\tilde{\psi}_n$ is bounded in H^1 and hence in L^6, interpolation implies that: $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^p strong for $2 \leq p < 6$ in particular for $p = \sigma + 2$, it follows that

$$I(\lambda,\mu) = -\int |\nabla\tilde{\psi}|^{\sigma+2}.$$

$*$In order to conclude, we still have to show that $\int |\Delta\tilde{\psi}|^2 = \mu$. We note: $\int |\Delta\tilde{\psi}|^2 = \nu$.
If $\nu = 0$, then $\tilde{\psi} = 0$, and this is a contradiction.
If $0 < \nu < \mu$, Gagliardo-Nirenberg's inequality implies:

$$-I(\lambda,\mu) \leq C\lambda^{1-\sigma/4}\nu^{3\sigma/4},$$

and

$$I(\lambda,\mu) \geq -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} > -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} = I(\lambda,\mu),$$

which is a contradiction, so that $\nu = \mu$, and $\Delta\tilde{\psi} \to \Delta\tilde{\psi}$ in L^2 strong. ■

We still have to prove the second part of Theorem 5. Indeed, we found ϕ satisfying

$$-div(|\nabla\psi|^\sigma\nabla\psi) = \alpha\Delta\phi + \beta\phi$$
$$\Delta\psi = \phi,$$

α and β being Lagrange multipliers. We introduce $\phi(x) = a\tilde{\phi}(bx)$ for $a, b \neq 0$. One obtains:

$$a(\alpha b^2 \Delta\tilde{\psi} + \beta\tilde{\phi}) = -a(\frac{|a|}{|b|})^\sigma div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}).$$

Let us take $a = b$ and $b^2 = \frac{1}{|\alpha|}$, we find:

$$\beta\tilde{\phi} + \nu\Delta\tilde{\phi} = -div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}),$$

where $\nu = \pm 1$.
We first prove a result of regularity:

Lemma 5. *If $\nabla\psi$ is a solution of:*

$$\begin{cases} \beta\tilde{\phi} + \nu\Delta\tilde{\phi} = -div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}) \\ \Delta\psi = \phi, \end{cases}$$

with $\psi \in H$, then $\nabla\psi \in \mathcal{C}^2(R^3)$.

Proof:Let us take $\nabla\psi \in H^s$ with $s \geq 1$, then $\nabla\psi \in L^z$ with $\frac{1}{z} = \frac{1}{2} - \frac{s}{3}$ so that

$$|\nabla\psi|^{\sigma+1} \in L^{\frac{6}{(3-2s)(\sigma+1)}} \subset H^{-s_0}$$

with $s_0 = \frac{(3-2s)(\sigma+1)-3}{2}$. Now $\nabla\psi$ satisfies:

$$\xi.\mathcal{F}(|\nabla\psi|^\sigma \nabla\psi) = \beta\mathcal{F}\phi + \nu|\xi|^2\mathcal{F}\phi$$

and

$$\nu|\xi|^{1-s_0}|\mathcal{F}\phi| \leq |\beta|\frac{|\mathcal{F}\phi|}{|\xi|^{1+s_0}} + |\xi|^{-s_0}|\mathcal{F}(|\nabla\psi|^\sigma \nabla\psi)$$

and all these terms are in $L^2(|\xi| \geq 1)$, hence $\phi \in H^{1-s_0}$ and $\nabla\psi \in H^{2-s_0} = H^{s_1}$. If now we define $s_j = \frac{4-3\sigma}{2} + s_{j-1}(\sigma+1)$ and $s_0 = 1$, then $s_j \to \infty$ therefore if j is sufficiently large, then $\nabla\psi \in H^{3/2+\epsilon}$ which is an algebra, we conclude that $div(|\nabla\psi|^\sigma \nabla\psi) \in H^{1/2+\epsilon}$ and $\phi \in H^{5/2+\epsilon}$; this leads to the result. ∎

Lemma 6. *We have:*

$$-\beta\int|\nabla\psi|^2+\nu\int|\phi|^2=\int|\nabla\psi|^{\sigma+2}.$$

Proof: Let us multiply

$$\beta\phi+\nu\Delta\phi=-div(|\nabla\psi|^{\sigma}\nabla\psi)$$

by $v\in\mathcal{D}(R^3)$, after integration, one obtains:

$$-\beta\int\nabla\psi.\nabla v+\nu\int\phi\Delta v=\int|\nabla\psi|^{\sigma}\nabla\psi.\nabla v.$$

Let $v_n\in\mathcal{D}(R^3)$ such that $v_n\to\psi$ in H, passing to the limit leads to the lemma. ■

Proposition 3. *ψ satisfies*

$$\beta/2\int|\nabla\psi|^2+\nu/2\int|\phi|^2=(1-\frac{3}{\sigma+2})\int|\nabla\psi|^{\sigma+2}$$

Proof: This identity is called Pohozaev Identity, and we proceed as follows: we multiply the equation by $(x.\nabla)\psi$ and then integrate. However, this is not directly possible since we do not know the exact behavior of ψ at infinity. We will use $(x.\nabla\psi)e^{-\epsilon x^2}$ and then we make $\epsilon\to 0$. We leave the detail of the proof in the Appendix. ■

The relationships given by lemma 6 and lemma 3 lead to:

$$\beta\int|\nabla\psi|^2=-\frac{4-\sigma}{2(\sigma+2)}\int|\nabla\psi|^{\sigma+2}$$

and $\beta<0$. On the other hand:

$$\nu\int|\phi|^2=3(\frac{1}{2}-\frac{1}{\sigma+2})\int|\nabla\psi|^{\sigma+2}>0$$

and $\nu=+1$. Let $\omega=-\beta$, we have found ϕ satisfying

$$\begin{cases}-\omega\phi+\Delta\phi=-div(|\nabla\psi|^{\sigma}\nabla\psi)\\ \Delta\psi=\phi\end{cases}$$

and the equalities

$$\begin{cases} \omega \int |\nabla\psi|^2 + \int |\phi|^2 = \int |\nabla\psi|^{\sigma+2} \\ -\omega \int |\nabla\psi|^2 + \int |\phi|^2 = (2 - \frac{6}{\sigma+2}) \int |\nabla\psi|^{\sigma+2}. \end{cases} \tag{14}$$

So we have proved Theorem 5 for **one** ω, by scalling, using the homogeneous properties of the equation, we have a solution for every $\omega > 0$. Moreover, thanks to the relationships (14), the values of $\int |\nabla\psi|^{\sigma+2}$, $\int |\nabla\psi|^2$, $\int |\Delta\psi|^2$ are the same for all the solutions of

$$inf\{-\int |\nabla\psi|^{\sigma+2}, \int |\nabla\psi|^2 = \lambda(\omega), \int |\Delta\psi|^2 = \mu(\omega)\}.$$

■

2.3 Radial standing waves.

The aim of this section is to prove that there exists some radial standing waves for the equation:

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \nabla\psi = \phi, \end{cases} \tag{15}$$

i.e. solution of the form $e^{i\omega t}\phi(x)$ with ϕ depending only on r. Then (15) becomes

$$\begin{cases} -\omega\phi + \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr}) = -\frac{1}{r^2}\frac{d}{dr}(r^2|\frac{d\psi}{dr}|^\sigma\frac{d\psi}{dr}) \\ \phi = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\psi}{dr}) \end{cases} \tag{16}$$

Multiplying by r^2 and then integrating leads to

$$-\omega\psi_r + \frac{d}{dr}(\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\psi}{dr})) = -|\psi_r|^\sigma\psi_r$$

Now let $f = \psi_r$, f satisfies:

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} - (\omega + \frac{2}{r^2})f = -|f|^\sigma f. \tag{17}$$

In order to solve (17), we consider the following minimization problem:

$$I(\lambda,\mu) = inf\{-\int_0^\infty f^{\sigma+2}r^2dr, \int_0^\infty f'^2r^2dr + 2\int_0^\infty f^2dr = \mu, \int_0^\infty f^2r^2dr = \lambda\} \tag{18}$$

About (18), we first prove:

Lemma 7. *$*$For $f \in H^1_r(R^3)$, $\int_0^\infty |f|^2 dr$ makes sense and*

$$\int_0^\infty |f|^2 dr \le 4 \int_0^\infty |\nabla f|^2 r^2 dr.$$

$*$ $I(\lambda, \mu) > -\infty$.

Proof: Since $f \in H^1_r(R^3)$, f as a function of one variable is in $H^1([\alpha, +\infty[)$ for all $\alpha > 0$, and

$$f^2(y) - f^2(x) = \int_x^y 2f(t)f'(t)dt \; \forall x, y.$$

Integrating for $x \in [\epsilon, y]$ leads to

$$f^2(y)(y-\epsilon) - \int_\epsilon^y f^2(x)dx = \int_\epsilon^y (\int_x^y 2f(t)f'(t)dt)dx.$$

We use Fubini's theorem in order to obtain

$$\int_\epsilon^y f^2(x)dx = f^2(y)(y-\epsilon) - \int_\epsilon^y 2f(t)f'(t)(t-\epsilon)dt.$$

Moreover,

$$\int_\epsilon^y |f^2(x)|dx \le |f^2(y)(y-\epsilon)| + 2(\int_\epsilon^y |f^2(t)|dt)^{1/2}(\int_\epsilon^y (t-\epsilon)^2|f'(t)|^2 dt)^{1/2}.$$

This implies that

$$\begin{array}{l}(\int_\epsilon^y |f^2(x)|dx)^{1/2} \le (\int_\epsilon^y (t-\epsilon)^2 |f'(t)|^2 dt)^{1/2} \\ +(\int_\epsilon^y (t-\epsilon)^2|f'(t)|^2 dt + |f^2(y)(y-\epsilon)|)^{1/2}.\end{array} \tag{19}$$

Now since $f \in H^1_r(R^3)$, we have $|f(y)| \le C|y|^{-1}|f|_{H^1}$ (see Berestycki-Lions [1]). Letting $\epsilon \to 0$ and $y \to \infty$ in (19), we get

$$\int_0^\infty f(x)^2 dx \le 4 \int_0^\infty |f'(t)|^2 t^2 dt$$

as claimed.
A Gagliardo-Nirenberg's inequality gives:

$$\int_0^\infty f^{\sigma+2} r^2 dr \le C(\int_0^\infty f^2 r^2 dr)^{1-\sigma/4}(\int_0^\infty f'^2 r^2 dr)^{3\sigma/2},$$

therefore

$$I(\lambda, \mu) \ge -C\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and $I(\lambda, \mu) > -\infty$. ∎

We now solve problem (18).

Proposition 4. *The minimum:*

$$I(\lambda,\mu)=inf\{-\int f^{\sigma+2}r^2dr,\ \int f'^2r^2dr+2\int f^2dr=\mu,\ \int f^2dr=\lambda\}$$

is attained.

Proof: Let us take a minimizing sequence f_n such that

$$-\int_0^\infty f_n^{\sigma+2}r^2dr\to I(\lambda,\mu)$$

and

$$\int_0^\infty f_n'^2r^2dr+2\int_0^\infty f_n^2dr=\mu,\ \int_0^\infty f_n^2r^2dr=\lambda.$$

Then we can suppose that $f_n\rightharpoonup f$ in $H_r^1(R^3)$ weakly and since the injection of $H_r^1(R^3)$ in $L^{\sigma+2}(R^3)$ is compact (see Strauss [8]), then $f_n\to f$ in $L^{\sigma+2}$ strong, so that

$$-\int_0^\infty f_n^{\sigma+2}r^2dr\to-\int_0^\infty f^{\sigma+2}r^2dr.$$

Moreover $f\mapsto\int_0^\infty f^2dr$ is convex and continuous on $H_r^1(R^3)$, thus it is weakly lower semi-continuous and

$$b\equiv\int_0^\infty f'^2r^2dr+2\int_0^\infty f^2dr\le\mu,$$

$$a\equiv\int_0^\infty f^2r^2dr\le\lambda.$$

We define $\tilde f(x)=\alpha f(\beta x)$, with α,β satisfying

$$\lambda=\int\tilde f^2r^2dr=\frac{\alpha^2}{\beta^3}a,$$

$$\mu=\int\tilde f'^2r^2dr+2\int\tilde f^2dr=\frac{\alpha^2}{\beta}b.$$

We obtain

$$-\int\tilde f^{\sigma+2}r^2dr=-\frac{\alpha^{\sigma+2}}{\beta^3}I(\lambda,\mu).$$

By definition of $I(\lambda,\mu)$ we get

$$-\int \tilde{f}^{\sigma+2}r^2dr \geq I(\lambda,\mu).$$

This implies that

$$\frac{\alpha^{\sigma+2}}{\beta^3} \leq 1, \tag{20}$$

since $I(\lambda,\mu)<0$. Now $\frac{\alpha^4}{\beta^4}=\frac{\lambda\mu}{ab}$ and $\beta=(\frac{a\mu}{b\lambda})^{1/2}$ and $\frac{\alpha^{\sigma+2}}{\beta^3}=(\frac{\mu}{b})^{3\sigma/4}(\frac{\lambda}{a})^{1-\sigma/4}\geq 1$. Together with (20), this inequality implies that $\lambda=a$ and $\mu=b$ and the minimum is attained. ■

For all $v\in\mathcal{D}(R^+)$

$$-\int |f|^\sigma f v r^2 dr = 2\alpha\int f v r^2 dr + 2\beta\int \frac{df}{dr}r^2v' dr + 4\beta\int f v dr,$$

so that we found a solution of (16).

We still have to prove Proposition 1.

Proposition 1. *Let $\sigma=4/3$, then for all $t_0>0$, there exists a radial function ψ_0 so that the corresponding solution to:*

$$\begin{cases} i\phi_t+\Delta\phi=-div(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi=\phi \end{cases}$$

blows up exactly at $t=t_0$.

Proof: If we restrict (1) to the radial functions, we obtain

$$iu_t+\Delta u-\frac{2}{|x|^2}u=-|u|^\sigma u, \tag{21}$$

where $u=\psi_r$. This last equation satisfies the pseudo-conformal transformation laws, i.e if $u(x,t)$ is a solution of (21), then the function

$$v(x,t)=\frac{1}{t^{3/2}}e^{\frac{i|x|^2}{4t}}\bar{u}(\frac{x}{t},\frac{1}{t})$$

is a solution to (21) too. If we apply this transformation to any radial standing waves that we found, we obtain the Proposition. ■

3 Stability of standing waves

In the previous section, we found some functions ϕ_ω satisfying

$$\begin{cases} \Delta\phi_\omega - \omega\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{cases} \tag{22}$$

$\nabla\psi_\omega$ are solutions of the following minimization problem

$$(P1)\ inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}.$$

Each solution of $(P1)$ satisfies:

$$\begin{cases} \omega \int |\nabla\psi_\omega|^2 = (\frac{3}{\sigma+2} - \frac{1}{2}) \int |\nabla\psi_\omega|^{\sigma+2}, \\ \int |\phi_\omega|^2 = 3(\frac{1}{2} - \frac{1}{\sigma+2}) \int |\nabla\psi_\omega|^{\sigma+2} \end{cases} \tag{23}$$

and since $\nabla\psi_\omega$ is a solution of $(P1)$, the quantities $\int |\nabla\psi_\omega|^2$, $\int |\nabla\psi_\omega|^{\sigma+2}$ and $\int |\phi_\omega|^2$ do not depend on the solution of $(P1)$ that we consider, but only on ω. The aim of this section is to investigate the stability of these standing waves. A few notations are in order:

$$\begin{cases} E(\psi) = \int (\frac{1}{2}|\Delta\psi|^2 - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx \\ Q(\psi) = \frac{1}{2}\int |\nabla\psi|^2. \end{cases} \tag{24}$$

So that (22) becomes $E'(\psi) + \omega Q'(\psi) = 0$. In order to prove the stability and instability properties, we introduce new minimization problems and we prove:

Proposition 5. *If $\sigma < 4/3$, then the following minimization problem has a solution*

$$(\tilde{P}2)\ min\{E(\psi),\ \psi \in H,\ \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2\} = -a$$

with $a > 0$.
Moreover, let $\psi_n \in H$ such that $E(\psi_n) \to -a$ and $\int |\nabla\psi_n|^2 \to \int |\nabla\psi_\omega|^2$, then there exists $y_n \in R^3$ such that $\psi_n(. - y_n)$ is compact in H.

Proposition 6. *If $0 < \sigma < 4$, then the following minimization problem has a solution:*

$$(\tilde{P}3) Min\{-\frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2,\ \psi \in H,\ \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\} = -b$$

with $b > 0$.

3.1 Proof of the propositions.

3.1.1 First proposition.

We fix $0 < \sigma < 4/3$. We first prove that the minimum is finite. Indeed, Gagliardo-Nirenberg's inequality gives:

$$E(\psi) \geq -C(\int |\Delta\psi|^2)^{3\sigma/4} + \frac{1}{2}\int |\Delta\psi|^2,$$

for $\psi \in H$ such that $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$. Since $\sigma < 4/3$,

$$-C(\int |\Delta\psi|^2)^{3\sigma/4} + \frac{1}{2}\int |\Delta\psi|^2$$

is bounded from below and $-a > -\infty$.
Let us now show that $-a < 0$. Indeed, consider $\psi \in H$ such that $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$ and $\tilde{\psi}(x) = \alpha\psi(\beta x)$ for $\alpha, \beta > 0$, then if $\alpha^2 = \beta$, $\int |\nabla\tilde{\psi}|^2 = \int |\nabla\psi|^2$ and

$$E(\tilde{\psi}) = -\beta^{3\sigma/2}\int \frac{|\nabla\psi|^{\sigma+2}}{\sigma+2} + \frac{\beta^2}{2}\int |\Delta\psi|^2.$$

Since $3\sigma/2 < 2$, for β small, $E(\tilde{\psi}) < 0$ and $-a < 0$.
Using the same technique as for the proof of Theorem 5, in particular the concentration-compactness lemma, one can prove that $(\tilde{P}2)$ has a solution. ■

3.1.2 Second proposition.

Gagliardo-Nirenberg's inequality yields:

$$-\frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2 \geq -C(\int |\nabla\psi|^2)^{1-\sigma/4} + \frac{\omega}{2}\int |\nabla\psi|^2,$$

for all $\psi \in H$ such that

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2.$$

Since $1 - \sigma/4 < 1$, we obtain that $-b > -\infty$.
On the other hand, for $\psi \in H$ with $\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2$, we define: $\tilde{\psi}(x) = \alpha\psi(\frac{x}{\alpha^2})$. We have $\int |\Delta\tilde{\psi}|^2 = \int |\Delta\psi_\omega|^2$, and

$$-\frac{1}{\sigma+2}\int |\Delta\tilde{\psi}|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\tilde{\psi}|^2 = -\frac{1}{\sigma+2}\alpha^{4-\sigma}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\alpha^4\int |\nabla\psi|^2.$$

Thus if α is small enough, this last quantity is negative.
To prove that problem $(\tilde{P}2)$ has a solution, we proceed as for the proof of theorem 5. ■

3.2 Equivalent problems

Problem $(\tilde{P}2)$ is equivalent to:

$$(P2) inf\{\frac{1}{2}\int|\Delta\psi|^2 - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,\ \int|\nabla\psi|^2 = \int|\nabla\psi_\omega|^2\}$$

for $\sigma < \frac{4}{3}$.
Problem $(\tilde{P}3)$ is equivalent to:

$$(P3) inf\{\frac{1}{2}\int|\Delta\psi|^2 - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,\ \int|\Delta\psi|^2 = \int|\Delta\psi_\omega|^2\}$$

for $\sigma < 4$.
We now investigate the relationships between $(P1), (P2)$ and $(P3)$.

Proposition 7. *Let $0 < \sigma < \frac{4}{3}$, then $(P1)$ and $(P2)$ are equivalent.*

Proof: Let us write the Euler-Lagrange equation corresponding to problem $(P2)$: there exists a Lagrange multiplier γ so that:

$$\Delta^2\psi + div(|\nabla\psi|^\sigma\nabla\psi) - \omega\Delta\psi = \gamma\Delta\psi. \qquad (25)$$

It is roughly the same equation as in the proof of Theorem 5, so that the corresponding result of regularity apply and if we multiply (25) by $\bar{\psi}$ and then integrate, we obtain

$$\int|\Delta\psi|^2 - \int|\nabla\psi|^{\sigma+2} + (\omega+\gamma)\int|\nabla\psi|^2 = 0. \qquad (26)$$

Multiplying (25) by $(x.\nabla)\psi$ and then integrating leads to:

$$\frac{1}{2}\int|\Delta\psi|^2 + (\frac{3}{\sigma+2} - 1)\int|\nabla\psi|^{\sigma+2} - \frac{\omega+\gamma}{2}\int|\nabla\psi|^2 = 0. \qquad (27)$$

We make a combination of (26) and (27):

$$2\int|\Delta\psi|^2 = 3(1 - \frac{2}{\sigma+2})\int|\nabla\psi|^{\sigma+2}, \qquad (28)$$

$$(1-\frac{6}{\sigma+2})\int|\nabla\psi|^{\sigma+2}=-2(\omega+\gamma)\int|\nabla\psi|^2. \tag{29}$$

We note $\lambda=\int|\nabla\psi_\omega|^2=\int|\nabla\psi|^2$, we have, using (28) and (29):

$$\int|\nabla\psi|^{\sigma+2}=\frac{2(\sigma+2)(\omega+\gamma)}{4-\sigma}\lambda \tag{30}$$

and

$$\int|\Delta\psi|^2=\frac{3\sigma(\omega+\gamma)}{4-\sigma}\lambda. \tag{31}$$

An analogue calculation, using the corresponding formulas for $\nabla\psi_\omega$ leads to

$$\int|\nabla\psi_\omega|^{\sigma+2}=\frac{2\omega(\sigma+2)}{4-\sigma}\lambda \tag{32}$$

and

$$\int|\Delta\psi_\omega|^2=\frac{3\sigma}{4-\sigma}\lambda. \tag{33}$$

Hence

$$-\frac{1}{\sigma+2}\int|\nabla\psi_\omega|^{\sigma+2}+\frac{1}{2}\int|\Delta\psi_\omega|^2=\frac{\omega\lambda(3\sigma-4)}{2(4-\sigma)} \tag{34}$$

while

$$-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{1}{2}\int|\Delta\psi|^2=\frac{(\omega+\gamma)\lambda(3\sigma-4)}{2(4-\sigma)}. \tag{35}$$

Since $3\sigma-4<0$ and $E(\psi_\omega)\geq E(\psi)$, (34) and (35) imply

$$\gamma\geq 0. \tag{36}$$

Set $\tilde{\psi}(x)=\alpha\psi(\alpha^2x)$, then $\int|\nabla\tilde{\psi}|^2=\int|\nabla\psi|^2$, and we choose α such that $\int|\Delta\tilde{\psi}|^2=\int|\Delta\psi|^2$, this gives:

$$\alpha^4=\frac{\omega}{\omega+\gamma} \tag{37}$$

Using (37), we obtain that

$$-\int|\nabla\tilde{\psi}|^{\sigma+2}=-\alpha^{3\sigma-4}\int|\nabla\psi_\omega|^{\sigma+2}.$$

If we use the definition of $(P1)$, we find

$$-\int |\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{3\sigma-4}\int |\nabla\psi_\omega|^{\sigma+2}$$

or equivalently

$$1 \geq \alpha,$$

and since $3\sigma - 4 < 0$, $\alpha \geq 1$ and $\gamma \leq 0$. Together with (36), this inequality implies

$$\gamma = 0.$$

In fact, we have proved that if ψ is a solution of $(P2)$, and ψ_ω is a solution of $(P1)$, then

$$\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2,$$

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2,$$

$$\int |\nabla\psi|^{\sigma+2} = \int |\nabla\psi_\omega|^{\sigma+2},$$

so that $(P1)$ and $(P2)$ are equivalent, and the proposition is proved. ■

Proposition 8. *Let* $0 < \sigma < 4$, *then* $(P1)$ *and* $(P2)$ *are equivalent.*

Proof: Let us write the Euler-Lagrange equation corresponding to problem $(P3)$

$$div(|\nabla\psi|^\sigma\nabla\psi) - \omega\Delta\psi + \Delta^2\psi = \delta\Delta^2\psi, \tag{38}$$

where δ is a Lagrange multiplier. Multiplying (38) by $\bar{\psi}$ and then integrating leads to:

$$(1-\delta)\int |\Delta\psi|^2 + \omega\int |\nabla\psi|^2 = \int |\nabla\psi|^{\sigma+2}. \tag{39}$$

Multiplying (38) by $(x.\nabla)\psi$ and integrating gives

$$\frac{1-\delta}{2}\int |\Delta\psi|^2 - \frac{\omega}{2}\int |\nabla\psi|^2 = (1 - \frac{3}{\sigma+2})\int |\nabla\psi|^{\sigma+2}. \tag{40}$$

Making linear combination of (39) and (40), we obtain

$$\omega\int |\nabla\psi|^2 = (\frac{3}{\sigma+2} - \frac{1}{2})\int |\nabla\psi|^{\sigma+2} \tag{41}$$

and

$$(1-\delta)\int|\phi|^2 = 3(\frac{1}{2}-\frac{1}{\sigma+2})\int|\nabla\psi|^{\sigma+2}. \tag{42}$$

We recall that for ψ_ω the solution of $(P1)$ we have

$$\omega\int|\nabla\psi_\omega|^2 = (\frac{3}{\sigma+2}-\frac{1}{2})\int|\nabla\psi|^{\sigma+2} \tag{43}$$

and

$$\int|\phi_\omega|^2 = 3(\frac{1}{2}-\frac{1}{\sigma+2})\int|\nabla\psi_\omega|^{\sigma+2}, \tag{44}$$

so that, using (43) and (44), we get that

$$-\frac{1}{\sigma+2}\int|\nabla\psi_\omega|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi_\omega|^2 = -\frac{1}{6}\int|\Delta\psi_\omega|^2.$$

On the other hand, using (41) and (42)

$$-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi|^2 = -\frac{1-\delta}{6}\int|\Delta\psi|^2 = -\frac{1-\delta}{6}\int|\Delta\psi_\omega|^2$$

and necessarily

$$\delta \leq 0. \tag{45}$$

If we let $\mu = \int|\Delta\psi|^2 = \int|\Delta\psi_\omega|^2$, we get:

$$\int|\nabla\psi|^{\sigma+2} = \frac{2(1-\delta)(\sigma+2)}{3\sigma}\mu,\ \int|\nabla\psi_\omega|^{\sigma+2} = \frac{2(\sigma+2)}{3\sigma}\mu \tag{46}$$

and

$$\int|\nabla\psi|^2 = \frac{(1-\delta)(4-\sigma)}{3\omega\sigma},\ \int|\nabla\psi_\omega|^2 = \frac{4-\sigma}{3\omega\sigma}\mu. \tag{47}$$

Set $\tilde{\psi}(x) = \alpha\psi(\frac{x}{\alpha^2})$, then $\int|\Delta\psi|^2 = \int|\Delta\tilde{\psi}|^2$, and we choose α such that $\int|\nabla\tilde{\psi}|^2 = \int|\nabla\psi_\omega|^2$. Using (46) and (47), we obtain $\alpha^4 = \frac{1}{1-\delta}$. Then

$$-\int|\nabla\tilde{\psi}|^{\sigma+2} = -\alpha^{-\sigma}\int|\nabla\psi_\omega|^{\sigma+2},$$

so that

$$-\int|\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{-\sigma}\int|\nabla\psi_\omega|^{\sigma+2}$$

and $\alpha \geq 1$ therefore

$$\delta \geq 0.$$

This inequality with (45) implies that $\delta = 0$, and we conclued as for the previous proposition. ■

3.3 Stability for $\sigma < 4/3$.

Adapting the method of Cazenave-Lions [2], we may prove the second part of Theorem 7. Let S_ω the set of standing waves, solutions of $(P1)$ or $(P2)$. Let us suppose that this set is unstable under the flow of

$$i(\nabla\psi)_t + \Delta(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi). \tag{48}$$

There exists $\psi_n^0 \in H$, $t_n \in R^+$ and $\epsilon > 0$ such that

$$inf_{\psi_\omega\in S_\omega}|\psi_n^0 - \psi_\omega|_H \to 0$$

and the solution $\psi_n(t)$ of (48) with $\psi_n(0) = \psi_0$ satisfies

$$inf_{\psi_\omega\in S_\omega}|\psi_n(t_n,.) - \psi_\omega|_H \geq \epsilon.$$

We have that
$$\begin{cases} \int|\nabla\psi_n^0|^2 \to \int|\nabla\psi_\omega|^2, \\ \int|\nabla\psi_n^0|^{\sigma+2} \to \int|\nabla\psi_\omega|^{\sigma+2}, \\ \int|\Delta\psi_n^0|^2 \to \int|\Delta\psi_\omega|^2. \end{cases}$$

Since E and Q are conserved by the flow of (48), we know that

$$Q(\psi_n(t_n)) \to Q(\psi_\omega)$$

and
$$E(\psi_n(t_n)) \to E(\psi_\omega) = -a.$$

By proposition 5, $\psi_n(t_n)$ is compact up to translation and

$$inf_{\psi_\omega\in S_\omega}|\psi_n(t_n,.) - \psi_\omega|_H \to 0$$

which is a contradiction. ■

3.4 Instability for $\sigma > 4/3$.

The aim of this section is to show how it is possible to adapt the proof of J.Shatah and W.Strauss [7] (which works for the Klein-Gordon equation), to our problem. We introduce

$$d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega).$$

Then, since $(P1)$ and $(P3)$ are equivalent

$$d(\omega) = inf\{E(u) + \omega Q(u),\ u \in H,\ \int |\Delta u|^2 = \int |\Delta \psi_\omega|^2\}.$$

Using the relationships (32) and (33), we obtain that

$$d(\omega) = \frac{1}{3} \int |\Delta \psi_\omega|^2.$$

Let us now find the dependence of $d(\omega)$ in ω. Let $\psi_\omega(x) = \alpha\tilde{\psi}(\beta x)$, $\tilde{\psi}$ satisfies:

$$-\frac{\omega}{\alpha^\sigma \beta^\sigma} \Delta \tilde{\psi} + \frac{1}{\alpha^\sigma \beta^{\sigma-2}} \Delta^2 \tilde{\psi} = -div(|\nabla \tilde{\psi}|^\sigma \nabla \tilde{\psi})$$

we take α and β such that $\alpha^\sigma \beta^{\sigma-2} = 1$ and $\alpha^\sigma \beta^\sigma = \omega$, this leads to:

$$\alpha = \omega^{\frac{1}{\sigma} - \frac{1}{2}},\ \beta = \omega^{1/2}.$$

Then $\tilde{\psi} = \psi_1$ and

$$d(\omega) = \frac{\omega^{\frac{2}{\sigma} - \frac{1}{2}}}{3} \int |\Delta \psi_1|^2.$$

This function is strictly concave if and only if $\frac{4}{3} < \sigma < 4$, and from now on, we fix such a σ.

Lemma 8.

$$d'(\omega) = Q(\psi_\omega).$$

Proof: Indeed, $d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega)$ and

$$d'(\omega) =< E'(\psi_\omega) + \omega Q'(\psi_\omega), \frac{\partial \psi_\omega}{\partial \omega} > +Q(\psi_\omega).$$

But

$$E'(\psi_\omega) + \omega Q'(\psi_\omega) = 0,$$

thereby proving the lemma. ■

Lemma 9. *Fix $\omega = \omega_0$ and $\psi_0 = \psi_{\omega_0}$. The for all C^2 curve $u(\lambda)$ such that $u(0) = \psi_0$ and $Q(u(\lambda)) = Q(\psi_0)$ then*

$$\frac{d^2}{dt^2} E(u(\lambda))|_{\lambda=0} =< (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 >,$$

where $y_0 = u'(0)$.

Proof:

$$\frac{d^2}{d\lambda^2} E(u(\lambda)) =< E''(u(\lambda))u', u' > + < E'(u), u'' > . \tag{49}$$

On the other hand, since Q is constant along the curve:

$$< Q''(u)u', u' > + < Q'(u), u'' >= 0 \tag{50}$$

and (49)+ω_0 (50)$|_{\lambda=0}$ gives the result, since $E'(\psi_0) + \omega_0 Q'(\psi_0) = 0$. ■

We introduce:

$$\nabla\chi(\omega, x) = (\nabla\psi_\omega)(\frac{x}{\lambda(\omega)}),$$

with

$$\lambda^3(\omega) = \frac{Q(\psi_0)}{Q(\psi_\omega)}. \tag{51}$$

Then we have $Q(\chi(\omega, x)) = Q(\psi_0)$.

Lemma 10.

$$\frac{d^2}{d\omega^2} E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0).$$

Proof: we define $\alpha(\omega) = E(\chi(\omega)) - d(\omega) + \omega d'(\omega_0)$, then $\alpha(\omega_0) = 0$ by definition of E, Q and d and lemma 8. Therefore by the definition of χ and since $Q(\chi(\omega)) = Q(\psi_0)$, we have:

$$E(\chi(\omega)) = -\omega Q(\psi_0) + \frac{\lambda}{2}\int |\Delta\psi_\omega|^2 - \frac{\lambda^3}{\sigma+2}\int |\nabla\psi_\omega|^{\sigma+2} + \frac{\lambda^3\omega}{2}\int |\nabla\psi_\omega|^2.$$

Using the relationships (32) and (33), one obtain:

$$\omega \int |\nabla\psi_\omega|^2 = \frac{4-\sigma}{3\sigma}\int |\phi_\omega|^2, \tag{52}$$

$$\int |\nabla\psi_\omega|^{\sigma+2} = \frac{2(\sigma+2)}{3\sigma}\int |\phi_\omega|^2, \tag{53}$$

whence

$$E(\chi(\omega)) = -\omega Q(\psi_0) + (\frac{\lambda}{2} - \frac{\lambda^3}{6})\int |\phi_0|^2.$$

Since $\frac{\lambda}{2} - \frac{\lambda^3}{6} \leq \frac{1}{3}$ for $\lambda \geq 0$ we have:

$$E(\chi(\omega)) \leq -\omega d'(\omega_0) + d(\omega_0)$$

and $\alpha(\omega) \leq 0$, therefore $\alpha(\omega)$ is maximum for $\omega = \omega_0$ and $\frac{\partial^2\alpha}{\partial\omega^2}(\omega_0) \leq 0$. The lemma follows. ■

Lemma 11. *We have:*

$$E(\chi(\omega)) < E(\psi_0),$$

for $\omega \neq \omega_0$, ω near ω_0.

Proof: First we note that

$$\frac{d}{d\omega}E(\chi(\omega))|_{\omega=\omega_0} =< E'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) > . \tag{54}$$

Since $Q(\chi(\omega)) = Cte$,

$$< Q'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) >= 0. \tag{55}$$

(54)+ω_0(55) leads to

$$\frac{d}{d\omega}E(\chi(\omega)) =< E'(\psi_0) + \omega_0 Q'(\psi_0), \frac{d\psi}{d\omega}(\omega_0) >= 0.$$

Moreover $\frac{d^2}{d\omega^2}E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0) < 0$ so that E is localy maximal in ψ_0 and the lemma is proved. ■

We define:

$$y_0 = \frac{\partial\nabla\chi(\omega)}{\partial\omega}|_{\omega=\omega_0}. \tag{56}$$

About y_0, we prove:

Lemma 12. *(a)*$< (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 > \leq d''(\omega_0)$.
(b) $< Q'(\psi_0), y_0 >= -\int \nabla\psi_0.\bar{y}_0 = 0$.
(c) $\int \Delta\psi_0 \ div(y_0) > 0$.

Proof: (a) follows from lemmas 9 and 10.
(b) is trivial.
(c) By definition of χ, $\int |\Delta\chi(\omega)|^2 dx = \lambda \int |\Delta\psi_\omega|^2$, hence by differentiation in $\omega = \omega_0$:

$$2\int \Delta\psi_0 \ div(y_0) = \lambda' \int |\Delta\psi_0|^2 + 3d'(\omega_0), \tag{57}$$

Since $d(\omega) = \frac{1}{3}\int |\Delta\psi_\omega|^2$. But $\lambda^3 = \frac{d'(\omega_0)}{d'(\omega)}$, therefore differentiation in $\omega = \omega_0$ leads to

$$3\lambda'(\omega_0) = -\frac{d''(\omega_0)}{d'(\omega_0)}.$$

But $d''(\omega_0) < 0$, so that $\lambda'(\omega_0) > 0$ and the lemma follows. ∎

Set

$$G = \{f \in (H^1(R^3))^3, \ \exists\psi \in \mathcal{D}' \ such \ that \ f = \nabla\psi\}.$$

We know that (see [4])

$$G = \{\nabla\psi, \ \psi \in L^6 \cap C_0, \ \nabla\psi \in H^1\}.$$

Let I the following injection of G in G':

$$G \to G'$$
$$\nabla u \to I(\nabla u)$$

defined by:

$$< I(\nabla u), \nabla v >= Re \int \nabla u.\nabla\bar{v}.$$

With this preliminaries, we are in the mathematical setting of Shatah-Strauss [7] and we have the result of Theorem 8.

Appendix.

A Proof of proposition 2.

Proposition 3. *ψ satisfies*

$$\beta/2\int|\nabla\psi|^2+\nu/2\int|\phi|^2=(1-\frac{3}{\sigma+2})\int|\nabla\psi|^{\sigma+2}.$$

Proof: We multiply the equation:

$$\beta\phi+\tilde{\epsilon}\Delta\phi=-div(|\nabla\psi|^{\sigma}\nabla\psi)$$

by $(x.\nabla\psi)e^{-\epsilon x^2}$, and we compute every terms:

i) $\int\phi(x.\nabla)\psi e^{-\epsilon x^2}$.
By numerous integrations by parts we get that

$$\int\phi(x.\nabla\psi)e^{-\epsilon x^2}=\frac{1}{2}\int|\nabla\psi|^2e^{-\epsilon x^2}+2\epsilon\int e^{-\epsilon x^2}(x.\nabla\psi)^2-\epsilon\int|\nabla\psi|^2x^2e^{-\epsilon x^2}. \tag{58}$$

When we make $\epsilon\to 0$, $\int|\nabla\psi|^2e^{-\epsilon x^2}\to\int|\nabla\psi|^2$ by Lebesgue's theorem.
The other terms will be treated by:

Lemma 13. *Let $f\in L^1_+(R^3)$ then*

$$\epsilon\int fx^2e^{-\epsilon x^2}dx\to 0,$$

as $\epsilon\to 0$.

Proof: Consider $g(x)=x^2e^{-\epsilon x^2}$, then $|g(x)|\leq\frac{1}{\epsilon}e^{-1}$. Define $r_\epsilon=\frac{1}{\epsilon^{1/4}}$, then $g(r_\epsilon)=\frac{1}{\epsilon^{1/2}}e^{-\epsilon^{1/2}}\leq\frac{1}{\epsilon^{1/2}}$. We have

$$\epsilon\int fx^2e^{-\epsilon x^2}dx=\epsilon\int_{|x|<r_\epsilon}fx^2e^{-\epsilon x^2}dx+\epsilon\int_{|x|>r_\epsilon}fx^2e^{-\epsilon x^2}dx.$$

Hence

$$\epsilon \int f x^2 e^{-\epsilon x^2} dx \le \epsilon^{1/2} \int_{|x|<r_\epsilon} f dx + \int_{|x|>r_\epsilon} f(x) e^{-1} dx$$
$$\le \epsilon^{1/2} \int_{R^3} f dx + \int_{|x|>r_\epsilon} f(x) e^{-1} dx \to 0,$$

as $\epsilon \to 0$ since $r_\epsilon \to \infty$. And the lemma is proved.

■

Using lemma 13, and passing to the limit in (58) leads to:

$$\lim_{\epsilon \to 0} \int \phi(x.\nabla\psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\nabla\psi|^2. \tag{59}$$

ii)$\int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2}$.
By numerous integrations by parts, one obtain:

$$\begin{array}{c} \int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2} = (\frac{3}{\sigma+2} - 1) \int |\nabla\psi|^{\sigma+2} \\ +2\epsilon \int (x.\nabla\psi)^2 |\nabla\psi|^\sigma e^{-\epsilon x^2} - \frac{1}{2}\epsilon \int x^2 |\nabla\psi|^{\sigma+2} e^{-\epsilon x^2}. \end{array} \tag{60}$$

Using lemma 13 and letting $\epsilon \to 0$ in (60) leads to:

$$\lim_{\epsilon \to 0} \int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2} = (\frac{3}{\sigma+2} - 1) \int |\nabla\psi|^{\sigma+2}. \tag{61}$$

iii)$\int \Delta\phi(x.\nabla\psi)e^{-\epsilon x^2}$.
Numerous integrations by parts and the use of lemma 13 leads to:

$$\lim_{\epsilon \to 0} \int \Delta\phi(x.\nabla\psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\phi|^2. \tag{62}$$

From (59), (61) and (62) we deduce

$$\frac{\beta}{2} \int |\nabla\psi|^2 + \frac{\tilde{\epsilon}}{2} \int |\phi|^2 = (1 - \frac{3}{\sigma+2}) \int |\nabla\psi|^{\sigma+2},$$

as claimed in the proposition. ■

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