On the Cauchy problem for a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics, Differential and Integral Equations

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Abstract

In this work, our goal is to study the Cauchy problem for some generalization of the following system.

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases}$$

In particular we shall construct solutions in the energy space associated to this system. We give some sufficient conditions on the initial data which ensure that the solutions are global, but we show that in some cases, finite time blow up occures.

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Contents

1 Introduction

1.1 Derivation of the equations

In a nonlinear plasma, one can observe two types of motions (see S.L. Musher, A.M. Rubenchick and V.E. Zakharov [11]): high-frequency electrons oscillations and low-frequency ones involving ions. We confine ourselves to the consideration of long wave oscillations. This makes it possible to consider low frequency motions as quasi-neutral. The interaction of high frequency oscillations will be neglected, which allows us to describe them using linearized hydrodynamical equations for an electron gaz. Using the Maxwell's equations, we obtain:

$$\frac{1}{c^2}(\frac{\partial^2}{\partial t^2}+\omega_p^2)\mathbf{E}+\mathbf{curl\ curl E}-\frac{3v_{T_e}^2}{c^2}\nabla div\mathbf{E}+\frac{\omega_p^2}{c^2 n_0}\delta_n\mathbf{E}=0,$$

where $\mathbf{E}$ is the electric field, n_0 the density of electron at the state of rest and ω_p the pulsation of the plasma. We consider oscillations with a frequency close to that of the plasma and an electric field of the form:

$$\mathbf{E}=e^{i\omega_p t}\tilde{\mathbf{E}},$$

with $\frac{\partial\tilde{\mathbf{E}}}{\partial t}<<\omega_p\tilde{\mathbf{E}}$. Neglecting the second derivative, we get:

$$2i\omega_p\frac{\partial\tilde{\mathbf{E}}}{\partial t}+c^2\mathbf{curl\ curl}\ \tilde{\mathbf{E}}-3v_{T_e}^2\nabla div\tilde{\mathbf{E}}+\frac{\omega_p^2}{n_0}\delta_n\tilde{\mathbf{E}}=0.$$

If we suppose that the characteristic time of the process is large enough, we may suppose that the electron distribution follows Bolzmann's law, so that:

$$\frac{\delta_n}{n_0}=\frac{|\tilde{\mathbf{E}}|^2}{16\pi n_0(T_e+T_i)}.$$

The equation becomes:

$$2i\omega_p\frac{\partial\tilde{\mathbf{E}}}{\partial t}+c^2\mathbf{curl\ curl}\ \tilde{\mathbf{E}}-3v_{T_e}^2\nabla div\tilde{\mathbf{E}}+\frac{\omega_p^2|\tilde{\mathbf{E}}|^2\tilde{\mathbf{E}}}{16\pi n_0(T_e+T_i)}=0.$$

If furthermore we are in the potential case, i.e. $\tilde{\mathbf{E}}=\nabla\psi$, taking the divergence of the above equation, we get:

$$\Delta(i\psi_t+\frac{3}{2}r_D^2\omega_p\Delta\psi)+\frac{\omega_p}{32\pi n_0(T_e+T_i)}div(|\nabla\psi|^2\nabla\psi)=0,$$

with $r_D=\frac{v_{T_e}}{\omega_p}$ characteristic radius.

By a scaling argument, we obtain finally:

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x). \end{cases} \qquad (1)$$

The goal of this article is to study from the mathematical point of view a larger class of equations:

$$\begin{cases} i\phi_t + \sum_{k,l=1}^{3} a_{kl}\frac{\partial^2\phi}{\partial x_k \partial x_l} = -div(|\nabla\psi|^\sigma\nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x), \end{cases} \qquad (2)$$

where $a_{kl} = a_{lk}$ are real constants, the matrix (a_{kl}) being nonsingular and $\sigma > 0$.

1.2 Conservation laws

If ϕ is a solution of (2), one can show that the following quantities are invariants of the motion:

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0), \qquad (3)$$

$$E(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx = E(0), \qquad (4)$$

where q is the following hermitian form:

$$q(\mathbf{u}) = \sum_{k,l=1}^{3} a_{kl} u_k \bar{u}_l.$$

Indeed, multiplying the first equation of (2) by $\bar{\psi}$ leads, after integration, to:

$$-i\int \nabla\psi_t.\nabla\bar{\psi} + \int a_{kl}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i} = \int |\nabla\psi|^{\sigma+2}. \qquad (5)$$

But $\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$ is real, indeed an integration by parts gives:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}=\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_i}.$$

Hence, taking the imaginary part of (5) leads to:

$$Re\int\nabla\psi_t.\nabla\bar{\psi}=0$$

and we obtain (3). On the other hand, multiplying the first equation of (2) by $\bar{\psi}_t$ and using the same method, we arrive at (with the summation convention):

$$-i\int\nabla\psi_t.\nabla\bar{\psi}_t+\int a_{kl}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i}=\int|\nabla\psi|^\sigma\nabla\psi.\nabla\bar{\psi}_t. \qquad (6)$$

But:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}=\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_i},$$

therefore,

$$Re(\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l})=\frac{1}{2}\int\frac{d}{dt}a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$$

$$=\frac{1}{2}\frac{d}{dt}\int\sum_i q(\nabla\frac{\partial\psi}{\partial x_i}).$$

Since

$$Re(\int|\nabla\psi|^\sigma\nabla\psi.\nabla\bar{\psi}_t)=(\sigma+2)\frac{d}{dt}(\int|\nabla\psi|^{\sigma+2}),$$

we now obtain (4) using (6). All these calculations are somewhat formal, we shall return to this point later on and make them rigorous.

1.3 Statement of the results

In this work, our goal is to study the Cauchy problem (2). In particular we shall construct solutions satisfying the conservation laws (4) and (3) (Theorem 1, 2, 3 and 4). We give sufficient conditions on the initial data which ensure that the solutions are global (see Theorem 4), but also we show that

in certain circumstances, some initial data can lead to finite time blow up (Theorem 5).
We introduce $K = \{\psi \in L^6, \nabla\psi \in L^2\}$ endowed with the norm $|\nabla\psi|_{L^2}$ (see also section 2.1.1).

Theorem 1. *Let $0 < \sigma < \frac{4}{3}$ and $\psi_0 \in K$. Let r be such that $\frac{2}{r} = 3(\frac{1}{2} - \frac{1}{\sigma+2})$, then there exists a unique maximal solution on $[0, T^*[$, $T^* > 0$ to:*

$$\left\{ \begin{array}{c} i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1} div(|\nabla\psi|^\sigma \nabla\psi) \\ \psi(x,0) = \psi_0(x) \end{array} \right.$$

such that $\psi \in C([0, T^*[, K) \cap L^r(0, t, L^{\sigma+2})$ for all $t < T^*$.

Moreover, $\nabla\psi \in L^q(0, t, L^p)$ for every (q, p) satisfying $\frac{2}{q} = 3(\frac{1}{2} - \frac{1}{p})$ with $2 < q \le \infty$ and $t < T^*$.

If $0 < \sigma < \frac{2}{3}$, the solution is unique in $C([0, T^*[, K)$.

Remark 1. *For $\sigma \ge 2/3$, the uniqueness statement holds in*

$$\{\psi \in L^\infty(0, T, K),\ \nabla\psi \in L^r(0, T, L^{\sigma+2})\}$$

and not in $C([0, T[, K)$. This theorem will be proved in third section.

Remark 2. *We can not apply this theorem to the initial system (1). We find here the classical critical value $\sigma = \frac{4}{3}$ for Schrödinger equations, see T.Kato [8].*

We now suppose that the initial value $\nabla\psi_0$ is more regular. We introduce

$$H = \{\psi \in L^6 \cap C_0(R^3), \nabla\psi \in H^1\},$$

endowed with the norm $||\nabla\psi||_{H^1}$ (see next section).

Theorem 2. *Let $0 < \sigma < 4$.*

$*$Let $\psi_0 \in H$, then there exists an unique maximal solution on $[0, T(\psi_0)[$ $\psi \in C([0, T(\psi_0)[, H)$ to:

$$\left\{ \begin{array}{c} i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1} div(|\nabla\psi|^\sigma \nabla\psi) \\ \psi(x,0) = \psi_0(x) \end{array} \right.$$

$*$Moreover, $\phi = \Delta\psi \in L^r(0,t,L^{\sigma+2})$ with $\frac{2}{r} + \frac{3}{\sigma+2} = \frac{3}{2}$ for all $t < T$.

$*$The function ψ is a solution to:

$$\begin{cases} i\phi_t + L\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \nabla\psi = \phi \\ \phi(x,0) = \phi_0(x) \end{cases}$$

and if $0 < \sigma < 3$, it is the only solution in $C([0,T[,H)$.

$*$The solution ψ depends continuously on ψ_0 in $C([0,T[,H)$ in the following sense: if $\psi_0^n \to \psi_0$ in H then for all $T < T(\psi_0)$, if n is sufficiently large, the corresponding solutions exist on a common interval $[0,T]$ and $\psi_n \to \psi$ in $C([0,T],H)$.

Remark 3. *This theorem applies in the physical case (1).*

We have a result of regularity:

Theorem 3. *If $\psi_0 \in H$ with $\nabla\psi_0 \in H^2$, then the solution given by Theorem 2 satisfies*

$$\nabla\psi \in C([0,T(\psi_0)[,H^2).$$

However, the expressions of $m(t)$ and $E(t)$ in (3) and (4) make sense for solutions which take their values in H. We exploit this fact in the following theorem.

Theorem 4. *a)Let $\psi_0 \in H$, then the solution of (2) satisfies:*

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0),$$

$$E(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx = E(0),$$

where q is the following hermitian form:

$$q(u) = \sum_{k,l=1}^{3} a_{kl} u_k \bar{u}_l.$$

b)If the matrix (a_{kl}) is negative, then $||\psi||_H$ remains bounded and the solution is global in time.

c)If the matrix (a_{kl}) is positive, then if $\sigma < \frac{4}{3}$, $||\psi||_H$ remains bounded and the solution is global in time. If $\sigma \geq \frac{4}{3}$, then if $||\psi_0||_H$ is sufficiently small, $||\psi||_H$ remains bounded and the solution is global in time.

d)If ψ is a solution of (1), if $\nabla\psi_0 \in H^{m+1} \cap W^{m+1,\frac{6}{5}}$ for $m \geq 4$, then there exists $\delta > 0$ such that, if

$$||\nabla\psi_0||_{H^{m+1}} + ||\nabla\psi_0||_{W^{m+1,\frac{6}{5}}} < \delta,$$

then the solution is global in time and there exists $\bar{\phi}$ satisfying

$$\begin{cases} i\bar{\phi}_t + \Delta\bar{\phi} = 0 \\ \Delta\bar{\psi} = \bar{\phi}, \end{cases}$$

such that:

$$\begin{cases} ||\nabla\psi||_{W^{m-2,6}} \leq \frac{C}{1+t} \\ ||\nabla\psi||_{H^{m+1}} \leq C \\ ||\nabla\psi - \nabla\bar{\psi}||_{H^{m+1}} \leq \frac{C}{1+t}. \end{cases}$$

Remark 4. *We restrict ourself to equation (1) in the last part of Theorem 4 to avoid technicalities in the estimates in the different spaces $W^{m+1,\frac{6}{5}}$...*

Theorem 3 and 4 will be proved in the next section.
We have the following finite-time blow-up result:

Theorem 5. *Let $\psi_0 \in H$ be such that $\int |x|^2|\nabla\psi|^2 < \infty$, then the solution of (2) with $L = \Delta$ satisfies*

$$|x|\ |\nabla\psi| \in L^\infty(0, t, L^2) \cap L^r(0, t, L^{\sigma+2}),$$

for all $t < T(\psi_0)$.

There exists some radial initial values such that the corresponding solutions blow up in finite time.

This theorem will be proved in next section using the "Virial identity".
The results of this paper were announced in [3].

2 The Cauchy problem with $\nabla\psi_0 \in H^1$.

We are going to solve the local and global Cauchy problems using the standard methods for Schrödinger-type equations (see T. Kato [9] and Ginibre and Velo [7]). However, the nonlinear term here is nonlocal and the equation is for vector valued functions; therefore (2) does not enter directly in the framework of [8] or [9].

2.1 Local Cauchy problem.

Through this section, we take $0 < \sigma < 4$.

2.1.1 The nonlocal term and the fixed point equation.

We need to solve the Poisson equation $\Delta\psi = \phi$ in all R^3. For this purpose, we use:

Lemma 1. *The completion of $\mathcal{D}(R^3)$ for the Dirichlet norm $(\int |\nabla v|^2)^{\frac{1}{2}}$ is exactly $\{v \in L^6, \nabla v \in L^2\}$.*

This result is due to Barros-Neto [1], for the sake of completeness, we give the proof in the Appendix.

Lemma 2. *Let $E = \{\phi \in L^2, \frac{\hat{\phi}}{|\xi|} \in L^2\}$.*

Then, if $\phi \in E$, then there exists a unique function $\psi \in \mathcal{C}_0(R^3) \cap L^6$ such that $\nabla\psi \in H^1$ and $\Delta\psi = \phi$.

Moreover, there exists $C > 0$ such that $(|\psi|_{L^6} + |\psi|_{L^\infty} + |\nabla\psi|_{H^1}) \leq C|\phi|_E$.

We give the proof of this lemma in the Appendix too. This lemma claims that

$$H = \{\psi \in L^6 \cap \mathcal{C}_0(R^3), \nabla\psi \in H^1\}$$

is exactly the space in which we can solve $\Delta\psi = \phi$ with ϕ given in E.

We now introduce:

$$V = \{f \in (\mathcal{D}'(R^3))^3, \exists\psi \in \mathcal{D}'(R^3), \nabla\psi = f\}.$$

We have the well-known characterization of V: $f \in V$ if and only if $\forall v \in (\mathcal{D}(R^3))^3$ with $div(v) = 0$, we have $< f, v >= 0$. With this characterization of V, it is clear that the intersection of V with all "reasonable" functions space is closed. We can now state:

Lemma 3. *We define:*

$$G = \{f \in (H^1)^3, \exists \psi \in \mathcal{D}', \nabla\psi = f\}.$$

Then $G = \{\nabla\psi, \psi \in H\}$.

Moreover, if $\tilde{\psi} \in \mathcal{D}'$ *with* $\nabla\psi \in H^1$*, then there exists* $\psi \in H$ *and a complex number* a *such that* $\tilde{\psi} = a + \psi$.

The proof of this lemma is left in the appendix too.

In order to solve (2):

$$\left\{\begin{array}{c} i\phi_t + \sum_{k,l=1}^{3} a_{kl} \frac{\partial^2\phi}{\partial x_k \partial x_l} = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x), \end{array}\right.$$

we will solve the following "equivalent" system

$$\left\{\begin{array}{c} i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1} div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \\ \psi(x,0) = \psi_0(x). \end{array}\right. \tag{7}$$

Now we give a precise signification to the term $\nabla(-\Delta)^{-1}div$ and to the equivalence between both systems.

Lemma 4. $*$*There exists an operator* B *continuous on every* L^p *for* $1 < p < \infty$*, such that for* $f \in \mathcal{D}(R^3)$ *we have* $Bf = \nabla(-\Delta)^{-1}divf$.

$*$*For* $f \in L^p$*, we have* $divBf = divf$ *and* $Bf \in V$.

Proof: If $f \in \mathcal{D}(R^3)$, we define $(-\Delta)^{-1}divf$ by:

$$(-\Delta)^{-1}divf = c\int \frac{(divf)(y)}{|x-y|}dy$$

where c is a constant. Moreover, $Bf = \nabla(\Delta)^{-1}divf$ makes sense and $divBf = divf$. Now thanks the Calderón-Zygmund theorem (see Folland [5] p138), since $\nabla(-\Delta)^{-1}div$ is homogeneous of order zero in Fourier variable, we have:

$$||Bf||_{L^p} \leq C||f||_{L^p}$$

for $1 < p < \infty$ and for all $f \in \mathcal{D}(R^3)$. We can now extend B to all L^p by continuity, and since the relationship $divBf = divf$ is true in $\mathcal{D}(R^3)$, it is still true in the distribution sense. Now since $V \cap L^p$, is closed in L^p, for all $f \in L^p$ we have that $Bf \in V$. ■

Lemma 5.

There exists an operator C such that if $f \in \mathcal{D}(R^3)$ then

$$\nabla(-\Delta)^{-1}f = Cf$$

and

$$C : L^p \mapsto L^q$$

for $1 < p < 3$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$.

If $g \in L^q$ and $divg \in L^p$ then $C(divg) = Bg$.

Proof: $*$If $f \in \mathcal{D}(R^3)$, we define:

$$\nabla(-\Delta)^{-1}f = c\int \frac{(x-y).f(y)}{|x-y|^3}dy.$$

Using the Riesz potential theory (see Stein [13]), we get that

$$||\nabla(-\Delta)^{-1}f||_{L^q} \leq C||f||_{L^p},$$

for q and p as in the lemma. So that we can extend $\nabla(-\Delta)^{-1}$ in C on all L^p into L^q.

$*$Now the property $\nabla(-\Delta)^{-1}divg = Bg$ is true on $\mathcal{D}(R^3)$, by density it still true for g with the convenient properties. ■

Of course, from now on, we do not use the letters B or C, but the expressions $\nabla(\Delta)^{-1}div$ or $\nabla(-\Delta)^{-1}$.

We have:

Proposition 1. $*$*If $\psi \in C([0,T], H)$ satisfies*

$$\begin{cases} i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi) \\ \Delta\psi = \phi \\ \psi(x,0) = \psi_0(x), \end{cases}$$

then ψ satisfies

$$\begin{cases} i\phi_t + L\phi = -div(|\nabla\psi|^\sigma\nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x). \end{cases}$$

$*$*The converse is true if* $0 < \sigma < 3$.

Proof: The first part is easy, we just have to take the divergence of (7). For the second part, we have to verify that we can apply $\nabla(-\Delta)^{-1}$ on $div(|\nabla\psi|^\sigma\nabla\psi)$, i.e that $div(|\nabla\psi|^\sigma\nabla\psi) \in L^p$ for one $1 < p < 3$. We fix now $0 < \sigma < 3$. We have that $\frac{\partial^2}{\partial x_i \partial x_j}\psi = R_i R_j(\Delta\psi)$, where R_i is the Riesz transform (see Stein [13]), which is continuous from L^p into itself for $1 < p < \infty$; so that we may say that $div(|\nabla\psi|^\sigma\nabla\psi)$ is a sum of terms which behave, in L^p norm, like $|\nabla\psi|^\sigma\Delta\psi$. Hölder's inequality implies that $div(|\nabla\psi|^\sigma\nabla\psi) \in L^{\frac{6}{\sigma+3}}$. Now since $|\nabla\psi|^\sigma\nabla\psi \in L^{\frac{6}{\sigma+1}}$ and $\frac{\sigma+1}{6} = \frac{\sigma+3}{6} - \frac{1}{3}$, we may apply the preceding lemmas and ψ satisfies (7). ■

Now in order to solve (7), we transform it into an integral equation:

$$\nabla\psi = S(t)\nabla\psi_0 + \Lambda\nabla A(\psi)(t) \tag{8}$$

where $\Lambda f(t) = -i\int_0^t S(t-s)f(s)ds$, $A(\psi) = (-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)$ and $S(t)$ is the group generated by the linear Schrödinger equation:

$$i\psi_t + L\psi = 0.$$

The following proposition prove that (8) and (7) are equivalent.

Proposition 2. *Let $\psi \in L^\infty(0, T, H)$, then ψ satisfies*

$$\nabla\psi = S(t)\nabla\psi_0 + \Lambda\nabla A(\psi)(t)$$

if and only if ψ satisfies

$$\begin{cases} i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi) \\ \Delta\psi = \phi \\ \psi(x,0) = \psi_0(x). \end{cases}$$

Proof: $\nabla\psi \in L^\infty(0, T, H^1)$ implies that $\nabla\psi \in L^\infty(0, T, L^6 \cap L^2)$ and $|\nabla\psi|^{\sigma+1} \in L^\infty(0, T, L^{\frac{6}{\sigma+1}})$.
If $\sigma \leq 2$, then $|\nabla\psi|^{\sigma+1} \in L^2$ and

$$\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi) \in L^\infty(0, T, L^2).$$

If $\sigma > 2$, then $\frac{6}{5} < \frac{6}{\sigma+1} < 2$ and there exists $0 < s < 1$ such that $L^{\frac{6}{\sigma+1}} \subset H^{-s}$. Therefore $\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi) \in L^\infty(0, T, H^{-s})$.
In both cases, there exists a $s \leq 0$ such that

$$\nabla(-\Delta)^{-1}div(|\nabla\psi|^{\sigma}\nabla\psi) \in L^{\infty}(0,T,H^{s}).$$

Now:

($\rightarrow$)If ψ is a solution of (7), then lemma 1.3 of Kato [8] implies that $\nabla\psi \in AC([0,T],H^{s-2})$ and ψ satisfies (8).

($\leftarrow$) If ψ is a solution of (8), then lemma 1.1 of Kato [8] implies that:

$$i(S(t)\nabla\psi_0)_t + L(S(t)\nabla\psi_0) = 0,$$

with $S(0)\nabla\psi_0 = \nabla\psi_0$ and

$$\Lambda\nabla A(\psi) \in C([0,T],H^{s}) \cap AC([0,T],H^{s-2}),$$

with

$$i(\Lambda\nabla A(\psi))_t + L(\nabla A(\psi)) = \nabla A(\psi).$$

Furthermore, we have

$$\Lambda(\nabla A(\psi))(0) = 0,$$

therefore $\nabla\psi$ satisfies (2). ■

2.1.2 Resolution of the integral equation

The basic idea of the proof is now to use contraction type arguments in spaces such as $L^{r}(0,T,L^{q})$. These arguments are based on $L^{p} - L^{q}$ estimates for the free propagator (i.e. S(t)) of the linear equation. A few notations are in order.

Let (r,p) two real numbers satisfying $1 \leq r \leq \infty$ and $1 < p < \infty$. The pair (r,p) is said to be admissible if $2 \leq p < 6$ and $\frac{2}{r} = 3(\frac{1}{2} - \frac{1}{p})$. If (r,p) is a pair of numbers, we denote by (r',p') its dual pair, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The following lemma is due to Strichartz [14] for the classical Schrödinger equation, and to Ghidaglia-Saut [6] for the linear equation corresponding to problem (2). For the proof see also Ginibre and Velo [7] or Cazenave and Haraux [2].

Lemma 6. *Let (r,p) and (ρ,γ) two admissible pairs, then:*

$$||\Lambda f||_{L^{r}(0,T,L^{p})} \leq C(r,\rho)||f||_{L^{\rho'}(0,T,L^{\gamma'})}$$

$$||S(.)g||_{L^r(0,T,L^p)} \leq C||g||_{L^2}.$$

The constants are independent of T and one may replace $L^\infty(0,T,L^2)$ by $C([0,T],L^2)$.

In order to apply the contraction principle on (8), we will use the $L^p - L^q$ estimates on $\nabla A(\psi)$. We take r such that $(r, \sigma+2)$ is admissible, and we improve the method of Kato [9]. We introduce the following spaces:

$$X = \{\nabla\psi \in L^\infty(0,T,L^2 \cap V) \cap L^r(0,T,L^{\sigma+2} \cap V)\},$$

$$X' = \{\nabla\psi \in L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}} \cap V)\},$$

$$X_0 = \{\nabla\psi \in L^\infty(0,T,L^2 \cap V) \cap L^\infty(0,T,L^{\sigma+2} \cap V)\},$$

endowed with their natural norms.
With these notations, we may state:

Lemma 7. *∇A maps X_0 in X' and, with $\alpha = \frac{1}{r'} - \frac{1}{r}$:*

$$||\nabla A(\psi_2) - \nabla A(\psi_1)||_{X'} \leq CT^\alpha ||\nabla\psi_1 - \nabla\psi_2||_X sup(||\nabla\psi_1||^\sigma_{X_0}, ||\nabla\psi_2||^\sigma_{X_0}).$$

Proof: The Calderón-Zygmund theorem implies that:

$$||\nabla A(\psi_1) - \nabla A(\psi_2)||_{L^{\frac{\sigma+2}{\sigma+1}}} \leq C||\nabla\psi_1|^\sigma\nabla\psi_1 - |\nabla\psi_2|^\sigma\nabla\psi_2|_{L^{\frac{\sigma+2}{\sigma+1}}}.$$

Using Hölder's inequality, we obtain that this expression is smaller than:

$$Csup(|\nabla\psi_1|^\sigma_{L^{\sigma+2}}, |\nabla\psi_2|^\sigma_{L^{\sigma+2}})|\nabla\psi_1 - \nabla\psi_2|_{L^{\sigma+2}}.$$

Integrating this last inequality with respect to the time, we obtain that:

$$||\nabla A(\psi_1) - \nabla A(\psi_2)||_{L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})}$$

$$\leq Csup(|\nabla\psi_1|^\sigma_{L^\infty(L^{\sigma+2})}, |\nabla\psi_2|^\sigma_{L^\infty(L^{\sigma+2})})|\nabla\psi_1 - \nabla\psi_2|_{L^{r'}(0,T,L^{\sigma+2})}.$$

Furthermore we have that $L^r(0,T,L^{\sigma+2}) \subset L^{r'}(0,T,L^{\sigma+2})$, the norm of the injection being $T^{\frac{1}{r}-\frac{1}{r'}} = T^\alpha$; the lemma follows. ■

This lemma leads to:

Corollary 1. *$\Lambda\nabla A$ maps X_0 into X continuously and boundedly. On each ball $B_R(X_0)$, $\Lambda\nabla A$ is a contraction in the X norm if T is sufficiently small.*

We now state:

Lemma 8. *If ψ_1 and ψ_2 are two solutions of (8) in X_0, then $\nabla\psi_1 = \nabla\psi_2$.*

Proof: We take R sufficiently large such that $\nabla\psi_1, \nabla\psi_2 \in B_R(X_0)$, then $\nabla\psi_1 - \nabla\psi_2 = \Lambda\nabla A(\psi_1) - \Lambda\nabla A(\psi_2)$ and $\Lambda\nabla A$ is a contraction in the X norm if T is sufficiently small. Therefore uniqueness is proved. ■

With these results, we can prove:

Proposition 3. *Let $\psi_0 \in H$, then there exists a unique $\psi \in C([0,T],H)$, solution of (7) and (8) if $0 < \sigma < 4$. Moreover, ψ is a solution of (2) and it is the only one if $\sigma < 3$.*

Proof: Let us introduce:

$$Y = \{\nabla\psi \in X, D^2\psi \in L^\infty(0,T,L^2) \cap L^r(0,T,L^{\sigma+2})\}$$
$$\subset L^\infty(0,T,H^1) \subset L^\infty(0,T,L^{\sigma+2} \cap V),$$
$$Y' = \{\nabla\psi \in X', D^2\psi \in L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})\}.$$

We need:

Lemma 9. *∇A maps Y in Y' boundedly and*

$$||\nabla A(\psi)||_{Y'} \leq CT^\alpha|\psi|_Y^{\sigma+1}.$$

Proof: Lemma 7 and the fact that $Y \subset X_0$ imply that

$$|\nabla A(\psi)|_{X'} \leq CT^\alpha|\nabla\psi|_Y^\sigma|\nabla\psi|_{L^r(0,T,L^{\sigma+2})}.$$

On the other hand, using the Calderón-Zygmund theorem and the continuity of the Riesz transform in the L^p spaces, we get that

$$||\frac{\partial}{\partial x_i}\nabla A(\psi)||_{X'} \leq C||\nabla\psi|^\sigma\Delta\psi|_{X'}.$$

Hölder's inequality in space and time show that this last expression is smaller than

$$CT^\alpha|\nabla\psi|_{X_0}^\sigma|\Delta\psi|_{L^{r'}(0,T,L^{\sigma+2})}.$$

The definition of Y and the fact that $Y \subset X_0$ complete the proof of the lemma. ■

For $\psi_0 \in H$, we note $||S(t)\nabla\psi_0||_Y \le C||\psi_0||_H \equiv R'$.

Lemma 10. *Choose a real number* R *such that* $R > R'$. *If* T *is sufficiently small, the map* $\mathcal{T}(\nabla\psi) = S(t)\nabla\psi_0 + \Lambda\nabla A(\psi)$ *maps* $B_R(Y)$ *into itself and is a contration in the* X *norm.*

Proof: First we have, using lemma 9:

$$||\mathcal{T}(\nabla\psi)||_Y \le C||\psi_0||_H + CT^{1-\alpha}|\psi|_Y^{\sigma+1} \le R' + CT^{1-\alpha}R^{\sigma+1}$$

and this last quantity is smaller than R if T is sufficiently small.

On the other hand, if $\nabla\psi_1, \nabla\psi_2 \in B_R(Y)$, then Corollary 1 and the fact that $Y \subset X_0$ implies

$$||\mathcal{T}(\nabla\psi_1) - \mathcal{T}(\nabla\psi_2)||_X \le k||\psi_1 - \psi_2||_X$$

thereby proving the lemma. ∎

Using the contraction principle, we find a solution of

$$i(\nabla\psi)_t + L(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi),$$

with $\nabla\psi \in C([0,T], H^1)$. Lemma 8 implies that this solution is unique. Since we have worked with functions which take their values in V, $\nabla\psi$ is the gradient of $\psi \in S'$ and Lemma 3 implies that we can choose $\psi \in H$, and the proposition is proved. ∎

To finish the proof of Theorem 2, we have to show the continuous dependence with respect to the initial data. We refer to Kato [9], the estimates that we need for this proof being roughtly the same as in [9].

2.2 The global Cauchy problem

The goal of this section is to prove Theorem 4.

2.2.1 Conservation laws

In the introduction, we made some formal calculations to obtain conservation laws. We want now make this calculations rigourous. Take $v \in \mathcal{D}(R^3)$, the solution of (2) given by Theorem 2 satisfies:

$$i < \phi_t, v > + < L\phi, v >= - < div|\nabla\psi|^\sigma\nabla\psi, v > .$$

An integration by part leads to:

$$-i < \nabla\psi_t, \nabla v > + < a_{kl}\frac{\partial^2\psi}{\partial x_k \partial x_i}, \frac{\partial^2 v}{\partial x_l \partial x_i} >=< |\nabla\psi|^\sigma \nabla\psi, \nabla v >, \qquad (9)$$

with the summation convention and since $\phi \in C^1([0,T], H^{-2})$. Let now $v_n \in \mathcal{D}(R^3)$ such that $\nabla v_n \to \nabla\bar{\psi}$ in $H^1(R^3)$, then passing to the limit and taking the imaginary part of (9) leads to (3).

If we try to apply the same method for the second conservation law (4), then we have to approximate $\nabla\psi_t$ by v_n and then to pass to the limit in the term $< \nabla\psi_t, v_n >$ and this is not possible since $\nabla\psi_t$ is only in H^{-1}. However, the expression of the conservation law in itself make sense for the solution that we found in Theorem 2. We are going to apply the following method: we take a regularization $\psi_0^n \in \mathcal{D}(R^3)$ of ψ_0, we show that the corresponding solutions are regular and that the existence time is the same as for the solution given by Theorem 2; then we pass to the limit using the continuous dependence with respect to the initial values. Let us begin with the regularity.
We introduce:

$$Z = \{\psi \in L^6, \nabla\psi \in H^2\},$$

endowed with the norm $|\nabla\psi|_{H^2}$ and

$$\mathcal{E} = \{\phi, \exists\psi \in Z, \Delta\psi = \phi\} = \{\phi \in H^1, \frac{\hat{\phi}}{|\xi|} \in L^2\},$$

endowed with the norm $|\nabla\phi|_{L^2} + |\frac{\hat{\phi}}{|\xi|}|_{L^2}$. The following proposition proves Theorem 3:

Proposition 4. *For all $\psi_0 \in Z$, there exists $T > 0$ depending only on $|\psi_0|_Z$ and there exists a unique $\phi \in C([0,T[,\mathcal{E})$ such that:*

$$\begin{cases} i\phi_t + L\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x). \end{cases}$$

Then $\phi \in C^1([0,T[,H^{-1})$.

Moreover, the existence time as a solution given by Theorem 2 is equal to T.

Proof: We split the proof in two parts:

i)*Existence and uniqueness.*

We are going to solve the integral equation.

Lemma 11. *Let $G : \mathcal{E} \to \mathcal{E}$ defined by:*

$$G : \phi \mapsto -div(|\nabla\psi|^\sigma \nabla\psi),$$

where $\Delta\psi = \phi$. G maps $\mathcal{E}$ into itself and is Lipschitzian on the bounded subsets of $\mathcal{E}$.

Proof: $H^2(R^3)$ is an algebra, hence $|\nabla\psi|^\sigma\nabla\psi \in H^2$. On the other hand

$$|\frac{\widehat{G(\phi)}}{|\xi|}| \leq |\widehat{|\nabla\psi|^\sigma \nabla\psi}| \in L^2$$

and $G(\phi) \in \mathcal{E}$.

We take $\phi_1, \phi_2 \in \mathcal{E}$ with $|\phi_i|_{\mathcal{E}} \leq A$ for $i = 1, 2$ where A is a positive number. The map $x \mapsto ||x||^\sigma x$ in R^3 is localy Lipschitz and since $H^2(R^3) \subset L^\infty(R^3)$, we have:

$$||\nabla\psi_1|^\sigma\nabla\psi_1 - |\nabla\psi_2|^\sigma\nabla\psi_2|^2 \leq K(A)|\nabla\psi_1 - \nabla\psi_2|^2$$

whence:

$$|\frac{1}{|\xi|}\mathcal{F}(G(\phi_1) - G(\phi_2))|^2_{L^2} \leq K(A)\int |\nabla\psi_1 - \nabla\psi_2|^2 dx,$$

where $\mathcal{F}$ is the Fourier transform.

The same method leads to:

$$\int |\xi|^2 |\mathcal{F}(G(\phi_1) - G(\phi_2))|^2 \leq \int |\xi|^4 K(A)|\mathcal{F}(\nabla\psi_1 - \nabla\psi_2)|^2$$

$$\leq K(A)\int |\nabla\phi_1 - \nabla\phi_2|^2.$$

The lemma follows. ■

By the result of Segal [12], there exists a unique maximal solution of (2), $\phi \in C([0, T(\phi_0)[, \mathcal{E})$ and if $T(\phi_0) < \infty$, then

$$\limsup_{t \to T(\phi_0)} ||\phi(t)||_{\mathcal{E}} = +\infty,$$

thereby proving the first part of the proposition.

ii)*Existence time.*

Since a solution in Z is a solution in H, we have to show that the solution given by the proposition exists, as long as the solution given by Theorem 2 exists. Let $T'(\phi_0)$ the existence time given by Theorem 2. If $T'(\phi_0) > T(\phi_0)$ then $\nabla\psi \in L^\infty(0, T(\phi_0), L^{\sigma+2})$ and $\phi \in L^r(0, T(\phi_0), L^{\sigma+2})$. Moreover

$$\lim_{s \to T(\phi_0)} |\nabla\phi|_{L^\infty(0,s,L^2)} = \infty.$$

We take $\tau < T < T(\phi_0)$. Then $V = \nabla\phi$ satisfies:

$$V(t) = S(t)\nabla\phi(\tau) + a(t) + F(t)[V], \tag{10}$$

with:

$$a(t) = \int_\tau^t S(t-s-\tau)[\text{sum of terms like } \frac{\partial^2\psi}{\partial x_i \partial x_j}\frac{\partial^2\bar{\psi}}{\partial x_i \partial x_k}|\nabla\psi|^{\sigma-1}]ds$$

and

$$F(t)[V] = \int_\tau^t S(t-s-\tau)[\text{sum of terms like } |\nabla\psi|^\sigma R_i R_j[\frac{\partial}{\partial x_k}\phi]]ds$$

where R_i is the Riesz transform (see Stein [13]) which is continuous on every L^p for $1 < p < \infty$, and we can confine ourselves to the consideration of the terms $|\phi|^2|\nabla\psi|^{\sigma-1}$ and $|\nabla\psi|^\sigma\nabla\phi$ in the definition of $a(t)$ and $F(t)$ respectively.

We define $P = L^r(\tau, T, L^{\sigma+2}) \cap L^\infty(\tau, T, L^2)$, endowed with its natural norm.

i)For the linear part, we have

$$|S(t)\nabla\phi(\tau)|_P \leq C|\nabla\phi(\tau)|_{L^2}. \tag{11}$$

ii)Estimate for $a(t)$:

$$|a(t)|_P \leq C|\ |\phi|^2|\nabla\psi|^{\sigma-1}|_{L^{r'}(\tau,T,L^{\frac{\sigma+2}{\sigma+1}})}$$

$$\leq C(T-\tau)^{\alpha}|\nabla\psi|^{\sigma-1}_{L^{\infty}(\tau,T,L^{\sigma+2})}|\phi|_{L^{r}(\tau,T,L^{\sigma+2})}|\phi|_{L^{\infty}(\tau,T,L^{\sigma+2})},$$

where we have used Hölder's inequality in space. Now since

$$\nabla\psi \in L^{\infty}(0,T(\phi_0),L^{\sigma+2})$$

and

$$\phi \in L^{r}(0,T(\phi_0),L^{\sigma+2}) \cap L^{\infty}(0,T(\phi_0),L^{2}),$$

we obtain that:

$$|a(t)|_P \leq C(T-\tau)^{\alpha}(K+|\nabla\phi|_P), \tag{12}$$

where the constants C and K do not depend on T and τ.
iii)Estimate for F:

$$|F(t)V|_P \leq C|\ |\nabla\psi|^{\sigma}\nabla\phi|_{L^{r'}(\tau,T,L^{\frac{\sigma+2}{\sigma+1}})}$$

$$\leq C(T-\tau)^{\alpha}|\nabla\psi|^{\sigma}_{L^{\infty}(\tau,T,L^{\sigma+2})}|\nabla\phi|_{L^{r}(\tau,T,L^{\sigma+2})}.$$

Since $\nabla\psi \in L^{\infty}(0,T(\phi_0),L^{\sigma+2})$, there exists a constant C independent of T and τ such that:

$$|F(t)V|_P \leq C(T-\tau)^{\alpha}|\nabla\phi|_P. \tag{13}$$

Therefore (10), (11), (12), (13) imply:

$$|V|_P \leq C|\nabla\phi(\tau)|_{L^2} + C + C(T-\tau)^{\alpha}|\nabla\phi|_P,$$

which is equivalent to:

$$(1-C(T-\tau)^{\alpha})|V|_P \leq C + C|\nabla\phi(\tau)|_{L^2}. \tag{14}$$

We choose τ such that $1-C(T(\phi_0)-\tau)^{\alpha} > 0$, hence (14) leads to:

$$|V|_{L^{\infty}(\tau,T,L^2)} \leq C.$$

Letting $T \to T(\phi_0)$, we obtain a contradiction and the proposition is proved. ■

Remark 5. *Of course if $\nabla\psi_0 \in H^m$ for a $m \geq 2$, one can prove that $\nabla\psi \in H^m$.*

We can prove:

Proposition 5. *For all $\psi_0 \in Z$, the solution ψ given by Proposition 4 satisfies the conservation of energy (4).*

Proof: For $v \in \mathcal{D}(R^3)$, we have:

$$-i < \nabla\psi_t, \nabla v > +a_{kl} < \frac{\partial^2\psi}{\partial x_k \partial x_i}, \frac{\partial^2 v}{\partial x_l \partial x_i} >=< |\nabla\psi|^\sigma \nabla\psi, \nabla v > . \qquad (15)$$

We choose $v_n \in \mathcal{D}(R^3)$ such that $\nabla v_n \to \nabla\bar{\psi}_t$ in L^2. Then $\frac{\partial^2 v_n}{\partial x_l \partial x_i} \to \frac{\partial^2 \bar{\psi}_t}{\partial x_l \partial x_i}$ in H^{-1}. Passing to the limit in (15) and taking the real part gives the result. ■

We can now show that the second conservation law is satisfied for the solution given by Theorem 2. Let $\psi_0 \in H$ and ψ be the solution given by Theorem 2. We take $\psi_0^n \in \mathcal{D}(R^3)$ such that $\psi_0^n \to \psi_0$ in H, and we denote by ψ^n the corresponding solution. Then:

$$E^n(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi^n}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi^n|^{\sigma+2})dx = E^n(0),$$

as long as ψ^n exists. Since $\psi_0^n \to \psi_0$ in H, the solutions in H (and hence in Z) exist on a common interval $[0, T]$. The result of continuity with respect to the initial data implies that:

$$E(t) \leftarrow E^n(t) = E^n(0) \to E(0)$$

on $[0, T]$ as $n \to \infty$. We conclude that $E(t) = E(0)$ and a) in Theorem 4 is proved.
In order to prove b) in Theorem 4, we remark that if the matrix (a_{kl}) is negative, $E(t)$ controls the L^2 norm of $\Delta\psi$, so that $|\psi|_H$ remains bounded and the solution is global in time.

We now prove c) in Theorem 4. If the matrix (a_{kl}) is positive, then we can use Δ instead of L and (4) becomes:

$$\int (\frac{1}{2}|\phi|^2 - \frac{|\nabla\psi|^{\sigma+2}}{\sigma+2})dx = M.$$

But using Gagliardo-Nirenberg's inequality, one obtains

$$\int |\Delta\psi|^2 \leq M + C(\int |\Delta\psi|^2)^{\frac{3\sigma}{4}} |\nabla\psi_0|_{L^2}^{2-\frac{\sigma}{2}},$$

this leads to an upper bound for $\int |\Delta\psi|^2$ if $\sigma < \frac{4}{3}$.
If $\sigma \geq \frac{4}{3}$, we introduce $f(y) = -C|\nabla\psi_0|_{L^2}\, y^{\frac{3\sigma}{2}} + y^2 - M$. Moreover if $|\nabla\psi_0|_H$ is sufficiently small, then there exists $a, b > 0$ such that $f(x) < 0$ for $0 < x < a$, $f(x) > 0$ if $a < x < b$. Since $f(\int |\Delta\psi|^2) < 0$ if $|\Delta\psi_0|_{L^2}$ is sufficiently small then $|\Delta\psi|_{L^2}$ remains bounded, and c) in Theorem 4 is proved.

2.2.2 Scattering

The aim of this subsection is to prove d) in Theorem 4. We will need the following lemma which is taken from S. Klainerman and G. Ponce [10].

Lemma 12. *If $\Sigma(t)$ is the Schrödinger's group on R^3 then:*

$$|\Sigma(t)u|_{L^q} \leq \frac{C}{(1+t)^{\frac{3}{2}-\frac{3}{q}}}|u|_{W^{N_p,p}}$$

with $N_p > \frac{3(2-p)}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $q \geq 2$.

Proof: the 3-dimensional group splits into three one-dimensional ones; since the one-dimensional group satisfies $|\Sigma(t)|_{L^\infty} \leq \frac{C}{t^{\frac{3}{2}}}|u|_{L^1}$ and since it is unitary in L^2, the lemma follows by interpolation and the Sobolev imbedding theorem (see [10]). ■

In our case, we take $q = 6$, $p = \frac{6}{5}$ and $N_p = 3$. We then have:

$$|\phi|_{W^{m-3,6}} \leq \frac{C}{1+t}|\phi_0|_{W^{m,6/5}} + \int_0^t \frac{C}{1+t-s}|div(|\nabla\psi|^2\nabla\psi)|_{W^{m,6/5}}ds \qquad (16)$$

for $m \geq 3$. In order to use (16), we state:

Lemma 13.

$$|div(|\nabla\psi|^2\nabla\psi)|_{W^{m,6/5}\cap H^m} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}$$

for $m \geq 4$.

Proof: Let z be a multi-index of length between 0 and m. Then:

$$(\phi\psi_{x_i}\bar{\psi}_{x_i})^{(z)} = \sum_{k_1+k_2+k_3=z} C\phi^{(k_1)}\psi_{x_i}^{(k_2)}\bar{\psi}_{x_i}^{(k_3)}.$$

i) We first estimate the norm in $W^{m,6/5}$:

$*$If $|k_2|,\ |k_3| \leq m-2$ then $\phi^{(k_1)} \in L^2$, $\psi^{(k_2)}_{x_i} \in L^6$ and $\bar{\psi}^{(k_2)}_{x_i} \in L^6$ and

$$|\phi^{(k_1)}\psi^{(k_2)}_{x_i}\bar{\psi}^{(k_3)}_{x_i}|_{L^{6/5}} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}.$$

$*$If $|k_3|$ or $|k_2| \geq m-1$ (for example $|k_3|$) then $\phi^{(k_1)} \in L^6$ by Sobolev imbedding, $\psi^{(k_2)}_{x_i} \in L^6$ and $\bar{\psi}^{(k_3)}_{x_i} \in L^6$ and

$$|\phi^{(k_1)}\psi^{(k_2)}_{x_i}\bar{\psi}^{(k_3)}_{x_i}|_{L^{6/5}} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}$$

and the first estimate of Lemma 13 is proved.

ii)Now, we estimate the norm in H^m:

$*$If $|k_1| = m$ then $\phi^{(m)} \in L^2$, $\psi_{x_i}, \bar{\psi}_{x_i} \in L^\infty$ and since for $m \geq 3$, $W^{m-2,6} \subset L^\infty$, we obtain:

$$|\phi^{(k_1)}\psi_{x_i}\bar{\psi}_{x_i}|_{L^2} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}.$$

$*$If $|k_1| \leq m-1$ and $|k_2|, |k_3| \leq m-2$ then:
$\phi^{(k_1)} \in L^6$, $\psi^{(k_2)}_{x_i} \in L^6$ and $\bar{\psi}^{(k_2)}_{x_i} \in L^6$ and

$$|\phi^{(k_1)}\psi^{(k_2)}_{x_i}\bar{\psi}^{(k_3)}_{x_i}|_{L^2} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}.$$

$*$If $|k_3|$ or $|k_2| \geq m-1$ (for example $|k_2|$) then $\phi^{(k_1)} \in L^\infty$ by Sobolev imbedding, $\psi^{(k_2)}_{x_i} \in L^2$ and $\bar{\psi}^{(k_3)}_{x_i} \in L^\infty$ and

$$|\phi^{(k_1)}\psi^{(k_2)}_{x_i}\bar{\psi}^{(k_3)}_{x_i}|_{L^2} \leq C|\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}}$$

and the second estimate of Lemma 13 is proved. ■

On the other hand, since $\Sigma(t)$ is unitary in H^m:

$$|\phi|_{H^m} \leq |\phi_0|_{H^m} + \int_0^t |div(|\nabla\psi|^2\nabla\psi)|_{H^m} ds,$$

hence

$$|\phi|_{H^m} \leq |\phi_0|_{H^m} + C\int_0^t |\phi|_{H^m}|\nabla\psi|^2_{W^{m-2,6}} ds.$$

We define: $M(T) = sup_{0\leq t\leq T}(1+t)|\nabla\psi|_{W^{m-2,6}}$. With this notation, we obtain:

$$|\phi|_{H^m} \leq |\phi_0|_{H^m} + C\int_0^t |\phi|_{H^m}\frac{M(t)^2}{(1+s)^2} ds$$

$$\leq |\phi_0|_{H^m} + C\int_0^t |\phi|_{H^m}\frac{M(T)^2}{(1+s)^2}ds.$$

Therefore Gronwall's lemma implies:

$$|\phi|_{H^m} \leq |\phi_0|_{H^m} exp(CM(T)^2). \tag{17}$$

Together with (16), (17) implies:

$$|\phi|_{W^{m-3,6}} \leq \frac{C}{1+t}|\phi_0|_{W^{m,6/5}} + \int_0^t \frac{C}{1+t-s}|\nabla\psi|^2_{W^{m-2,6}}|\phi_0|_{H^m} exp(C\ M(T)^2)ds \tag{18}$$

If we suppose $|\phi_0|_{H^m} + |\nabla\psi_0|_{W^{m+1,6/5}} \leq \delta$, we get:

$$|\phi|_{W^{m-3,6}} \leq \frac{C}{1+t}|\phi_0|_{W^{m,6/5}} + \int_0^t \frac{C}{1+t-s}\delta\frac{M(T)^2}{(1+s)^2}exp(C\ M(T)^2)ds.$$

In order to conclude, we have to estimate $|\nabla\psi|_{L^6}$; we recall that $\nabla\psi$ satisfies the integral equation (8):

$$\nabla\psi = S(t)\nabla\psi_0 + \Lambda\nabla(-\Delta)^{-1}div(|\nabla\psi|^2\nabla\psi)(t).$$

Using the same estimates than above, we obtain:

$$|\nabla\psi|_{L^6} \leq \frac{C}{1+t}|\nabla\psi_0|_{W^{3,6/5}} + \int_0^t \frac{C}{1+t-s}|\phi|_{H^m}|\nabla\psi|_{W^{m-2,6}}ds,$$

this last estimate and (18) leads to:

$$|\nabla\psi|_{W^{m-2,6}} \leq \frac{C}{1+t}|\nabla\psi_0|_{W^{m+1,6/5}} + \int_0^t \frac{C\delta M(T)^2}{(1+t-s)(1+s)^2}exp(CM(T)^2)ds.$$

Thus:

$$M(T) \leq c_0\delta + C\delta M(T)^2 exp(CM(T)^2),$$

for all T such that the solution exists on $[0,T]$. We define:

$$f(x) = c_0\delta + C\delta x^2 exp(Cx^2) - x$$

if δ is sufficiently small, then there exists $\eta > 0$ such that $f(x) > 0$ on $[0,\eta[$ and $f(x) < 0$ on an interval $]\eta,\eta+\epsilon]$. Since $f(M(T)) \geq 0$ and $M(0) = |\nabla\psi_0|_{W^{m-3,6}}$.Therefore, if δ is sufficiently small, $M(T) < \eta$ for all $T > 0$ and

$$|\nabla\psi|_{W^{m-2,6}} \leq \frac{C}{1+t},$$

$$|\nabla\psi|_{H^{m+1}} \leq C$$

and the solution is global.
We define a function $\tilde{\phi}$ by

$$\tilde{\phi} = \Sigma(t)\phi_0 + \int_0^\infty \Sigma(t-s)div(|\nabla\psi|^2\nabla\psi)ds,$$
$$= \Sigma(t)(\phi_0 + \int_0^\infty \Sigma(-s)div(|\nabla\psi|^2\nabla\psi)ds),$$

so that $\tilde{\phi}$ satisfies

$$i\tilde{\phi}_t + \Delta\tilde{\phi} = 0$$

and

$$|\nabla\psi - \nabla\tilde{\psi}|_{H^{m+1}} \leq \int_t^\infty |\,|\nabla\psi|^2\nabla\psi|_{H^{m+1}}ds$$
$$\leq \int_t^\infty |\phi|_{H^m}|\nabla\phi|^2_{W^{m-2,6}}$$
$$\leq C\int_t^\infty \frac{ds}{(1+s)^2} = \frac{C}{1+t} \to 0$$

and Theorem 4 is proved. ■

2.3 Finite time blow up.

2.3.1 Radial solutions and Virial indentity.

Let us return to the general system in nonlocal form with $L = \Delta$:

$$i(\nabla\psi)_t + \Delta(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi).$$

From now on, we call this equation $NLS\nabla$.
First, we remark:

Lemma 14. *The operator $P = -\nabla(-\Delta)^{-1}div$ is the projector on the set of the gradients of $(L^2(R^3))^3$.*

Proof: The set of grandients is closed and contains the range of P. If we take $\nabla v \in (L^2(R^3))^3$, then:

$$< \nabla(-\Delta)^{-1}div\ u, \nabla v >= - < (-\Delta)^{-1}div\ u, \Delta v >$$
$$=< div\ u, v >= - < u, \nabla v >,$$

the lemma follows. ■

Now every radial vector in $(L^2(R^3))^3$ is a gradient; and so, if we restrict $NLS\nabla$ to radial functions, we obtain

$$i(\nabla\psi)_t + \Delta(\nabla\psi) = -|\nabla\psi|^\sigma \nabla\psi, \tag{19}$$

with $\nabla\psi(0) = \frac{\partial}{\partial r}\psi_0 e_r$, where e_r is the radial unit vector.

Lemma 15. *The solutions of (19) satisfy*

$$\frac{d^2}{dt^2}\int |x|^2|\nabla\psi|^2 dx = -16E(\nabla\psi_0) + \frac{16}{\sigma+2}(1-\frac{3\sigma}{4})\int |\nabla\psi|^{\sigma+2},$$

where E is the energy defined by (4).

The proof is a direct calculus using numerous integrations by part. If $\sigma \geq \frac{4}{3}$ and if the initial value is such that $E(\nabla\psi_0) > 0$, then the solution blows up in finite time. In order to make this argument rigorous, we have to show that the quantity $\int |x|^2|\nabla\psi|^2$ persists, wich will complete the proof of Theorem 5.

2.3.2 Persistance of $\int |x|^2|\nabla\psi|^2$.

Following Ginibre and Velo [7], we introduce the operators

$$J_k(t)\psi_{x_j} = 2it\psi_{x_k x_j} + x_k\psi_{x_j}.$$

These operators commute with the Schrödinger group $S(t)$:

$$J_k(t)S(t)\frac{\partial\psi_0}{\partial x_j} = S(t)(x_k\frac{\partial\psi_0}{\partial x_j}). \tag{20}$$

We use the notation of the proof of Theorem 2 and we denote by $\mathcal{T}$ the following map:

$$\mathcal{T}(\nabla\psi) = S(t)\nabla\psi_0 + \int_0^t S(t-s)\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi).$$

Also,

$$Z = \{\nabla\psi \in Y, |x|\ |\nabla\psi| \in L^r(0,T,L^{\sigma+2}) \cap L^\infty(0,T,L^2)\},$$

endowed with the norm

$$|\nabla\psi|_Y + \sum_{j,k} |J_k\psi_{x_j}|_{L^r(0,T,L^{\sigma+2})\cap L^\infty(0,T,L^2)}.$$

The relevance of this kind of space was pointed out by Ginibre and Velo [7].

Lemma 16. *Let $R' > ||x||\nabla\psi_0||_{L^2}$. If T is sufficiently small then $\mathcal{T}$ maps*

$$B_R(Y) \cap \{|\,|x|\,|\nabla\psi|\,|_{L^2} < R'\}$$

into itself and is a contraction in the X-norm.

Proof: We consider:

$$J_k(t)\mathcal{T}(\nabla\psi) = S(t)x_k\nabla\psi_0$$

$$+\int_0^t S(t-s)[2is\frac{\partial}{\partial x_k}(-P)(|\nabla\psi|^\sigma\nabla\psi) + x_k(-P)|\nabla\psi|^\sigma\nabla\psi]ds.$$

In order to have an estimate in $L^r(L^{\sigma+2}) \cap L^\infty(L^2)$ we need to study three terms:

i)The linear term:

$$|S(t)(x_k\nabla\psi_0)|_{L^r(0,T,L^{\sigma+2})\cap L^\infty(0,T,L^2)} \leq C|x_k\nabla\psi_0|_{L^2}$$

ii) The term:

$$|\int_0^t 2isS(t-s)\frac{\partial}{\partial x_k}((-P)|\nabla\psi|^\sigma\nabla\psi)ds|_{L^r(L^{\sigma+2})\cap L^\infty(L^2)}$$

$$\leq C|s\frac{\partial}{\partial x_k}((-P)(|\nabla\psi|^\sigma\nabla\psi)|_{L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})}.$$

Using the Calderón-Zygmund theorem and Hölder's inequality, we obtain that this last quantity is smaller than

$$CTT^\alpha|\nabla\psi|^\sigma_{L^\infty(0,T,L^{\sigma+2})}|\Delta\psi|_{L^r(0,T,L^{\sigma+2})},$$

where $\alpha = \frac{1}{r'} - \frac{1}{r}$.

iii) The term:

$$|\int_0^t S(t-s)[x_k(-P)|\nabla\psi|^\sigma\nabla\psi]ds|_{L^r(0,T,L^{\sigma+2})\cap L^\infty(0,T,L^2)}$$

$$\leq C inf\{|x_k(-P)|\nabla\psi|^\sigma\nabla\psi|_{L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})}\ ;\ |x_k(-P)|\nabla\psi|^\sigma\nabla\psi|_{L^1(0,T,L^2)}\}.$$

We need to estimate the commutator of x_k and P, wich we do in the following lemma.

Lemma 17. *We have:*

$$inf\{|x_k(-P)|\nabla\psi|^\sigma\nabla\psi|_{L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})}\ ;\ |x_k(-P)|\nabla\psi|^\sigma\nabla\psi|_{L^1(0,T,L^2)}\}$$

$$\leq CT|\nabla\psi|^{\sigma+1}_{L^\infty(0,T,H^1)} + CT^\alpha|\nabla\psi|^\sigma_{L^\infty(0,T,L^{\sigma+2})}|\nabla\psi|_{L^r(0,T,L^{\sigma+2})}$$
$$+CT^\alpha|\nabla\psi|^\sigma_{L^\infty(0,T,H^1)}\ |x_k\nabla\psi|_{L^r(0,T,L^{\sigma+2})}.$$

Assuming this lemma, we get that

$$|J_k(t)\mathcal{T}(\nabla\psi)|_{L^r(0,T,L^{\sigma+2})\cap L^\infty(0,T,L^2)} \leq C|\ |x|\ |\nabla\psi_0|\ |_{L^2}$$

$$+CT^\alpha|\nabla\psi|^\sigma_{L^\infty(0,T,H^1)}(|x|\ |\nabla\psi|\ |_{L^r(0,T,L^{\sigma+2})} + |\Delta\psi|_{L^r(0,T,L^{\sigma+2})});$$

and the proof of Lemma 16 is now complete. ■

We still have to prove Lemma 17.

Proof of lemma 17 :

Set $R = \mathcal{F}(|\nabla\psi|^\sigma\nabla\psi)$. Now the jth component of $\mathcal{F}(x_k(-P)|\nabla\psi|^\sigma\nabla\psi)$ is

$$\frac{\partial}{\partial\xi_k}(\frac{\xi_i\xi_jR_i}{|\xi|^2}) = \delta_{jk}\frac{\xi_iR_i}{|\xi|^2} + \frac{\xi_jR_i}{|\xi|^2} + \frac{\xi_j\xi_i\frac{\partial}{\partial x_k}R_i}{|\xi|^2} - \frac{2\xi_i\xi_j\xi_kR_i}{|\xi|^4}.$$

Since we are interested in the L^p norm of this quantities, it is sufficient (modulo the use of the Riesz transforms) to estimate the three terms

$$(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi),\ |\nabla\psi|^\sigma\nabla\psi\ and\ x_k|\nabla\psi|^\sigma\nabla\psi.$$

i) $|\ |\nabla\psi|^\sigma\nabla\psi|_{L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})}$

$$\leq |\nabla\psi|^\sigma_{L^\infty(0,T,L^2)}|\nabla\psi|_{L^{r'}(0,T,L^{\sigma+2})}$$

$$\leq T^\alpha|\nabla\psi|^\sigma_{L^\infty(0,T,L^2)}|\nabla\psi|_{L^r(0,T,L^{\sigma+2})}.$$

ii) $|\ |x_k|\ |\nabla\psi|^\sigma\nabla\psi|_{L^{r'}(0,T,L^{\sigma+2})}$

$$\leq C|\nabla\psi|^\sigma_{L^\infty(0,T,L^{\sigma+2})}|x_k\nabla\psi|_{L^r(0,T,L^{\sigma+2})}.$$

iii) $(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)$; we know that

$$|\nabla\psi|^\sigma\nabla\psi \in L^\infty(0,T,L^{6/5}).$$

On the other hand,

$$|(-\Delta)^{-1} div f(x)| \leq \int \frac{|f(y)|}{|x-y|^2} dy.$$

The theory of Riesz potentials (see Stein [13]) implies that:

$$|(\Delta)^{-1} div f|_{L^q} \leq C|f|_{L^p},$$

for $1 < p < 3$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Here, we take $p = 6/5$ and $q = 2$ and we obtain

$$|(-\Delta)^{-1} div(|\nabla\psi|^\sigma \nabla\psi)|_{L^\infty(0,T,L^2)} \leq C|\nabla\psi|^{\sigma+1}_{L^\infty(0,T,H^1)},$$

thereby proving Lemma 17. ■

Proposition 6. *Let $\nabla\psi_0 \in H^1$ such that $\int |x|^2|\nabla\psi_0|^2 < \infty$ then the solution $\nabla\psi$ of $NLS\nabla$ satisfies:*

$$|x|\ |\nabla\psi| \in L^\infty(0,t,L^2) \cap L^r(0,t,L^{\sigma+2}) \quad \forall t < T(\psi_0).$$

Proof: We only have to prove that the existence time of the solution in the space Z is the same as the time given by Theorem 2. Let us suppose the contrary, ie $\lim_{t\to T} \int |x|^2|\nabla\psi|^2(t)dx = +\infty$ for some $T < T(\psi_0)$. Taking $\tau < T$ and $\epsilon < T - \tau$, we have that

$$\nabla\psi(T-\epsilon) = S(T-\epsilon-\tau)\nabla\psi(\tau) - i\int_\tau^{T-\epsilon} S(T-\epsilon+\tau-s)(-P)|\nabla\psi|^\sigma\nabla\psi ds.$$

Applying $J_k(T-\epsilon)$, we obtain that

$$|\ |x|\ |\nabla\psi|\ |_{L^\infty(\tau,T-\epsilon,L^2)\cap L^r(\tau,T-\epsilon,L^{\sigma+2})} \leq C|\ |x|\ |\nabla\psi(\tau)|\ |_{L^2}$$

$$+|\int_\tau^{T-\epsilon} J_k(T-\epsilon)S(T-\epsilon+\tau-s)(-P)|\nabla\psi|^\sigma\nabla\psi ds|_{L^r(\tau,T-\epsilon,L^{\sigma+2})\cap L^\infty(\tau,T-\epsilon,L^2)}$$

$$= I\ +\ II.$$

Now using the estimate in the proof of Lemma 16, we see that

$$II \leq C|T-\epsilon-\tau|^\alpha(|\ |x|\ |\nabla\psi| + |\Delta\psi|\ |_{L^r(\tau,T-\epsilon,L^{\sigma+2})})|\nabla\psi|^\sigma_{L^\infty(\tau,T-\epsilon,L^2)}. \quad (21)$$

On the other hand, since

$$|\Delta\psi|_{L^r(\tau,T-\epsilon,L^{\sigma+2})} \leq |\Delta\psi|_{L^r(0,T,L^{\sigma+2})} < \infty$$

and

$$|\nabla\psi|_{L^\infty(\tau,T-\epsilon,L^2)} \leq |\nabla\psi|_{L^\infty(0,T,L^2)} < \infty,$$

we can choose τ such that

$$C|T-\tau|^\alpha |\nabla\psi|_{L^\infty(0,T,L^2)} \leq k < 1.$$

Then (21) implies

$$(1-k)|\ |x|\ |\nabla\psi|\ |_{L^\infty(\tau,T-\epsilon,L^2)\cap L^r(\tau,T-\epsilon,L^{\sigma+2})}$$

$$\leq C|\ |x|\ |\nabla\psi(\tau)|\ |_{L^2} + k|\Delta\psi|_{L^r(0,T,L^{\sigma+2})}.$$

Letting $\epsilon \to 0$, we obtain

$$|\ |x|\ |\nabla\psi|\ |_{L^\infty(\tau,T,L^2)\cap L^r(\tau,T,L^{\sigma+2})} < \infty,$$

which is a contradiction, and the proposition is proved. ■

Remark 6 *In [4], we prove that for $0 < \sigma < 4$, for all $\omega > 0$ there exists a radial function $\psi(r)$ such that $e^{i\omega t}\psi(r)$ is solution to (2); Theses solutions are called standing waves. We also prove that $NLS\nabla$ restricted to the radial functions with $\sigma = 4/3$ satisfies the pseudo-conformal concervation law. Therefore we can build radial initial data such that the corresponding solutions blow up in finite time, at any prescribe time t_0.*

3 Local Cauchy problem with $\nabla\psi_0 \in L^2$.

The goal of this section is to prove Theorem 1. We will use the same technics as for Theorem 2. We recall that

$$X = \{\nabla\psi \in L^\infty(0,T,L^2) \cap L^r(0,T,L^{\sigma+2})\}$$

and

$$X' = \{\nabla\psi \in L^{r'}(0,T,L^{\frac{\sigma+2}{\sigma+1}})\},$$

where $\frac{2}{r} = 3(\frac{1}{2} - \frac{1}{\sigma+2})$. As for Theorem 2 we introduce the integral equation associated to problem (7):

$$(INT)\ \ \mathcal{T}(\nabla\psi) \equiv S(t)\nabla\psi_0 - i\int_0^t S(t-s)\nabla(-\Delta)^{-1}div|\nabla\psi|^\sigma\nabla\psi ds = \nabla\psi$$

Now, as in the proof of theorem 2, $\nabla\psi \in X$ is a solution of (INT) if and only if ψ is a solution of (7).

Lemma 18 *We have:*

$$|\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)|_{X'} \leq CT^\gamma|\nabla\psi|_X^{\sigma+1}$$

and

$$|\nabla(-\Delta)^{-1}div(|\nabla\psi_2|^\sigma\nabla\psi_2 - |\nabla\psi_1|^\sigma\nabla\psi_1)|_{X'}$$

$$\leq CT^\gamma(|\nabla\psi_1|_X^\sigma + |\nabla\psi_2|_X^\sigma)|\nabla\psi_2 - \nabla\psi_1|_X,$$

with $\gamma = \frac{1}{r'} - \frac{\sigma+1}{r} > 0$.

Proof: It is just Hölder's inequality in space and time, and $\gamma > 0$ since $\sigma < \frac{4}{3}$. ∎

Now we take R such that $|S(t)\nabla\psi_0|_X \leq C|\psi_0|_K \equiv R$. We choose $R' > R$ and we have:

Proposition 7 *If T is sufficiently small, $\mathcal{T}$ maps $B_R(X)$ into itself and is a contraction.*

The proof is easy using $L^p - L^q$ estimates and the first part of Theorem 1 is proved.

Now if $0 < \sigma < \frac{2}{3}$ and if $\psi \in C([0,T[,K)$ is a solution of (7) then $|\nabla\psi|^\sigma\nabla\psi \in L^\infty(0,t,L^{6/5+\epsilon})$ for an $\epsilon > 0$ and for all $t < T$. With this regularity, we may deduce that $\nabla\psi$ satisfies (INT), and $L^p - L^q$ estimates imply that $\nabla\psi \in L^r(0,t,L^{\sigma+2})$ for all $t < T$ and therefore it is the only solution. ∎

Appendix.

A Proof of lemma 1.

Lemma 1. *The completion of $\mathcal{D}(R^3)$ for the Dirichlet norm $(\int |\nabla v|^2)^{\frac{1}{2}}$ is exactly $\{v \in L^6, \nabla v \in L^2\}$.*

Proof:
We first note the Sobolev imbedding:

$$|\psi|_{L^6} \leq C|\nabla\psi|_{L^2} \tag{22}$$

for $\psi \in \mathcal{D}(R^3)$. From this inequality, it is clear that the completion of $\mathcal{D}(R^3)$ is inclued in $\{v \in L^6,\ \nabla v \in L^2\}$. For the converse assertion, we have to proceed by truncation and regularization. ■

B Proof of lemma 2.

Lemma 2. *Let $E = \{\phi \in L^2, \frac{\hat{\phi}}{|\xi|} \in L^2\}$.
Then, if $\phi \in E$, there exists a unique function $\psi \in \mathcal{C}_0(R^3) \cap L^6$ such that $\nabla\psi \in H^1$ and $\Delta\psi = \phi$.
Moreover, there exists $C > 0$ such that $(|\psi|_{L^6} + |\psi|_{L^\infty} + |\nabla\psi|_{H^1}) \leq C|\phi|_E$.*

Proof:
We set as in section 3: $K = \{\psi \in L^6, \nabla\psi \in L^2\}$ and take $\phi \in E$. We study the following variationnal problem:
Find ψ such that, for all $v \in K$,

$$\int \nabla\psi.\nabla v = -\int \phi v.$$

If $v \in \mathcal{D}(R^3)$, then $\int \phi v = \int \hat{\phi}\hat{v}$ and since $\phi \in E$,

$$|\int \hat{\phi}\hat{v}| = |\int \frac{\hat{\phi}}{|\xi|}|\xi||\hat{v}|\ | \leq |\phi|_E|\nabla v|_{L^2},$$

so that $v \mapsto \int \phi v$ is a linear form on K. The Lax-Milgram lemma implies that there exists ψ in K satisfying $\Delta\psi = \phi$. Moreover using the Fourier transform, it is easy to see that ψ is in $\mathcal{F}^{-1}L^1$, and Lemma 2 is proved. ■

C Proof of lemma 3.

Lemma 3. *We define:*

$$G = \{f \in (H^1)^3, \exists \psi \in \mathcal{D}', \nabla\psi = f\}.$$

Then $G = \{\nabla\psi, \psi \in H\}$.
Moreover, if $\tilde{\psi} \in \mathcal{D}'$ *with* $\nabla\psi \in H^1$ *then there exists* $\psi \in H$ *and a complex number* a *such that* $\tilde{\psi} = a + \psi$.

Proof:
We take $f \in G$, if $g = \Delta\tilde{\psi} = div f$, then $g \in E$ and thanks to lemma 2, there exists an unique $\psi \in H$ such that $g = \Delta\psi$. It follows that $\Delta(\psi - \tilde{\psi}) = 0$ and Lemma 3 is proved. ■

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