Stability of Piecewise Affine Systems with State-Dependent Delay, and Application to Congestion Control

Christophe Fiter, Emilia Fridman

Stability of Piecewise Affine Systems with State-Dependent Delay, and Application to Congestion Control

Christophe Fiter, Emilia Fridman

Abstract— **In this work, we consider the exponential stability of piecewise affine systems with time- and state-dependent delay, and delayed-state-dependent switching. The stability analysis is based on the use of Lyapunov-Krasovskii functionals, and is divided into two parts. First, global stability conditions are proposed in the case of systems with (state-independent) time-varying delay. Then, local stability conditions are derived in the case of systems with time- and state-dependent delay. In the latter case, estimations of the domain of attraction are also proposed. The theoretical results are applied to the congestion control problem, which can be modelled by such systems.**

I. Introduction

Systems with state-dependent delay appear in different applications, such as biological systems, metal rolling systems, or communication networks [14], [23]. Compared to systems with time-varying delay, they bring new challenges since their state-dependency creates nonlinearities even in the case of linear systems with delay.

These systems have drawn a lot of attention recently especially because of their applications in fluid flow models of communication networks [10], [1], [16], [17], [2], [5]. However, most of these works ignore the state-dependency of the delay: in most of these approaches, the delay is considered constant, and local stability conditions are provided, without any information on the domain of attraction. In [16], it was shown that ignoring the varying part (*i.e.* the state-dependent part) of the delay may lead to conservative results or, worse, to false stability results.

Although some works concerning the stability analysis or the stabilization of systems with state-dependent delay exist ([23], [20], [7], [24], [3]), to our knowledge, none has yet concerned the problem of congestion control of communication networks.

In the present work, we want to provide tools to guarantee the local stability of systems with state-dependent delay and provide estimations of the domain of attraction that could be used for congestion control.

We consider a model of communication network that has been introduced in [1]. In this model, data sent via a computer network is represented by a fluid, and the case of a single bottleneck router is considered. The buffer of the router is modeled as a bucket of infinite capacity which is filled with the fluid with a variable rate. This variable rate represents the rate at which the data source sends its data to the destination. The fluid flows out of the bucket at the constant rate μ which corresponds to the service rate of the router. The rate at which the data is injected into the network is controlled at the source. Then, if the clock is placed at the entrance of the router's buffer, the dynamics of the amount of data in the buffer $y(t)$ can be described by the following system:

$$\begin{aligned} \dot{y}(t) &= \begin{cases} z(t-y(t)/\mu-d)-\mu, & y(t)>0,\\ \max\{0, z(t-y(t)/\mu-d)-\mu\}, & y(t)=0, \end{cases}\\ \dot{z}(t) &= u(t), \end{aligned} \tag{1}$$

where $z(t)$ is the sending rate of the data source, and d is a propagation delay.

Denoting $x_1(t)=y(t)-y_d$ and $x_2(t)=z(t)-\mu$, with y_d the desired amount of data in the router's buffer, and using the approach presented in [2], it is possible to embed the system (1) into the following switched system:

$$\begin{aligned} &\begin{cases} \dot{x}_1(t) = x_2(t-(x_1(t)+y_d)/\mu-d),\\ \dot{x}_2(t) = u(t), \end{cases}\\ &\quad \text{for } x_1(t)>-y_d \text{ or } x_2(t-(x_1(t)+y_d)/\mu-d)\ge 0,\\ &\begin{cases} \dot{x}_1(t) = -x_1(t)-y_d,\\ \dot{x}_2(t) = u(t), \end{cases}\\ &\quad \text{for } x_1(t)\le -y_d \text{ and } x_2(t-(x_1(t)+y_d)/\mu-d)\le 0. \end{aligned} \tag{2}$$

This representation motivates us to analyse the stability of congestion control with tools adapted from works about piecewise affine systems such as [12], [8], [13], or [18]. However, the state dependency of the delay, as well as the delayed-state dependency of the switching law constitute new challenges.

The contribution of this work is twofold:

1) We introduce a generic stability analysis for piecewise affine systems with both a state-dependent delay and a delayed-state-dependent switching law;
2) We provide new stability tools for the congestion control problem which take into account the state-dependency of the delay.

The paper is organized as follows. First, in Section II, we present the class of systems that will be considered throughout the paper, and we formulate the objectives. Then, in Sections III and IV, we present the stability analysis in the case of time-varying delay, and in the case of time- and state-dependent delay respectively. Finally, the application of the theoretical results to the problem of congestion control is presented in Section V, before concluding in Section VI.

This work was partially supported by Israel Science Foundation (grant No 754/10)

The authors work at the Department of Electrical Engineering - Systems, Tel-Aviv University, Tel-Aviv, 69978, Israel. Christophe Fiter is also with Laboratoire d'Automatique, Génie Informatique et Signal (CNRS UMR 8219), Université Lille 1, 59655, Villeneuve dAscq, France.
christophe.fiter@univ-lille1.fr, emilia@eng.tau.ac.il

Notations: $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices. $P\succeq 0_{n\times n}$ (resp. $P\succ 0_{n\times n}$) means that the symmetric matrix $P\in\mathbb{R}^{n\times n}$ is positive (resp. positive definite). $\mathbb{S}_n$, $\mathbb{S}_n^+$, and $\mathbb{S}_n^{+*}$ denote respectively the set of symmetric, symmetric positive, and symmetric positive definite real $n\times n$ matrices. The set of symmetric real $n\times n$ matrices with nonnegative entries is called $\mathbb{SP}_n$. The largest (resp. lowest) eigenvalue of a matrix $M\in\mathbb{S}_n$ is denoted $\lambda_{\max}(M)$ (resp. $\lambda_{\min}(M)$). $\text{rank}(M)$ represents the rank of a matrix $M\in\mathbb{R}^{n\times m}$, and $\text{Ker}(M)$ its kernel. The symmetric elements of a symmetric matrix are written as $*$. $\text{col}\{x_1,\cdots,x_n\}$ represents the vector composed of vectors $x_1,\ \cdots,\ x_n$. The boundary (resp. the closure) of a subset X of $\mathbb{R}^n$ is denoted $\text{bd}(X)$ (resp. $\text{cl}(X)$). The vector inequality $z\geq 0_{n\times 1}$ means that each entry of the vector $z\in\mathbb{R}^n$ is nonnegative. $0_{0\times m}$ (resp. $0_{n\times 0}$ or $0_{0\times 0}$) denotes the empty matrix from $\mathbb{R}^{0\times m}$ (resp. $\mathbb{R}^{n\times 0}$ or $\mathbb{R}^{0\times 0}$). $\|x\|$ denotes the Euclidean norm of $x\in\mathbb{R}^n$, while $\|x\|_L$, for a given matrix $L\succeq 0_{n\times n}\in\mathbb{R}^{n\times n}$, denotes the seminorm (or norm if $L\succ 0_{n\times n}$) $\sqrt{x^TLx}$. The space of functions $\phi:[-h,0]\to\mathbb{R}^n$ which are absolutely continuous on $[-h,0)$, have a finite $\lim_{\theta\to 0^-}\phi(\theta)$, and have square integrable first-order derivatives, is denoted by $\mathcal{W}$ with the norm $\|\phi\|_{\mathcal{W}}=\max_{\theta\in[-h,0]}\|\phi(\theta)\|+\left[\int_{-h}^{0}\|\dot{\phi}(s)\|^2ds\right]^{1/2}$. We also denote $x_t(\theta)=x(t+\theta)$ $(\theta\in[-h,0])$, and $\dot{x}_t(\theta)=\dot{x}(t+\theta)$ $(\theta\in[-h,0])$.

II. Problem statement

We consider the piecewise affine system with a time- and state-dependent delay and delayed-state-dependent switching

$$\dot{x}(t)=A_ix(t)+A_{d_i}x(t-\tau(t,x(t)))+b_i,\ \text{for } G\xi(t)\in X_i, \tag{3}$$

with $x(t)\in\mathbb{R}^n$, $\xi(t)=\begin{bmatrix}x(t)\\ x(t-\tau(t,x(t)))\end{bmatrix}$, matrices A_i, A_{d_i}, b_i of appropriate dimensions, a matrix $G\in\mathbb{R}^{n_G\times 2n}$, and a covering $\{X_i\}_{i\in\mathcal{I}}$ of the space $\mathbb{R}^{n_G}$ into a finite number of (possibly unbounded) polyhedral cells with pairwise disjoint interiors. Without loss of generality, it is considered that $n_G\leq 2n$ and $\text{rank}(G)=n_G$.

We denote $\mathcal{I}_0=\{i\in\mathcal{I}\mid 0_{n_G\times 1}\in\text{cl}(X_i)\}$, and $\mathcal{J}_0=\{i\in\mathcal{I}\mid b_i=0_{n\times 1}\}$.

The objective of the present work is to find conditions that guarantee the local exponential stability of the origin of system (3), for a given decay-rate, and to provide an under-approximation of its domain of attraction. To this aim, we make the following assumptions.

Assumption 1:
The system is linear, or piecewise linear around the origin: $b_i=0_{n\times 1}$ for any $i\in\mathcal{I}_0$ (*i.e.* $\mathcal{I}_0\subseteq\mathcal{J}_0$).

Assumption 2:
The delay is lower-bounded and positive: there exists a scalar $h_0\geq 0$ such that

$$\forall t\geq 0,\ \forall x\in\mathbb{R}^n,\ \tau(t,x)\geq h_0\geq 0. \tag{4}$$

Assumption 3:
The variations of the delay are norm-state-bounded in a neighbourhood of the origin: there exist scalars $c\geq 0$ and $r\geq 0$, a reference delay at the origin $\tau_0\geq h_0$, and matrices L and $\Psi\in\mathbb{S}_n^+$ such that

$$\forall t\geq 0,\forall x\in\mathbb{R}^n,\ \|x\|_L\leq 1\implies|\tau(t,x)-\tau_0|\leq c+\|x\|_\Psi^r. \tag{5}$$

Assumption 1 ensures that the system's origin is an equilibrium, while Assumptions 2 and 3 extend the classic assumptions in the case of constant or time-varying delay, to the more general case of time- and state-dependent delay. For simplicity, no assumption is made regarding the derivative of the delay.

The stability analysis in the next sections will be divided into two main parts. First, in Section III, we will consider the case of bounded time-varying delay and provide global exponential stability conditions. Then, in Section IV, we will consider the case of time- and state-dependent delay, and derive local exponential stability conditions, as well as under-approximations of the domain of attraction.

III. Stability analysis in the case of time-varying delay

In this section, we consider the case of piecewise affine systems with bounded time-varying (but state-independent) delay. Before starting the stability analysis, we need to go through some technical preliminaries.

A. Technical preliminaries

Following the approach introduced in [12] and [8], and further developped in more recent studies [21], [13], or [18], the stability analysis in this work is based on a Lyapunov function with a piecewise quadratic part.

Unlike in the works cited previously however, here the switching occurs not only depending on the state $x(t)$, but also on the delayed state $x(t-\tau(t,x(t)))$, since it is linked to the position of $G\xi(t)$ in the space $\mathbb{R}^{n_G}$. This brings a new difficulty to the stability analysis. Indeed, in order to be able to guarantee the continuity of the Lyapunov function at the switching instants, the existing approaches require the use of a piecewise quadratic part that is based on the extended state $\xi(t)$ which is used in the switching law:

$$\bar{V}_0(t)=\begin{bmatrix}\xi(t)\\ 1\end{bmatrix}^T\bar{P}_i\begin{bmatrix}\xi(t)\\ 1\end{bmatrix},\ \text{for all } G\xi(t)\in X_i,\ \ i\in\mathcal{I}.$$

However in that case, it is clear that the derivative of $\bar{V}_0$ contains the derivative of the delay $\frac{d}{dt}(\tau(t,x(t)))$, which is unknown and potentially unbounded.

In order to deal with this issue, the approach we propose consists in *designing a switching law for the Lyapunov function, that is independent of the delayed state*, by using a piecewise quadratic part of the form

$$\bar{V}_0(t)=\begin{bmatrix}x(t)\\ 1\end{bmatrix}^T\bar{P}_i\begin{bmatrix}x(t)\\ 1\end{bmatrix},\ \text{for all } G'x(t)\in {X'}_i,\ \ i\in\mathcal{I},$$

with a matrix $G'\in\mathbb{R}^{n_{G'}\times n}$ and a covering $\{{X'}_i\}_{i\in\mathcal{I}}$ of the space $\mathbb{R}^{n_{G'}}$ into a finite number of (possibly unbounded) polyhedral cells, designed to satisfy

$$\forall(x,y)\in\mathbb{R}^n\times\mathbb{R}^n,\ \forall i\in\mathcal{I},\ G\begin{bmatrix}x\\y\end{bmatrix}\in X_i\ \Rightarrow\ G'x\in X'_i. \tag{6}$$

1) Design of suitable matrix G' and polyhedral cells X'_i: Let us first rewrite G as $\begin{bmatrix}G_1 & G_2\end{bmatrix}$, with matrices G_1 and G_2 in $\mathbb{R}^{n_G\times n}$. G_1 represents the part of the system's switching law that involves the non-delayed state $x(t)$, while G_2 represents the part that involves the delayed state $x(t-\tau(t,x(t)))$. Let $(\lambda_j)_{j\in\{1,\cdots,n_G-\text{rank}(G_2)\}}$ be a basis of $\text{Ker}(G_2^T)$, and design $G'=\begin{bmatrix}\lambda_1^T\\ \vdots\\ \lambda_{n_G-\text{rank}(G_2)}^T\end{bmatrix}G_1$.

Let us now consider the orthogonal projection from the space generated by $G\xi$ (*i.e.* $\mathbb{R}^{n_G}$) to the space generated by $G'x$ (*i.e.* $\mathbb{R}^{n_{G'}}$) defined as:

$$\begin{aligned} p\ &:\ \mathbb{R}^{n_G}\longrightarrow\mathbb{R}^{n_{G'}}\\ &z=G\begin{bmatrix}x\\y\end{bmatrix}\longmapsto p(z)=\begin{bmatrix}\lambda_1^T\\ \vdots\\ \lambda_{n_{G'}}^T\end{bmatrix}z=G'x\end{aligned}$$

Then, if we design the polyhedral cells X'_i in the space $\mathbb{R}^{n_{G'}}$ as the image of the polyhedral cells X_i in the space $\mathbb{R}^{n_G}$ by the orthographic projection p, it is clear by construction that the relation (6) is satisfied.

Such a construction is presented for the congestion control problem (Section V), in Figures 1 and 2.

2) Introduction of some notions and notations: First, we introduce some elements that will be used to relax the LMI stability conditions, as in the framework of [12].

Since the cells defining the covering of $\mathbb{R}^{n_G}$ are polyhedra, we can construct matrices $\bar{E}_i=\begin{bmatrix}E_i & e_i\end{bmatrix}$, with $E_i\in\mathbb{R}^{n_{E_i}\times n_G}$ and $e_i\in\mathbb{R}^{n_{E_i}}$, such that

$$\bar{E}_i\begin{bmatrix}G\xi\\1\end{bmatrix}\geq 0_{n_{E_i}\times 1},\ \text{for } G\xi\in X_i,\ i\in\mathcal{I}. \tag{7}$$

It is assumed that the matrices $\bar{E}_i$ are designed such that $e_i=0_{n_{E_i}\times 1}$ for all $i\in\mathcal{J}_0$.

Similarly, we can construct matrices $\bar{E}'_i=\begin{bmatrix}E'_i & e'_i\end{bmatrix}$, with $E'_i\in\mathbb{R}^{n_{E'_i}\times n_{G'}}$ and $e'_i\in\mathbb{R}^{n'_{E_i}}$, with $e'_i=0_{n_{E'_i}\times 1}$ for all $i\in\mathcal{I}'_0\cup\mathcal{J}_0$, such that

$$\bar{E}'_i\begin{bmatrix}G'x\\1\end{bmatrix}\geq 0_{n_{E'_i}\times 1},\ \text{for } G'x\in X'_i,\ i\in\mathcal{I},$$

with the notation $\mathcal{I}'_0=\{i\in\mathcal{I}\mid 0_{n_{G'}\times 1}\in\text{cl}(X'_i)\}$.

We also introduce some elements that will be used to ensure the continuity of the Lyapunov function at the switching instants, as in the framework of [8]. An important difference here is that the cells X'_i that we designed previously are not necessarily of disjoint interiors.

We denote $\bar{\mathcal{N}}'_i$ the cells whose interior is not disjoint with the interior of X'_i, and $\mathcal{N}'_i$ the cells whose boundary share a common facet with the boundary of the cell X'_i. Such a facet between the boundaries of the cells X'_i and X'_j (with $i\in\mathcal{I}$ and $j\in\mathcal{N}'_i$), is contained in the hyperplan $\{z\in\mathbb{R}^{n_{G'}}\mid c'^T_{ij}z-d'_{ij}=0\}$, where $c'_{ij}\in\mathbb{R}^{n_{G'}}$ and $d'_{ij}\in\mathbb{R}$. Following [8], we use a parametric description of these hyperplans of $\mathbb{R}^{n_{G'}}$:

$$\text{bd}(X'_i)\cap\text{bd}(X'_j)\subseteq\{l'_{ij}+F'_{ij}s\mid s\in\mathbb{R}^{n_{G'}-1}\}, \tag{8}$$

for all $i\in\mathcal{I}$ and $j\in\mathcal{N}'_i$, where $F'_{ij}\in\mathbb{R}^{n_{G'}\times(n_{G'}-1)}$ is a full rank matrix whose columns span the null space of c'^T_{ij} (they can be designed following the Gram-Schmidt process for example), and $l'_{ij}\in\mathbb{R}^{n_{G'}}$ is given by $l'_{ij}=c'_{ij}(c'^T_{ij}c'_{ij})^{-1}d'_{ij}$.

B. Main stability conditions

Theorem 1: Consider scalars $h_0\leq\bar{h}_1\leq\bar{h}_2$ and $\alpha>0$. If there exist $P\in\mathbb{S}_n^{+*}$, S_1, S_2, R_1, $R_2\in\mathbb{S}_n^{+}$, $S_{12}\in\mathbb{R}^{n\times n}$, $P_i\in\mathbb{S}_{n_{G'}}$, $W_i\in\mathbb{SP}_{n_{E'_i}}$, $U_i\in\mathbb{SP}_{n_{E_i}}$, $q_i\in\mathbb{R}^{n_{G'}}$, and $r_i\in\mathbb{R}$, for $i\in\mathcal{I}$ ($q_i=0_{n_{G'}\times 1}$ and $r_i=0$ if $i\in\mathcal{I}'_0\cup\mathcal{J}_0$), satisfying the LMIs $\begin{bmatrix}R_2 & S_{12}\\ * & R_2\end{bmatrix}\succeq 0_{2n\times 2n}$,

$$\begin{bmatrix}\Omega_{11}+\bar{G}^T\bar{E}_i^TU_i\bar{E}_i\bar{G} & \Omega_{12} & \Omega_{13} & \begin{bmatrix}A_i^TH\\A_{d_i}^TH\\b_i^TH\end{bmatrix}\\ * & \Omega_{22} & \Omega_{23} & 0_{n\times n}\\ * & * & \Omega_{33} & 0_{n\times n}\\ * & * & * & -H\end{bmatrix}\preceq 0_{(5n+1)\times(5n+1)}, \tag{9}$$

$$\bar{G'}^T(\bar{P}_i-\bar{E'}_i^TW_i\bar{E}'_i)\bar{G}'\succeq 0_{(n+1)\times(n+1)}, \tag{10}$$

for all $i\in\mathcal{I}$, as well as the equality constraints

$$F'^T_{ij}(P_i-P_j)F'_{ij}=0_{(n_{G'}-1)\times(n_{G'}-1)}, \tag{11a}$$

$$F'^T_{ij}(P_i-P_j)l'_{ij}+F'^T_{ij}(q_i-q_j)=0_{(n_{G'}-1)\times 1}, \tag{11b}$$

$$l'^T_{ij}(P_i-P_j)l'_{ij}+2(q_i-q_j)^Tl'_{ij}+(r_i-r_j)=0, \tag{11c}$$

for all $i\in\mathcal{I},\ j\in\mathcal{N}'_i\ (i<j)$,

$$\bar{P}_i=\bar{P}_j,\ \text{for all } i\in\mathcal{I},\ j\in\bar{\mathcal{N}}'_i\ (i<j), \tag{11d}$$

with the notations

$$\bar{G}=\begin{bmatrix}G & 0_{n_G\times 1}\\ 0_{1\times 2n} & 1\end{bmatrix},\ \bar{G}'=\begin{bmatrix}G' & 0_{n_{G'}\times 1}\\ 0_{1\times n} & 1\end{bmatrix},$$

$$\bar{P}_i=\begin{bmatrix}P_i & q_i\\ * & r_i\end{bmatrix},\ \Omega_{11}=\begin{bmatrix}\Omega_{11}^{(11)} & \mathcal{P}_iA_{d_i} & \Omega_{11}^{(13)}\\ * & \Omega_{11}^{(22)} & A_{d_i}^TG'^Tq_i\\ * & * & \Omega_{11}^{(33)}\end{bmatrix},$$

$$\Omega_{11}^{(11)}=A_i^T\mathcal{P}_i+\mathcal{P}_iA_i+2\alpha\mathcal{P}_i+S_1-\mathrm{e}^{-2\alpha\bar{h}_1}R_1,$$

$$\Omega_{11}^{(13)}=\mathcal{P}_ib_i+A_i^TG'^Tq_i+2\alpha G'^Tq_i,$$

$$\Omega_{11}^{(22)}=\mathrm{e}^{-2\alpha\bar{h}_2}(-2R_2+S_{12}+S_{12}^T),$$

$$\Omega_{11}^{(33)}=b_i^TG'^Tq_i+q_i^TG'b_i+2\alpha r_i,$$

$$\Omega_{12}=\begin{bmatrix}\mathrm{e}^{-2\alpha\bar{h}_1}R_1\\ \mathrm{e}^{-2\alpha\bar{h}_2}(R_2-S_{12})^T\\ 0_{1\times n}\end{bmatrix},\ \Omega_{13}=\begin{bmatrix}0_{n\times n}\\ \mathrm{e}^{-2\alpha\bar{h}_2}(R_2^T-S_{12})\\ 0_{1\times n}\end{bmatrix},$$

$$\Omega_{22}=\mathrm{e}^{-2\alpha\bar{h}_1}(S_2-S_1-R_1)-\mathrm{e}^{-2\alpha\bar{h}_2}R_2,$$

$$\Omega_{23}=\mathrm{e}^{-2\alpha\bar{h}_2}S_{12},\ \Omega_{33}=-\mathrm{e}^{-2\alpha\bar{h}_2}(R_2+S_2),$$

$$\mathcal{P}_i=P+G'^TP_iG',\ H=\bar{h}_1^2R_1+(\bar{h}_2-\bar{h}_1)^2R_2, \tag{12}$$

then the origin of system (3) with state-independent time-varying delay $\bar{\tau}(t)=\tau(t,x(t))$ in $[\bar{h}_1,\bar{h}_2]$ is globally exponentially stable with a decay rate α.

Proof: Consider the switched Lyapunov-Krasovskii functional (LKF)

$$\bar{V}(t) \triangleq V(t, x_t, \dot{x}_t) = \bar{V}_0(t) + \bar{V}_1(t), \tag{13}$$

with

$$\bar{V}_0(t) \triangleq V_0(x(t)) = \bar{x}^T(t)\bar{\mathcal{P}}_i\bar{x}(t), \text{ for } G'x(t) \in X'_i,\ i \in \mathcal{I}, \tag{14a}$$

$$\bar{V}_1(t) \triangleq V_1(t, x_t, \dot{x}_t) = \sum_{k=1}^{2} \int_{t-\bar{h}_k}^{t-\bar{h}_{k-1}} \mathrm{e}^{2\alpha(s-t)} x^T(s) S_k x(s) ds + \sum_{k=1}^{2} (\bar{h}_k - \bar{h}_{k-1}) \int_{-\bar{h}_k}^{-\bar{h}_{k-1}} \int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)} \dot{x}^T(s) R_k \dot{x}(s) ds d\theta, \tag{14b}$$

where $\bar{x}(t) = \mathrm{col}\{x(t), 1\}$, $\bar{h}_0 = 0$, and
$\bar{\mathcal{P}}_i = \bar{P} + \bar{G}'^T \bar{P}_i \bar{G}' = \begin{bmatrix} \mathcal{P}_i & G'^T q_i \\ * & r_i \end{bmatrix}$, with $\bar{P} = \begin{bmatrix} P & 0_{n\times 1} \\ * & 0 \end{bmatrix}$.

The condition (10) (along with $P \in \mathbb{S}_n^{+*}$) ensures the positive definiteness of V_0 (and thus of $\bar{V}$, since S_1, S_2, R_1, and $R_2 \in \mathbb{S}_n^{+}$), while the conditions (11) guarantee its continuity at the switching instants.

Furthermore, it is clear that since $q_i = 0_{n_{G'}\times 1}$ and $r_i = 0$ when $i \in \mathcal{I}'_0$ (*i.e.* in a neighbourhood of the origin), then one can design scalars $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 \|x(t)\|^2 \le \bar{V}(t) \le \delta_2 \|x_t\|^2_{\mathcal{W}}$, and therefore global exponential stability with a decay rate α will be guaranteed if $\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \le 0$ for all $t \ge 0$.

By computing $\dot{\bar{V}}(t) + 2\alpha\bar{V}(t)$ and by using some classic upper-bounds such as Jensen's inequality ([6], Appendix B) and the convex approach from [19], we can show that

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \le \pi^T(t) \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix} \pi(t) + [A_i x(t) + A_{d_i} x(t-\bar{\tau}(t)) + b_i]^T H [A_i x(t) + A_{d_i} x(t-\bar{\tau}(t)) + b_i], \tag{15}$$

with the notations in (12), and $\pi(t) = \mathrm{col}\{x(t), x(t-\bar{\tau}(t)), 1, x(t-\bar{h}_1), x(t-\bar{h}_2)\}$. Further using the Schur complement ([4], Section 2.1) and the S-procedure ([4], Section 2.6.3), we can show that LMI (9) guarantees that $\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \le 0$ for all $G\xi(t) \in X_i$ and $i \in \mathcal{I}$. ∎

IV. Regional stability in the case of state-dependent delay

Now that the case of systems with time-varying delay has been treated, we will start to study the case of systems with time-and state-dependent delay. Note that unlike in the time-varying case, here, the state dependency in the delay introduces a nonlinearity in the system. Therefore, we may only be able to guarantee local stability.

If the state x belongs to a neighbourhood of the origin, then, thanks to Assumptions 2 and 3, it is possible to find bounds $\bar{h}_1$ and $\bar{h}_2$ on the delay:

$$\tau(t, x) \in [\bar{h}_1, \bar{h}_2]. \tag{16}$$

Because of the symmetry in (5) with respect to τ_0, we can design such bounds as

$$\bar{h}_1 = \max(h_0, \tau_0 - \eta),\ \bar{h}_2 = \tau_0 + \eta, \tag{17}$$

for some scalar $\eta \ge c$ which represents the upper-bound on the difference between $\tau(t, x)$ and the delay of reference at the origin τ_0.

Now, let us consider a neighbourhood of the origin in the form of an ellipsoid defined by

$$\mathcal{X}_\beta(P) = \{x \in \mathbb{R}^n \mid x^T P x \le \beta\}, \tag{18}$$

for some matrix $P \in \mathbb{S}_n^{+*}$ and scalar $\beta > 0$. We have the following property.

Lemma 2: Consider $\eta \ge c$, $\bar{h}_1$ and $\bar{h}_2$ defined in (17), $\beta > 0$, and $P \in \mathbb{S}_n^{+*}$. Under Assumptions 2 and 3, if there exist scalars $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfying the LMIs

$$\gamma_1 \begin{bmatrix} -L & 0_{n\times 1} \\ * & 1 \end{bmatrix} - \begin{bmatrix} -P & 0_{n\times 1} \\ * & \beta \end{bmatrix} \succeq 0_{(n+1)\times(n+1)}, \tag{19a}$$

$$\gamma_2 \begin{bmatrix} -\Psi & 0_{n\times 1} \\ * & (\eta - c)^{\frac{2}{r}} \end{bmatrix} - \begin{bmatrix} -P & 0_{n\times 1} \\ * & \beta \end{bmatrix} \succeq 0_{(n+1)\times(n+1)}, \tag{19b}$$

then for any $t \ge 0$ and $x \in \mathcal{X}_\beta(P)$, $\tau(t, x) \in [\bar{h}_1, \bar{h}_2]$.

Proof: Consider a scalar $t \ge 0$. The S-procedure ([4], Section 2.6.3) ensures that if there exists a scalar $\gamma_1 > 0$ such that (19a) is satisfied, then $x^T L x \le 1$ for all $x \in \mathcal{X}_\beta(P)$ (*i.e.* such that $x^T P x \le \beta$), and thus $\|x\|_L \le 1$. Then, using Assumption 3, one shows that $|\tau(t, x) - \tau_0| \le c + \|x\|^r_\Psi$ for any $x \in \mathcal{X}_\beta(P)$.
Let us now analyse the terms $\bar{h}_2 - \tau(t, x)$ and $\tau(t, x) - \bar{h}_1$ using this property. One can show that $\bar{h}_2 - \tau(t, x) \ge \bar{h}_2 - \tau_0 - c - \|x\|^r_\Psi = \eta - c - \|x\|^r_\Psi$, which is positive for any $x \in \mathcal{X}_\beta(P)$ according to (19b) and the S-procedure. For the second term, two cases may occur. If $\eta > \tau_0 - h_0$, then $\bar{h}_1 = h_0$ and $\tau(t, x) - \bar{h}_1$ is positive according to Assumption 2. Otherwise, with a similar analysis as with the first term, one shows that $\tau(t, x) - \bar{h}_1 \ge \tau_0 - \bar{h}_1 - c - \|x\|^r_\Psi = \eta - c - \|x\|^r_\Psi$, which has been proved to be positive for any $x \in \mathcal{X}_\beta(P)$. Therefore, for any $t \ge 0$ and $x \in \mathcal{X}_\beta(P)$, $\tau(t, x) \in [\bar{h}_1, \bar{h}_2]$. ∎

A. Main results

Theorem 3: Consider a parameter $\eta \ge c$ and a scalar $\alpha > 0$. Consider $\bar{h}_0 = 0$, and $\bar{h}_1$ and $\bar{h}_2$ designed in (17). Under Assumptions 2 and 3, if there exist parameters satisfying the conditions of Theorem 1 and scalars $\gamma_1 > 0$, $\gamma_2 > 0$, $\beta > 0$, such that the LMIs (19) are satisfied, then the origin of system (3) is locally exponentially stable with a decay rate α. Furthermore, an under-approximation of the domain of attraction is provided by

$$\mathcal{D}_{\mathcal{W}} = \{\phi \in \mathcal{W} \mid V(0, \phi, \dot{\phi}) \le \beta\}, \tag{20}$$

with V the Lyapunov-Krasovskii functional defined in (13). In the case of constant initial conditions $\phi \in \mathcal{W}$ (*i.e.* such that $\phi(t) = x_0$ for all $t \in [-\bar{h}_2, 0]$), an under-approximation of the domain of attraction can be written explicitly as

$$\mathcal{D} = \bigcup_{i\in\mathcal{I}} \left(\{x_0 \in \mathbb{R}^n \mid G'x_0 \in X'_i\} \cap \left\{ x_0 \in \mathbb{R}^n \mid \begin{bmatrix} x_0 \\ 1 \end{bmatrix}^T B_i \begin{bmatrix} x_0 \\ 1 \end{bmatrix} \le \beta \right\} \right), \tag{21}$$

with

$$B_i = \begin{bmatrix} \mathcal{P}_i + \sum_{k=1}^{2} \kappa_{S_k} S_k & {G'}^T q_i \\ * & r_i \end{bmatrix}, \tag{22}$$

and

$$\kappa_{S_k} = \frac{\mathrm{e}^{-2\alpha \bar{h}_{k-1}} - \mathrm{e}^{-2\alpha \bar{h}_k}}{2\alpha}. \tag{23}$$

Proof: Theorem 1 guarantees that the system is stable and converges towards the origin with a decay rate α if $\bar{\tau}(t) \triangleq \tau(t, x(t)) \in [\bar{h}_1, \bar{h}_2]$ for all $t \geq 0$. Considering the LKF $\bar{V}$ defined in (13), we find that if the initial conditions $\phi \in \mathcal{W}$ of the system are such that $V(0, \phi, \dot{\phi}) = \bar{V}(0) \leq \beta$, then $x^T(0) P x(0) \leq \bar{V}(0) \leq \beta$, and thus $\bar{\tau}(0) \in [\bar{h}_1, \bar{h}_2]$, according to Lemma 2. Then, according to the analysis from the time-varying delay case, one gets $x^T(t) P x(t) \leq \bar{V}(t) \leq \bar{V}(0) \leq \beta$, and thus $\bar{\tau}(t) \in [\bar{h}_1, \bar{h}_2]$, for all $t \geq 0$, which proves the local exponential stability with a domain of attraction containing the under-approximation $\mathcal{D}_{\mathcal{W}}$.
In the case of constant initial condition, simple computations show that $\bar{V}(0) = \begin{bmatrix} x(0) \\ 1 \end{bmatrix}^T B_i \begin{bmatrix} x(0) \\ 1 \end{bmatrix}$, with B_i defined in (22), and it is clear that $\bar{V}(0) \leq \beta$ if $x(0) \in \mathcal{D}$. ■

Theorem 3 provides an under-approximation of the domain of attraction in the general case ($\mathcal{D}_{\mathcal{W}}$), and in the particular case of constant initial conditions ($\mathcal{D}$), which arises often in practice. The approach can be adapted to other kinds of initial conditions, such as discontinuous ones for example (see [15]).

Remark 1: One can obtain good estimates of the domain of attraction $\mathcal{D}$ (in the case of constant initial conditions) by using a linear search on the scalar η, and convex optimization algorithms similar to the ones used in [9], [11] and [22].

V. Application to Congestion Control

A. System description

We consider the fluid flow model of communication network (1), with a linear controller

$$u(t) = -k_1(y(t) - y_d) - k_2(z(t) - \mu). \tag{24}$$

Using a similar approach to the one presented in the introduction, and denoting $x_1(t) = y(t) - y_d$ and $x_2(t) = z(t) - \mu$, we embed (1) and (24) in the switched system (3), with

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2, \; G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; \mathcal{I} = \{1, 2, 3\},$$

$$\tau(t, x(t)) \triangleq \tau(x(t)) = \max\left(\frac{x_1(t) + y_d}{\mu} + d, d \right),$$

polyhedral cells presented in Figure 1 and defined by

$$X_1 = \{G\xi(t) \mid x_1(t) \leq -y_d \text{ and } x_2(t - \tau(t, x(t))) \leq 0\},$$
$$X_2 = \{G\xi(t) \mid x_1(t) \leq -y_d \text{ and } x_2(t - \tau(t, x(t))) \geq 0\},$$
$$X_3 = \{G\xi(t) \mid x_1(t) > -y_d\},$$

and system matrices

$$A_1 = A_2 = \begin{bmatrix} -1 & 0 \\ -k_1 & -k_2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix},$$

$$A_{d_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_{d_2} = A_{d_3} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$b_1 = b_2 = \begin{bmatrix} -y_d \\ 0 \end{bmatrix}, \; b_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The parameters from the polyhedral covering of $\mathbb{R}^{n_G}$ are designed as:

$$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; E_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; E_3 = \begin{cases} 0_{0\times 2} & \text{if } y_d > 0, \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \text{if } y_d = 0, \end{cases}$$
$$e_1 = \begin{bmatrix} -y_d \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} -y_d \\ 0 \end{bmatrix}, \; e_3 = \begin{cases} 0_{0\times 1} & \text{if } y_d > 0, \\ \begin{bmatrix} 0 \end{bmatrix} & \text{if } y_d = 0, \end{cases}$$

and the parameters from the polyhedral covering of $\mathbb{R}^{n_{G'}}$ (which are presented in Figure 2) are chosen as:

$$G' = [1 \; 0],$$
$$X'_1 = X'_2 = \{G'x(t) \mid x_1(t) \leq -y_d\},$$
$$X'_3 = \{G'x(t) \mid x_1(t) > -y_d\},$$
$$E'_1 = E'_2 = [-1], \; E'_3 = \begin{cases} 0_{0\times 1} & \text{if } y_d > 0, \\ \begin{bmatrix} 1 \end{bmatrix} & \text{if } y_d = 0, \end{cases}$$
$$e'_1 = [-y_d], \; e'_2 = [-y_d], \; e'_3 = \begin{cases} 0_{0\times 1} & \text{if } y_d > 0, \\ \begin{bmatrix} 0 \end{bmatrix} & \text{if } y_d = 0, \end{cases}$$
$$l'_{13} = l'_{23} = [-y_d], \; F'_{13} = F'_{23} = 0_{1\times 0}.$$

Finally, the parameters related to the delay (in Assumptions 2 and 3) are set as

$$h_0 = d, \; \tau_0 = \frac{y_d}{\mu} + d, \; c = 0, \; r = 1, \; \Psi = \begin{bmatrix} \frac{1}{\mu^2} & 0 \\ * & 0 \end{bmatrix}.$$

Since the condition $|\tau(t, x) - \tau_0| \leq c + \|x\|_\Psi^r$ in Assumption 3 is satisfied for all $t \geq 0$ and $x \in \mathbb{R}^n$, the matrix L in (5) can be taken arbitrarily. In that case, fixing $L = \frac{1}{\beta} P$ helps reducing the complexity, since the LMI (19a) in Lemma 2 becomes automatically satisfied with $\gamma_1 = \beta$.

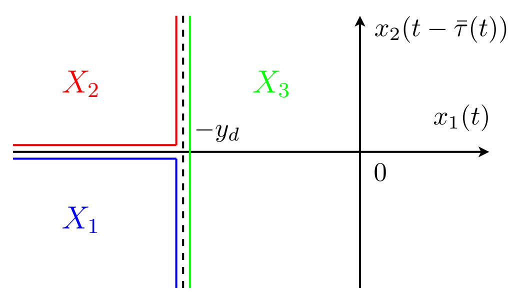

Fig. 1. Delay-dependent regions on which the system's switching law is defined, in the space $\mathbb{R}^{n_G}$ generated by $G\xi(t)$

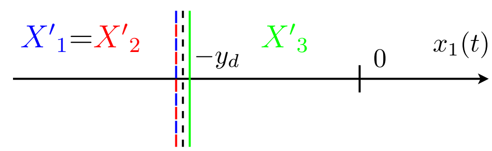

Fig. 2. Delay-free regions on which the switching law of the Lyapunov function's matrices is defined, in the space $\mathbb{R}^{n_{G'}}$ generated by $G'x(t)$

B. Results

For the congestion control problem, in addition to the system's stability, we need to verify that the amount of data in the buffer $y(t) = x_1(t) + y_d \geq 0$ and that the sending rate of the data source $z(t) = x_2(t) + \mu \geq 0$. The first condition is guaranteed by the system's dynamics, while the second can be enforced by imposing the positively invariant set $\mathcal{X}_\beta(P)$ to be restricted to the half plane $\mathcal{H} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_2 \geq -\mu \right\}$, by adding the LMI condition

$$\begin{bmatrix} \mu\beta & \begin{bmatrix} 0 & -\beta \end{bmatrix} \\ * & P \end{bmatrix} \succeq 0. \tag{25}$$

For our simulations, we have chosen a desired decay rate $\alpha = 0.01$, and system parameters $d = 0.8$, $\mu = 4$, $y_d = 0.2$, and $k_1 = k_2 = 1$.

The local stability conditions from Theorem 3 (with additional LMI (25)) have solutions for any $\eta \in [c, \eta^* = 0.161]$. Using a linear search on η and a convex optimization algorithm (see remark 1), we can therefore obtain an under-approximation $\mathcal{D}$ of the domain of attraction, as well as the associated positive-invariant set $\mathcal{X}_\beta(P)$. These two domains, as well as simulation results, are shown in Figure 3.

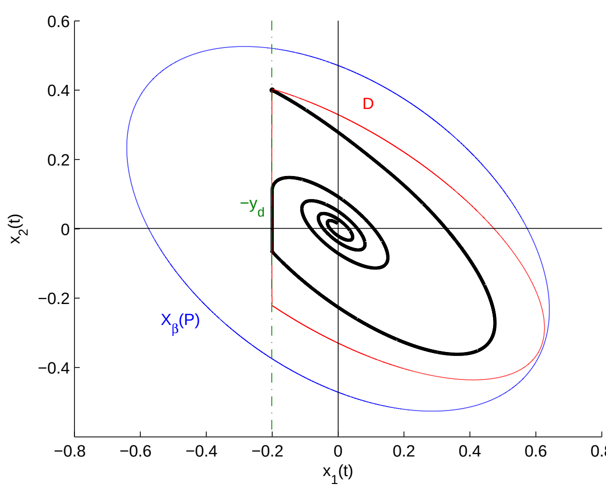

Fig. 3. Estimation $\mathcal{D}$ of the domain of attraction, ellipsoid $\mathcal{X}_\beta(P)$, and evolution of the system's state

VI. Conclusion

We have proposed a novel stability analysis for piecewise affine systems with time- and state-dependent delay, and delayed-state-dependent switching. The approach is based on a Lyapunov-Krasovskii functional with piecewise quadratic terms that allows for obtaining exponential stability conditions under the form of LMIs. The case of (possibly fast) time-varying delay is treated, before considering the case of time- and state-dependent delay. In that latter case, the stability domain is approximated by a union of ellipsoids, whose volume can be enlarged thanks to a linear optimization problem. An application to congestion control is then presented.

References

[1] K. Avrachenkov and W. Paszke. State-dependent delay system model for congestion control. In *1st International Conference on Informatics in Control, Automation, and Robotics*, Setbal, Portugal, 2004.

[2] H. Balakrishnan, N. Dukipati, N. McKeown, and C.-J. Tomlin. Stability analysis of explicit congestion control protocols. *IEEE Communication Letters*, 11(10):823–825, 2007.

[3] N. Bekiaris-Liberis and M. Krstic. Compensation of state-dependent input delay for nonlinear systems. *IEEE Transactions on Automatic Control*, 58(2):275–289, 2013.

[4] S. Boyd, L. El-Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. SIAM, Studies in Applied Mathematics, 1994.

[5] C. Briat, H. Hjalmarsson, K.-H. Johansson, U.-T. Jönsson, G. Karlsson, and H. Sandberg. Nonlinear state-dependent delay modeling and stability analysis of internet congestion control. In *49th IEEE Conference on Decision and Control*, pages 1484–1491, Atlanta, Georgia, USA, 2010.

[6] K. Gu, V. Kharitonov, and J. Chen. *Stability of time-delay systems*. Boston: Birkhauser, 2003.

[7] I. Gyori and F. Hartung. Exponential stability of a state-dependent delay system. *Discrete and Continuous Dynamical Systems*, 18(4):773–791, 2007.

[8] A. Hassibi and S. Boyd. Quadratic stabilization and control of piecewise-linear systems. In *American Control Conference*, Philadelphia, Pennsylvania, USA, 1998.

[9] H. Hindi and S. Boyd. Analysis of linear systems with saturation using convex optimization. In *37th IEEE Conference on Decision and Control*, Tampa, Florida, USA, 1998.

[10] C.-V. Hollot, V. Misra, D. Towley, and W. Gong. Analysis and design of controllers for AQM routers supporting TCP flows. *IEEE Transactions on Automatic Control*, 47(6):945–959, 2002.

[11] M. Johansson. Piecewise quadratic estimates of domains of attraction for linear systems with saturation. In *IFAC World Congress*, Barcelona, Spain, 2002.

[12] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, 1998.

[13] V. Kulkarni M. Jun and J. Hespanha. Piecewise quadratic Lyapunov functions for piecewise affine time-delay systems. In *American Control Conference*, Boston, Massachusetts, USA, 2004.

[14] V. Kolmanovskii and A. Myshkis. *Applied theory of functional differential equations*. Kluwer, 1999.

[15] K. Liu and E. Fridman. Delay-dependent methods: when does the first delay interval need a special analysis? In *11th Workshop on Time-Delay Systems*, Grenoble, France, 2013.

[16] S. Liu, T. Basar, and R. Srikant. Pitfalls in the fluid modeling of RTT variations in window-based congestion control. In *INFOCOM*, Miami, Florida, USA, 2005.

[17] W. Michiels, D. Melchior-Aguilar, and S.-I. Niculescu. Stability of some classes of TCP/AQM networks. *International Journal of Control*, 79(9):1136–1144, 2006.

[18] K. Moezzi, L. Rodrigues, and A.-G. Aghdam. Stability of uncertain piecewise affine systems with time-delay. In *American Control Conference*, St. Louis, Missouri, USA, 2009.

[19] P. Park, J. Ko, and C. Jeong. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1):235–238, 2011.

[20] P. Ranjan, R.-J. La, and E.-H. A. Global stability with a state-dependent delay in rate control. In *5th IFAC Workshop on Time Delay Systems*, Leuven, Belgium, 2004.

[21] L. Rodrigues and J.-P. How. Observer-based control of piecewise-affine systems. *International Journal of Control*, 76(5):459–477, 2003.

[22] S. Tarbouriech and J.-M. Gomes da Silva Jr. Synthesis of controllers for continuous-time delay systems with saturating controls via lmis. *IEEE Transactions on Automatic Control*, 45(1):105–110, 2000.

[23] E. Verriest. Stability of systems with state-dependent and random delays. *IMA Journal of Mathematical Control and Information*, 19(1–2):103–114, 2002.

[24] H.-O. Walther. A periodic solution of a differential equation with state-dependent delay. *Journal of Differential Equations*, 244:1910–1945, 2008.