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# On countably skewed Brownian motion with accumulation point

Youssef Ouknine <sup>1</sup>, Francesco Russo <sup>2</sup>, Gerald Trutnau <sup>3</sup>

**Abstract.** In this work we connect the theory of Dirichlet forms and direct stochastic calculus to obtain strong existence and pathwise uniqueness for Brownian motion that is perturbed by a series of constant multiples of local times at a sequence of points that has exactly one accumulation point in  $\mathbb{R}$ . The considered process is identified as special distorted Brownian motion  $X$  in dimension one and is studied thoroughly. Besides strong uniqueness, we present necessary and sufficient conditions for non-explosion, recurrence and positive recurrence as well as for  $X$  to be semimartingale and possible applications to advection-diffusion in layered media.

Mathematics Subject Classification (2010): primary; 31C25, 60J60, 60J55; secondary: 31C15, 60B10.

Key words: Skew Brownian motion, Local time, strong existence, Pathwise uniqueness, transience, recurrence, positive recurrence.

## 1 Introduction

In this paper we are concerned with a special distorted Brownian motion in dimension one. Distorted BM in dimension  $d$  was first introduced in [1]. It is roughly speaking the Hunt process associated to a regular Dirichlet form

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} \nabla f \cdot \nabla g d\mu, \quad f, g \in D(\mathcal{E}),$$

on  $L^2(\mathbb{R}^d, \mu)$ , where  $\mu$  is a Radon measure on  $\mathbb{R}^d$  with full support. If  $\mu = \rho dx$  is absolutely continuous w.r.t. the Lebesgue measure  $dx$ , then the conditions on  $\rho$  for an extension of  $(\mathcal{E}, C_0^\infty(\mathbb{R}^d) \subset D(\mathcal{E}))$  to be associated to a Hunt process are quite weak (cf. [7, Theorem 3.1.3] or [17, (2.4)]). In particular, if the partial derivatives of  $\rho$  are absolutely continuous and sufficiently regular, we obtain using integration by parts

$$\mathcal{E}(f, g) = - \int_{\mathbb{R}^d} \left( \frac{1}{2} \Delta f + \frac{\nabla \rho}{2\rho} \cdot \nabla f \right) g d\mu,$$

and we can see that the process associated to  $(\mathcal{E}, D(\mathcal{E}))$  is a  $d$ -dimensional BM with drift  $\frac{\nabla \rho}{2\rho}$  (cf. [8]).

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In this work, we will take a particularly probabilistic viewpoint on distorted BM. For a given a.e. strictly positive nice function  $\rho$ , distorted BM in dimension one with initial condition  $x_0 \in \mathbb{R}$  may be regarded (whenever it makes sense) as a solution to

$$X_t = x_0 + W_t + \int_{\mathbb{R}} \ell_t^a(X) \frac{d\rho(a)}{2\rho(a)}, \quad (1)$$

where  $W$  is a standard BM and  $\ell^a(X)$  the symmetric semimartingale local time of  $X$  at  $a \in \mathbb{R}$ . (1) makes in particular sense, when  $\rho$  is weakly differentiable with derivative  $\rho' \in L_{loc}^1(\mathbb{R}, dx)$ , and  $\frac{1}{\rho}$  is not too singular. In this case, we obtain by the occupation times formula

$$\int_{\mathbb{R}} \ell_t^a(X) \frac{d\rho(a)}{2\rho(a)} = \int_0^t \frac{\rho'}{2\rho}(X_s) ds,$$

so that (1) is a BM with logarithmic derivative as a drift. The Bessel processes of dimension  $\delta \in (1, 2)$  fall into this category with  $\rho(x) = |x|^{\delta-1}$ . In this paper, however, we will consider a very special  $\rho$  whose logarithmic derivative has no absolutely continuous component. More precisely, we consider a concrete simple function  $\rho$  whose derivative is an infinite sum of Dirac measures such that (1) can be rewritten as

$$X_t = x_0 + W_t + \sum_{k \in \mathbb{Z}} (2\alpha_k - 1) \ell_t^{z_k}(X), \quad (2)$$

where  $(z_k)_{k \in \mathbb{Z}}$  is an unbounded sequence of real numbers that may have an accumulation point, say in zero, and  $\alpha_k \in (0, 1)$ ,  $k \in \mathbb{Z}$  are real numbers (see Remark 2.6(i) for the concrete  $\rho$ ). Actually, one can think of considering a more elaborate version of (2). For instance, one can consider (2) with diffusion coefficient in front of the BM and with absolutely continuous drift component, or even w.r.t. sequences  $(z_k)_{k \in \mathbb{Z}}$  that could have more than only one accumulation point. But as we think, this will not provide more insight or will lead us too far from the main object of this note (see Remark 2.6(ii)).

To our knowledge, an equation of the form (2) first occurs explicitly in [33], [32] as special case. There, weak existence and pathwise uniqueness for (2) w.r.t a sequence  $(z_k)_{k \in \mathbb{Z}}$  is shown under the following assumptions:  $z_k < z_{k+1}$ ,  $k \in \mathbb{Z}$ , and  $\inf_{k \in \mathbb{Z}} (z_{k+1} - z_k) > 0$ ,  $\sup_{k \in \mathbb{Z}} (z_{k+1} - z_k) < \infty$ . In particular, no accumulation points are allowed. Earlier, in [15] Le Gall obtained weak existence and pathwise uniqueness of (1) in a general global setting, where  $\frac{d\rho(a)}{2\rho(a)}$  is replaced by an arbitrary signed measure  $\nu(da)$  with globally bounded total variation and whose absolute value on atoms is strictly less than one (in Le Gall's setting there is also some diffusion coefficient, see (23) below). Although Le Gall's global condition applied to (2) is equivalent to  $\sum_{k \in \mathbb{Z}} |2\alpha_k - 1| < \infty$  and is quite strong, it does not exclude the possibility of accumulation points. In case  $(z_k)_{k \in \mathbb{Z}}$  has no accumulation points Ramirez constructs in [23] a solution to (2) as regular diffusion (cf. [12]) and presents interesting applications to advection-diffusions in layered media. Our work includes all the mentioned cases. For a more detailed discussion on previous work, we refer to Remark 2.9.

We discover at least two interesting phenomena that seem to be generic for equations with a drift as in (2) and we fully characterize these. First, a non-explosive solution to

(2), which by definition is a semimartingale and continuous up to infinity may exist, even if  $\sum_{\{k | z_k \in U_0\}} |2\alpha_k - 1| = \infty$ , where  $U_0$  is any neighborhood of the accumulation point 0 (see Remarks 2.7 and 2.10). In fact the semimartingale property is equivalent to  $\rho(a)$  being locally of bounded variation (see (5) below). In particular, we are able to consider (1), even if  $\frac{d\rho(a)}{\rho(a)}$  is not locally of bounded total variation. Furthermore, (2) is not automatically non-explosive. We present an example with explosion in finite time where the sequence  $(z_k)_{k \in \mathbb{Z}}$  has an accumulation point at infinity, but none in  $\mathbb{R}$  (see Remark 3.6(ii)). So, there appears to be an inconsistency in [23].

For the proof of strong existence and pathwise uniqueness of (2) we need two types of conditions. The first one is the local condition  $\sum_{\{k | z_k \in U_0\}} |2\alpha_k - 1| < \infty$  and the second is a global condition that ensures non-explosion (see Proposition 3.9 and discussions in Remarks 3.10 and 3.6(i) where we relate our work to [14] and [15]). Both conditions are explicit. The local condition is optimal in the sense that it is equivalent to the existence of a nice scale function (see Remark 3.1) and the global condition on non-explosion is sharp (see Proposition 3.3, Corollary 3.4 and Remark 3.6(i)). We emphasize that the global conditions (32) and (33) are directly readable from the density  $\rho$  in (3) of the underlying Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  determined by (4). In fact, the construction of a solution to (2) is performed via Dirichlet form theory. The key point is to identify (2) as distorted BM. This is done in Proposition 2.11 where starting from (2), the density  $\rho$  for which (2) is a distorted BM w.r.t. the Dirichlet form given by (4) is determined. The identification of the distorted BM (7) in Theorem 2.3 with a solution to equation (2) (see (14) in Theorem 2.5) is done with the help of Remark 2.4(ii). Note that the approach through distorted BM, i.e. through the process associated to the Dirichlet form (4), is more general than the approach through (2), since distorted BM does even not need to be semimartingale. Necessary and sufficient conditions for the latter are presented in Theorem 2.3.

In addition to the above mentioned results, we present necessary and sufficient conditions for transience, recurrence and positive recurrence, as well as a sufficient condition for the existence of a unique invariant distribution (see Theorem 3.11, Theorem 3.14 and Corollary 3.16). These results are quite standard from the existence of a scale function  $h$  as in Remark 3.1 and similar results were also presented in [23]. However, we insist on explicitly pointing out that in each of these statements, additionally to the statements corresponding to the scale function, an equivalent condition for the Dirichlet form (4) is presented. This underlines our bidirectional approach.

In section 3.3, we use the theory of generalized Dirichlet forms as applied in [30], as well as the results of this work to propose a generalization for the longitudinal and transversal directions of advection-diffusion in layered media considered in [23] and [24] (see Remark 3.18).

Finally, we want to say a few words on skew reflected diffusions and corresponding uniqueness results. If all  $\alpha_k$  except  $\alpha := \alpha_1$  are  $\frac{1}{2}$  in (2) and  $z_1 = 0$ , then  $X$  is called the  $\alpha$ -skew Brownian motion (see Remark 2.8). It was first considered by Itô and McKean (see e.g. [11, Section 4.2, Problem 1]) and strong uniqueness was derived in [10]. Skew reflected diffusions and strong uniqueness results have been considered by many authors then. Additionally to [33, 32, 15] the existence and uniqueness results of [4], [28] and [29] are particularly close to ours. A good survey on skew reflected diffusions is [16].

## 2 Construction and basic properties of countably skewed Brownian motion

We consider two sequences of real numbers  $(l_k)_{k \in \mathbb{Z}}$  and  $(r_k)_{k \in \mathbb{Z}}$  such that

$$l_k < l_{k+1} < 0 < r_k < r_{k+1}, \quad \forall k \in \mathbb{Z},$$

$$\text{and } \lim_{k \rightarrow \infty} l_k = 0 = \lim_{k \rightarrow -\infty} r_k.$$

We suppose further, that zero is the sole accumulation point of the sequences  $(l_k)_{k \in \mathbb{Z}}$ ,  $(r_k)_{k \in \mathbb{Z}}$ . In particular

$$\lim_{k \rightarrow -\infty} l_k = -\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} r_k = \infty.$$

Let  $(\gamma_k)_{k \in \mathbb{Z}}$  and  $(\bar{\gamma}_k)_{k \in \mathbb{Z}}$  be another two sequences of arbitrary, but strictly positive real numbers. Let

$$\rho(x) := \sum_{k \in \mathbb{Z}} \{ \gamma_{k+1} 1_{(l_k, l_{k+1})} + \bar{\gamma}_{k+1} 1_{(r_k, r_{k+1})} \} (x), \quad (3)$$

where  $1_A$  is the indicator function of the set  $A$  and  $(a, b)$  the open interval from  $a$  to  $b$ . Since  $\rho$  appears as density to the Lebesgue measure we do not have to care about the values of  $\rho$  at the boundary points  $l_k, r_k, 0$  at the moment. However, whenever we have to choose a pointwise version, we will choose its ‘‘symmetric’’ version  $\tilde{\rho}$  (cf. e.g. Remark 2.6(i) below). We always assume that

$$\rho \in L^1_{loc}(\mathbb{R}; dx).$$

By its structure,  $\rho$  then automatically satisfies a Hamza type condition on  $\mathbb{R} \setminus \{0\}$  (see [17, (2.4) on p. 44]), and so by the results of [17]

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}} f'(x) g'(x) \rho(x) dx, \quad f, g \in C_0^\infty(\mathbb{R}) \quad (4)$$

is closable in  $L^2(\mathbb{R}; \rho dx)$  and the closure  $(\mathcal{E}, D(\mathcal{E}))$  is a (quasi-)regular symmetric Dirichlet form (see [17], [7]). Here  $C_0^\infty(\mathbb{R})$  denotes the infinitely often differentiable functions with compact support on  $\mathbb{R}$ , and  $f'$  is the derivative of  $f$ .

Let  $(T_t)_{t \geq 0}$  be the strongly continuous submarkovian contraction semigroup on  $L^2(\mathbb{R}; \rho dx)$  that is associated to  $(\mathcal{E}, D(\mathcal{E}))$ . By general Dirichlet form theory (see [7]) there exists a Hunt process with life time  $\zeta$  and cemetery  $\Delta$

$$((X_t)_{t \geq 0}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \zeta, (P_x)_{x \in \mathbb{R} \cup \{\Delta\}})$$

such that  $x \mapsto E_x[f(X_t)]$  is a quasi-continuous  $dx$ -version of  $T_t f$  for any (measurable)  $f \in L^2(\mathbb{R}; \rho dx)$ , and  $E_x$  denotes the expectation w.r.t.  $P_x$ .

$(\mathcal{E}, D(\mathcal{E}))$  is called *conservative*, if its associated semigroup satisfies  $T_t 1_{\mathbb{R}}(x) = 1$  for  $dx$ -a.e.  $x \in \mathbb{R}$  and any  $t \geq 0$ . Throughout this section, we **assume** that

$$(\mathcal{E}, D(\mathcal{E})) \text{ is conservative.}$$

For instance, if

$$\int_1^\infty \frac{r}{\log v(r)} dr = +\infty,$$

where  $v(r) := \int_{B_r(0)} \rho(x) dx$ , or if there exists some  $T > 0$  such that for any  $R > 0$

$$\liminf_{r \rightarrow \infty} \frac{e^{-\frac{r^2}{T}}}{r} \int_{\{|x| < \sqrt{e^{R+r}-1}\}} \rho(x) dx = 0,$$

then  $(\mathcal{E}, D(\mathcal{E}))$  is conservative (see [31, Theorem 4] and [34, Theorem 2.2]). Necessary and sufficient conditions are presented a posteriori in Proposition 3.3, Corollary 3.4 and Remark 3.6(i), in case (5) and (29) below hold.

Let  $\text{cap}$  be the capacity related to  $(\mathcal{E}, D(\mathcal{E}))$  as defined in [7, p.64]. Since  $C_0^\infty(\mathbb{R})$  is a special standard core for  $(\mathcal{E}, D(\mathcal{E}))$ , and  $(\mathcal{E}, D(\mathcal{E}))$  is except in zero locally comparable with the Dirichlet form  $\frac{1}{2} \int_{\mathbb{R}} f'(x)g'(x)dx$  of the Brownian motion, it follows from [7, Lemma 2.2.7 (ii), and Theorem 4.4.3] that

$$\text{cap}(\{x\}) > 0 \text{ for any } x \in \mathbb{R} \setminus \{0\}.$$

We will consider the following assumption on  $(\mathcal{E}, D(\mathcal{E}))$ :

**(H1)**  $\text{cap}(\{0\}) > 0$ .

**Remark 2.1** **(H1)** holds if for instance  $\exists \lim_{k \rightarrow \infty} \gamma_k, \lim_{k \rightarrow -\infty} \bar{\gamma}_k > 0$ , because then  $(\mathcal{E}, D(\mathcal{E}))$  is locally comparable with the Dirichlet form  $\frac{1}{2} \int_{\mathbb{R}} f'(x)g'(x)dx$  of the Brownian motion. It has hence the same exceptional sets (see [7, Lemma 2.2.7 (ii), and Theorem 4.4.3]).

**Proposition 2.2** Under **(H1)**, the Hunt process  $((X_t)_{t \geq 0}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \zeta, (P_x)_{x \in \mathbb{R} \cup \{\Delta\}})$  associated to  $(\mathcal{E}, D(\mathcal{E}))$  is a conservative diffusion, i.e. we have:

(i) The process has infinite life time, namely

$$P_x[\zeta = \infty] = P_x[X_t \in \mathbb{R}, \forall t \geq 0] = 1 \text{ for all } x \in \mathbb{R}.$$

(ii) The process is a diffusion, namely

$$P_x[t \mapsto X_t \text{ is continuous on } [0, \infty)] = 1 \text{ for all } x \in \mathbb{R}.$$

If **(H1)** does not hold, then in general (i), (ii) are only valid for all  $x \in \mathbb{R} \setminus \{0\}$ .

**Proof** We have  $P_x[X_t \in \mathbb{R}] = T_t 1_{\mathbb{R}}$   $dx$ -a.e. Thus by conservativeness  $R_1 1_{\mathbb{R}}(x) := \int_0^\infty e^{-t} P_x[X_t \in \mathbb{R}] dt = 1$  for  $dx$ -a.e.  $x \in \mathbb{R}$ . Since  $R_1 1_{\mathbb{R}}$  is 1-excessive, it follows that  $R_1 1_{\mathbb{R}}(x) = 1$  for all  $x \in \mathbb{R} \setminus N_1$ , where  $N_1 \subset \mathbb{R}$  satisfies  $\text{cap}(N_1) = 0$ . It follows  $P_x[\zeta = \infty] = P_x[X_t \in \mathbb{R}, \forall t \geq 0] = 1$  for all  $x \in \mathbb{R} \setminus N_1$ .

Since  $(\mathcal{E}, D(\mathcal{E}))$  is (strongly) local we obtain by [7, Theorem 4.5.1 (ii)] that

$$P_x[t \mapsto X_t \text{ is continuous on } [0, \zeta]] = 1 \text{ for all } x \in \mathbb{R} \setminus N_2$$

where  $N_2 \subset \mathbb{R}$  satisfies  $\text{cap}(N_2) = 0$ . By considering  $N := N_1 \cup N_2$  if necessary, we may assume that  $N := N_1 = N_2$ . Since  $\{0\}$  is the only non-trivial subset of  $\mathbb{R}$  which might have zero capacity we obtain that  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is a conservative diffusion for any  $x \in \mathbb{R} \setminus \{0\}$ , and under **(H1)** for any  $x \in \mathbb{R}$ . □

**Theorem 2.3** (i) Under **(H1)**, the family  $\mathbb{M} := ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  associated to  $(\mathcal{E}, D(\mathcal{E}))$  satisfies: for  $x \in \mathbb{R}$ ,  $((X_t)_{t \geq 0}, P_x)$  is a semimartingale, iff

$$\sum_{k \leq 0} |\bar{\gamma}_{k+1} - \bar{\gamma}_k| + \sum_{k \geq 0} |\gamma_{k+1} - \gamma_k| < \infty. \quad (5)$$

In particular (5) implies

$$\exists \lim_{k \rightarrow \infty} \gamma_k =: \gamma \in [0, \infty) \quad \text{and} \quad \exists \lim_{k \rightarrow -\infty} \bar{\gamma}_k =: \bar{\gamma} \in [0, \infty). \quad (6)$$

(ii) If **(H1)** does not hold, then  $\mathbb{M}$  is (always) a semimartingale for all  $x \in \mathbb{R} \setminus \{0\}$ .

(iii) Suppose **(H1)** and (5) hold. Then we have for all  $x \in \mathbb{R}$

$$X_t = x + W_t + \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \ell_t^{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \ell_t^{r_k} \right\} + \frac{\bar{\gamma} - \gamma}{2} \ell_t^0, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (7)$$

where  $(\ell_t^a)_{t \geq 0}$  is the unique positive continuous additive functional (PCAF) of  $\mathbb{M}$  (cf. [7, Chapter 5.1]) that is associated via the Revuz correspondence (cf. [7, Theorem 5.1.3]) to the smooth measure  $\delta_a$ ,  $a \in \mathbb{R}$ , and  $((W_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, P_x)$  is a Brownian motion starting from zero for all  $x \in \mathbb{R}$ .

(iv) If **(H1)** does not hold, then

$$X_t = x + W_t + \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \ell_t^{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \ell_t^{r_k} \right\}, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (8)$$

for all  $x \in \mathbb{R} \setminus \{0\}$ .

**Proof** (i) Let  $id(x) := x$  for  $x \in \mathbb{R}$ . Then  $id \in D(\mathcal{E})_{loc}$  (cf [7, p. 117] for the definition), and for any  $v \in C_0^\infty(\mathbb{R})$  we calculate

$$\begin{aligned} -\mathcal{E}(id, v) &= \lim_{k \rightarrow \infty} \left\{ -\frac{1}{2} \int_{-\infty}^{l_{k+1}} v'(x) \rho(x) dx - \frac{1}{2} \int_{r_{-k}}^{\infty} v'(x) \rho(x) dx \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \sum_{k \leq n} \frac{\gamma_{k+1} - \gamma_k}{2} \int_{\mathbb{R}} v(x) \delta_{l_k}(dx) - \frac{\gamma_{n+1}}{2} \int_{\mathbb{R}} v(x) \delta_{l_{n+1}}(dx) \right. \\ &\quad \left. + \sum_{k \geq -n} \frac{\bar{\gamma}_{k+2} - \bar{\gamma}_{k+1}}{2} \int_{\mathbb{R}} v(x) \delta_{r_{k+1}}(dx) + \frac{\bar{\gamma}_{-n+1}}{2} \int_{\mathbb{R}} v(x) \delta_{r_{-n}}(dx) \right\} \quad (9) \end{aligned}$$

where  $\delta_x$  denotes the Dirac measure in  $x \in \mathbb{R}$ . Thus in an informal way, we can write

$$-\mathcal{E}(id, v) = \int_{\mathbb{R}} v(x) \nu(dx), \quad (10)$$

with

$$\nu = \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \delta_{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \delta_{r_k} \right\} + \frac{\lim_{k \rightarrow -\infty} \bar{\gamma}_k - \lim_{k \rightarrow \infty} \gamma_k}{2} \delta_0, \quad (11)$$

Under **(H1)** the notion of smooth measure is equivalent to the notion of Radon measure, i.e. a positive measure on  $\mathbb{R}$  is smooth, iff it is locally finite in  $\mathbb{R}$ . Thus  $\nu$  in (11) is a signed smooth measure, iff its positive and negative parts are locally finite in  $\mathbb{R}$ . The last is the case, iff (5) holds, because (5) clearly implies (6). Now the statement follows easily by [7, Theorem 5.5.4].

(ii) If **(H1)** does not hold, then  $\nu$  is a signed smooth measure, iff it is locally finite in  $\mathbb{R} \setminus \{0\}$ . But  $\nu$  is always locally finite in  $\mathbb{R} \setminus \{0\}$  and so the assertion follows from [7, Theorem 5.5.4].

(iii) By [7, Theorem 5.5.1] we only have to calculate the local martingale part  $M_t^{[id]}$ , and the local zero energy part  $N_t^{[id]}$  appearing in the local Fukushima decomposition for  $A_t^{[id]} = X_t - X_0 = M_t^{[id]} + N_t^{[id]}$ . By (9), (5), and (6) it immediately follows with [7, Corollary 5.5.1] that

$$N_t^{[id]} = \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \ell_t^{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \ell_t^{r_k} \right\} + \frac{\bar{\gamma} - \gamma}{2} \ell_t^0, t \geq 0. \quad (12)$$

Under **(H1)** the equality in (12) is strict, i.e. it holds  $P_x$ -a.s. for all  $x \in \mathbb{R}$ . Since  $M_t^{[id]}$  is a continuous local martingale it suffices to show that for its quadratic variation, we have  $\langle M^{[id]} \rangle_t = t$ . The Revuz measure  $\mu_{\langle M^{[id]} \rangle}$  of  $\langle M^{[id]} \rangle$  satisfies

$$\mu_{\langle M^{[id]} \rangle} = \rho dx$$

which is the same than the Revuz measure of the additive functional  $A_t = t$ . Thus the equality  $\langle M^{[id]} \rangle_t = t$  is strict by **(H1)**.

(iv) If **(H1)** does not hold, then using (ii) and the same line of arguments as in (iii) but with test functions  $v \in D(\mathcal{E})_{b, F_n}$ ,  $n \geq 1$ , where  $(F_n)_{n \geq 1}$  is a generalized nest, we obtain by [7, Theorem 5.5.4] that

$$N_t^{[id]} = \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \ell_t^{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \ell_t^{r_k} \right\}, t \geq 0. \quad (13)$$

$P_x$ -a.s. for all  $x \in \mathbb{R} \setminus \{0\}$ . As in (iii) we obtain  $\langle M^{[id]} \rangle_t = t$  in the sense of equivalence of PCAFs. Thus  $\langle M^{[id]} \rangle_t = t$   $P_x$ -a.s. for all  $x \in \mathbb{R} \setminus \{0\}$ . This completes our proof.  $\square$



**Remark 2.4** (i) If we do not assume conservativeness of  $(\mathcal{E}, D(\mathcal{E}))$ , then we obtain Theorem 2.3 exactly as before, except that the semimartingale property and the identification of the associated process only hold up to the lifetime  $\zeta$ , i.e. for  $t < \zeta$ . Indeed, the corresponding process is then a diffusion up to lifetime, i.e. Theorem 2.2(ii) holds with  $\infty$  replaced by  $\zeta$  (see proof of Theorem 2.2) and the semimartingale property, as well as the identification of the process can be worked out up to lifetime exactly as in the proof of Theorem 2.3.

(ii) The PCAFs  $(\ell_t^a)_{t \geq 0}$  in Theorem 2.3 can be uniquely determined up to a constant. If  $((X_t)_{t \geq 0}, P_x)$  is a semimartingale, then

$$\ell^{l_k} = \frac{2}{\gamma_{k+1} + \gamma_k} \ell^{l_k}(X), \quad \ell^{r_k} = \frac{2}{\bar{\gamma}_{k+1} + \bar{\gamma}_k} \ell^{r_k}(X), \quad \ell^0 = \frac{2}{\bar{\gamma} + \gamma} \ell^0(X)$$

$P_x$ -a.s. for any  $k \in \mathbb{Z}$  and for  $\bar{\gamma} + \gamma \neq 0$ , where  $\ell^a(X)$ ,  $a \in \mathbb{R}$ , denotes the symmetric semimartingale local time at  $a$  of  $((X_t)_{t \geq 0}, P_x)$  as defined in [25, VI. (1.25) Exercise]. Once the process  $X_t$  is a semimartingale, this can be carried out by comparing the symmetric Tanaka formula (see [25, VI. (1.2) Theorem] for the left version of it) for  $|X_t - a|$  with the local Fukushima decomposition (cf. [7, Theorem 5.5.1]) for  $|X_t - a|$ , where we choose  $a = 0$  for  $\bar{\gamma} + \gamma \neq 0$ , and  $a = l_k, r_k, k \in \mathbb{Z}$ . This is done in all details for the point  $a = 0$  in the introduction of [26], but the procedure is exactly the same for any other point. So, we omit the proof.

In view of Remark 2.4(ii) the following theorem follows immediately from Theorem 2.3.

**Theorem 2.5** (i) Suppose that  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is a semimartingale. If **(H1)** holds, then for any  $x \in \mathbb{R}$

$$X_t = x + W_t + N_t, \quad t \geq 0, \quad P_x\text{-a.s.}, \quad (14)$$

with

$$N_t = \sum_{k \in \mathbb{Z}} \left\{ (2\alpha_k - 1) \ell_t^{l_k}(X) + (2\bar{\alpha}_k - 1) \ell_t^{r_k}(X) \right\} + (2\alpha - 1) \ell_t^0(X), \quad (15)$$

where  $(\ell_t^a(X))_{t \geq 0}$  is the symmetric semimartingale local time of  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  at  $a$ , and

$$\alpha = \frac{\bar{\gamma}}{\bar{\gamma} + \gamma} \quad \text{if } \bar{\gamma} \neq \gamma \quad \text{otherwise } \alpha = \frac{1}{2}, \quad \alpha_k = \frac{\gamma_{k+1}}{\gamma_{k+1} + \gamma_k}, \quad \bar{\alpha}_k = \frac{\bar{\gamma}_{k+1}}{\bar{\gamma}_{k+1} + \bar{\gamma}_k}, \quad k \in \mathbb{Z}.$$

(ii) If **(H1)** does not hold, then (14) holds for any  $x \in \mathbb{R} \setminus \{0\}$  with

$$N_t = \sum_{k \in \mathbb{Z}} \left\{ (2\alpha_k - 1) \ell_t^{l_k}(X) + (2\bar{\alpha}_k - 1) \ell_t^{r_k}(X) \right\}, \quad (16)$$

where  $(\ell_t^a(X))_{t \geq 0}$  is the symmetric semimartingale local time of  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R} \setminus \{0\}})$  at  $a$ .

Moreover, we have  $P_x$ -a.s.  $\langle X \rangle_t = \langle W \rangle_t = t$  for any  $t \geq 0$  and for all  $x \in \mathbb{R}$  (resp. for all  $x \in \mathbb{R} \setminus \{0\}$  in case of (ii)). Thus by the occupation times formula [25] we have  $\int_0^t \mathbf{1}_{\{y\}}(X_s) ds = \int_{\mathbb{R}} \mathbf{1}_{\{y\}}(a) \ell_t^a(X) da = 0$   $P_x$ -a.s. for any  $x, y \in \mathbb{R}$  (resp. for any  $x, y \in \mathbb{R} \setminus \{0\}$ ), and so also

$$\int_0^t \mathbf{1}_{\{y\}}(X_s) dW_s = 0. \quad (17)$$

$P_x$ -a.s. for any  $x, y \in \mathbb{R}$  (resp. for any  $x, y \in \mathbb{R} \setminus \{0\}$ ), and  $t \geq 0$ .

**Remark 2.6** (i) Let us choose a “symmetric” pointwise version of  $\rho$

$$\tilde{\rho} := \sum_{k \in \mathbb{Z}} \left\{ \gamma_{k+1} \mathbf{1}_{(l_k, l_{k+1})} + \bar{\gamma}_{k+1} \mathbf{1}_{(r_k, r_{k+1})} + \frac{\gamma_{k+1} + \gamma_k}{2} \mathbf{1}_{\{l_k\}} + \frac{\bar{\gamma}_{k+1} + \bar{\gamma}_k}{2} \mathbf{1}_{\{r_k\}} \right\} + \frac{\bar{\gamma} + \gamma}{2} \mathbf{1}_{\{0\}}.$$

Condition (5) implies that  $\tilde{\rho}$  is locally of bounded variation and so  $d\tilde{\rho}$  is a signed Radon measure that is locally of bounded total variation. In particular, it can be written as

$$d\tilde{\rho}(a) = \sum_{k \in \mathbb{Z}} \left\{ (\gamma_{k+1} - \gamma_k) \delta_{l_k}(da) + (\bar{\gamma}_{k+1} - \bar{\gamma}_k) \delta_{r_k}(da) \right\} + (\bar{\gamma} - \gamma) \delta_0(da).$$

Then clearly  $N_t$  in (15) equals

$$\frac{1}{2} \int_{\mathbb{R}} \ell_t^a(X) \frac{d\tilde{\rho}(a)}{\tilde{\rho}(a)}.$$

and so (14) has the form

$$X_t = x + W_t + \frac{1}{2} \int_{\mathbb{R}} \ell_t^a(X) \frac{d\tilde{\rho}(a)}{\tilde{\rho}(a)}. \quad (18)$$

We will see below in Remark 2.7 and Example 2.10 that the signed measure  $\frac{d\tilde{\rho}(a)}{\tilde{\rho}(a)}$  needs not to be locally of bounded total variation.

(ii) Let  $\tilde{\rho}$  be as in (i). Instead of (18), we could have considered the more general equation

$$X_t = x + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds + \frac{1}{2} \int_{\mathbb{R}} \ell_t^a(X) \frac{d\tilde{\rho}(a)}{\tilde{\rho}(a)}. \quad (19)$$

Indeed, this is easily possible for very general  $\sigma$  and  $b$  by considering instead of the bilinear form (4) on  $L^2(\mathbb{R}, \rho dx)$ , the bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}} \sigma^2(x) f'(x) g'(x) \rho(x) \varphi(x) dx, \quad f, g \in C_0^\infty(\mathbb{R}) \quad (20)$$

on  $L^2(\mathbb{R}, \rho \varphi dx)$ , where

$$\varphi(x) := \frac{1}{\sigma(x)^2} e^{\int_0^x \frac{2b}{\sigma^2}(y) dy}.$$

(19) is easily derived by the same techniques as presented here for  $\varphi \equiv 1$ . However, we do not do this for several reasons. First, we do not expect any new phenomena (cf. e.g. [32],

[33]), except if  $\sigma$  and  $b$  are very singular as for instance in [5, 6, 27, 2]. In [5, 6, 27] well-posedness of a weak solution of the corresponding martingale problem to (19) (with  $\tilde{\rho} \equiv 1$ ) is discussed when  $b$  is the derivative of a continuous function and  $\Sigma(x) := \int_0^x \frac{2b(y)}{\sigma^2}(y)dy$  is defined via a regularization procedure. That solution may fail of course to be a semimartingale; it is a semimartingale if and only if  $\Sigma$  has locally bounded variation, see Corollary 5.11 of [6]. Observe that this condition is equivalent to  $e^\Sigma$  being locally of bounded variation. In fact when  $\sigma \equiv 1$ ,  $e^{\Sigma(x)} = e^{\int_0^x 2b(y)dy}$  is the density of the invariant measure and it shows the same qualitative behavior as in Remark 2.6(i), where the semimartingale property takes place when  $\tilde{\rho}$  (which is again the invariant measure density) has locally bounded variation. Such an analysis however, mixing the just mentioned case and the countably skewed Brownian motion framework would lead us too far and is more suitably performed in a subsequent work.

Second, considering a drift like in (18) seems to us interesting for its own because to our knowledge such kind of drifts were up to now not studied in a detailed manner. Moreover, solutions to equations of type (18) find applications in advection diffusions (see Remark 3.18).

Finally, it does not deliver more insight to consider multiple (even countably many) isolated accumulation points, because the same phenomena that occur for one accumulation point then occur locally around each accumulation point.

**Remark 2.7** Assume that **(H1)** holds. It may happen that

$$\sum_{k \leq 0} |\bar{\gamma}_{k+1} - \bar{\gamma}_k| + \sum_{k \geq 0} |\gamma_{k+1} - \gamma_k| < \infty,$$

thus  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is a semimartingale by Theorem 2.3 (i), but for the  $\alpha_k$  and  $\bar{\alpha}_k$  corresponding to the  $l_k$ ,  $k \geq 0$ , and  $r_k$ ,  $k \leq 0$ , we have

$$\sum_{k \geq 0} |2\alpha_k - 1| = \sum_{k \geq 0} \left| \frac{\gamma_{k+1} - \gamma_k}{\gamma_{k+1} + \gamma_k} \right| = \infty \quad \text{or} \quad \sum_{k \leq 0} |2\bar{\alpha}_k - 1| = \sum_{k \leq 0} \left| \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{\bar{\gamma}_{k+1} + \bar{\gamma}_k} \right| = \infty.$$

If the last both series are divergent, then typically  $\ell_t^0(X) \equiv 0$ , because then  $\lim_{k \rightarrow \infty} \ell_t^k(X) = 0$  and  $\lim_{k \rightarrow -\infty} \ell_t^k(X) = 0$ . This is for instance the case in Example 2.10 below with  $\delta \in (1, 2)$ .

On the other hand  $\sum_{k \geq 0} |2\alpha_k - 1| + \sum_{k \leq 0} |2\bar{\alpha}_k - 1| < \infty$  is stronger than (5) and **(H1)** together as it also implies (29) below (cf. proof of Proposition 3.9 below).

**Example 2.8** ( $\alpha$ -skew Brownian motion)

Let  $\alpha \in (0, 1)$ , and  $\gamma_k = \frac{1-\alpha}{\alpha}$ ,  $\bar{\gamma}_k = 1$ , for all  $k \in \mathbb{Z}$ , i.e.

$$\rho(x) = \frac{1-\alpha}{\alpha} 1_{(-\infty, 0)}(x) + 1_{(0, \infty)}(x).$$

Then, since the corresponding Dirichlet (form) norm is equivalent to the one of Brownian motion, we obtain that the corresponding process is conservative (even recurrent), and **(H1)** holds. Thus the corresponding process is a conservative diffusion by Theorem 2.2.

Moreover, clearly (5) and (6) hold with  $\gamma = \frac{1-\alpha}{\alpha}$  and  $\bar{\gamma} = 1$ . Hence by Theorem 2.3  $((X_t)_{t \geq 0}, P_x)$  is a semimartingale for any  $x \in \mathbb{R}$ . By Theorem 2.5 we have  $N_t = (2\alpha - 1)\ell_t^0(X)$ , since  $\alpha_k = \bar{\alpha}_k = \frac{1}{2}$  for all  $k \in \mathbb{Z}$ . Hence  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is the  $\alpha$ -skew Brownian motion (cf e.g. [10], [11]).

**Remark 2.9** (Previously obtained results by other authors and countably skewed Brownian motion without accumulation point).

Let  $(x_k)_{k \in \mathbb{Z}}$  be sequence of real numbers without accumulation point and such that  $x_k < x_{k+1}$  for all  $k \in \mathbb{Z}$ , and let  $x_0 = 0$ . Let  $(\alpha_k)_{k \in \mathbb{Z}} \subset (0, 1) \setminus \frac{1}{2}$ . In [23] a solution to

$$X_t = x + W_t + \sum_{k \in \mathbb{Z}} (2\alpha_k - 1)\ell_t^{x_k}(X), \quad t \geq 0, \quad P_x\text{-a.s.} \quad (21)$$

for all  $x \in \mathbb{R}$  is constructed as regular diffusion and interesting applications to advection diffusion in layered media were presented. The author names a solution to (21) multi-skewed Brownian motion although it seems that no uniqueness of (21) is derived. If  $\inf_{k \in \mathbb{Z}} |x_{k+1} - x_k| > 0$  and  $\sup_{k \in \mathbb{Z}} |x_{k+1} - x_k| < \infty$ , existence and uniqueness of (21) was derived in [32], [33].

By our method, we can construct a conservative diffusion  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  that is a semimartingale and that weakly solves (21), thus recovering partially results of [23] by a completely different method: Indeed, let  $\gamma_0 > 0$  be arbitrary, and  $\gamma_k = 1$  for all  $k \geq 0$ ,  $\bar{\gamma}_k = \bar{\gamma}_1$  for all  $k \leq -1$ , and

$$\gamma_k = \prod_{j=k}^{-1} \frac{1 - \alpha_j}{\alpha_j} \gamma_0, \quad k \leq -1, \quad \bar{\gamma}_k = \prod_{j=0}^{k-1} \frac{\alpha_j}{1 - \alpha_j} \gamma_0, \quad k \geq 1. \quad (22)$$

Let further  $l_k = x_k$ ,  $k \leq -1$ ,  $r_k = x_k$ ,  $k \geq 1$ , and all remaining  $l_k, r_k$  arbitrary but satisfying the conditions at the beginning of Section 2. Set

$$\kappa_k := \gamma_k, \quad k \leq 0, \quad \bar{\kappa}_k := \bar{\gamma}_k, \quad k \geq 1.$$

Then we can see that

$$\rho(x) := \sum_{k \in \mathbb{Z}} \kappa_{k+1} 1_{(x_k, x_{k+1})}(x) = \sum_{k \in \mathbb{Z}} \{ \gamma_{k+1} 1_{(l_k, l_{k+1})} + \bar{\gamma}_{k+1} 1_{(r_k, r_{k+1})} \}(x),$$

satisfies our conditions and we can construct a solution  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  to (21) as in Example 2.8. Conservativeness of (21) is equivalent to the conditions given in Corollary 3.4 below. These hold if e.g.  $\inf_{k \in \mathbb{Z}} |x_{k+1} - x_k| > 0$  (see Remark 3.5 below). On the other hand, constructing a solution to (21) means implicitly that the solution is a continuous semimartingale, hence conservative. (21) is not automatically conservative. We provide a counterexample with explosion in finite time in case there is an ‘‘accumulation point at infinity’’ in Remark 3.6(ii). So it appears to occur a minor inconsistency in [23].

It follows from Theorem 3.8 below that strong uniqueness trivially holds for (21) (provided we have conservativeness). In fact, in this case the sums in (29) below can always be chosen to have only finitely many summands, since there is no accumulation point. Moreover, we

recover the existence and uniqueness results for (21) derived in [32], [33]. On the other hand, in [15] Le Gall considered equations of type

$$X_t = x + \int_0^t \sigma(X_s) dW_s + \int_{\mathbb{R}} \ell_t^a(X) \mu(da) \quad (23)$$

where  $\sigma$  is of bounded variation, bounded away from zero and right continuous, and  $\mu$  is a signed measure of bounded total variation such that  $|\mu(\{a\})| < 1$  for any  $a \in \mathbb{R}$ . Under these global assumptions (that imply in particular conservativeness) Le Gall showed weak existence and pathwise uniqueness for (23). Hence by Le Gall's results we know that weak existence and pathwise uniqueness holds for (21), if

$$\sum_{k \in \mathbb{Z}} |2\alpha_k - 1| < \infty. \quad (24)$$

Le Gall's results do not cover in whole generality equation (21), since in equation (21) no assumption on the finiteness of  $\sum_{k \in \mathbb{Z}} |2\alpha_k - 1|$  is made. On the other hand, the results in [15] allow for some accumulation point of the sequence  $(x_k)_{k \in \mathbb{Z}}$  in (21), because (24) is sufficient for weak existence and pathwise uniqueness of (23) with  $\sigma \equiv 1$ . But (24) is qualitatively stronger than our local assumptions (5) and **(H1)** (see Remark 3.10(i) and (ii) below and Remarks 2.1 and 2.7).

**Example 2.10** (Resemblance to Bessel processes)

In this example, we show that there is a solution to (14) (which by definition is a conservative diffusion that is a semimartingale) with

$$\sum_{k \geq 0} |2\alpha_k - 1| + \sum_{k \leq 0} |2\bar{\alpha}_k - 1| = \infty.$$

Let  $-l_k = r_{-k} = \frac{1}{k}$  for  $k \geq 1$ ,  $-l_k = r_{-k} = -k + 2$  for  $k \leq 0$ , and  $\gamma_k = (-l_k)^{\delta-1}$ ,  $\bar{\gamma}_k = (r_k)^{\delta-1}$ ,  $k \in \mathbb{Z}$ ,  $\delta \in (0, 1) \cup (1, 2) \cup [2, \infty)$ . (The case  $\delta = 1$  corresponds to Brownian motion.) Then  $\rho(x)$  is the upper Riemann step function of  $\varphi(x) := |x|^{\delta-1}$  corresponding to the partition  $(l_k)_{k \in \mathbb{Z}}$  on  $(-\infty, 0)$ , and the lower Riemann step function of  $\varphi$  corresponding to the partition  $(r_k)_{k \in \mathbb{Z}}$  on  $(0, \infty)$ . We can hence easily see from [7, Example 2.2.4] that

$$\text{cap}(0) > 0 \Leftrightarrow \delta \in (0, 2).$$

By comparing the underlying Dirichlet form with the Dirichlet form of the Bessel processes (in this case  $\rho(x) = |x|^{\delta-1}$ ), and using [31, Theorems 4], we can see the conservativeness of  $(\mathcal{E}, D(\mathcal{E}))$  and so Theorem 2.2 applies. Moreover

$$\sum_{k \leq -2} |\bar{\gamma}_{k+1} - \bar{\gamma}_k| = |1 - \lim_{k \rightarrow \infty} k^{1-\delta}| \quad \text{and} \quad \sum_{k \geq 1} |\gamma_{k+1} - \gamma_k| = |\lim_{k \rightarrow \infty} k^{1-\delta} - 1|.$$

Thus by Theorem 2.3 the corresponding process is not a semimartingale if  $\delta \in (0, 1)$ , and a semimartingale w.r.t. to all starting points that have positive capacity, if  $\delta \geq 1$ .

However (cf. Remark 2.7), if  $\delta \in (1, 2)$ , then by the mean value theorem for some  $\vartheta_k \in [k, k+1]$ ,  $k \geq 1$ , we have multiplying the nominator and denominator by  $(k(k+1))^{\delta-1}$

$$\begin{aligned} \sum_{k \leq -2} \left| \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{\bar{\gamma}_{k+1} + \bar{\gamma}_k} \right| &= \sum_{k \geq 1} \frac{(k+1)^{\delta-1} - (k)^{\delta-1}}{(k+1)^{\delta-1} + (k)^{\delta-1}} \\ &= \sum_{k \geq 1} \frac{(\delta-1)\vartheta_k^{\delta-2}}{(k+1)^{\delta-1} + (k)^{\delta-1}} \geq \sum_{k \geq 1} \frac{\delta-1}{2} (k+1)^{-1} = +\infty. \end{aligned}$$

Exactly in the same way we can show

$$\sum_{k \geq 0} \left| \frac{\gamma_{k+1} - \gamma_k}{\gamma_{k+1} + \gamma_k} \right| = \infty.$$

Note that in this case  $\ell_t^0(X) \equiv 0$ , and  $\ell_t^a(X)$ ,  $a \in \{0, l_k, r_k, k \in \mathbb{Z}\}$  is uniquely associated to its Revuz measure  $a^{\delta-1} \delta_a$ . Moreover,  $a_n^{\delta-1} \delta_{a_n} \rightarrow 0$  weakly whenever  $a_n \rightarrow 0$ .

Our strategy to construct a solution to (14) was first to construct a solution to the basic equation (7) via the underlying Dirichlet form determined by (4), and then to rewrite (7) as (14) using Remark 2.4(ii). Now, we ask under which assumptions on the underlying parameters a solution to (14) exists.

**Proposition 2.11** *Let  $(\alpha_k)_{k \in \mathbb{Z}}, (\bar{\alpha}_k)_{k \in \mathbb{Z}} \subset (0, 1)$ ,  $\alpha \in (0, 1)$  be arbitrarily given. Let  $(l_k)_{k \in \mathbb{Z}}, (r_k)_{k \in \mathbb{Z}}$ , be a partition of  $\mathbb{R}$  as described at the beginning of section 2. For  $\gamma_0 > 0$  and  $\bar{\gamma}_0 > 0$  define*

$$\bar{\gamma}_k = \prod_{j=k}^{-1} \frac{1 - \bar{\alpha}_j}{\bar{\alpha}_j} \bar{\gamma}_0, \quad k \leq -1, \quad \bar{\gamma}_k = \prod_{j=0}^{k-1} \frac{\bar{\alpha}_j}{1 - \bar{\alpha}_j} \bar{\gamma}_0, \quad k \geq 1, \quad (25)$$

and

$$\gamma_k = \prod_{j=k}^{-1} \frac{1 - \alpha_j}{\alpha_j} \gamma_0, \quad k \leq -1, \quad \gamma_k = \prod_{j=0}^{k-1} \frac{\alpha_j}{1 - \alpha_j} \gamma_0, \quad k \geq 1. \quad (26)$$

Suppose that (5) holds for  $(\gamma_k)_{k \geq 0}$  and  $(\bar{\gamma}_k)_{k \leq 0}$ . Then the bilinear form in (4) with  $\rho$  defined through  $(\gamma_k)_{k \geq 0}, (\bar{\gamma}_k)_{k \leq 0}, (l_k)_{k \in \mathbb{Z}},$  and  $(r_k)_{k \in \mathbb{Z}}$ , is closable in  $L^2(\mathbb{R}; \rho dx)$ . Suppose that the closure  $(\mathcal{E}, D(\mathcal{E}))$  is conservative and satisfies **(H1)**. Then there exists a conservative diffusion  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ , which is a semimartingale and which weakly solves (14), i.e. for any  $x \in \mathbb{R}$

$$X_t = x + W_t + \sum_{k \in \mathbb{Z}} \left\{ (2\alpha_k - 1) \ell_t^{l_k}(X) + (2\bar{\alpha}_k - 1) \ell_t^{r_k}(X) \right\} + (2\alpha - 1) \ell_t^0(X), \quad t \geq 0, \quad (27)$$

$P_x$ -a.s. where  $\alpha = \frac{\bar{\gamma}}{\bar{\gamma} + \gamma}$  if  $\bar{\gamma} := \lim_{k \rightarrow -\infty} \bar{\gamma}_k \neq \gamma := \lim_{k \rightarrow \infty} \gamma_k$ , and  $\alpha = \frac{1}{2}$  otherwise.

**Proof** Condition (5) implies that  $\rho$  defined through  $(\gamma_k)_{k \geq 0}$ ,  $(\bar{\gamma}_k)_{k \leq 0}$ , and  $(l_k)_{k \in \mathbb{Z}}$ ,  $(r_k)_{k \in \mathbb{Z}}$  is in  $L^1_{loc}(\mathbb{R}, dx)$ , and so the bilinear form (4) is closable in  $L^2(\mathbb{R}; \rho dx)$ . Since the closure  $(\mathcal{E}, D(\mathcal{E}))$  is regular and by assumption conservative and satisfies **(H1)**, we can apply Theorem 2.5(i) to obtain the result.  $\square$

**Remark 2.12** *Suppose that all the conditions of Proposition 2.11 are satisfied and  $\gamma, \bar{\gamma} > 0$  in Proposition 2.11. Then we can obtain a solution to (27) for any  $\alpha \in (0, 1)$  by varying  $\gamma_0, \bar{\gamma}_0$ .*

### 3 Pathwise uniqueness and ergodic properties

In what follows, we will assume that  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is a semimartingale (see Theorem 2.3(i)), but we do not assume conservativeness at the beginning. We further suppose

$$\gamma, \bar{\gamma} > 0,$$

hence  $\frac{1}{\rho} \in L^1_{loc}(\mathbb{R}, dx)$  and **(H1)** holds (see Remark 2.1). Moreover, by Remark 2.4(ii)  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is a diffusion up to lifetime  $\zeta = \inf\{t > 0 \mid X_t \notin \mathbb{R}\}$  and for all  $x \in \mathbb{R}$  it holds that

$$X_t = x + W_t + \sum_{k \in \mathbb{Z}} \left\{ \frac{\gamma_{k+1} - \gamma_k}{2} \ell_t^{l_k} + \frac{\bar{\gamma}_{k+1} - \bar{\gamma}_k}{2} \ell_t^{r_k} \right\} + \frac{\bar{\gamma} - \gamma}{2} \ell_t^0, \quad t < \zeta, \quad P_x\text{-a.s.} \quad (28)$$

#### 3.1 Conservativeness and pathwise uniqueness

Let  $\alpha := \frac{\bar{\gamma}}{\bar{\gamma} + \gamma}$ . Suppose  $h : \mathbb{R} \rightarrow \mathbb{R}$  is the difference of two convex functions and piecewise linear with slope  $\frac{\alpha\gamma}{\gamma_{k+1}}$  on the interval  $(l_k, l_{k+1})$  and slope  $\frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}}$  on the interval  $(r_k, r_{k+1})$ ,  $k \in \mathbb{Z}$ . In particular  $h$  is continuous and uniquely determined up to a constant. In order to fix a version, we let

$$h(0) = 0.$$

Let  $h'(x) = \frac{h'(x+) + h'(x-)}{2}$  denote the symmetric derivative of  $h$ . In particular

$$h'(0) = \frac{\lim_{k \rightarrow -\infty} \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} + \lim_{k \rightarrow \infty} \frac{\alpha\gamma}{\gamma_{k+1}}}{2} = \frac{1}{2}.$$

**Remark 3.1**  *$h$  with the properties stated above exists, iff  $h'$  is locally of bounded variation, that is, iff*

$$\sum_{k \geq 0} \left| \frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right| + \sum_{k \leq 0} \left| \frac{1}{\bar{\gamma}_k} - \frac{1}{\bar{\gamma}_{k+1}} \right| < \infty. \quad (29)$$

*Note further that all our assumptions so far are satisfied, iff (5) and (29) hold.*

According to Remark 3.1 we will assume (5) and (29) and fix  $h$  like above from now on.

**Lemma 3.2**  $(Y_t := h(X_t))_{t \geq 0}$  is a continuous local martingale up to  $\zeta$  with quadratic variation

$$\langle Y \rangle_t = \int_0^t (h' \circ h^{-1})^2(Y_s) ds, \quad t < \zeta, \quad P_x\text{-a.s.} \quad (30)$$

for all  $x \in \mathbb{R}$ .

**Proof** (Cf. proof of Theorem 2.3) Note that  $h \in D(\mathcal{E})_{loc}$ , since it can be approximated locally in the Dirichlet space by its convolution with a standard Dirac sequence. For any  $f \in C_0^\infty(\mathbb{R})$  we then calculate

$$\begin{aligned} -\mathcal{E}(h, f) &= -\frac{1}{2} \lim_{n \rightarrow \infty} \sum_{-n \leq k \leq n} \left\{ \int_{l_k}^{l_{k+1}} \alpha \gamma f'(x) dx + \int_{r_k}^{r_{k+1}} (1 - \alpha) \bar{\gamma} f'(x) dx \right\} \\ &= -\frac{1}{2} \lim_{n \rightarrow \infty} \{ \alpha \gamma (f(l_{n+1}) - f(l_{-n})) + (1 - \alpha) \bar{\gamma} (f(r_{n+1}) - f(r_{-n})) \} \\ &= -\frac{1}{2} \{ \alpha \gamma f(0) - (1 - \alpha) \bar{\gamma} f(0) \} = 0. \end{aligned} \quad (31)$$

Therefore the drift  $N^{[h]}$  in the Fukushima decomposition of  $h(X_t) - h(X_0)$  vanishes on account of [7, Theorem 5.5.4]. It then follows from [7, Theorem 5.5.1] that  $h(X_t) - h(X_0) = M_t^{[h]}$  is a continuous local martingale up to lifetime. Under **(H1)** the equality is strict, i.e. it holds  $P_x$ -a.s. for all  $x \in \mathbb{R}$ . The quadratic variation  $\langle M^{[h]} \rangle$  of the local martingale  $M^{[h]}$  can be identified by calculating its Revuz measure  $\mu_{\langle M^{[h]} \rangle}$ . We have

$$\mu_{\langle M^{[h]} \rangle} = h'(x)^2 \rho dx,$$

which is the same than the Revuz measure of the additive functional  $A_t = \int_0^t h'(X_s)^2 ds$ . We hence obtain by the uniqueness of the Revuz correspondence and **(H1)** that

$$\langle M^{[h]} \rangle_t = \int_0^t h'(X_s)^2 ds, \quad t < \zeta$$

$P_x$ -a.s. for all  $x \in \mathbb{R}$ . Writing  $X_t = h^{-1}(Y_t)$  we obtain the final result. □

Although, in our case we do not have a classical Itô-equation, we shall call the function  $h$  in Lemma 3.2 *scale function* of the diffusion  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ , and then the corresponding *speed measure* is

$$\mu(dy) = \frac{2}{h'(y)} dy.$$

We let further for  $x \in \mathbb{R}$

$$\Phi(x) := \frac{1}{2} \int_0^x h'(z) \int_0^z \mu(dy) dz = \int_0^x \frac{h(x) - h(y)}{h'(y)} dy.$$



Note that  $\Phi$  is well defined and continuous, since  $h$  is strictly increasing and continuous, and  $h'$  is locally bounded and locally bounded away from zero by the assumption  $\gamma, \bar{\gamma} > 0$ . Indeed the latter implies  $h'(0) = \frac{1}{2}$ .

For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we define

$$f(\infty) := \lim_{x \nearrow \infty} f(x) \quad \text{and} \quad f(-\infty) := \lim_{x \searrow -\infty} f(x)$$

whenever the limits exist in  $\mathbb{R} \cup \{\pm\infty\}$

**Proposition 3.3** *The following are equivalent:*

- (i)  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is conservative, i.e.  $P_x(\zeta = \infty) = 1 \quad \forall x \in \mathbb{R}$ , where  $\zeta = \inf\{t > 0 \mid X_t \notin (-\infty, \infty)\} = \inf\{t > 0 \mid Y_t \notin (h(-\infty), h(\infty))\}$
- (ii)  $\Phi(-\infty) = \Phi(\infty) = \infty$ , i.e.  $-\infty$  and  $\infty$  are non-exit (inaccessible) boundaries.
- (iii) There exist  $u_n \in D(\mathcal{E})$ ,  $n \geq 1$ ,  $0 \leq u_n \nearrow 1$   $dx$ -a.e. as  $n \rightarrow \infty$  such that  $\mathcal{E}(u_n, G_1 w) \rightarrow 0$  as  $n \rightarrow \infty$  for some  $w \in L^2(\mathbb{R}, \rho dx) \cap L^1(\mathbb{R}, \rho dx)$  such that  $w > 0$  a.e. (Here  $(G_\alpha)_{\alpha > 0}$  is the resolvent of  $(\mathcal{E}, D(\mathcal{E}))$ , see [7]).

**Proof** (i)  $\Leftrightarrow$  (ii) is the well-known *Feller's test of non-explosions*. Although, in our case we do not have a classical Itô-equation, it can be carried out exactly as in [3, Section 6.2]. Indeed, for its proof we mainly need the existence of a good scale function and speed measure, which is here the case. Further, it is well-known in the theory of Dirichlet forms that (iii) implies  $P_x(\zeta = \infty) = 1 \quad \forall x \in \mathbb{R} \setminus N$ , where  $\text{cap}(N) = 0$  (see [7, Theorem 1.6.6 (iii)]). Under **(H1)** we must have  $N = \emptyset$ , hence (iii)  $\Rightarrow$  (i). In order to see (ii)  $\Rightarrow$  (iii), we can define  $(u_n)_{n \geq 1}$  as follows. Let  $a_n := \int_{-n}^0 \frac{h(y)-h(x)}{h'(y)} dy$ ,  $b_n := \int_0^n \frac{h(x)-h(y)}{h'(y)} dy$  and for  $n \geq 1$

$$u_n(x) := \begin{cases} 1 - \frac{1}{a_n} \int_x^0 \frac{h(y)-h(x)}{h'(y)} dy & \text{if } x \in [-n, 0], \\ 1 - \frac{1}{b_n} \int_0^x \frac{h(x)-h(y)}{h'(y)} dy & \text{if } x \in [0, n], \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly  $0 \leq u_n \nearrow 1$   $dx$ -a.e. as  $n \rightarrow \infty$ . Fix a standard Dirac sequence  $(\varphi_\varepsilon)_{\varepsilon > 0}$  and define  $u_n^k := \varphi_{\frac{1}{k}} * u_n$ ,  $k \geq 1$ . Then  $u_n^k \in C_0^\infty(\mathbb{R})$  and by standard properties of the convolution product one can easily see that  $u_n^k \rightarrow u_n$  in  $D(\mathcal{E})$  as  $k \rightarrow \infty$ . Hence  $u_n \in D(\mathcal{E})$ . For

$$\lim_{n \rightarrow \infty} \mathcal{E}(u_n, G_1 w) = 0,$$

see e.g. [20, Lemma 3.1].

□

**Corollary 3.4** *Property (ii) of Proposition 3.3 holds, iff*

$$\lim_{n \rightarrow \infty} \sum_{l \leq n} (r_{l+1} - r_l) \left\{ \frac{1}{2} (r_{l+1} - r_l) + \frac{1}{\bar{\gamma}_{l+1}} \sum_{k \leq l-1} \bar{\gamma}_{k+1} (r_{k+1} - r_k) \right\} = \infty$$

and

$$\lim_{n \rightarrow \infty} \sum_{m \geq -n} (l_{m+1} - l_m) \left\{ \frac{1}{2} (l_{m+1} - l_m) + \frac{1}{\gamma_{m+1}} \sum_{k \geq m+1} \gamma_{k+1} (l_{k+1} - l_k) \right\} = \infty,$$

*i.e. in this case we have non-explosion for every starting point.*

**Proof** We get for any  $n \in \mathbb{Z}$

$$\begin{aligned} \Phi(r_{n+1}) &= \int_0^{r_{n+1}} h'(z) \int_0^z \frac{1}{h'(y)} dy dz \\ &= \sum_{l \leq n} \int_{r_l}^{r_{l+1}} \bar{s}_l \left\{ \sum_{k \leq l-1} \int_{r_k}^{r_{k+1}} \frac{1}{h'(y)} dy + \int_{r_l}^z \frac{1}{h'(y)} dy \right\} dz \\ &= \sum_{l \leq n} \left\{ \frac{1}{2} (r_{l+1} - r_l)^2 + \sum_{k \leq l-1} \frac{\bar{\gamma}_{k+1}}{\bar{\gamma}_{l+1}} (r_{k+1} - r_k) (r_{l+1} - r_l) \right\} \end{aligned}$$

and similarly

$$\begin{aligned} \Phi(l_n) &= \int_{l_n}^0 h'(z) \int_z^0 \frac{1}{h'(y)} dy dz \\ &= \sum_{m \geq n} \left\{ \frac{1}{2} (l_{m+1} - l_m)^2 + \sum_{k \geq m+1} \frac{\gamma_{k+1}}{\gamma_{m+1}} (l_{k+1} - l_k) (l_{m+1} - l_m) \right\}. \end{aligned}$$

Hence

$$\Phi(\infty) = \lim_{n \rightarrow \infty} \Phi(r_{n+1}) = \infty \quad \text{and} \quad \Phi(-\infty) = \lim_{n \rightarrow -\infty} \Phi(l_n) = \infty$$

hold, iff two conditions stated in the lemma are satisfied. □

**Remark 3.5** *For  $l, m \in \mathbb{Z}$  let*

$$\bar{v}_l := \left\{ \frac{1}{2} (r_{l+1} - r_l) + \frac{1}{\bar{\gamma}_{l+1}} \sum_{k \leq l-1} \bar{\gamma}_{k+1} (r_{k+1} - r_k) \right\}$$

and

$$v_m := \left\{ \frac{1}{2} (l_{m+1} - l_m) + \frac{1}{\gamma_{m+1}} \sum_{k \geq m+1} \gamma_{k+1} (l_{k+1} - l_k) \right\}.$$

It follows immediately from Corollary 3.4 that a sufficient condition for conservativeness is given by:  $\exists \delta > 0$  such that

$$\text{either } r_{l+1} - r_l \geq \delta \text{ for infinitely many } l \quad \text{or} \quad \exists l_0 \in \mathbb{Z} \text{ with } \bar{v}_l \geq \delta \text{ for all } l \geq l_0$$

and

$$\text{either } l_{k+1} - l_k \geq \delta \text{ for infinitely many } k \quad \text{or} \quad \exists m_0 \in \mathbb{Z} \text{ with } v_m \geq \delta \text{ for all } m \leq m_0.$$

**Remark 3.6** (i) The conditions (5) and (29) are local conditions as they depend only on the local behavior of  $\rho$  around the accumulation point zero. In particular (29) is crucial for deriving pathwise uniqueness properties (see Theorem 3.8 below) and we note that the conservativeness assumption (resp. (5)) in Theorem 3.8 can be seen as formal conditions that are used to ensure uniqueness up to infinity (resp. the semimartingale property). Under the local conditions (5) and (29) the conditions in Corollary 3.4 are global ones and depend only on the behavior of  $\rho$  outside arbitrarily large compact sets that contain the accumulation point. In fact, for any  $n_0 \in \mathbb{N}$ , (5) and (29) imply that  $(\gamma_k)_{k > -n_0}$  and  $(\bar{\gamma}_k)_{k < n_0}$  are bounded below and above by strictly positive constants and moreover  $\sum_{l < n_0} (r_{l+1} - r_l) = r_{n_0}$ ,  $\sum_{k > -n_0} (l_{k+1} - l_k) = l_{-n_0+1}$ . From this it is then not difficult to see that the conditions of Corollary 3.4 are equivalent to the following ones:

$$\lim_{n \rightarrow \infty} \sum_{l=n_0}^n \frac{r_{l+1} - r_l}{\bar{\gamma}_{l+1}} \left( \sum_{k=n_0}^l \bar{\gamma}_{k+1} (r_{k+1} - r_k) \right) = \infty \quad (32)$$

and

$$\lim_{n \rightarrow \infty} \sum_{m=-n}^{-n_0} \frac{l_{m+1} - l_m}{\gamma_{m+1}} \left( \sum_{k=m}^{-n_0} \gamma_{k+1} (l_{k+1} - l_k) \right) = \infty. \quad (33)$$

where as usually  $\sum_{k=l}^m := 0$  for  $m < l$  is defined to be the empty sum.

(ii) Let us give an example where we have explosion. Let  $r_l := \sum_{k=1}^l \frac{1}{k}$ ,  $l \geq 1$  and  $\bar{\gamma}_{k+1} = C^k(k+1)$ ,  $k \geq 1$ , where  $C > 1$  is some constant, and let the remaining  $r_l, \bar{\gamma}_{k+1}, l_k, \gamma_{k+1}$  be just chosen such that conditions (5) and (29) are satisfied. Then

$$\sum_{l=1}^{\infty} \frac{r_{l+1} - r_l}{\bar{\gamma}_{l+1}} \left( \sum_{k=1}^l \bar{\gamma}_{k+1} (r_{k+1} - r_k) \right) = \sum_{l=1}^{\infty} \frac{1}{C^l(l+1)^2} \left( \frac{C^{l+1} - C}{C - 1} \right) < \infty,$$

and so according to (i) with  $n_0 = 1$  it follows that we cannot have conservativeness. Note that in this example, we have that  $(r_l)_{l \in \mathbb{Z}}$  has an accumulation point at “infinity” and that the skew reflection is with  $\bar{\alpha}_k \approx \frac{1}{1+C} > \frac{1}{2} + \varepsilon$  for  $k \geq N$  for some  $N \in \mathbb{N}$ .

**Lemma 3.7** Suppose that additionally to (5) and (29), we have that  $(\mathcal{E}, D(\mathcal{E}))$  is conservative. Then  $(Y_t := h(X_t))_{t \geq 0}$  satisfies  $P_x$ -a.s

$$Y_t = h(x) + \int_0^t \sum_{k \in \mathbb{Z}} \left( \frac{\alpha \gamma}{\gamma_{k+1}} 1_{[l_k, l_{k+1})} + \frac{(1-\alpha) \bar{\gamma}}{\bar{\gamma}_{k+1}} 1_{[r_k, r_{k+1})} \right) \circ h^{-1}(Y_s) dW_s \quad (34)$$

for all  $t \geq 0$  and all  $x \in \mathbb{R}$ .

**Proof** Let  $h''(da)$  the signed measure that is induced by the second derivative of  $h$ . Then applying the symmetric version of [25, VI. (1.5) Theorem] with  $h$  and (14) we obtain  $P_x$ -a.s.

$$\begin{aligned}
h(X_t) &= h(x) + \int_0^t h'(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} \ell_t^\alpha(X) h''(da) \\
&= h(x) + \int_0^t h'(X_s) dW_s + (2\alpha - 1) \int_0^t h'(X_s) d\ell_s^0(X) \\
&+ \sum_{k \in \mathbb{Z}} \left\{ (2\alpha_k - 1) \int_0^t h'(X_s) d\ell_s^{l_k}(X) + (2\bar{\alpha}_k - 1) \int_0^t h'(X_s) d\ell_s^{r_k}(X) \right\} \\
&+ \sum_{k \in \mathbb{Z}} \left\{ \frac{\frac{\alpha\gamma}{\gamma_{k+1}} - \frac{\alpha\gamma}{\gamma_k}}{2} \ell_t^{l_k}(X) + \frac{\frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} - \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_k}}{2} \ell_t^{r_k}(X) \right\} + \frac{(1-\alpha) - \alpha}{2} \ell_t^0(X) \\
&= h(x) + \int_0^t h'(X_s) dW_s + \left\{ (2\alpha - 1) \frac{1}{2} + \frac{1 - 2\alpha}{2} \right\} \ell_t^0(X) \\
&+ \sum_{k \in \mathbb{Z}} \left( (2\alpha_k - 1) \frac{\frac{\alpha\gamma}{\gamma_{k+1}} + \frac{\alpha\gamma}{\gamma_k}}{2} + \frac{\frac{\alpha\gamma}{\gamma_{k+1}} - \frac{\alpha\gamma}{\gamma_k}}{2} \right) \ell_s^{l_k}(X) \\
&+ \sum_{k \in \mathbb{Z}} \left( (2\bar{\alpha}_k - 1) \frac{\frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} + \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_k}}{2} + \frac{\frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} - \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_k}}{2} \right) \ell_s^{r_k}(X) \\
&= h(x) + \int_0^t h'(X_s) dW_s. \tag{35}
\end{aligned}$$

Now the statement follows from (17). □

**Theorem 3.8** *Suppose that additionally to (5) and (29), we have that  $(\mathcal{E}, D(\mathcal{E}))$  is conservative. Then strong uniqueness holds for (7) and (14), i.e pathwise uniqueness holds for (7) and (14) and there exists a unique strong solution to (7) and (14).*

**Proof** Let  $\tilde{\sigma} := \sum_{k \in \mathbb{Z}} \left( \frac{\alpha\gamma}{\gamma_{k+1}} 1_{[l_k, l_{k+1})} + \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} 1_{[r_k, r_{k+1})} \right) \circ h^{-1}$ ,  $\sigma := \tilde{\sigma} \circ h$ , and  $h$  like in Lemma 3.7. By [14, Remarques: b), p. 21] (see also [25, IX.(3.5) Theorem iii) and (3.13) Exercise], or even [19] that we could use with a localization procedure), we know that pathwise uniqueness holds for (34), if  $\tilde{\sigma}$  is locally bounded away from zero and locally of finite quadratic variation. Of course, it is enough to check this in a neighborhood of zero and for  $\sigma$  instead of  $\tilde{\sigma}$ , since  $h^{-1}$  is strictly increasing and continuous in a neighborhood of zero and  $h^{-1}(0) = 0$ . Since

$$\lim_{k \rightarrow -\infty} \frac{(1-\alpha)\bar{\gamma}}{\bar{\gamma}_{k+1}} = 1 - \alpha, \quad \lim_{k \rightarrow \infty} \frac{\alpha\gamma}{\gamma_{k+1}} = \alpha,$$

and  $\alpha \in (0, 1)$ , we clearly have that  $\sigma$  is locally bounded away from zero in any neighborhood of zero. If (29) is satisfied, then  $\sigma$  is locally of finite variation around zero, hence in

particular locally of finite quadratic variation around zero. Thus the result follows by [14, Remarques: b), p. 21]. Since  $h$  is a continuous bijection on its image with  $h(0) = 0$ , and  $X := h^{-1}(Y)$  with  $Y$  like in (34) solves (14), pathwise uniqueness also holds for (14). By the Yamada-Watanabe Theorem there exists a unique strong solution  $Y$  to (34), hence strong existence and pathwise uniqueness also holds for  $X := h^{-1}(Y)$ . Since (14) is just (7) rewritten with the symmetric local times, strong uniqueness also holds for (7).  $\square$

In the following proposition, we do not assume from the beginning (5) and (29), which were in force throughout the subsection. We also do not assume that  $(\mathcal{E}, D(\mathcal{E}))$  is conservative from the beginning.

**Proposition 3.9** *Let  $(\alpha_k)_{k \in \mathbb{Z}}, (\bar{\alpha}_k)_{k \in \mathbb{Z}} \subset (0, 1)$ . Let  $(l_k)_{k \in \mathbb{Z}}, (r_k)_{k \in \mathbb{Z}}$  be a partition of  $\mathbb{R}$  as described at the beginning of section 2. Suppose*

$$\sum_{k \geq 0} |2\alpha_k - 1| + \sum_{k \leq 0} |2\bar{\alpha}_k - 1| < \infty \quad (36)$$

and that (32), (33) are satisfied for  $(\gamma_k)_{k \in \mathbb{Z}}, (\bar{\gamma}_k)_{k \in \mathbb{Z}}$  given by (25), (26). Then for any  $\alpha \in (0, 1)$  there exists a unique strong solution to (27).

**Proof** Assume we can show (5) and (29) for  $(\gamma_k)_{k \in \mathbb{Z}}, (\bar{\gamma}_k)_{k \in \mathbb{Z}}$  given by (25), (26). Then  $\exists \lim_{k \rightarrow \infty} \gamma_k, \exists \lim_{k \rightarrow -\infty} \bar{\gamma}_k > 0$  and so **(H1)** holds by Remark 2.1 for the regular Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  corresponding to  $\rho$  in (3) with the above data. Conditions (32), (33) are equivalent to the conservativeness of  $(\mathcal{E}, D(\mathcal{E}))$  according to Remark 3.6(i). Hence we obtain existence of a solution to (27) for any  $\alpha \in (0, 1)$  by Proposition 2.11 and Remark 2.12. Strong uniqueness then follows from Theorem 3.8.

Now, we show that (5) and (29) hold. By symmetry it is enough to show that  $\sum_{k \geq 0} |2\alpha_k - 1| < \infty$  implies  $\sum_{k \geq 0} \left( |\gamma_{k+1} - \gamma_k| + \left| \frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} \right| \right) < \infty$ . We have

$$\gamma_k = \gamma_0 \prod_{j=0}^{k-1} (1 + \beta_j), \quad \text{where } \beta_j := \frac{2\alpha_j - 1}{1 - \alpha_j}, \quad j \geq 0.$$

Since  $\sum_{k \geq 0} |2\alpha_k - 1| < \infty$  it follows easily  $\sum_{k \geq 0} |\beta_k|^2 < \infty$ . Let  $N \in \mathbb{N}$  be such that  $|\beta_k| < 1$  for all  $k \geq N$ . For  $|z| < 1$  we have

$$\log(1+z) = z + z^2 \underbrace{\left( -\frac{1}{2} + \frac{z}{3} - \frac{z^2}{4} + \dots \right)}_{:=f(z)}$$

and  $f$  is continuous at 0 with  $\lim_{z \rightarrow 0} f(z) = -\frac{1}{2}$ . Thus  $(f(\beta_k))_{k \geq N}$  converges to  $-\frac{1}{2}$  and is therefore bounded. It follows that  $\sum_{k \geq N} \beta_k^2 f(\beta_k)$  converges absolutely. Since  $\log(1 + \beta_j) = \beta_j + \beta_j^2 f(\beta_j)$  for  $j \geq N$ , we have that  $\sum_{j \geq N} \log(1 + \beta_j)$  converges absolutely. In particular  $(\gamma_k)_{k \geq 0}$  converges. But then

$$\sum_{k \geq 0} |\gamma_{k+1} - \gamma_k| = \sum_{k \geq 0} |2\alpha_k - 1| (\gamma_{k+1} + \gamma_k) < \infty.$$

Since

$$\frac{1}{\gamma_k} = \frac{1}{\gamma_0} \prod_{j=0}^{k-1} \frac{1 - \alpha_j}{\alpha_j} = \frac{1}{\gamma_0} \prod_{j=0}^{k-1} \left(1 - \frac{1 - 2\alpha_j}{\alpha_j}\right),$$

we obtain similarly that  $\left(\frac{1}{\gamma_k}\right)_{k \geq 0}$  converges and then

$$\sum_{k \geq 0} \left| \frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} \right| = \frac{1}{\gamma_0} \sum_{k \geq 0} \left| \prod_{j=0}^{k-1} \left(\frac{1 - \alpha_j}{\alpha_j}\right) \left(1 - \frac{1 - 2\alpha_k}{\alpha_k} - 1\right) \right| = \sum_{k \geq 0} \frac{1}{\gamma_k} \left| \frac{1 - 2\alpha_k}{\alpha_k} \right| < \infty.$$

□

**Remark 3.10** (i) If (36) holds globally, i.e. if

$$\sum_{k \in \mathbb{Z}} (|2\alpha_k - 1| + |2\bar{\alpha}_k - 1|) < \infty, \quad (37)$$

then (32), (33) automatically hold, because  $(\gamma_k)_{k \in \mathbb{Z}}$ ,  $(\bar{\gamma}_k)_{k \in \mathbb{Z}}$  are bounded below and above by strictly positive constants. Indeed, this can be shown exactly as in the proof of the previous proposition. Hence we recover qualitatively Le Gall's strong uniqueness results according to (24) by Proposition 3.9. Here, we used the word "qualitatively" because of the following. Condition (24) covers also the case of multiple accumulation points, as long as only the sum in (24) remains finite. However, we could have easily covered this situation even with no finiteness condition on the sums in a straightforward manner. But since no new phenomena will occur locally by considering even countably many isolated accumulation points, we excluded the case of multiple accumulation points for the convenience of the reader.

(ii) It can be seen from the proof of Proposition 3.9 that (5) together with (29) are equivalent to (36). Therefore, Theorem 3.8 and Proposition 3.9 are equivalent. But Le Gall's global condition (37) is stronger than our assumptions in Proposition 3.9. One can say that the assumptions in Proposition 3.9 consist of two types of assumptions. A local assumption (36), to ensure pathwise uniqueness, and a global assumption (32) together with (33) to ensure non-explosion of the solution. Indeed, our strategy is similar to the one used in [14]. With the help of a nice function, we transform our equation into a local martingale (see (34)) and then obtain uniqueness (cf. proof of Theorem 3.8). Since our assumptions are only local, we need some global control, i.e. non-explosion. This is our additional contribution to the work of Le Gall in [14].

## 3.2 Recurrence and transience

In this subsection, we assume throughout that (5) and (29) hold. We define

$$D_y := \inf\{t \geq 0 \mid X_t = y\}, \quad y \in \mathbb{R}.$$

Under the assumptions (5) and (29), the scale function  $h$  always exists. Therefore, exactly as in [3, Chapter 6, Lemma (3.1)] we can show that

$$P_x(D_a \wedge D_b < \infty) = 1, \quad \forall x \in (a, b). \quad (38)$$

It follows in particular that  $((h(X_{t \wedge D_a \wedge D_b}))_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ , with  $h$  like in Lemma 3.7, is a uniformly bounded local martingale and by standard calculations it is well-known that for any  $x \in (a, b)$

$$P_x(D_a < D_b) = \frac{h(b) - h(x)}{h(b) - h(a)} \quad (39)$$

and

$$P_x(D_b < D_a) = \frac{h(x) - h(a)}{h(b) - h(a)}. \quad (40)$$

**Theorem 3.11** *The following are equivalent:*

- (i)  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is recurrent, i.e.  $P_x(D_y < \infty) = 1 \quad \forall x, y \in \mathbb{R}$ .
- (ii)  $h(-\infty) = -\infty$  and  $h(\infty) = \infty$ .
- (iii)  $\sum_{k \in \mathbb{Z}} \frac{l_{k+1} - l_k}{\gamma_{k+1}} = \infty$  and  $\sum_{k \in \mathbb{Z}} \frac{r_{k+1} - r_k}{\bar{\gamma}_{k+1}} = \infty$ .
- (iv)  $\int_{-\infty}^0 \frac{1}{\rho(x)} dx = \infty$  and  $\int_0^{\infty} \frac{1}{\rho(x)} dx = \infty$ .
- (v) There exist  $u_n \in D(\mathcal{E})$ ,  $n \geq 1$ ,  $0 \leq u_n \nearrow 1$  dx-a.e. as  $n \rightarrow \infty$  such that  $\mathcal{E}(u_n, u_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proof** (H1) implies that  $P_x$ -a.s.  $D_a \rightarrow \infty$  as  $a \rightarrow +\infty$  or  $a \rightarrow -\infty$ . Hence by (39)  $h(\infty) = \infty$  is equivalent to  $P_x(D_a < \infty) = 1$  for any  $x \in (a, b)$ , and by (40)  $h(-\infty) = -\infty$  is equivalent to  $P_x(D_b < \infty) = 1$  for any  $x \in (a, b)$ . This is clearly equivalent to the recurrence of  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ , hence (i)  $\Leftrightarrow$  (ii). (ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv) is obvious. (iv)  $\Rightarrow$  (v) is a special case of [9, Theorem 2.2 (i)]. For the readers convenience, we include the proof. Let  $a_n := \int_{-n}^0 \frac{1}{\rho(x)} dx$ ,  $b_n := \int_0^n \frac{1}{\rho(x)} dx$  and for  $n \geq 1$

$$u_n(x) := \begin{cases} 1 - \frac{1}{a_n} \int_x^0 \frac{1}{\rho(x)} dx & \text{if } x \in [-n, 0], \\ 1 - \frac{1}{b_n} \int_0^x \frac{1}{\rho(x)} dx & \text{if } x \in [0, n], \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly  $0 \leq u_n \nearrow 1$  dx-a.e. as  $n \rightarrow \infty$ . Fix a standard Dirac sequence  $(\varphi_\varepsilon)_{\varepsilon > 0}$  and define  $u_n^k := \varphi_{\frac{1}{k}} * u_n$ ,  $k \geq 1$ . Then  $u_n^k \in C_0^\infty(\mathbb{R})$  and by standard properties of the convolution product one can easily see that  $u_n^k \rightarrow u_n$  in  $D(\mathcal{E})$  as  $k \rightarrow \infty$ . Hence  $u_n \in D(\mathcal{E})$ . Furthermore

$$\mathcal{E}(u_n, u_n) = \frac{1}{2} \int_{-n}^0 \frac{1}{a_n^2} \frac{1}{\rho(x)} dx + \frac{1}{2} \int_0^n \frac{1}{b_n^2} \frac{1}{\rho(x)} dx = \frac{1}{2} \left( \frac{1}{a_n} + \frac{1}{b_n} \right) \rightarrow 0$$

as  $n \rightarrow \infty$ . (v)  $\Rightarrow$  (i) is well known (see e.g. [7]).

□

**Lemma 3.12** *Let one of the conditions of Theorem 3.11 be satisfied. Let  $(\theta_t)_{t \geq 0}$  be the shift operator of  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ . Then for any  $x, y \in \mathbb{R}$*

$$\limsup_{t \rightarrow \infty} \sup_{A \in \mathcal{F}} |P_x \circ \theta_t^{-1}(A) - P_y \circ \theta_t^{-1}(A)| = 0. \quad (41)$$

**Proof** By Theorem 3.11(i)  $((X_t)_{t \geq 0}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \zeta, (P_x)_{x \in \mathbb{R}})$  is a regular, recurrent diffusion in the sense of [12]. Therefore the statement follows from [12, Lemma 23.17].

□

**Remark 3.13** *The Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  is irreducible (see [7] for the definition). Therefore, by [7, Lemma 1.6.4.(iii)], it is either recurrent or transient. Thus Theorem 3.11 provides also sharp conditions about transience in the sense of [7].*

Let  $(P_t(x, dy))_{t \geq 0}$  be the transition kernels corresponding to  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ . Let  $A \subset \mathbb{R}$  be Borel measurable such that  $\int_A \rho(x) dx < \infty$ . Since **(H1)** holds, (we may assume that)  $p_t 1_A(x) := p_t(x, A) \in D(\mathcal{E})$  is continuous in  $x$  for any  $t > 0$ . (If not we could choose continuous versions and construct a process via Kolmogorov's method. This process would then be indistinguishable from  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$ ). In particular the transition kernels have a density w.r.t. reference measure  $m(dx) := \rho(x) dx$ , since  $m(A) = 0$  implies  $p_t(x, A) = 0$  for  $m$ -a.e.  $x$ , hence every  $x$  by continuity and full support of  $m$ .

Let  $\mathcal{B}(\mathbb{R})$  be the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . For a positive measure  $\mu$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $t > 0$ , we define

$$\mu p_t(A) := \int_{\mathbb{R}} p_t(x, A) \mu(dx), \quad A \in \mathcal{B}(\mathbb{R}).$$

$\mu$  is called an *invariant measure*, if  $\mu p_t = \mu$  for any  $t > 0$ . It is called an *invariant distribution*, if additionally  $\mu(\mathbb{R}) = 1$ . Clearly, the reference measure  $m = \rho dx$  is invariant since by symmetry of  $(p_t)_{t \geq 0}$  w.r.t.  $m$  and conservativeness

$$m p_t(A) = \int_{\mathbb{R}} p_t 1_A(x) m(dx) = \int_{\mathbb{R}} 1_A(x) P_x(X_t \in \mathbb{R}) m(dx) = m(A), \quad t > 0.$$

Suppose  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is recurrent. Then  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is called *null-recurrent* if

$$\lim_{t \rightarrow \infty} p_t 1_K(x) = \lim_{t \rightarrow \infty} P_x(X_t \in K) = 0$$

for any  $x \in \mathbb{R}$  and any compact set  $K$  with non-empty interior. Otherwise it is called *positive recurrent*.

It follows from the proof of Theorem 3.14 (iv)  $\Rightarrow$  (i) below, that if  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is recurrent, then it is positive recurrent, iff  $P_x(X_t \in K)$  does not converge to zero as  $t \rightarrow \infty$  for any  $x \in \mathbb{R}$  and any compact set  $K$  with non-empty interior.

**Theorem 3.14** *Suppose  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is recurrent. Then the following are equivalent:*



(i)  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is positive recurrent.

(ii)  $\int_{-\infty}^{\infty} \frac{1}{h'(x)} dx < \infty$ .

(iii) The invariant measure  $\rho dx$  is finite, i.e.  $\sum_{k \in \mathbb{Z}} \{\gamma_{k+1}(l_{k+1} - l_k) + \bar{\gamma}_{k+1}(r_{k+1} - r_k)\} < \infty$ .

(iv)  $p_t(x, dy) = P_x(X_t \in \cdot)$  converges weakly to the invariant distribution  $\frac{\rho dx}{\int_{\mathbb{R}} \rho(x) dx}$  as  $t \rightarrow \infty$  for any  $x \in \mathbb{R}$ .

(v)  $E_x[D_y] < \infty \forall x, y \in \mathbb{R}$ .

**Proof** (ii)  $\Leftrightarrow$  (iii) is obvious. (ii)  $\Leftrightarrow$  (iv) follows from [18, IV.4. Theorem 7]. (In order to facilitate comparison we note that the  $m$  of [18] writes as  $m(s) = \int_0^s \frac{2}{h'(x)} dx$ , and that the  $p$  of [18] is just our  $h$ ). For any compact set  $K$  with non-empty interior  $G$  we have using (iv) and the Portemanteau-Theorem

$$\liminf_{t \rightarrow \infty} P_x(X_t \in K) \geq \liminf_{t \rightarrow \infty} P_x(X_t \in G) \geq \frac{\int_G \rho(x) dx}{\int_{\mathbb{R}} \rho(x) dx} > 0.$$

Hence (iv)  $\Rightarrow$  (i). For  $-\infty < a < x < b < \infty$  we have (cf. [13, Chapter 5.5. D, Exercise 5.40 (i)], or [18, IV.4 (55), IV.3 (46)])

$$E_x[D_a] = \int_a^x \int_y^{\infty} \frac{2}{h'(z)} dz dh(y) = (h(x) - h(a)) \int_a^{\infty} \frac{2}{h'(y)} dy + 2 \int_a^x \frac{h(y) - h(x)}{h'(y)} dy,$$

and

$$E_x[D_b] = \int_x^b \int_{-\infty}^y \frac{2}{h'(z)} dz dh(y) = (h(b) - h(x)) \int_{-\infty}^b \frac{2}{h'(y)} dy + 2 \int_x^b \frac{h(y) - h(x)}{h'(y)} dy.$$

Thus (ii)  $\Leftrightarrow$  (v).

If (i) is satisfied then we can find  $t_n \nearrow \infty$  as  $n \rightarrow \infty$ ,  $x \in \mathbb{R}$ , and a compact set  $K_0$  such that  $\inf_{n \geq 1} p_{t_n}(x, K_0) > 0$ . By Helly's Theorem we can find another subsequence, again denoted by  $(t_n)_{n \geq 1}$  and a subprobability measure  $\mu$ , such that

$$p_{t_n}(x, \cdot) \longrightarrow \mu \quad \text{weakly as } n \rightarrow \infty.$$

The weak convergence holds indeed for any  $x \in \mathbb{R}$  by Lemma 3.12. Thus for any open set  $U$  and any compact set  $K$ , we have by the Portemanteau-Theorem that  $\liminf_{n \rightarrow \infty} p_{t_n} 1_U(x) \geq \mu(U)$ , and  $\limsup_{n \rightarrow \infty} p_{t_n} 1_K(x) \leq \mu(K)$  for any  $x \in \mathbb{R}$ . In particular,  $\mu(U_0) \geq \mu(K_0) > 0$  for any relatively compact (open) set  $U_0$  containing  $K_0$ . Then, by Fatou's lemma, conservativeness, and symmetry of  $(p_t)_{t \geq 0}$  w.r.t.  $\rho dx$

$$\begin{aligned} \int_{\mathbb{R}} 1_U \rho(x) dx &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} p_{t_n} 1_U(x) \rho(x) dx \\ &\geq \int_{\mathbb{R}} \liminf_{n \rightarrow \infty} p_{t_n} 1_U(x) \rho(x) dx \\ &\geq \mu(U) \int_{\mathbb{R}} \rho(x) dx. \end{aligned} \tag{42}$$

Applying (42) with  $U = U_0$  we conclude that  $\int_{\mathbb{R}} \rho(x) dx < \infty$  and then  $\mu(U) \leq \frac{\int_U \rho(x) dx}{\int_{\mathbb{R}} \rho(x) dx}$  for any open set  $U$ . Similarly to (42) we derive

$$\mu(K) \geq \frac{\int_K \rho(x) dx}{\int_{\mathbb{R}} \rho(x) dx} \quad (43)$$

for any compact set  $K$ . Hence by inner regularity of the measures it follows  $\mu(B) \geq \frac{\int_B \rho(x) dx}{\int_{\mathbb{R}} \rho(x) dx}$  for any Borel set  $B$ , which further implies that  $\mu = \rho dx$ . Since our arguments hold for any subsequence  $(t_n)_{n \geq 1}$  it follows

$$p_t(x, \cdot) \longrightarrow \frac{\rho dx}{\int_{\mathbb{R}} \rho(x) dx}$$

weakly as  $t \rightarrow \infty$  for any  $x \in \mathbb{R}$ . Hence (i)  $\Rightarrow$  (iv) and our proof is complete.  $\square$

**Remark 3.15** *Similarly to Remark 3.6(i) one can see that under (5) and (29), properties (iii), (iv) of Theorem 3.11 and properties (ii), (iii) of Theorem 3.14 are global assumptions and hence do not depend on the local behavior around the accumulation point.*

**Corollary 3.16** *Assume  $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}})$  is positive recurrent. Then  $\frac{\rho dx}{\int_{\mathbb{R}} \rho(x) dx}$  is the unique invariant distribution.*

**Proof** Let  $\nu$  be an invariant distribution. Then using Theorem 3.14(iv), (42), (43) with  $\rho dx$  replaced by  $\nu$ , and  $\mu$  replaced by  $\frac{\rho dx}{\int_{\mathbb{R}} \rho(x) dx}$ , we obtain similarly to the proof of (i)  $\Rightarrow$  (iv) in Theorem 3.14 that

$$\nu(B) = \frac{\int_B \rho(x) dx}{\int_{\mathbb{R}} \rho(x) dx}$$

for any Borel set  $B$ . The result hence follows.  $\square$

### 3.3 Advection-diffusion in layered media

Let  $(l_k)_{k \in \mathbb{Z}}, (r_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$  be as at the beginning of section 2. For  $\alpha \in (0, 1)$  consider the sequences

$$\gamma_{k+1} := c_\alpha \sqrt{D_k}, \quad \bar{\gamma}_{k+1} := \bar{c}_\alpha \sqrt{\bar{D}_k}, \quad k \in \mathbb{Z},$$

where  $(D_k)_{k \in \mathbb{Z}}, (\bar{D}_k)_{k \in \mathbb{Z}} \subset (0, \infty)$  and  $c_\alpha, \bar{c}_\alpha > 0$  are some constants that will be stated precisely below. We suppose that (5), (29), (32), and (33) hold. Then

$$\exists D := \lim_{k \rightarrow \infty} D_k, \quad \exists \bar{D} := \lim_{k \rightarrow -\infty} \bar{D}_k \quad \text{and} \quad D, \bar{D} > 0.$$

Let

$$\alpha_k := \frac{\sqrt{D_k}}{\sqrt{D_k} - \sqrt{D_{k-1}}}, \quad \bar{\alpha}_k := \frac{\sqrt{\bar{D}_k}}{\sqrt{\bar{D}_k} - \sqrt{\bar{D}_{k-1}}}, \quad k \in \mathbb{Z},$$

and define

$$c_\alpha := \frac{\alpha}{\sqrt{D}}, \quad \bar{c}_\alpha := \frac{1 - \alpha}{\sqrt{\bar{D}}}.$$

By Remark 3.10, we know that (5) together with (29) are equivalent to (36). Then, by Proposition 3.9 there exists a unique strong solution  $Z^\alpha$  to

$$Z_t^\alpha = x + W_t + \sum_{k \in \mathbb{Z}} \left\{ (2\alpha_k - 1) \ell_t^{l_k}(Z^\alpha) + (2\bar{\alpha}_k - 1) \ell_t^{r_k}(Z^\alpha) \right\} + (2\alpha - 1) \ell_t^0(Z^\alpha),$$

which is constructed with the help of the Dirichlet form that is determined by (3) and (4).

We now fix  $\alpha \in (0, 1)$ . By (5) and (29), there exists  $\Psi : \mathbb{R} \rightarrow \Psi(\mathbb{R})$  which is the difference of two convex functions, (continuous), piecewise linear with  $\Psi(0) = 0$  such that

$$\Psi'(x) = \begin{cases} c_\alpha \sqrt{D_k} & \text{on } (l_k, l_{k+1}) \\ \bar{c}_\alpha \sqrt{\bar{D}_k} & \text{on } (r_k, r_{k+1}). \end{cases}$$

Applying the Itô-Tanaka-formula and formulas about local times from [22], we obtain after a long calculation that  $X := \Psi(Z^\alpha)$  is a strong solution to

$$X_t = \Psi(x) + M_t + N_t, \tag{44}$$

where

$$N_t = \sum_{k \in \mathbb{Z}} \left\{ \frac{D_k - D_{k-1}}{D_k + D_{k-1}} \ell_t^{\Psi(l_k)}(X) + \frac{\bar{D}_k - \bar{D}_{k-1}}{\bar{D}_k + \bar{D}_{k-1}} \ell_t^{\Psi(r_k)}(X) \right\} + \frac{\bar{D} - D}{\bar{D} + D} \ell_t^0(X)$$

and

$$M_t = \int_0^t \sum_{k \in \mathbb{Z}} \left( c_\alpha \sqrt{D_k} 1_{[\Psi(l_k), \Psi(l_{k+1}))} + \bar{c}_\alpha \sqrt{\bar{D}_k} 1_{[\Psi(r_k), \Psi(r_{k+1}))} \right) (X_s) dW_s$$

Note that we do not have to worry about the endpoints of the intervals  $(\Psi(l_k), \Psi(l_{k+1}))$  and  $(\Psi(r_k), \Psi(r_{k+1}))$  by (17). Define

$$\sigma_1(x) := \sum_{k \in \mathbb{Z}} \left( c_\alpha \sqrt{D_k} 1_{[\Psi(l_k), \Psi(l_{k+1}))} + \bar{c}_\alpha \sqrt{\bar{D}_k} 1_{[\Psi(r_k), \Psi(r_{k+1}))} \right).$$

Then  $\sigma_1^2$  is locally uniformly strictly elliptic and so by results of [7], [17], he have that

$$\mathcal{A}^0(f, g) := \frac{1}{2} \int_{\Psi(\mathbb{R})} \sigma_1^2 f' g' dx, \quad f, g \in C_0^\infty(\Psi(\mathbb{R})) \tag{45}$$

is closable in  $L^2(\Psi(\mathbb{R}); dx)$ . Denote the closure by  $(\mathcal{A}^0, D(\mathcal{A}^0))$ . Following the lines of arguments in this article (as for the Dirichlet form defined through (3), (4)) one can verify that the unique solution  $X$  to (44) is associated to the (quasi-)regular Dirichlet form  $(\mathcal{A}^0, D(\mathcal{A}^0))$ . Let  $(L^{\mathcal{A}^0}, D(L^{\mathcal{A}^0}))$  be its generator as defined in [17, I.Proposition 2.16].

**Remark 3.17** (i) *The state space of  $X$  is  $\Psi(\mathbb{R})$  and might be different from  $\mathbb{R}$  if  $\Psi$  has a low growth rate when approaching to  $\pm\infty$ . As an example consider the case where  $r_k = k$  and  $\sqrt{\bar{D}_k} = \frac{1}{k^2}$  for  $k \geq 1$ . Additionally, as  $\Psi : \mathbb{R} \rightarrow \Psi(\mathbb{R})$  is a homeomorphism we have that  $x \in \mathbb{R}$  is an accumulation point for  $(l_k)_{k \in \mathbb{Z}}$  or  $(r_k)_{k \in \mathbb{Z}}$ , iff  $\Psi(x) \in \Psi(\mathbb{R})$  is an accumulation point of  $(\Psi(l_k))_{k \in \mathbb{Z}}$  or  $(\Psi(r_k))_{k \in \mathbb{Z}}$ . Thus  $\Psi(0) = 0$  is the only accumulation point of  $(\Psi(l_k))_{k \in \mathbb{Z}}$  and  $(\Psi(r_k))_{k \in \mathbb{Z}}$ . Considering this, we could not follow [23, Remark 3.1].*

(ii) *An invariant measure for  $X$ , is the Lebesgue measure restricted to  $\Psi(\mathbb{R})$ . This is directly visible from the definition of the corresponding Dirichlet form in (45). It will be finite, iff  $\Psi(\mathbb{R})$  is bounded. It is evident from (45) that a scale function  $\bar{h}$  for  $X$ , i.e. a piecewise linear function  $\bar{h} : \Psi(\mathbb{R}) \rightarrow \mathbb{R}$  with  $\bar{h}(0) = 0$  and  $\mathcal{A}^0(\bar{h}, g) = 0$  for all  $g$  is given by defining its derivatives as*

$$\bar{h}'(x) = \begin{cases} \frac{1}{D_k} & \text{on } (l_k, l_{k+1}) \\ \frac{1}{\bar{D}_k} & \text{on } (r_k, r_{k+1}). \end{cases}$$

Then, exactly as in Theorems 3.11 and 3.14, we can see that  $X$  is recurrent, iff

$$\sum_{k \leq 0} \frac{\Psi(l_{k+1}) - \Psi(l_k)}{D_k} = \sum_{k \leq 0} \frac{\Psi(r_{k+1}) - \Psi(r_k)}{\bar{D}_k} = \infty, \quad (46)$$

that is  $-\bar{h}(\Psi(-\infty)) = \infty = \bar{h}(\Psi(\infty))$ , and positive recurrent, if additionally to (46)

$$\sum_{k \leq 0} D_k(\Psi(l_{k+1}) - \Psi(l_k)) + \sum_{k \leq 0} \bar{D}_k(\Psi(r_{k+1}) - \Psi(r_k)) < \infty. \quad (47)$$

If (46) and (47) hold, then the normalized Lebesgue measure, i.e. the uniform distribution on (the bounded set !)  $\Psi(\mathbb{R})$ , is the unique invariant probability measure for  $X$ . This statement can be shown analogously to Corollary 3.16. This seems to contradict what is stated in [23, Remark 3.2].

Take an independent copy  $(B_t)_{t \geq 0}$  of  $(W_t)_{t \geq 0}$  and let  $\sigma_2, B_2 : \Psi(\mathbb{R}) \rightarrow \mathbb{R}$  be locally bounded Borel-measurable functions. Assume additionally that  $\sigma_2$  is strictly positive and locally uniformly bounded away from zero. For  $y \in \mathbb{R}$  consider the Itô-process

$$Y_t^y := y + \int_0^t \sigma_2(X_s) dB_s + \int_0^t B_2(X_s) ds, \quad t \geq 0.$$

Clearly,  $Y := (Y^y)_{y \in \mathbb{R}}$  is non-explosive, since the paths of  $X$  are continuous.

Below, we will show that  $(X, Y)$  is associated to a generalized Dirichlet form, stationary

and “reversible” (see Remark 3.18 and in particular [37] and [35]) w.r.t. the two dimensional Lebesgue measure  $dxdy$ . For this, we need some preparations.

Let

$$A = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}. \quad (48)$$

Let  $\partial_x f(x, y)$  denote the partial derivative in the  $x$ -coordinate and  $\partial_y f(x, y)$  denote the partial derivative in the  $y$ -coordinate. Since  $A$  is locally uniformly strictly elliptic it is well-known, that

$$\begin{aligned} \mathcal{E}^0(f, g) &:= \frac{1}{2} \int_{\Psi(\mathbb{R})} \int_{\mathbb{R}} \langle A \nabla f, \nabla g \rangle dxdy \\ &= \frac{1}{2} \int_{\Psi(\mathbb{R})} \int_{\mathbb{R}} \sigma_1^2 \partial_x f \partial_x g dxdy + \frac{1}{2} \int_{\Psi(\mathbb{R})} \int_{\mathbb{R}} \sigma_2^2 \partial_y f \partial_y g dxdy, \end{aligned} \quad (49)$$

with  $f, g \in C_0^\infty(\Psi(\mathbb{R}) \times \mathbb{R})$  is closable in  $L^2(\Psi(\mathbb{R}) \times \mathbb{R}; dxdy)$  and that the closure  $(\mathcal{E}^0, D(\mathcal{E}^0))$  is a ((quasi-)regular) symmetric Dirichlet form (see [17], [7]). Let  $(L^0, D(L^0))$  be the corresponding generator.

Moreover, we know that the vector field  $B = (B_1, B_2) : \Psi(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}^2$ , with  $B_1 \equiv 0$  is divergence free w.r.t.  $dxdy$ , i.e.

$$\int_{\Psi(\mathbb{R})} \int_{\mathbb{R}} \langle B, \nabla f \rangle dxdy = \int_{\Psi(\mathbb{R})} \int_{\mathbb{R}} B_2(x) \partial_y f dxdy = 0 \quad \forall f \in C_0^\infty(\Psi(\mathbb{R}) \times \mathbb{R}) \quad (50)$$

Clearly, (50) extends to all  $f$  in

$$D(\mathcal{E}^0)_{0,b} := D(\mathcal{E}^0) \cap L^\infty(\Psi(\mathbb{R}) \times \mathbb{R}; dxdy) \cap \{f \text{ has compact support in } \Psi(\mathbb{R}) \times \mathbb{R}\}.$$

Let

$$U_n := (\Psi(l_{-n}), \Psi(r_n)) \times (-n, n), \quad n \geq 1.$$

Then the  $U_n$  are relatively compact open subsets of  $\Psi(\mathbb{R}) \times \mathbb{R}$  and we can consider the part Dirichlet forms  $(\mathcal{E}^{0,U_n}, D(\mathcal{E}^{0,U_n}))$ ,  $n \geq 1$ , as given in [7, Theorem 4.4.3]. Let  $(L^{0,U_n}, D(L^{0,U_n}))$  be the corresponding generators. Furthermore, since  $D(\mathcal{E}^{0,U_n})_b \subset D(\mathcal{E}^0)_{0,b}$ , (50) holds for all  $f \in D(\mathcal{E}^{0,U_n})_b$ . Following the line of arguments in [30], there exists for each  $n \geq 1$ , a closed extension  $(\bar{L}^{U_n}, D(\bar{L}^{U_n}))$  on  $L^1(U_n, dxdy)$  of

$$L^{U_n} u := L^{0,U_n} u + B_2 \partial_y u, \quad u \in D(L^{0,U_n})_b \quad (51)$$

that generates a submarkovian  $C_0$ -semigroup of contractions on  $L^1(U_n, dxdy)$ . The part  $(L^{U_n}, D(L^{U_n}))$  of  $(\bar{L}^{U_n}, D(\bar{L}^{U_n}))$  on  $L^2(U_n, dxdy)$  is then associated to generalized Dirichlet form (cf. [30, Section 1a]). Let  $D(L^0)_{0,b} := D(L^0)_b \cap \{f \text{ has compact support in } \Psi(\mathbb{R}) \times \mathbb{R}\}$ . Then using a localization procedure by following [30, Section 1b), Theorem 1.5] one can show, that there exists a closed extension  $(\bar{L}, D(\bar{L}))$  on  $L^1(\Psi(\mathbb{R}) \times \mathbb{R}, dxdy)$  of

$$Lu := L^0 u + B_2 \partial_y u, \quad u \in D(L^0)_{0,b}, \quad (52)$$

that generates a submarkovian  $C_0$ -semigroup of contractions on  $L^1(\Psi(\mathbb{R}) \times \mathbb{R}, dx dy)$  and whose resolvent can be approximated by the resolvents of  $(\bar{L}^{U_n}, D(\bar{L}^{U_n}))$ ,  $n \geq 1$ . Note that for this one has to verify that  $D(L^0)_{0,b} \subset L^2(\Psi(\mathbb{R}) \times \mathbb{R}; dx dy)$  densely, which holds since  $D(L^{0,U_n})_b \subset D(L^0)_{0,b}$  for any  $n$ , hence  $D(L^0)_{0,b}$  is dense in  $L^2(U_n; dx dy)$  for any  $n$ . Then again analogously to the line of arguments in [30]), one shows that the part  $(L, D(L))$  of  $(\bar{L}, D(\bar{L}))$  on  $L^2(\Psi(\mathbb{R}) \times \mathbb{R}, dx dy)$  is associated to a quasi-regular generalized Dirichlet form that satisfies D3. The identification of the associated process can then be performed as in [26], [36].

**Remark 3.18** *The process  $(X, Y)$  has been constructed in [23] under the stronger additional assumptions that  $\sigma_1, \sigma_2$ , and  $\frac{B_2}{\sigma_2}$  are bounded. In particular  $B_2$  is then also bounded. The components  $X$ , resp.  $Y$ , represent the transversal, resp. longitudinal directions of advection-diffusion in layered media and the fundamental solution corresponding to the underlying Kolmogorov operator serves as a model for the concentration of a solute undergoing advection-diffusion there (see [24], [23, section 3]). Having constructed  $(X, Y)$  in a more general setting, one can study the asymptotic properties of  $X$  and  $Y$  as in [23] and [24]. Our generalization for  $X$  may be interpreted as increased heterogeneity of the layered media. The weaker assumptions on  $B_2$  allow higher speed of transportation (advection) of the solute particles in the respective layers. We have seen in Remark 3.17(ii) that  $X$  has a unique invariant distribution if (46) and (47) hold. So in this case one may hope to obtain a central limit theorem for  $(X, Y)$  as in [24] (cf. also [23, Remark 3.2]), i.e. one may hope to solve the Taylor-Aris problem. Heterogeneous dispersion in a longitudinal flow was carried out in [24, section 2, 3] w.r.t. a compact domain  $G$  with finitely many layers and normal reflecting boundary condition at  $\partial G$ . In this regard, it could also be interesting to investigate the effect of replacing  $G$  with the bounded domain  $\Psi(\mathbb{R})$ , thus allowing increased heterogeneity and inaccessible boundaries. Finally, we note that the (non-sectorial) Lyons-Zheng decomposition holds for  $(X, Y)$ . Hence one can use it as an additional tool to derive ergodic properties (cf. e.g. [37] and [35]).*

## References

- [1] Albeverio, S., Høegh-Krohn, R., Streit, L.: Energy forms, Hamiltonians, and distorted Brownian paths. J. Mathematical Phys. 18 (1977), no. 5, 907-917.
- [2] Bass, R.F.; Chen, Z.-Q.: Stochastic differential equations for Dirichlet processes. Probab. Theory Related Fields 121 (2001), no. 3, 422-446.
- [3] Durrett, R.: Stochastic calculus. A practical introduction. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996. x+341 pp. ISBN: 0-8493-8071-5.
- [4] Engelbert, H. J., Schmidt, W.: On one-dimensional stochastic differential equations with generalized drift. Stochastic differential systems (Marseille-Luminy, 1984), 143-155, Lecture Notes in Control and Inform. Sci., 69, Springer, Berlin, 1985.

- [5] Flandoli, F., Russo, F. Wolf, J.: Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.* 40 (2003), no. 2, 493–542.
- [6] Flandoli, F., Russo, F. Wolf, J.: Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations* 12 (2004), no. 2, 145–184.
- [7] Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet forms and Symmetric Markov processes*. Berlin-New York: Walter de Gruyter 1994.
- [8] Fukushima, M.: On a stochastic calculus related to Dirichlet forms and distorted Brownian motions. *New stochastic methods in physics*. *Phys. Rep.* 77 (1981), no. 3, 255-262.
- [9] Gim, M., Trutnau, G.: Explicit recurrence criteria for symmetric gradient type Dirichlet forms satisfying a Hamza type condition, arXiv:1308.0234.
- [10] Harrison, J.M., Shepp, L.A.: On skew Brownian motion, *Ann. Prob.*, Vol. 9, No.2, 309–313, (1981).
- [11] Itô, K.; McKean, H. P.: *Diffusion processes and their sample paths*. Second printing, corrected. *Die Grundlehren der mathematischen Wissenschaften, Band 125*. Springer-Verlag, Berlin-New York, 1974. xv+321 pp.
- [12] Kallenberg, Olav: *Foundations of modern probability*. Second edition. *Probability and its Applications (New York)*. Springer-Verlag, New York, 2002. xx+638 pp. ISBN: 0-387-95313-2.
- [13] Karatzas, I.; Shreve, S.E.: *Brownian motion and stochastic calculus*. Second edition. *Graduate Texts in Mathematics, 113*. Springer-Verlag, New York, 1991. xxiv+470 pp. ISBN: 0-387-97655-8.
- [14] Le Gall, J.-F.: Applications du temps local aux équations différentielles stochastiques unidimensionnelles, *Séminaire de probabilités, XVII*, 15-31, *Lecture Notes in Math.*, 986, Springer, Berlin, 1983.
- [15] Le Gall, J.-F.: One-dimensional stochastic differential equations involving the local times of the unknown process. *Stochastic analysis and applications (Swansea, 1983)*, 51-82, *Lecture Notes in Math.*, 1095, Springer, Berlin, 1984.
- [16] Lejay, A.: On the constructions of the skew Brownian motion. *Probab. Surv.* 3 (2006), 413–466 (electronic).
- [17] Ma, Z.M., Röckner, M.: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Berlin: Springer 1992.

- [18] Mandl, P.: Analytical treatment of one-dimensional Markov processes. Die Grundlehren der mathematischen Wissenschaften, Band 151 Academia Publishing House of the Czechoslovak Academy of Sciences, Prague; Springer-Verlag New York Inc., New York 1968 xx+192 pp.
- [19] Nakao, S.: On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations, Osaka J. Math. 9 (1972), 513-518.
- [20] Oshima, Y.: On conservativeness and recurrence criteria for Markov processes. Potential Anal. 1 (1992), no. 2, 115-131.
- [21] Ouknine, Y.: Le “Skew-Brownian motion” et les processus qui en dérivent. [Skew-Brownian motion and associated processes] (French) Teor. Veroyatnost. i Primenen. 35 (1990), no. 1, 173–179; translation in Theory Probab. Appl. 35 (1990), no. 1, 163–169 (1991).
- [22] Ouknine, Y., Rutkowski, M.: Local times of functions of continuous semimartingales, Stochastic Anal. Appl. 13 (1995), no. 2, 211–231.
- [23] Ramirez, J.M.: Multi-skewed Brownian motion and diffusion in layered media. Proc. Amer. Math. Soc. 139 (2011), no. 10, 3739-3752.
- [24] Ramirez, J.M.; Thomann, E.A.; Waymire, E.C.; Haggerty, R.; Wood, Brian.: A generalized Taylor-Aris formula and skew diffusion. (English summary) Multiscale Model. Simul. 5 (2006), no. 3, 786-801.
- [25] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Springer Verlag, 2005.
- [26] Russo, F., Trutnau, G.: About a construction and some analysis of time inhomogeneous diffusions on monotonely moving domains, Journal of Functional Analysis 221, (2005), pp 37–82.
- [27] Russo, F., Trutnau, G.: Some parabolic PDEs whose drift is an irregular random noise in space. Ann. Probab. 35 (2007), no. 6, 2213–2262.
- [28] Rutkowski, M.: Strong solutions of stochastic differential equations involving local times. Stochastics 22 (1987), no. 3-4, 201–218.
- [29] Rutkowski, M.: Stochastic differential equations with singular drift. Statist. Probab. Lett. 10 (1990), no. 3, 225-229.
- [30] Stannat, W.: (Nonsymmetric) Dirichlet operators on  $L^1$ : Existence, uniqueness and associated Markov processes, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 99–140.
- [31] Sturm, K.-T.: Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and  $L^p$ -Liouville properties. J. Reine Angew. Math. 456 (1994), 173–196



- [32] Takanobu, S.: On the existence of solutions of stochastic differential equations with singular drifts, *Probab. Theory Related Fields* 74 (1987), no. 2, 295-315.
- [33] Takanobu, S.: On the Uniqueness of Solutions of Stochastic Differential Equations with Singular Drifts, *Publ. RIMS, Kyoto Univ.*, **22** (1986), 813-848.
- [34] Takeda, M.: On a martingale method for symmetric diffusion processes and its applications. *Osaka J. Math.* 26 (1989), no. 3, 605–623.
- [35] Takeda, M., Trutnau, G.: Conservativeness of non-symmetric diffusion processes generated by perturbed divergence forms. *Forum Math.* 24 (2012), no. 2, 419-444.
- [36] Trutnau, G.: On a class of non-symmetric diffusions containing fully non-symmetric distorted Brownian motions, *Forum Math.* 15, No.3, 409-437 (2003).
- [37] Trutnau, G.: A short note on Lyons-Zheng decomposition in the non-sectorial case. *Proceedings of RIMS Workshop on Stochastic Analysis and Applications*, 237–245, *RIMS Kôkyûroku Bessatsu, B6*, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

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