Hyperbolic Delaunay triangulations and Voronoi diagrams made practical

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Abstract

We show how to compute Delaunay triangulations and Voronoi diagrams of a set of points in hyperbolic space in a very simple way. While the algorithm follows from [7], we elaborate on arithmetic issues, observing that only rational computations are needed. This allows an exact and efficient implementation.

1 Introduction

As D. Eppstein states: "Hyperbolic viewpoint may help even for Euclidean problems" [8].[1] He gives two examples: the computation of 3D Delaunay triangulations of sets lying in two planes [4], and optimal Möbius transformation and conformal mesh generation [3]. Hyperbolic geometry is also used in applications like graph drawing [11, 9].

Several years ago, we showed that the hyperbolic Delaunay triangulation and Voronoi diagram can easily be deduced from their Euclidean counterparts [7, 4]. As far as we know, this was the first time when the computation of hyperbolic Delaunay triangulations and Voronoi diagrams was addressed. Since then, the topic appeared again in many publications. Onishi and Takayama write that they "rediscover the algorithm of [7]", in a way that they consider as more natural (ie. their proofs rely only on computations) [13]. Nielsen and Nock transform the computation of the Voronoi diagram in the Klein model to the computation of an Euclidean power diagram [12]; however, even when the input sites have rational coordinates, the weighted points on which the power diagram is computed have algebraic coordinates. Other references can be found in [15] (which does not mention [7], though).

None of the above papers shows interest in practical aspects, especially arithmetic aspects, which are well known to be crucial for exactness and efficiency of implementations. In this paper, we show that our earlier approach allows to use only very simple arithmetic computations. Moreover, the proofs are purely geometric, and avoid any computation. We first recall some background on hyperbolic geometry (Section 2.1) and on the space of circles (Section 2.2). Section 3 details

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[1] see also `http://www.ics.uci.edu/~eppstein/pubs/geom-hyperbolic.html`

algorithmic and arithmetic aspects of the computation of hyperbolic Delaunay triangulations and Voronoi diagrams. Section 4 gives quick overview of the implementation.

Due to lack of space, the current presentation is restricted to the 2D case, but the results hold in any dimension.

2 Background

2.1 The Poincaré disk

We refer the reader to basic geometry books for an introduction to hyperbolic geometry (eg. [2, Chapter 19] [16, 17]). The transformations between the various models of the hyperbolic space are recalled in [12].

In the Poincaré disk model, the hyperbolic plane $\mathbb{H}^2$ is represented as the unit disk of $\mathbb{R}^2$. $\mathcal{H}_\infty$, the set of points of $\mathbb{H}^2$ at infinity, is represented as the unit circle of $\mathbb{R}^2$. The set of finite points of $\mathbb{H}^2$ is the interior of the unit disk.

Hyperbolic lines are represented either as portions of lines of $\mathbb{R}^2$ that are orthogonal to $\mathcal{H}_\infty$ or as portions of circles of $\mathbb{R}^2$ that are orthogonal to $\mathcal{H}_\infty$. Hyperbolic circles are circles of $\mathbb{R}^2$ contained in the unit disk and that do not intersect $\mathcal{H}_\infty$. The hyperbolic center of a circle C is the unique point (or circle with null radius) in $\mathbb{H}^2$ of the linear pencil of circles of $\mathbb{R}^2$ defined by C and $\mathcal{H}_\infty$.

2.2 The space of circles

The space of circles states a correspondence between circles of $\mathbb{R}^2$ and points of $\mathbb{R}^3$ [2, Chapter 20] [7]: the circle C of $\mathbb{R}^2$ with center $c = (x_c, y_c)$ and radius r is associated to the point $\sigma_C = (x_c, y_c, z_c = x_c^2 + y_c^2 - r^2)$ in $\mathbb{R}^3$. A point $p = (x_p, y_p)$ of $\mathbb{R}^2$, seen as a circle of null radius, is thus associated to the point $\sigma_p = (x_p, y_p, x_p^2 + y_p^2)$ on the unit paraboloid $\Pi : (z = x^2 + y^2)$ of $\mathbb{R}^3$.

In the space of circles, we are considering polarity relatively to Π, ie. orthogonality with respect to the symmetric bilinear form $\phi_\Pi\left((x_1, y_1, z_1), (x_2, y_2, z_2)\right) = x_1 x_2 + y_1 y_2 - \frac{z_1+z_2}{2}$ associated to the quadratic form $Q_\Pi(x, y, z) = x^2 + y^2 - z$. In this setting, for a circle C of $\mathbb{R}^2$, π_C denotes the polar hyperplane of σ_C in $\mathbb{R}^3$; each point $\sigma_{C'}$ of π_C represents a circle C' that is orthogonal to C. For a point $p \in \mathbb{R}^2$, π_p is the hyperplane tangent to Π at point $\sigma_p \in \Pi$; each point $\sigma_{C'}$ of π_p represents a circle C' that passes through p. The unit circle $\mathcal{H}_\infty$ of $\mathbb{R}^2$ (ie. the infinite line of $\mathbb{H}^2$) is represented as the point σ_∞ of coordinates $(0, 0, -1)$ in $\mathbb{R}^3$. Its polar hyperplane is the plane π_∞ of equation $(z = 1)$ in $\mathbb{R}^3$.

3 Algorithmic and arithmetic aspects

Let $\mathcal{P}$ be a set of points in $\mathbb{H}^2$, ie. in the unit disk of $\mathbb{R}^2$. We assume coordinates of points in $\mathcal{P}$ to be rational.

To compute the hyperbolic Delaunay triangulation $DT_\mathbb{H}(\mathcal{P})$ of $\mathcal{P}$, it is enough to compute the Euclidean Delaunay triangulation $DT_\mathbb{R}(\mathcal{P})$, and to filter out the triangles of $DT_\mathbb{R}(\mathcal{P})$ whose circumscribing circle intersects $\mathcal{H}_\infty$. It directly follows that the complexity of computing $DT_\mathbb{H}(\mathcal{P})$ is of the same order as $DT_\mathbb{R}(\mathcal{P})$, $\Theta(n \log n)$.

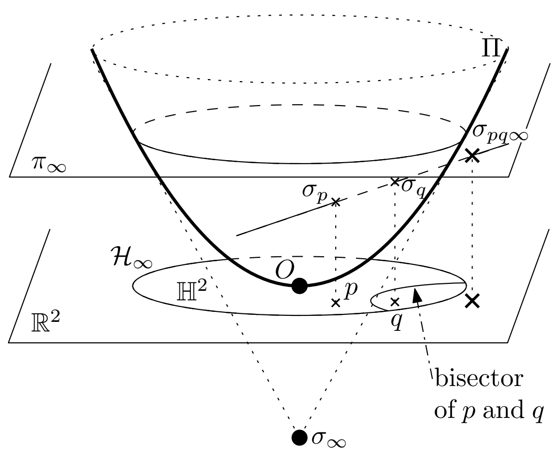
Many standard algorithms allow to compute $DT_\mathbb{R}(\mathcal{P})$ using only rational computations, since the two elementary predicates *orientation*(p, q, r) and *in_circle*(p, q, r, s) boild down to computing signs of determinants on the coordinates of the points p, q, r, s. Testing whether a circle defined by two rational points intersects the unit circle $\mathcal{H}_\infty$ is also done by rational computations only. So, the combinatorial part of $DT_{\mathbb{H}}(\mathcal{P})$ is easily computed using only rational computations.

Let us now focus on the geometric embedding of $DT_{\mathbb{H}}(\mathcal{P})$ and of the dual Voronoi diagram $VD_{\mathbb{H}}(\mathcal{P})$. We first remark that, straightforwardly from the above definition, a circle $C \subset \mathbb{R}^2$ has rational equation *iff* its associated point σ_C in the space of circles has rational coordinates.

For a triangle pqr of $DT_{\mathbb{H}}(\mathcal{P})$, the hyperbolic edge $e_{\mathbb{H}}(pq)$ is the arc between p and q of the hyperbolic line through p and q, ie. the circle through p and q that is orthogonal to $\mathcal{H}_\infty$. The set of circles passing through p and q is the line $\delta_{pq} = \pi_p \cap \pi_q$ in the space of circles. δ_{pq} is also the polar line of line $(\sigma_p\sigma_q)$. The circle through p and q that is orthogonal to $\mathcal{H}_\infty$ is thus associated to the intersection $\delta_{pq} \cap \pi_\infty$. So, $DT_{\mathbb{H}}(\mathcal{P})$ can be geometrically embedded using only rational computations.

Proposition. *The bisector of two points of $\mathcal{P}$ is a hyperbolic line, whose equation in $\mathbb{R}^2$ is rational.*

Proof. Let p and q be two points of $\mathcal{P}$, ie. rational points of $\mathbb{R}^2$ inside the unit disk. We are going to construct their hyperbolic bisector as the locus of hyperbolic centers of circles passing through p and q.

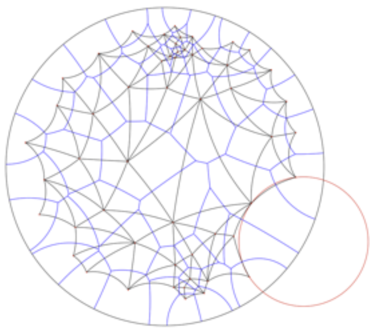


The hyperbolic center of a given circle $\sigma_C \in \delta_{pq}$ is the intersection of the line $\sigma_C\sigma_\infty$ with Π, so, the locus of such centers is the intersection $E_{pq\infty}$ with Π of the plane $P_{pq\infty}$ containing δ_{pq} and σ_∞. The polar point $\sigma_{pq\infty}$ of $P_{pq\infty}$ represents a circle, so $E_{pq\infty}$ is in fact associated to the set of points of $\mathbb{R}^2$ on this circle. And $\sigma_{pq\infty}$ is the intersection of the polars of σ_∞ and δ_{pq}, ie. the intersection $\pi_\infty \cap (\sigma_p\sigma_q)$.
To complete the proof, it is enough to notice that all geometric constructions above manipulate only rational objects: rational points, and intersection of a rational plane with a rational line. □

A vertex of $VD_{\mathbb{H}}(\mathcal{P})$ is the intersection of two hyperbolic bisectors, ie. the intersection of two circles whose equation is rational, so, its coordinates are algebraic numbers of degree 2. A Voronoi vertex can alternatively be computed in the following way: it is the hyperbolic center of a circle C_{pqr} circumscribing a Delaunay triangle pqr, associated with point σ_{pqr} in the space of circles; the hyperbolic center of C_{pqr} is thus the intersection $(\sigma_{pqr}\sigma_\infty) \cap \Pi$ (from the last sentence of Section 2.1). Infinite edges of the Voronoi diagram intersect $\mathcal{H}_\infty$ in points whose coordinates are also algebraic numbers of degree 2. So, all edges of $VD_{\mathbb{H}}(\mathcal{P})$ are arcs of rational circles whose endpoints are algebraic numbers of degree 2.

4 Implementation

The algorithm was implemented using CGAL [1]. The class `Delaunay_hyperbolic_triangulation_2` derives from the class `CGAL::Delaunay_triangulation_2<GT>`, which computes $DT_{\mathbb{R}}(\mathcal{P})$ [18]. It just has to mark all triangles of $DT_{\mathbb{R}}(\mathcal{P})$ that are not in $DT_{\mathbb{H}}(\mathcal{P})$. The template parameter `GT` is the *geometric traits*, which provides the algorithm with elementary predicates and constructions.



Only predicates are used for the computation of $DT_{\mathbb{R}}(\mathcal{P})$, they don't need to be modified to compute $DT_{\mathbb{H}}(\mathcal{P})$, which is thus computed exactly and efficiently using filtered rational arithmetics. Our preliminary experiments show an extra cost of 9% to extract $DT_{\mathbb{H}}(\mathcal{P})$ from $DT_{\mathbb{R}}(\mathcal{P})$ (ie., $DT_{\mathbb{H}}(\mathcal{P})$ is constructed in less than 1 sec for 1 million points, on a MacBookPro 2.6 GHz). Only geometric embeddings need constructions, which must be replaced in `GT` by constructions presented in Section 3; if they are only used for drawing, they don't need to be exact, but some CGAL, CORE or LEDA exact algebraic numbers can be used if necessary.

One of our goals is to compute hyperbolic periodic Delaunay triangulations, which appears much more difficult than in the Euclidean setting [6], even for the simplest case of a group of four hyperbolic translations in $\mathbb{H}^2$ [5, Section 4.4], which would already be useful for various applications [14, 10].

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