Optimization of running strategies based on anaerobic energy and variations of velocity

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OPTIMIZATION OF RUNNING STRATEGIES BASED ON ANAEROBIC ENERGY AND VARIATIONS OF VELOCITY. REVISED VERSION 08 MAY 2014

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Abstract. The aim of this work is to present a model relying on a system of ordinary differential equations describing the evolution of the anaerobic energy and the velocity of a runner. We perform numerical simulations and rigorous analysis to deduce an optimal running strategy. Our model relies on the fundamental principle of dynamics, energy conservation, a hydraulic analogy, and control conditions. We take into account the resistive force, the propulsive force and the variations in the volume of oxygen used per unit time. Our main result is to show that varying one's velocity rather than running at a constant velocity allows to run longer. Using optimal control theory, we present proofs on the structure of the optimal race and relate the problem to a relaxed formulation, where the propulsive force represents a probability distribution rather than a function of time. Our mathematical analysis leads us to introduce a bound on the variations of the propulsive force to obtain a more realistic model which displays oscillations of the velocity. We also present numerical simulations of our system which qualitatively reproduce quite well physiological measurements on real runners. We show how, by optimizing over a period, we recover the oscillations of speed.

Key words. Running race, anaerobic energy, energy recreation, optimal control, singular arc, state constraint, optimality conditions.

1. Introduction.

In sports training, the use of various measurements devices to track speed and calories and to identify the way bodies react to efforts has greatly spread. The issue of the optimal strategy of a runner, given the distance or time to run, is still a major challenge. The motivation of this paper is to provide a reliable system of differential equations describing the evolution of the anaerobic energy and the velocity of a runner. We are going to use the optimal control theory and algorithms.

A pioneering mathematical work is that of Keller [13] relying on Newton's law of motion and energy conservation. For sufficiently long races (> 291m), his analysis leads to an optimal race in three parts

1. initial acceleration with a propulsive force at its maximal value,
2. constant speed during the major part of the race,
3. final small part with constant energy equal to zero.

Though this analysis reproduces quite well the record times for distances up to 10km, it has some weaknesses. First, physiological measurements [3, 11] show that runners do not keep a constant speed but vary their speed by an order of 10%. Next, Keller assumes that the runner keeps a constant value of $\dot{V}O2_{max}$, the maximal oxygen uptake, whereas this value gradually increases to its maximal value, and then drops at the end of the race (Figure 1 of [3] and Figure 1 of [11]). Some authors [1, 14, 26] have tried to improve Keller's model, but still relying on the same strategy and mathematical arguments, leading to an almost constant speed. Other references concerning the optimality of a run include [16, 18, 25].

In this paper, we will rely on Keller's equations [13] and we will try to improve them using a hydraulic analogy and physiological indications described in [17]. Nevertheless, the formula of [17] yield averaged values while we want to make instantaneous

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energy balance taking into account optimal control theory. We aim at fully accounting for measurements of [3, 11]. We will also provide a proof that Keller's optimal race has exactly the three parts: indeed, Keller makes up a race with three optimal pieces that he computes, without proving the optimality.

The paper is organized as follows: in section 2, we present our new models together with numerical simulations of optimal velocity and anaerobic energy. In section 3, we describe our mathematical analysis and proofs while a conclusion is derived in section 4.

2. Numerical presentation of the models. In this section, we will present our ideas for an improved modeling of races and our numerical findings based on the Bocop toolbox for solving optimal control problems [6]. This software combines a user friendly interface, general Runge-Kutta discretization schemes described in [10, 5], and the numerical resolution of the discretized problem using the nonlinear programming problems solver IPOPT [24].

Our first equation is the equation of motion, as in Keller's paper:

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t) \tag{2.1}$$

where t is the time, $v(t)$ is the instantaneous velocity, $f(t)$ is the propulsive force and v/τ is a resistive force per unit mass. The resistive force can be modified to include another power of v. Note that we could take into account a changing altitude, by adding to the right hand side a term of the form $-g \sin \alpha(d(t))$, where $\alpha(d(t))$ is the slope at distance $d(t)$. We can relate $\sin \alpha(d)$ to $A(d)$, the altitude of the center of mass of the runner at distance d, by

$$\sin \alpha(d(t)) = \frac{A'(d(t))}{\sqrt{1 + (A'(d(t)))^2}}.$$

For most races, one can assume that $\sin \alpha(d(t)) \sim A'(d(t))$ and here, we assume for simplicity that A is constant along the race.

Next, we have to establish an equation governing the energy. In fact, human energy can be split into aerobic energy called e_{ae}, which is the energy provided by oxygen consumption, and anaerobic energy e_{an}, which is provided by glycogen and lactate. A very good review on different types of modeling can be found in [17] and a more general reference is [2]. In his paper [13], Keller claims that his energy equation only deals with aerobic energy: he speaks of oxygen supplies. In fact, as we will show below, using a hydraulic analogy, we believe that it well describes the accumulated oxygen deficit: $e_{an}^0 - e_{an}(t)$, where e_{an}^0 is the value at $t = 0$ of $e_{an}(t)$ the anaerobic energy.

Constraints have to be imposed; the force is controlled by the runner but it cannot exceed a maximal value:

$$0 \le f(t) \le f_{max}. \tag{2.2}$$

The aim is to minimize the time T, given the distance $d = \int_0^T v(t)\, dt$, with the initial conditions:

$$v(0) = 0, \quad e_{an}(0) = e_{an}^0 \quad \text{under the constraint } e_{an}(t) \ge 0. \tag{2.3}$$

The rest of this section will be devoted to finding a good equation for $e_{an}(t)$.

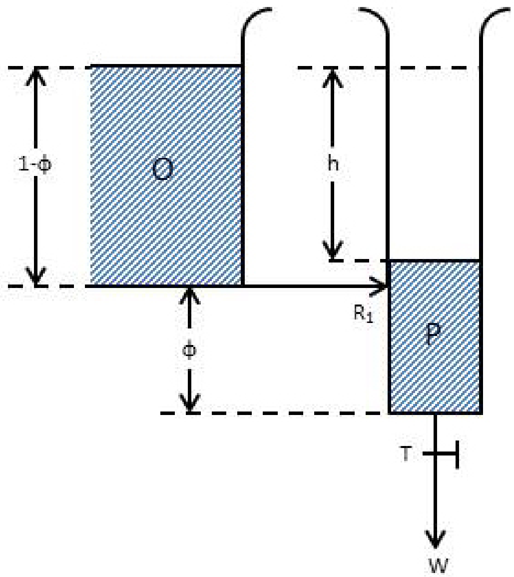

FIG. 2.1. *Scheme of the container modeling.*

2.1. First model: how Keller's model describes the accumulated oxygen deficit. Let us use a hydraulic analogy to account for Keller's equations and justify our improvements. This hydraulic analogy is described in [16, 17] in order to develop a three parameter critical power model: the equations in [16, 17] are on averaged values of the energy and the power, while we will use instantaneous values. We assume that the anaerobic energy has finite capacity modeled by a container of height 1 and surface A_p. When it starts depleting by a height h, then what is called in physiology the accumulated oxygen deficit is $e^0_{an} - e_{an}$ and we relate

$$e^0_{an} - e_{an} = A_p h, \tag{2.4}$$

where $e^0_{an} = A_p$ is the initial supply of anaerobic energy, therefore, A_p has in fact the dimension of an energy and h is a nondimensionalized height.

We assume that the aerobic energy is of infinite capacity and flows at a maximal rate of $\bar{\sigma}$ through R_1 (a connecting tube of fixed diameter illustrated in Figure 2.1). Here, O is the infinite aerobic container, P is the finite capacity anaerobic container, h is the height of depletion of the anaerobic container. An important point is the height at which the aerobic container is connected to the anaerobic one. If we assume in this first model that it is connected at height 1 (at the top of the anaerobic container, and not φ for the moment as on Figure 2.1), then it means that the aerobic energy always flows at rate $\bar{\sigma}$. Note that $\bar{\sigma}$ is the energetic equivalent per unit time of $\dot{V}O2$, the volume of oxygen used by unit of time. This equivalent can be determined thanks to the Respiratory Exchange Ratio and depends on the intensity of effort. Nevertheless, a reasonable average value is that $1l$ of oxygen produces $20kJ$ [21]. The available flow

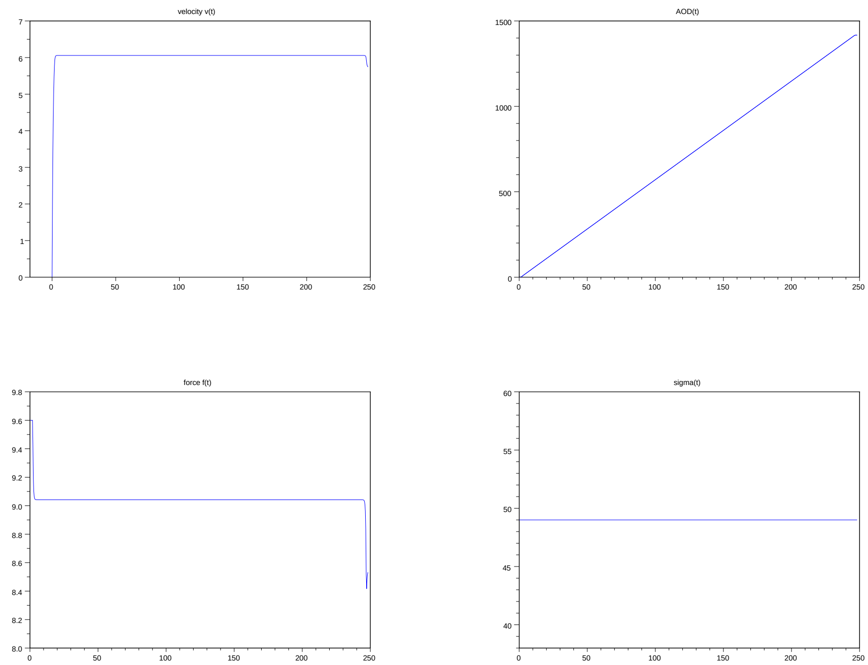

FIG. 2.2. *Simulations of (2.1)-(2.2)-(2.3)-(2.5). Plot of the velocity v, the accumulated oxygen deficit (AOD) $e_{an}^0 - e_{an}$, the propulsive force f, and $\bar{\sigma}$ vs time.*

at the bottom of the anaerobic container, through T is

$$W = \bar{\sigma} + A_p \frac{dh}{dt} = \bar{\sigma} - \frac{de_{an}}{dt}.$$

Since the energy is used at a rate fv, where v is the velocity and f is the propulsive force, we have that W is equal to the available work capacity hence to fv. This allows us to find the equation governing the evolution of the anaerobic energy

$$\frac{de_{an}}{dt} = \bar{\sigma} - fv. \tag{2.5}$$

We point out that this is exactly the energy equation studied by Keller, except that we have explained that it models the accumulated oxygen deficit, while Keller describes it as the aerobic energy.

Some improvements are needed for this model to better account for the physiology:

- change the height where the aerobic container is connected. This implies that the flow of aerobic energy is not constant at the beginning of the race.
- take into account that when the energy supply is low, then a physiological control mechanism implies that the flow of energy drops significantly. In other words, $\dot{V}O2$ does not keep its maximal value at the end of the race.

Before improving the model, we describe our numerical simulations of (2.1)-(2.2)-(2.3)-(2.5) using bocop. We plot, in figure 2.2, the velocity v, the force f, the accumulated oxygen deficit (AOD) $e_{an}^0 - e_{an}$. We have added $\bar{\sigma}$, though it is constant, just to be consistent with the next figures. We take $\bar{\sigma} = 49m^2\ s^{-3}$, $f_{max} = 9.6m\ s^{-2}$, $e_{an}^0 = 1400m^2\ s^{-3}$, $\tau = 0.67s$ and $d = 1500m$. The optimal time is $248.21s$, and we have 2000 discretization steps, i.e. the time step is close to $0.12s$. We display in figure 3.1 a detailed view of what happens at the end of the race for the AOD and the force. We point out that Keller [13] and Woodside [26] choose $\bar{\sigma} = 41.56m^2\ s^{-3}$, $f_{max} = 12.2m\ s^{-2}$, $e_{an}^0 = 2409m^2\ s^{-3}$ and $\tau = 0.892s$. They choose these values by minimizing the sum of the squares of the relative errors compared to world records for a series of races. If we take for $\bar{\sigma}$ either the value of Keller, or the experimental value of [11] corresponding to a $\dot{V}O2$ of $66ml\ mn^{-1}\ kg^{-1}$, we do not match as well the velocity curve of [11] for our final model. So we have adapted our values of $\bar{\sigma}$, f_{max}, e_{an} and τ to minimize the error on the velocity curve of Figure 2.3 compared to [11].

We observe that the race splits into three parts

- The race starts with a strong acceleration, the velocity increases quickly and the force is at its maximal value,
- for the major part of the race, the force is at an intermediate constant value, the velocity is constant with value close to 6.06 m/s. We will see that this corresponds to what is called, in the optimal control theory, a singular arc.
- during the last part corresponding to the last two seconds, the velocity decreases, the force sharply decreases, the energy reaches 0 and then stays at the zero level (AOD is constant equal to e_{an}^0), and the force slightly increases again.

We insist on the fact that this is the first simulation not based on the hypothesis that there are three parts of the above type. Also, we can optimize either on the time to run or the distance to run, where all previous simulations had to fix the time to run and optimize on the distance. Even if this is mathematically equivalent in terms of optimization, fixing the distance requires an extra parameter in the simulations. The next models introduce improvements.

2.2. Second model: improving the initial phase to reach $\dot{V}O2_{max}$. The experimental results of [3, 11] show that the rate of oxygen uptake $\dot{V}O2$ is not constant throughout the race but rises steadily from an initial value of about $10ml\ min^{-1}\ kg^{-1}$ to its maximum value $66ml\ min^{-1}\ kg^{-1}$ over the first 20 to 40 seconds of the race. To model this effect, we now assume that the aerobic container is connected to the anaerobic container at a height $\varphi \in (0, 1)$. This implies that there is an initial phase of the race where the flow from the aerobic container to the anaerobic one is no longer $\bar{\sigma}$, but is proportional to the difference of fluid heights in the containers. We assume additionally that there is a residual value at σ_r so that,

$$\sigma(h) = \begin{cases} \sigma_r + (\bar{\sigma} - \sigma_r)\frac{h}{1-\varphi} \text{ when } h < 1 - \varphi \\ \bar{\sigma} \text{ when } h \geq 1 - \varphi. \end{cases} \tag{2.6}$$

This is illustrated in Figure 2.1. We still have the same balance on total work capacity namely

$$W = fv = \sigma(h) + A_p \frac{dh}{dt}. \tag{2.7}$$

So this and (2.4) lead to the following equations for e_{an}:

$$\frac{de_{an}}{dt} = (\sigma_r + \lambda(\bar{\sigma} - \sigma_r)(e_{an}^0 - e_{an})) - fv \quad \text{when} \quad \lambda(e_{an}^0 - e_{an}) < 1, \tag{2.8}$$

where $1/\lambda = A_p(1-\varphi)$. To match the results of [3, 11], we expect that $\lambda(e^0_{an} - e_{an})$ reaches 1 in about 20 to 40 seconds, so that we choose $\varphi = 0.7$, and $\sigma_r = 10W\ kg^{-1}$.

In the second phase, when $\lambda(e^0_{an} - e_{an})$ has reached 1, we are back to equation (2.5), that is

$$\frac{de_{an}}{dt} = \bar\sigma - fv \quad \text{when} \quad \lambda(e^0_{an} - e_{an}) > 1. \tag{2.9}$$

2.3. Third model: drop in $\dot V O2$ at the end of the race. We want to keep the same initial phases as in the previous model, but take into account that there are limitations when the energy supply is small. The experimental results of [11] show that the rate of oxygen uptake $\dot V O2$ falls slightly at the end of the race, dropping from $66ml\ min^{-1}\ kg^{-1}$ to about $60ml\ min^{-1}\ kg^{-1}$ over the last $200m$ to $250m$. Morton [17] models this by supposing that the work capacity is proportional to the anaerobic energy e_{an} when e_{an} is small. We prefer to assume that there is a drop in the rate of oxygen uptake $\dot V O2$ (and therefore the aerobic power σ) when e_{an} is too small. So we add a last phase to the run: when $e_{an}/e^0_{an} < e_{crit}$, then

$$\bar\sigma \text{ is replaced by } \bar\sigma \frac{e_{an}}{e^0_{an} e_{crit}} + \sigma_f \Big(1 - \frac{e_{an}}{e^0_{an} e_{crit}}\Big).$$

We choose for instance $e_{crit} = 0.15$ and $\sigma_f = 0.75\bar\sigma$ to fit the drop in [11]. We add the final stage:

$$\frac{de_{an}}{dt} = \bar\sigma \frac{e_{an}}{e^0_{an} e_{crit}} + \sigma_f \Big(1 - \frac{e_{an}}{e^0_{an} e_{crit}}\Big) - fv \quad \text{when} \quad \frac{e_{an}}{e^0_{an}} < e_{crit}. \tag{2.10}$$

The coupling of the 3 equations (2.8), (2.9) when $\frac{e_{an}}{e^0_{an}} > e_{crit}$ on the one hand, and (2.10) on the other hand, leads to a better running profile. This model encompasses physiological observations that σ or $\dot V O2$ is not constant and provides a velocity profile with initial acceleration, then deceleration and acceleration again before the final sprint. This is closer to real races than the constant velocity. It takes care of fatigue with a much better physiological description than [1, 14]. This model can be summarized as follows

$$\frac{de_{an}}{dt} = \sigma(e_{an}) - fv \tag{2.11}$$

where

$$\sigma(e_{an}) = \begin{cases} \bar\sigma \frac{e_{an}}{e^0_{an} e_{crit}} + \sigma_f \big(1 - \frac{e_{an}}{e^0_{an} e_{crit}}\big) \text{ if } \frac{e_{an}}{e^0_{an}} < e_{crit} \\ \bar\sigma \text{ if } \frac{e_{an}}{e^0_{an}} \geq e_{crit} \text{ and } \lambda(e^0_{an} - e_{an}) \geq 1 \\ (\sigma_r + \lambda(\bar\sigma - \sigma_r)(e^0_{an} - e_{an}) \text{ if } \lambda(e^0_{an} - e_{an}) < 1 \end{cases} \tag{2.12}$$

together with (2.1)-(2.2)-(2.3). Numerically, in order to avoid piecewise linear functions, we make a regularization in order to have a C^∞ function.

Let us now describe our numerical findings. The results are displayed in Figure 2.3. Since σ is not constant, the singular arc has no longer a constant velocity. Let us be more specific:

- The very first part of the race is still at maximal force with a strong acceleration,
- then the force smoothly decreases to its minimal value at the middle of the race, and so does the velocity,

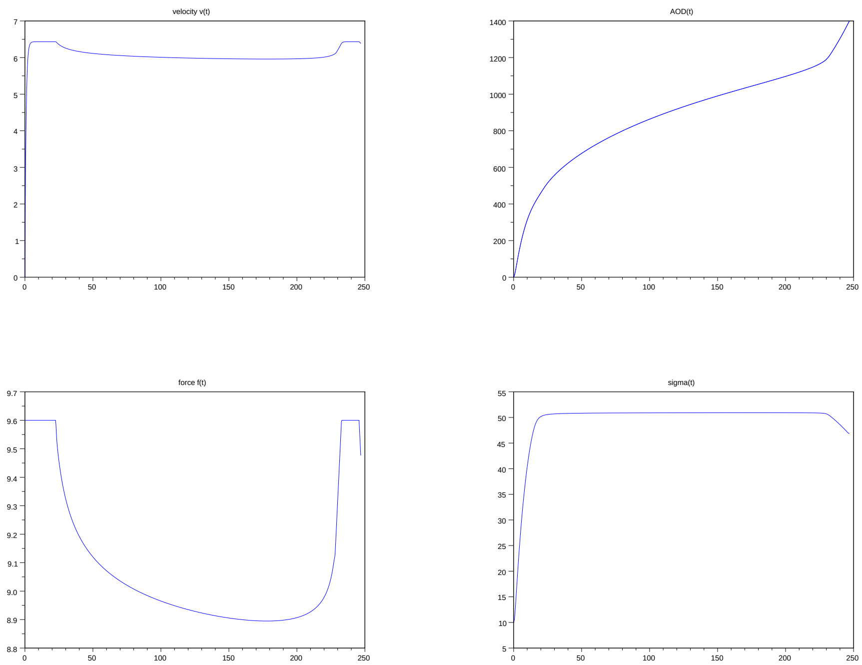

FIG. 2.3. *Variable σ: Plot of the velocity v, the accumulated oxygen deficit (AOD) $e_{an}^0 - e_{an}$, the propulsive force f, and σ vs time.*

- then the velocity and force smoothly increase again
- the last part of the race is at maximal force, corresponding to the final sprint, with a final zero energy arc.

The final time is 246.961.

2.4. Fourth model: energy recreation when slowing down. On sufficiently long races, it has been observed that slowing down recreates anaerobic energy. This is the motivation of intermittent training. Morton and Billat [19] find evidence of energy recreation in intermittent exercises. This can be understood using the hydraulic analogy: if the tap T is closed, then vessel A_n refills by virtue of the flow through R_1. Therefore, on a regular run, the mathematical model has to encompass a new term recreating energy when slowing down. The experimental results for the velocity profiles in [3, 11] show that the velocity is not constant during the bulk of the race, as predicted by the previous models, but oscillates about a constant value. The recreation of anaerobic energy when slowing down can produce such oscillations, but only if the total Hamiltonian is non-convex. This can be achieved by replacing (2.11) by

$$\frac{de_{an}}{dt} = \sigma(e_{an}) + \eta(a) - fv \tag{2.13}$$

where $a = \frac{dv}{dt}$ is the acceleration, if the energy recreation term $\eta(a)$ is a power of a strictly larger than 1 when a is negative. We choose $\eta(a) = ca_-^2$ where a_- is the

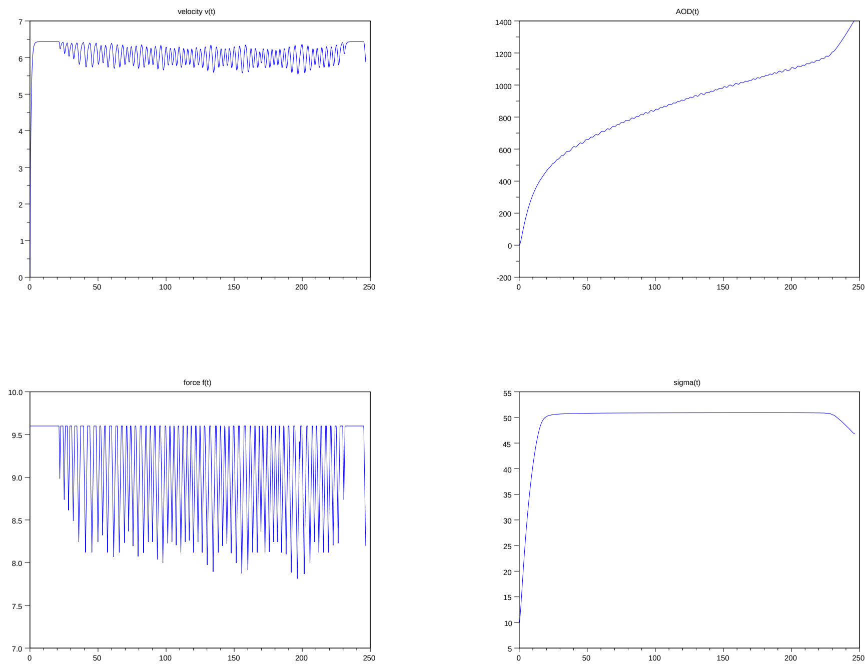

FIG. 2.4. *Recreation when slowing down: speed, energy, force, σ.*

negative part of a. In other words,

$$\eta(a) = 0 \text{ if } a \geq 0 \text{ and } \eta(a) = c|a|^2 \text{ if } a \leq 0. \tag{2.14}$$

We will see that this provides a simulation of velocity which is consistent with real runs.

Because of the term $\eta(a)$, the hamiltonian gets non convex, so that by Pontryagin's maximum principle (see our analysis in section 3.3.2) the optimal solution oscillates between the maximal and minimal value of the force (i.e. f_{max} and 0). This is in fact to be understood in a relaxed sense, as a probability of taking the maximal and minimal values of the force. However, a runner cannot change his propulsive force instantaneously for several reasons: the information to vary its propulsive force takes some time to reach the brain, and the dynamics of the bone and muscles take some time. We choose to take this into account by bounding the derivative of f:

$$|\frac{df}{dt}| \leq C. \tag{2.15}$$

The simulations of (2.1)-(2.2)-(2.3)-(2.12)-(2.13) are illustrated in figure 2.4. We take $C = 1$ in (2.15) and $c = 4$ in (2.14). The optimal time is 246.596.

We see that the force, having a bounded derivative, does not oscillates between its maximal value and 0, but between its maximal value, and some lower value, the derivative of the force reaching its bounds. Consequently, the velocity oscillates and so does the energy which gets recreated. These oscillation reproduce qualitatively the measurements of [3, 11].

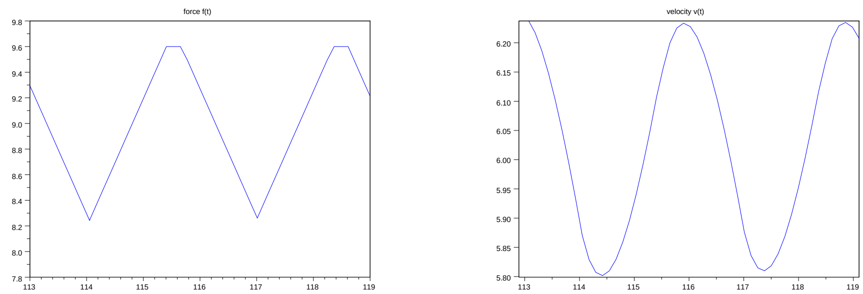

FIG. 2.5. *Zoom of the case with recreation when slowing down (2000 time steps)*

2.5. A periodic pattern. The previous experiments show a behavior of the optimal control which looks, in the time interval $[20, 240]$, i.e., except for the initial and final part of the trajectory, close to a periodic one. We have approximately $e_{an}(20) = 950$ and $e_{an}(240) = 100$, and so the average decrease per unit time is $e_d = 3.86$. We observe that over this time interval the speed varies between 5.8 and 6.3 m/s, and the force varies between 8.2 and 9.6.

This leads us to consider the problem of maximizing the average speed over a period T (the period itself being an optimization parameter): the periodicity conditions apply to the speed and force, and the energy is such that $e(0) = e(T) + Te_d$. In other words, we wish to solve the following optimal control problem:

$$\left\{ \begin{array}{ll} \text{Min} -\dfrac{1}{T}\displaystyle\int_0^T v(t)\mathrm{d}t; & v(0) = v(T); \quad e(0) = e(T) + Te_d; \quad f(0) = f(T). \\ & \dot{v}(t) = f(t) - \frac{v(t)}{\tau};\ \dot{e}(t) = \sigma + \eta(a(t)) - f(t)v(t); \\ & 0 \le f(t) \le f_M;\ |\dot{f}(t)| \le 1, \quad \text{for a.e. } t \in (0,T). \end{array} \right. \tag{2.16}$$

We can fix the initial energy to 0.

Now we can compare figure 2.5, where we made a zoom on the solution computed before over the time interval $[113, 119]$, with the solution of the periodic problem, displayed in figure 2.6, with period 3.5. We observe a good agreement between those two figures, which indicates that computing over a period may give a good approximation of the optimal trajectory.

3. Mathematical analysis. We have to study optimal control problems with a scalar state constraint and a scalar control, which, in some of the models enters linearly into the state equation. We mention among others the related theoretical studies by Bonnans and Hermant [4] about state constrained problems, Maurer [15], who considers problems with bounded state variables and linear control, Felgenhauer [9] about the stability of singular arcs, and the two recent books by Osmolovskii and Maurer[20] and Schättler and Ledzewicz [23].

3.1. Statement of the model. we consider the following state equation

$$\dot{d} = v; \quad \dot{v} = f - \phi(v); \quad \dot{e} = \sigma(e) - fv, \tag{3.1}$$

where the *drag function* ϕ satisfies

$$\phi \text{ is a } C^2 \text{ function; } \phi(0) = 0,\ \phi' \text{ positive, } v\phi'(v) \text{ nondecreasing.} \tag{3.2}$$

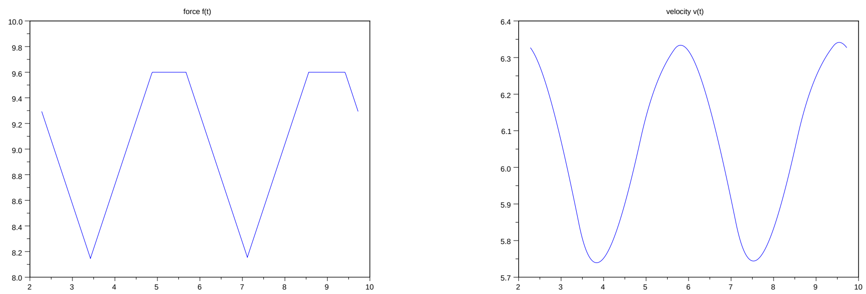

Fig. 2.6. *Optimization over a period*

For $v \geq 1$, we have that $\phi'(v) \geq \phi'(1)/v$, so that $\phi(v) \geq \phi(1) + \phi'(1) \log v$. Consequently,

$$\phi(v) \uparrow +\infty \text{ when } v \uparrow +\infty. \tag{3.3}$$

We assume for the moment (in section 3.3 we will discuss a more general recreation model) that the *recreation function* $\sigma(e)$ satisfies

$$\sigma(e) \text{ is } C^2 \text{ and nonnegative.} \tag{3.4}$$

We will often mention Keller's model which corresponds to the case where

$$\phi(v) = v/\tau \text{ and } \sigma(e) \text{ is a positive constant.} \tag{3.5}$$

As before, the initial condition is

$$d(0) = 0; \quad v(0) = 0; \quad e(0) = e^0 > 0, \tag{3.6}$$

and the constraints are

$$0 \leq f(t) \leq f_M; \quad e(t) \geq 0; \;\; t \in (0, T); d(T) = D. \tag{3.7}$$

The optimal control problem is to minimize the final time:

$$\text{Min}\, T; \quad \text{s.t. (3.1) and (3.6)-(3.7).} \tag{3.8}$$

Note that we could as well take the final constraint as $-d(T) \leq -D$. Writing an inequality in this way yields the sign of the Lagrange multiplier.

We establish in Appendix A that the optimal solutions (for a given distance D) with corresponding time T are also solutions of the problem of maximizing the distance over a time interval T. So, in the following, we will rather use this second formulation. Some useful tools of the optimal control theory are introduced and discussed in Appendix D.

3.2. Main results. If $0 \leq a < b \leq T$ is such that $e(t) = 0$ for $t \in [a, b]$, but $e(\cdot)$ does not vanish over an interval in which $[a, b]$ is strictly included, then we say that (a, b) is an *arc with zero energy*. Similarly, if $f(t) = f_M$ a.e. over (a, b) but not over an open interval strictly containing (a, b), we say that (a, b) is an *arc with maximal* *force*. We define in a similar way an *arc with zero force*. We say that a (resp. b) is the *entry* (resp. *exit*) time of the arc. A *singular arc* is one over which the bound constraints are not active (the associated multiplier is a.e. equal to 0).

If the distance is small enough, then the optimal strategy consists in setting the force to its maximal value. Let $D_M > 0$ be the supremum (assumed to be finite) of the distance for which this property holds. Using standard arguments based on minimizing sequences and weak topology (based on the fact that the control enters linearly into the state equation), the following can be easily proved:

LEMMA 3.1. *The above problem has at least one optimal solution.*

THEOREM 3.2. *Assume that $D > D_M$. Then:* (i) *An optimal trajectory starts with a maximal force arc, and is such that $e(T) = 0$.* (ii) *If σ is a positive constant, an optimal trajectory has the following structure: a maximal force arc, followed or not by a singular arc, and a zero energy arc.*

This will be a consequence of Theorem 3.16 and Remark 3.3.

Since, by proposition A.1, the optimal solutions are solutions of problems of maximizing the achieved distance in a given time, we consider in the sequel problems with a given final time T such that the strategy of constant maximal force is not feasible.

3.3. Recreation when decreasing the speed.

3.3.1. Framework. We next consider a variant of the previous model where the dynamics of the energy is a sum of functions of the energy and the acceleration $a(t) = f(t) - \phi(v(t))$. We can write the dynamics are as follow:

$$\dot{v} = f - \phi(v), \ \dot{e} = \sigma(e) + \eta(a) - fv, \tag{3.9}$$

with initial conditions

$$v(0) = 0; \quad e(0) = e^0 > 0, \tag{3.10}$$

The constraints are

$$0 \le f(t) \le f_M; \quad e(t) \ge 0; \ \ t \in (0,T). \tag{3.11}$$

The recreation optimal control problem is as follows:

$$\text{Min} - \int_0^T v(t)\mathrm{d}t; \quad \text{s.t. (3.9)-(3.11) hold.} \tag{3.12}$$

We will assume that

$$\eta \text{ is a convex and } C^1 \text{ function, that vanishes over } \mathbb{R}_+. \tag{3.13}$$

This implies that η in nonincreasing. A typical example is $\eta(a) = c|a_-|^\beta$, with $c \ge 0$, $\beta \ge 1$, and $a_- := \min(a, 0)$. Let us denote the *maximal recovery function* (obtained with a zero force) by

$$R(v) := \eta(-\phi(v)). \tag{3.14}$$

This is a C^1 and nondecreasing function of v, with value 0 at 0. The following function appears in the analysis of singular arcs :

$$\begin{aligned} Q(v) &:= v\phi'(v) + \phi(v) - R'(v)(1 - \phi(v)/f_M) + \phi'(v)R(v)/f_M. \\ &= (v\phi(v))' + (\phi(v)R(v))'/f_M - R'(v). \end{aligned} \tag{3.15}$$

Let v_M denote the supremum of speeds that can be reached by taking $f(t) = f_M$ for all t, defined in (B.1). In some of the results, we need to assume that

$$Q \text{ is an increasing function of } v \text{ over } [0, v_M]. \tag{3.16}$$

In Keller's model, we have that $R(v) = 0$ and $Q(v) = 2v/\tau$, so that the previous hypothesis holds.

REMARK 3.3. *More generally, if η has the following structure*

$$\eta = c\bar{\eta}, \text{ with } c \geq 0,\ \bar{\eta}(s) > 0 \text{ for all } s > 0, \tag{3.17}$$

since by (3.2) *we have that $v\phi'(v) + \phi(v)$ is increasing, we have that* (3.16) *holds if $c > 0$ is small enough.*

A solution of (3.12) called an optimal control solution does not necessarily exist:

THEOREM 3.4. *Assume that* (3.16) *holds, that* (3.17) *holds with $c > 0$, and that $\sigma(\cdot)$ is a positive constant. Then the optimal control problem* (3.12) *has no solution.*

This will be proved at the end of the section, as a consequence of the analysis of the relaxed problem that we now perform. The theorem motivates the introduction of a relaxed problem.

3.3.2. Relaxed problem. In the relaxed formulation for which we refer to [8], we replace the control $f(t)$ with a probability distribution $\Xi(t, f)$ with values in $[0, f_M]$. Denoting by $\mathbb{E}_{\Xi(t)}$ the expectation associated with this probability measure, and by $\bar{\Xi}(t)$ the expectation of f at time t, the state equation becomes

$$\left\{ \begin{array}{ll} \dot{v}(t) & = \bar{\Xi}(t) - \phi(v(t)), \\ \dot{e}(t) & = \sigma(e(t)) - \bar{\Xi}(t)v(t) + \mathbb{E}_{\Xi(t)}\eta(f - \phi(v(t))). \end{array} \right. \tag{3.18}$$

The relaxed optimal control problem is

$$\text{Min} - \int_0^T v(t)\text{d}t; \quad \text{s.t. (3.18) and (3.10)-(3.11) hold.} \tag{3.19}$$

The Hamiltonian function is the same as for the non relaxed version, and its expression is

$$H[p](f, v, e) := -v + p_v(f - \phi(v)) + p_e(\sigma(e) - fv + \eta(f - \phi(v)). \tag{3.20}$$

The costate equation is therefore, omitting time arguments:

$$\left\{ \begin{array}{ll} -\dot{p}_v & = -1 - p_v\phi'(v) - p_e\bar{\Xi} - p_e\phi'(v)\mathbb{E}_{\Xi(t)}\eta'(f - \phi(v)), \\ -\text{d}p_e & = p_e\sigma'(e)\text{d}t - \text{d}\mu, \end{array} \right. \tag{3.21}$$

with final conditions

$$p_v(T) = p_e(T) = 0. \tag{3.22}$$

Here $\text{d}\mu$, identified to the bounded variation function μ on $[0, T]$, is a Borel measure (that can be interpreted as a Lagrange multiplier) associated to the state constraint $-e \leq 0$. It satisfies the properties of nonnegativity and complementarity, i.e.,

$$\text{d}\mu \geq 0; \quad \int_0^T e(t)\text{d}\mu(t) = 0. \tag{3.23}$$

Note that equivalent relations are, denoting by $\text{supp}(\cdot)$ the support of a measure:

$$\mathrm{d}\mu \geq 0; \quad \text{supp}(\mathrm{d}\mu) \subset \{t \in [0,T];\ e(t) = 0\}. \tag{3.24}$$

The variables p_v and p_e are called costate variables. By standard arguments based on minimizing sequences, we obtain that

LEMMA 3.5. *For $c \geq 0$ small enough, the relaxed optimal control problem* (3.19) *has at least one solution.*

By lemma D.2, any feasible point of the relaxed optimal control problem (3.19) is qualified.

LEMMA 3.6. *For $c \geq 0$ small enough, the relaxed optimal control problem* (3.19) *has the same value as the (non relaxed) problem* (3.12)*, and therefore, any solution of* (3.12) *is solution of* (3.19).

Proof. Let Ξ be a feasible point of the relaxed problem. For $c \geq 0$ small enough, by lemma D.2, this point is qualified, so that for any $\varepsilon > 0$, there exists a feasible point Ξ' of the relaxed problem such that $\|\Xi' - \Xi\| \leq \varepsilon$ in the norm of $L^\infty(0,T,M([0,T]))$, with associated state (v', e') so that the state constraint $e(t) \geq 0$ holds with strict inequality for all $t \in [0,T]$. It follows [8] that Ξ' can be approximated by a classical control f whose state (v^f, e^f) satisfies in the uniform norm $\|v^f - v'\| + \|e^f - e'\| \leq \varepsilon_f$, for arbitrary $\varepsilon_f > 0$. When $\varepsilon_f \downarrow 0$ we have that f is feasible for the (unrelaxed) optimal control problem, and the associated cost converges to the one associated with Ξ'. The conclusion follows. □

We have then that Pontryagin's principle holds in qualified form, i.e., with each optimal trajectory (f, v, e) is associated at least one multiplier (p, μ) such that the relaxed control minimizes the Hamiltonian, in the sense that

$$\Xi(t,\cdot) \text{ has support in } \operatorname{argmin}\{H[p(t)](f, v(t), e(t));\ f \in [0, f_M]\}, \text{ for a.a. } t. \tag{3.25}$$

LEMMA 3.7. *Let $t_1 \in (0,T]$ be such that $e(t_1) = 0$. Then:* (i) *we have that*

$$v(t_1) \geq \sigma(0)/f_M. \tag{3.26}$$

(ii) *An arc of maximal force cannot start at time t_1, or include time t_1.*

Proof. (i) If (3.26) does not hold, then $\sigma(0) - v(t_1) f_M > 0$, and therefore $\dot{e}(t) \geq \sigma(t) - v(t) f_M > 0$ for t close enough to t_1, contradicting the fact that $e(t)$ reaches its minimum at time t_1.

(ii) Let t_1 contradict point (ii). By point (i), $\dot{e}(t_1+) = \sigma(0) - f_M v(t) \leq 0$, but since the energy must be nonnegative, this is an equality, and so $\dot{e}(t_+) = 0$. Therefore,

$$\ddot{e}(t_1^+) = \sigma'(0)\dot{e}(t_1^+) - f_M \dot{v}(t_1^+) = -f_M \dot{v}(t_1^+) < 0, \tag{3.27}$$

implying that the energy cannot be positive after time t_1. This gives the desired contradiction. □

LEMMA 3.8. *We have that* (i) $T \in \text{supp}(\mathrm{d}\mu)$*, so that* $e(T) = 0$*, and*

$$\begin{cases} \text{(ii)} & p_e(t) < 0, \quad t \in [0,T). \\ \text{(iii)} & p_v(t) < 0, \quad t \in [0,T). \end{cases} \tag{3.28}$$

Proof. (i) If $T \notin \text{supp}(\mathrm{d}\mu)$, let $t_e := \max \text{supp}(\mathrm{d}\mu)$. Then $t_e \in (0,T)$ and $e(t_e) = 0$: indeed, if $t_e = T$, then $\min_t e(t) > 0$ and, since by assumption the strategy of maximal force is not feasible, one can easily construct by slightly increasing the

force a feasible trajectory that reaches a smaller cost. We analyze what happens over (t_e, T). Since p_e has derivative $\dot{p}_e = -p_e\sigma'(e(t))$ and is continuous with zero value at time T, it vanishes, and so, $\dot{p}_v = 1 + p_v\phi'(v)$. Since $p_v(T) = 0$, this implies that p_v has negative values. As p_e vanishes and p_v is negative, the Hamiltonian is equal to $-v + p_v(f - \phi(v))$ and has a unique minimum at f_M. It follows that (t_e, T) is included in a maximal force arc, which, since $e(t_e) = 0$, is in contradiction with lemma 3.7(ii). Point (i) follows.
(ii) If $p_e(t_a) \geq 0$ for some $t_a \in (0, T)$, then, by the costate equation, p_e should vanish on $(t_a, T]$ and so (t_a, T) would not belong to the support of $\mathrm{d}\mu$, in contradiction with (i). This proves (ii).
(iii) Let on the contrary $t_c \in [0, T)$ be such that $p_v(t_c) \geq 0$. Then the Hamiltonian is a sum of nondecreasing functions of f and has a unique minimum point at 0, and since $p_e(t_c) < 0$, the same holds for t close enough to t_c. Therefore t_c belongs to a zero force arc, along which

$$\dot{p}_v = 1 + p_v\phi'(v) + p_e\eta'(-\phi(v))\phi'(v) \tag{3.29}$$

remains positive (remember that $p_e(t) < 0$ and that η is nonincreasing), and so this arc cannot end before time T. But then we cannot meet the final condition $p_v(T) = 0$. The conclusion follows. □

Since $p_e(t) < 0$ over $(0, T)$, we deduce by (3.25) that when $\eta(a) > 0$ for all $a < 0$, H is a concave function of f with minima over $[0, f_M]$ at $\{0, f_M\}$ for a.a. t, and so, by (3.25):

Corollary 3.9. *If $\eta(a) > 0$ for all $a < 0$, an optimal control for the relaxed control problem* (3.19) *has support over $\{0, f_M\}$ for a.a. t.*

3.3.3. Reformulation of the relaxed problem. In view of the previous corollary, when $\eta(a) > 0$ for all $a < 0$, we may as well restrict the study of the relaxed optimal control problem (3.19) to the case when the relaxed control has values in $\{0, f_M\}$. Such a relaxed control can be parameterized by its expectation $f(t)$ at any time t: the probability to take the value 0 is $1 - f(t)/f_M$. Remembering the definition of the recovery obtained with a zero force in (3.14), the dynamics can now be written in the form, since $\eta(f_M - \phi(v)) = 0$ along the trajectory:

$$\begin{cases} \dot{v}(t) &= f(t) - \phi(v(t)), \\ \dot{e}(t) &= \sigma(e(t)) - f(t)v(t) + \left(1 - \dfrac{f(t)}{f_M}\right) R(v(t)), \end{cases} \tag{3.30}$$

with initial conditions (3.10). The optimal control problem is

$$\mathrm{Min} - \int_0^T v(t)\mathrm{d}t; \quad \text{s.t. (3.30) and (3.10) hold, and } 0 \leq f \leq f_M \text{ a.e., } e \geq 0 \text{ on } [0, T]. \tag{3.31}$$

Remark 3.10. *When η identically vanishes, the above problem still makes sense and coincides with the formulation of the original model of section 3.1. So we will be able to apply the results of this section to Keller's problem.*

The Hamiltonian is

$$H^R = -v + p_v(f - \phi(v)) + p_e(\sigma(e) - fv) + p_e\left(1 - \frac{f}{f_M}\right) R(v). \tag{3.32}$$

The costate equations are

$$\begin{cases} -\dot{p}_v & = -1 - p_v\phi'(v) - p_e f + p_e\left(1 - \dfrac{f}{f_M}\right)R'(v), \\ -\mathrm{d}p_e & = p_e\sigma'(e)\mathrm{d}t - \mathrm{d}\mu. \end{cases} \tag{3.33}$$

Of course we recover the expressions obtained in section 3.3.2 in the particular case of relaxed controls with support in $\{0, f_M\}$, and therefore all lemmas of this section are still valid. By construction, the Hamiltonian is an affine function of the control f, and the switching function is

$$\Psi^R = H^R_f = p_v - p_e(v + R(v)/f_M). \tag{3.34}$$

LEMMA 3.11. *Any optimal trajectory starts with a maximal force arc.*

Proof. Since $v(0) = 0$, we have that $\Psi^R(0) = p_v(0)$ is negative by lemma 3.8(iii). The conclusion follows from the fact that the optimal control minimizes the Hamiltonian, see e.g. [4]. □

The following hypothesis implies that the energy is nonzero along a zero force arc.

$$\text{Either } \sigma(0) > 0, \text{ or } \eta \text{ is nonzero over } \mathbb{R}^-. \tag{3.35}$$

LEMMA 3.12. *Let* (3.35) *hold. Then μ has no jump over $[0,T]$.*

Proof. a) Let $t \in [0,T)$ be such that $[\mu(t)] > 0$. Necessarily $e(t) = 0$, and so $t \in (0,T)$, and $\Psi^R(t_-) \leq 0$ (since otherwise t belongs to a zero force arc and then by (3.35) we cannot have $e(t) = 0$). We have that $[p_e(t)] = [\mu(t)] > 0$, and so $[\Psi^R(t)] = -(v(t) + R(v)/f_M)[p_e(t)] < 0$, implying $\Psi^R(t_+) < 0$. Therefore, for some $\varepsilon > 0$, $(t, t+\varepsilon)$ is included in a maximal force arc, in contradiction with lemma 3.7(ii).
b) If $[\mu(T)] \neq 0$, since $\lim p_v(t) = 0$ and $\lim p_e(t) = -[\mu(T)]$ when $t \to T$, and $v(T) > 0$, we get $\lim_{t\uparrow T} \Psi^R(t) = [\mu(T)](v(T) + R(T)/f_M) > 0$, meaning that the trajectory ends with a zero force arc, but then by (3.35), the energy cannot vanish at the final time, contradicting lemma 3.8(i). □

By lemma 3.12, the switching function is continuous. When the state constraint is not active, its derivative satisfies

$$\begin{aligned} \dot{\Psi}^R \quad = \quad & 1 + p_v\phi'(v) + p_e f - p_e\left(1 - \frac{f}{f_M}\right)R'(v) \\ & + p_e\sigma'(e)\left(v + R(v)/f_M\right) - p_e\left(1 + R'(v)/f_M\right)(f - \phi(v)). \end{aligned} \tag{3.36}$$

The coefficient of f vanishes as expected and we find that

$$\begin{aligned} \dot{\Psi}^R \quad = \quad & 1 + p_v\phi'(v) + p_e\phi(v) + p_e\sigma'(e)(v + R(v)/f_M) \\ & - p_e R'(v)(1 - \phi(v)/f_M). \end{aligned} \tag{3.37}$$

Subtracting $\Psi^R\phi'(v)$ in order to cancel the coefficient of p_v, we obtain that

$$\dot{\Psi}^R - \Psi^R\phi'(v) = 1 + p_e\sigma'(e)\left(v + R(v)/f_M\right) + p_e Q(v), \tag{3.38}$$

where Q was defined in (3.15).

LEMMA 3.13. *For given $0 \leq t_1 < t_2 \leq T$, let (t_1, t_2) be included in a singular arc over which $\sigma'(e(t))$ is equal to 0. Then, over (t_1, t_2), $Q(v)$ is constant and, if the function Q is not constant on any interval, v is constant.*

Proof. Along a singular arc, $\mathrm{d}\mu$ vanishes, so that p_e is constant. By (3.38), so is also $Q(v) = -1/p_e$. The conclusion follows. ◻

REMARK 3.14. *If η is analytic over $\mathbb{R}_-$ and ϕ is analytic over $\mathbb{R}_+$, then Q is analytic over $\mathbb{R}_+$, and therefore is either constant over $\mathbb{R}$, or not constant on any interval of $\mathbb{R}_+$.*

We say that $\bar{t} \in (0,T)$ is a *critical time* if $\Psi^R(t) = 0$, and we say that $\bar{t}_\pm$ is *energy free* if $e(t) > 0$ for t in $(\bar{t}, \bar{t} \pm \varepsilon)$, for $\varepsilon > 0$ small enough (but it may happen that $e(\bar{t}_\pm) = 0$. As for (3.38), we have that, for such times the existence of left or right derivatives:

$$\dot{\Psi}^R_{t_\pm} = 1 + p_e(t_\pm)\sigma'(e(t))\,(v(t) + R(v(t))/f_M) + p_e(t_\pm)Q(v(t)), \tag{3.39}$$

where $p_e(t_\pm)$ is the right or left limit of p_e at time t, and $Q(\cdot)$ was defined in (3.15).

LEMMA 3.15. *Let σ be a positive constant, and* (3.16) *hold. Then, along an optimal trajectory:* (i) *There is no zero force arc, and hence, $\Psi^R(t) \le 0$, for all $t \in [0,T]$.* (ii) *The only maximal force arc is the one starting at time 0.*

Proof. (i) Let (t_a, t_b) be a zero force arc, over which necessarily Ψ^R is nonnegative. By lemma 3.11, $t_a > 0$, and so $\Psi^R(t_a) = 0$ and $\dot{\Psi}^R(t_{a+}) \ge 0$. Since σ is constant and positive, over the arc, $e(t) > 0$, p_e is constant, the speed is decreasing, and so by (3.39), we have that $\dot{\Psi}^R(t_{b-}) > \dot{\Psi}^R(t_{a+}) \ge 0$ meaning that the zero force arc cannot end before time T, contradicting the final condition $e(T) = 0$.
(ii) On a maximal force arc (t_a, t_b) with $t_a > 0$, since the speed increases, (3.39) implies $\dot{\Psi}^R(t_{b-}) < \dot{\Psi}^R(t_{a+}) \le 0$, and since $\Psi^R_t \le 0$ along the maximal force arc, it follows that $\Psi^R(t_b) < 0$, meaning that the maximal force arc ends at time T. But then $[p_e(T)] = [\mu(T)] > 0$, in contradiction with lemma 3.12. ◻

THEOREM 3.16. *Let σ be a positive constant, and* (3.16) *hold. Then an optimal trajectory has the following structure: maximal force arc, followed or not by a singular arc, and a zero energy arc.*

Proof. The existence of a maximal force arc starting at time 0 is established in lemma 3.11. Let $t_a \in (0,T)$ be its exit point ($t_a = T$ is not possible since $T > T_M$), and let $t_b \in (0,T)$ be the first time at which the energy vanishes (that $t_b < T$ follows from lemmas 3.8(i) and 3.12). If $t_a < t_b$, over (t_a, t_b), by lemma 3.15, Ψ^R is equal to zero and hence, (t_a, t_b) is a singular arc. Finally let us show that the energy is zero on (t_b, T). Otherwise there would exist t_c, t_d with $t_b \le t_c < t_d \le T$ such that $e(t_c) = e(t_d) = 0$, and $e(t) > 0$, for all $t \in (t_c, t_d)$. By lemma 3.15, (t_c, t_d) is a singular arc, over which $\dot{e} = \sigma - fv$ is constant, which gives a contradiction since the energy varies along this arc. The result follows. ◻

Proof. [Proof of theorem 3.4] By lemma 3.6, any solution of the classical problem is solution of the relaxed one. By theorem 3.16, the trajectory must finish with an arc of zero energy, over which $0 = \dot{e}(t) = \sigma - f(t)v(t) + \eta(f(t) - \phi(v(t))$. The r.h.s. is a strictly decreasing function of $f(t)$. We deduce that $f(t)$ is a continuous function of $v(t)$, and hence, of time over this arc. On the other hand, since $p_e < 0$ a.e., the Hamiltonian is a concave function of f which is not affine on $[0, f_M]$, and so attains its minima at either 0 or f_M. Therefore $f(t)$ is constant and equal to either 0 or f_M over the zero energy arc. For $f(t) = 0$ we have that $\dot{e}(t)$ is positive. That $f(t) = f_M$ is not possible since we know that the trajectory has only one maximal force arc. We have obtained the desired contradiction. ◻

REMARK 3.17. *We plot in figure 3.1 a zoom on the end of the race. While we have proved that for the continuous problem, there is a switching time from the singular arc, with constant speed, to the zero energy arc, we observe in the discretized problem a progressive transition between 246 and 248 seconds.*

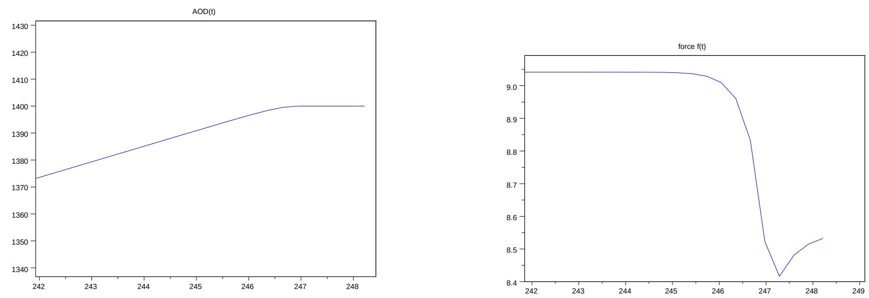

FIG. 3.1. *Race problem with Keller's model: zoom on end of race: AOD and force.*

3.4. Bounding variations of the force. It seems desirable to avoid discontinuities of the force that occur with the previous model, and for that we introduce bounds on $\dot{f}$. The force becomes then a state and the new control $\dot{f}$ is denoted by g. So the state equation is (note that we have taken here $\sigma = 0$)

$$\dot{v} = f - \phi(v); \quad \dot{e} = \sigma(e) + \eta(a) - fv; \quad \dot{f} = g, \tag{3.40}$$

with constraints

$$0 \leq f \leq f_M; \quad e \geq 0; \quad g_m \leq g \leq g_M. \tag{3.41}$$

We minimize as before $-\int_0^T v(t)\mathrm{d}t$. The Hamiltonian is

$$H = -v + p_v(f - \phi(v)) + p_e(\eta(f - \phi(v)) - fv) + p_f g. \tag{3.42}$$

The costate equation $-\dot{p} = H_y$ are now

$$\begin{cases} -\dot{p}_v & = -1 - p_v\phi'(v) - p_e(\eta'(a)\phi'(v) + f), \\ -\mathrm{d}p_e & = \sigma'(e)p_e\mathrm{d}t - \mathrm{d}\mu, \\ -\dot{p}_f & = p_v + p_e(\eta'(a) - v). \end{cases} \tag{3.43}$$

The state constraint $e \geq 0$ is of second order, and we may expect a jump of the measure μ at time T. The final condition for the costate are therefore

$$p_v(T) = 0; \quad p_e(T) = 0; \quad p_f(T) = 0. \tag{3.44}$$

We may expect and will assume that the above two inequalities are strict. By the analysis of the previous section we may expect that the optimal trajectory is such that g is bang-bang (i.e., always on its bounds), except if a state constraint is active (the state constraints now include bound constraints on the force), as is confirmed in our numerical experiments.

The bounds on the derivative of the force modify the last part of the race compared to Keller's (see figure 3.1): the drop in force cannot be so sharp.

4. Conclusion. We have established a system of ordinary differential equations governing the evolution of the velocity v, the anaerobic energy e_{an}, and the propulsive force f. This is based on the equation of motion (relating the acceleration $a = dv/dt$

to the propulsive force and the resistive force) and a balance of energy. Several constraints have to be taken into account: the propulsive force is positive and less than a maximal value, its derivative has to be bounded, the anaerobic energy is positive. Keller [12, 13] used in his model the evolution of the aerobic energy, which is not satisfactory. Here, using a hydraulic analogy initiated by Morton [16, 17], we manage to write an equation for the instantaneous accumulated oxygen deficit instead. In our model, in difference with respect to Keller's, we introduce variations in σ, modeling the oxygen uptake, $\dot{V}O2$: indeed, one of the roles of the anaerobic energy is to compensate the deficit in oxygen uptake, $\dot{V}O2$, which has not reached its maximal value at the beginning of the race. Conversely, when the anaerobic energy gets too low, the oxygen uptake $\dot{V}O2$ cannot be maintained to its maximal value. We make two further extensions: we introduce a physiological observation that energy is recreated when the acceleration is negative, that is when decreasing the speed, and the fact that the derivative of the propulsive force has to be bounded.

Our model could be used, in the simulations, given the velocity profile of a runner, to compute the evolution of his anaerobic energy. This is an important challenge for sportsmen to determine instantaneous anaerobic energy consumption.

In this paper, we use our system for the optimization of strategy in a race: given a distance, we want to find the optimal velocity leading to the shortest run. Our main results are illustrated in Figures 2.3, 2.4. Without the energy recreation term, we find that the race starts with a strong acceleration to achieve a peak velocity, then the race is run at an average velocity, first with a decreasing pace, and then an increasing pace, before the final sprint. Our numerical simulations on the final model where energy recreation is involved provide oscillations of the velocity that qualitatively reproduce the physiological measurements of [3, 11].

Using optimal control theory, we manage to get rigorous proofs of most of our observations. We prove in particular that in the case of Keller, the race is made up of exactly three parts: run at maximal propulsive force, run at constant speed (corresponding to a singular arc), run at zero energy. It cannot be made of any other arcs. For this purpose, we relate the problem to a relaxed formulation, where the propulsive force represents a probability distribution rather than a function of time. We also find that the concavity of the Hamiltonian results in speed oscillations and we show how, by reducing the problem on optimizing over a period, we recover the latter.

Appendix A. Abstract distance and time functions. In this section, we establish, in a general setting, the relation between the distance and time functions, defined in (3.1), (3.6), (3.7). Let the following be the control and state spaces :

$$\mathcal{U}_T := L^\infty(0,T); \quad \mathcal{Y}_T := L^\infty(0,T;\mathbb{R}^n). \tag{A.1}$$

Given $\delta : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $K_T \subset \mathcal{U}_T \times \mathcal{Y}_T$, we consider the "abstract" problems of minimal time

$$(P_D) \qquad \begin{array}{l} \operatorname{Min} T; \quad \dot{y}(t) = F(f(t), y(t)), \ \text{ for a.a. } t \in (0,T), \ y(0) = y^0; \\ (f,y) \in K_T; \ \delta(y(T)) = D, \end{array}$$

and of maximal distance

$$(\tilde{P}_T) \qquad \operatorname{Max} \delta(y(T)); \quad \dot{y}(t) = F(f(t), y(t)), \ t \in (0,T), \ y(0) = y^0; \quad (f,y) \in K_T.$$

In our examples, $y = (h, v, e)$ and $\delta(y) = y_1$ is the distance.

Proposition A.1. *Denote by $\mathcal{T}(D)$ and $\mathcal{D}(T)$ the optimal values of the above problems. Assume that these functions are finitely valued, increasing and continuous over $\mathbb{R}_+$ with value 0 at 0. Then* (a) *$T(D)$ is the inverse function of $D(T)$, and* (b) *any optimal solution of $(\tilde{P}_T)$ (resp. (P_D)) is a solution of $(P_{\mathcal{D}(T)})$ (resp. $(\tilde{P}_{\mathcal{T}(D)})$).*

Proof. (a1) Given $\varepsilon > 0$, let $(\hat{f}, \hat{y})$ be as above and such that $\hat{T} \leq \mathcal{T}(D) + \varepsilon$. Then

$$D \leq \mathcal{D}(\hat{T}) \leq \mathcal{D}(\mathcal{T}(D) + \varepsilon). \tag{A.2}$$

The first inequality is due to the fact that the trajectory $(\hat{f}, \hat{y})$ is feasible for $(\tilde{P}_{\hat{T}})$, and the second one holds since $\mathcal{D}$ is nondecreasing. Passing to the limit when $\varepsilon \downarrow 0$ and using the continuity of $\mathcal{D}$, we deduce that

$$D \leq \mathcal{D}(\mathcal{T}(D)). \tag{A.3}$$

(a2) Given $\varepsilon > 0$, let $(\bar{f}, \bar{y})$ be as above and such that $\mathcal{D}(T) - \varepsilon \leq \bar{D}$. Then

$$\mathcal{T}(\mathcal{D}(T) - \varepsilon) \leq \mathcal{T}(\bar{D}) \leq T. \tag{A.4}$$

The first inequality holds since $\mathcal{T}$ is nondecreasing, and the second one is due to the fact that the trajectory $(\bar{f}, \bar{y})$ is feasible for $(P_{\bar{D}})$. Passing to the limit when $\varepsilon \downarrow 0$ and using the continuity of $\mathcal{T}$, we deduce that $\mathcal{T}(\mathcal{D}(T)) \leq T$.
(a3) Combining with (A.3), we get $\mathcal{T}(D) \leq \mathcal{T}(\mathcal{D}(\mathcal{T}(D))) \leq \mathcal{T}(D)$, so that for all $T = \mathcal{T}(D)$, we have that $T = \mathcal{T}(\mathcal{D}(T))$. Point (a) follows.
Point (b) is an easy consequence of point (a). □

Appendix B. Strategy of maximal force. The strategy of *maximal force* is the one for which the force always has its maximal value. Then speed is an increasing function of time, with positive derivative, and asymptotic value

$$v_M = \phi^{-1}(f_M) \quad (v_M = \tau f_M \text{ in Keller's model}). \tag{B.1}$$

Note that, by (3.2)-(3.3), $\phi^{-1}(f_M)$ is a locally Lipschitz function $\mathbb{R} \to \mathbb{R}$. So we have that

$$v(t) < v_M, \ \dot{v}(t) > 0, \ \text{and } v(t) \uparrow v_M \text{ if } f(t) = f_M \text{ for all } t \geq 0. \tag{B.2}$$

We first discuss the existence of a critical distance D_M at which the energy vanishes, under the following hypothesis, implying that, when reaching the maximal speed with a maximal force, the energy decreases :

$$\sup_{e \geq 0} \sigma(e) < f_M v_M = f_M \phi^{-1}(f_M). \tag{B.3}$$

Lemma B.1. *(i) With the strategy of maximal force, if* (B.3) *holds, the energy cannot remain nonnegative for all time $t \geq 0$. (ii) If the energy vanishes at time t_M, then the maximal force strategy does not respect the contraint of nonnegative energy over $[0, t]$ for any $t > t_M$.*

Proof. (i) By (B.3), there exists $\varepsilon_M > 0$ such that $\sup_{e \geq 0} \sigma(e)) + \varepsilon < f_M v_M$. For large enough time, $\dot{e}(t) \leq -\varepsilon_M$ so that $e(t) \to -\infty$; point (i) follows.
(ii) If the conclusion does not hold, then $e(t)$ attains a local minimum over $(0, \tau)$ at

time t_M, and so we have $e(t_M) = 0$ and $\dot{e}(t_M) = \sigma(0) - f_M v(t_M) = 0$. Since the speed has a positive derivative, it follows that

$$\ddot{e}(t_M) = \sigma'(0)\dot{e}(t_M) - f_M \dot{v}(t_M) = -f_M \dot{v}(t_M) < 0, \tag{B.4}$$

and therefore in any case $e(t) < 0$ for $t > t_M$, close to t_M, which gives the desired contradiction. □

Appendix C. Link with the Bellman function.

Consider the following extension of the relaxed problem (3.19), where the problem is still to compute the maximal distance one can run in time T, but with general intial conditions $v_0 = v^0$, $e_0 = e^0 > 0$. The value function $V(T, v^0, e^0)$ (equal to the opposite of the optimal distance) is called in this setting the Bellman value. The analysis of section 3 is based on the (negative) sign of the costate variables. It is known that, under some hypotheses, the costate is equal to the the gradient of the Bellman value. So, it is of interest to check in a direct way (without using the optimal control theory) the following result.

LEMMA C.1. *The function $V(T, v^0, e^0) : \mathbb{R}^3_+ \to \mathbb{R}$ is a decreasing function of each of its three arguments.*

Proof. That D is a increasing function of T is easy to prove. When changing the initial energy from e^0 to $\hat{e}^0 > e^0$, given an optimal control and state $(\bar{f}, \bar{v}, \bar{e})$, we have that $\bar{f}$ is feasible for the new problem. Indeed, the state has the same speed $\bar{v}$ and a new energy $\hat{e}$ that must satisfy $\hat{e}(t) \geq \bar{e}(t)$ for all $t \geq 0$, since if equality holds at some time $t_e > 0$, then $\hat{e}(t) = \bar{e}(t)$ for all $t > t_e$. It follows that D is an increasing function of e^0. Finally, let us change the initial speed to $\hat{v}^0 > v^0$. If, for a zero force strategy, the corresponding speed $\hat{v}$ is always greater than $\bar{v}$ over $(0, T)$, the conclusion holds. Otherwise, let $t_a \in (0, T)$ be such that when applying the zero force over $(0, t_a)$, we have that $\hat{v}(t_a) = v(t_a)$. Define the strategy $\hat{f}$ to have value 0 over $(0, t_a)$, and to be equal to f otherwise.

Clearly, the distance at time t_a is greater than the corresponding one for the original strategy, and the energy denoted by $\hat{e}$ satisfies $\hat{e}(t_a) > \bar{e}(t_a)$ (equality is not possible since it would mean that $\bar{f}(t) = 0 = \hat{f}(t)$ for all $t \in (0, t_a)$, but then $\hat{v}(t_a) > \bar{v}(t_a)$). Since we know that D is an increasing function of energy, the conclusion follows. □

Appendix D. Qualification. We consider the model with energy recreation of section 3.3. We assume that the functions σ, Φ and $\bar{\eta}$ are of class C^1. Let as before $\eta = c\bar{\eta}$ for some $c \geq 0$. The mapping $(v[f], e[f])$ is of class C^1 and the directional derivative in the direction $\delta f \in \mathcal{U}$ is a solution of the *linearized state equation*, i.e.,

$$\left\{ \begin{array}{lcl} \dot{\delta v} & = & \delta f - \phi'(v)\delta v, \quad t \geq 0, \\ \dot{\delta e} & = & \sigma'(e)\delta e - \delta f v - f \delta v + \eta'(a)(\delta f - \phi'(v)\delta v) \quad t \geq 0, \\ \delta v(0) = \delta e(0) & = & 0. \end{array} \right. \tag{D.1}$$

We denote the solution of this system by $(\delta v[\delta f], \delta e[\delta f])$. Let us write the constraints in the form

$$f \in \mathcal{U}_M \text{ and } e[f] \in K, \tag{D.2}$$

where

$$\mathcal{U}_M := \{f \in \mathcal{U}; \quad 0 \leq f(t) \leq f_M \text{ a.e.}\}; \quad K = C([0, T])_+. \tag{D.3}$$

Let $f \in \mathcal{U}_M$ be a feasible control, i.e., which is such that $f \in \mathcal{U}_M$ and $e[f] \in K$. The constraints are said to be *qualified* at f (see [22] or [7, section 2.3.4]) if there exists $\delta f \in \mathcal{U}$ such that

$$f + \delta f \in \mathcal{U}_M; \quad e[f] + \delta e[\delta f] \in \operatorname{int}(K). \tag{D.4}$$

In other word, the variation δf of the control is compatible with the control constraints, and the linearized state δe allows to reach the interior of the set of feasible states. Remember that $e(0) > 0$.

LEMMA D.1. *If c is small enough, the optimal control problem* (3.12) *is qualified.*

Proof. a) We first obtain the result when $c = 0$. If $e(t)$ is always positive the qualification holds with $\delta f = 0$. Otherwise, let t_a be the smaller time at which the energy vanishes with $\delta f = -f$. Obviously $f + \delta f \in \mathcal{U}_M$, and since δf is a.e. nonpositive, so is δv. Next, since $c = 0$, we have that

$$\dot{\delta e}(t) = \sigma'(e(t))\delta e(t) - \delta f(t)v(t) - f(t)\delta v(t), \quad t \in (0,T); \quad \delta e(0) = 0, \tag{D.5}$$

which implies $\delta e \geq 0$. Let t_b be the essential supremum of times for with $f(t) = 0$ for a.a. $t \in (0, t_b)$. Clearly, $t_b < t^a$, and for any $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$v(t) \geq \alpha \text{ and } -\delta v(t) \geq \alpha, \text{ for all } t \in [t_b + \varepsilon, T]. \tag{D.6}$$

Let $C := \max\{|\sigma'(e(t))|;\ t \in [0,T]\}$. Then for t in $(t^f + \varepsilon, T]$:

$$\dot{\delta e}(t) \geq \sigma'(e(t))\delta e(t) - f(t)\delta v(t) \geq -C\delta e(t) + 2\alpha f(t). \tag{D.7}$$

Taking $\varepsilon \in (0, t^a - t^f)$, it follows that $\delta e(t) > 0$ over $[t^a, T]$, and so $e(t) + \delta e(t)$ is positive over $[0,T]$, and hence uniformly positive as was to be shown.
b) We now show that for $c > 0$ small enough the qualification can be obtained, again by taking $\delta f = -f$. Given a sequence c_k of positive number converging to 0 and $f_k \in \mathcal{U}_M$, we may extract a subsequence such that f_k converges to f in L^∞ weak$*$, and since $c_k \downarrow 0$, the associated states (v_k, e_k) uniformly converge to the associated state (v, e). Since δf_k converges to δf in L^∞ weak$*$, we deduce that $(\delta v_k, \delta e_k)$ uniformly converge to $(\delta v, \delta e)$. By point (a), $e_k + \delta e_k$ is (uniformly) positive over $[0,T]$, as was to be shown. □

We now consider the relaxed formulation of section 3.3.2.

LEMMA D.2. *If c is small enough, the optimal control problem* (3.19) *is qualified.*

Proof. The proof is essentially the same, up to technical details (the main point is that for the variation of the control we still take the opposite of the control), and is left to the reader. □

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