An empirical analysis of heavy-tails behavior of financial data: The case for power laws

Nicolas Champagnat[b,c,d,†] Madalina Deaconu[b,c,d,†]
Antoine Lejay[b,c,d,†] Nicolas Navet[e,‡] Souhail Boukherouaa[a,b,c,d]

August 14, 2013

Abstract

This article aims at underlying the importance of a correct modelling of the heavy-tail behavior of extreme values of financial data for an accurate risk estimation. Many financial models assume that prices follow normal distributions. This is not true for real market data, as stock (log-)returns show heavy-tails. In order to overcome this, price variations can be modeled using stable distribution, but then, as shown in this study, we observe that it over-estimates the Value-at-Risk. To overcome these empirical inconsistencies for normal or stable distributions, we analyze the tail behavior of price variations and show further evidence that power-law distributions are to be considered in risk models. Indeed, the efficiency of power-law risk models is proved by comprehensive backtesting experiments on the Value-at-Risk conducted on NYSE Euronext Paris stocks over the period 2001-2011.

Keywords: model for log-returns of assets; Value-at-Risk; risk management; power tail distribution; stable distribution; normal distribution; Hill estimator; backtesting.

AMS classification (2010): 62G32

JEL classification: C14, C18, G11

1 Introduction

The Value-at-Risk (VaR) is one of the main indicators for risk management of financial portfolios [47]. It is expressed as the threshold that a loss over a chosen time horizon occurs with at most a given level of confidence.

The VaR may be estimated either by parametric or non-parametric approach. The non-parametric one uses only the empirical distributions (historical, resampling) without

[a]Alphability, F-54500 Vandoeuvre-lès-Nancy, France.
[b] Université de Lorraine, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France.
[c] CNRS, Institut Elie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France.
[d] Inria, Villers-lès-Nancy, F-54600, France.
[e] FSTC/CSC Research Unit/Lassy lab., Université du Luxembourg, L-1359 Luxembourg.
[†] Email: {Nicolas.Champagnat,Madalina.Deaconu,Antoine.Lejay}@inria.fr
[‡] Email: nicolas.navet@uni.lu

fitting a model. Due to the small amount of available data, it does not provide an accurate way to deal with extreme events. On the other hand, the parametric approach consists in fitting the parameters of a model on historical data and compute afterwards the VaR, either by analytic or numerical methods.

The RiskMetric methodology [64] is widely used to estimate the risk associated to a Portfolio through the computations relating variations of the risk indicator to variations of risk factors (stocks and prices derivatives for example). Nowadays this methodology incorporates heavy tail distributions, but it was initially developed in the Gaussian context, which is still prevalent is risk management and enforced by Basel Accord [77].

However, actual regulations and standard ways of computing the VaR, mainly related to the Gaussian world, have been invalidated by studies (see *e.g.* [2, 7]) because they severely underestimate the risk observed in the market.

The successive financial crisis since 1987 have led to a greater attention to tail behavior of the induced (log-)returns distributions, and using Extreme Value Theory (EVT) has been pushed forward as a central concept in Risk Management (See *e.g.* [13, 24]).

Using a model for the tail distributions overrides partially the problem induced by the lack of data for computing the VaR with historical distributions. Once a model for the (log-)returns is specified and calibrated, the VaR, as it is related to the quantiles of the distribution, can be computed through tractable expressions or simulations. Other risk indicators as the CVaR or expected shortfall may be computed as well.

There is no single model nor statistical methodology which are acknowledged as standard for dealing with heavy-tails. In this study, we choose to stick on the simplest possible models. Calibrating a complex model leads to high uncertainties on the parameters which decreases its intricate qualities for practical use, especially with small data samples.

Let us point that we focus here only on univariate distributions. The case of several assets leads to higher complexity, as the notion of VaR itself should be properly defined (See *e.g.* [73]). The multivariate case will be subject to further studies.

As the Gaussian models underestimate extreme losses, we consider heavy tails distributions for the log-returns L with distribution function F, typically, generalized Pareto (power laws):

$$1 - F(x) = \mathbb{P}[L \geq x] = \frac{\ell(x)}{x^\alpha} \text{ for } x \geq x_0, \tag{1}$$

where ℓ is a function of regular variation, α is the *tail index* and x_0 is a threshold above which the power law holds. This formula deals only with the tail distribution. We give in Section 2 a literature review of heavy-tailed models, together with empirical evidences of their relevance in finance and estimation procedures.

This work aims at studying the performance of VaR estimators for three classes of models (Gaussian, stable and Pareto tails) on historical market data from NYSE Euronext Paris. We focus only on daily log-returns of single stock prices.

Outline. In Section 2, we give an overview of models with heavy-tails used in finance, the markets on which they have been applied and the estimators for power laws. In Section 3, we introduce our methodology for estimating the parameters and performing the backtesting. Finally, in Section 4, we present our dataset as well as the conclusions we drawn from the study over a selection of 71 assets from NYSE Euronext Paris. Some perspectives and conclusions are then presented in Section 5.

2 Overview of heavy tails: models, empirical evidences and estimations

2.1 Some commonly used models for heavy tails

Normal distributions and Brownian motion in finance go back to Bachelier's thesis [5] and were popularized by F. Black and M. Scholes in the 70's [8].

There are several reasons making log-normal returns appealing. The first one is that they could be simply interpreted and estimated. Second, closed-form expression exists for several options. Third, they could be embedded in a continuous time process, as the geometric Brownian motion, which models the evolution of the stock over the time.

Indeed, many theories, for example the Capital Asset Pricing Model (CAPM) for portfolio management [61], take their roots in the Gaussian world.

However, starting with B. Mandelbrot [59] in the 60's, it has been evidenced that normal or log-normal distributions do not fit some of the stylized facts, mainly in regard to skewness and heavy tail of observed distributions.

A large amount of literature in quantitative finance deals with the problem of finding tractable models which reproduce heavy-tails returns or log-returns.

Stable distributions and processes have been proposed first by B. Mandelbrot [59] and E.F. Fama [30]. This popular approach is referred to as the "Paretian world". As for the Gaussian world, the log-returns of a given stock may be embedded in a stochastic discontinuous process while having a heavy-tail decreasing like a power law of index α. The CAPM can also be extended in some directions. Yet choosing this model imposes that $\alpha < 2$. This parameter is however difficult to estimate, especially when close to 2 [27]. In addition to the references and empirical studies cited above, which backed that α may be greater than 2, let us cite [9, 12, 49] as containing critics on the stable model.

Thus, many alternatives have been sought. There is a huge literature on this subject [71], so that we give the main lines with a focus only on univariate returns. Regarding stochastic processes in continuous time, let us cite jump diffusion models [19, 50], variance Gamma processes [60] and subordinated processes [16], and SDE whose invariant distributions are fat-tailed [67, 74] or present moments explosions [38, Chapter 7].

Along with continuous time stochastic processes, chronological series play a very important role in the development of such financial models for stock prices. The (log-)returns are solutions to equations of the form $r_{t+1} = \mu r_t + \sigma_t \epsilon_t$, where σ_t is itself described by an equation of similar form. One of the most popular model is the General Autoregressive Conditional Heteroskedasticity (GARCH) model which captures clustering effects. Here, the innovation ϵ_t is a noise, that could be Gaussian or follow a given distribution. Among them, Student's t distributions have been widely studied. We refer for example to [21, 52] and their introduction to references in this field. In [76], A.K. Singh *et al.* give a specific account on such model in relation with power laws.

Finally, another approach consists in separating the tail of the distribution from its bulk through hybrid models. Several approaches may be found in [72]. Furthermore, mixtures of models may also lead to heavy tails [14].

2.2 Evidences for the power law in financial markets

It is widely acknowledged that prices and returns of assets obey to general laws usually called "stylized facts" [17]. Skewness and heavy-tail are the two main properties of observed prices which are not verified by the Black & Scholes models.

Although stable distributions have been proposed since the 60's [30, 59] and an alternative model (finite variance subordinated log-normal distributions) has been proposed in 1973 by P. Clark [16], the systematic use of Extreme Value Theory (EVT) is more recent, as the crash of 1987 urged for a better understanding of large losses. For the first occurrences of the use of EVT focusing only on the tail distributions, let us cite [43, 54, 56].

Soon, the availability of large datasets and the failure of normal distributions to replicate the extreme movements led to a blossoming of empirical studies about EVT.

Of course, crises such as the Asian crisis, have been subjects of particular importance [41, 48]. Another body of works concerns emerging markets: Asia [36, 48], MENA region [4], Turkey [36, 78], Latin America [36, 44], ...

Developed markets have been investigated as well, mainly through market indices: S&P 500, Dow Jones and Nasdaq [37, 54, 57, 58, 76], German DAX Stocks [25], Australian ASX-ALL [76], Nikkei and Eurostoxx 50 [37], ...

Finally, the EVT has also been applied to other prices, such as exchanges rates (See *e.g.* [45, 49]), futures margins [20], ...

In all these situations, the normal hypotheses is rejected as a model for the return distributions and heavy tails should be taken into account. However, the stable hypothesis seems too strong.

In many situations, it is recorded that the variance of the returns is finite [4, 36, 37, 48] and the tail index lies between 2 and 5. Some authors note that regarding extreme events, markets in emerging and developed countries present similar features [48]. In [35], X. Gabaix summarizes various studies on power law in finance and defend the notion of "universality" of a tail index around 3 for short terms returns.

In addition, to these empirical evidences, a growing body of literature also focuses on economic and agent based models which could explain fat tails and phenomena leading to them, such as volatility clustering. This subject is beyond the scope of this article and we refer for example only to [18, 33] and related references therein.

2.3 Estimators for the tail index

The tail index α summarizes the heaviness of the tail distributions, and characterizes also the existence of moments. Hence, let us consider that the (log-)return X at a given time satisfies (1) and that n successive (log-)returns $(X_1, \dots, X_n)$ are independent or at least stationary.

It is a crucial and complex problem to estimate α and $\ell(x)$ written in a parametric or semi-parametric form (for example, $\ell(x) = C$ or $\ell(x) = C_1 + C_2/x^\beta + \mathrm{o}(x^{-\beta})$), as well as x_0. For studying the tail of the distribution of X, we use the *order statistics* $(X_{(1)}, \dots, X_{(n)})$ of $(X_1, \dots, X_n)$ with $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

Estimating α is a difficult problem in general due to the lack of observed extreme events, by their very definition.

The literature is too wide to be cited here. Estimators may in general be attached to one of the following families, none of them superseding the others:

- *Hill type estimator.* The Hill estimator [40] provides us with

$$H_{k,n} = k^{-1} \sum_{j=1}^{k} \ln X_{(n-j+1)} - \ln X_{(n-k)}$$

as an estimator of $\gamma = 1/\alpha$. There are several ways to interpret this estimator (maximum likelihood, least squares, ...). The main difficulty for its implementation consists in choosing the optimal index k, which gave rise to a huge literature. A large part of it consists

in assuming that $\ell(x)$ is not constant and balancing between the Monte Carlo error and the bias. Generalizations of Hill estimator include least-square estimators. See *e.g.* [42] for an application to financial data. On the Hill estimator and its variants, we refer to the book [6] and references therein.
The Hill method gave rise to a graphical procedure, the *Hill Plot* [26], which consists in plotting $(k, H_{k,n})$. The lack of stability of the produced graph leads to a lot of critics, the Hill Plot being dubbed as the "Hill horror plot".

• The *Pickands estimator* [70], estimates $\alpha = 1/\gamma$ from a slope using three points. Again, a tail index k should be carefully chosen.

• The *Peak over Threshold* (POT) method [29], is popular in the hydrogeology community and also led to a graphical procedure. It relies on the convergence of the distribution function of the renormalized maximum of the data toward a Generalized Extreme Value distribution. Applications to financial market may be found in [3, 37].

• *Block maxima*, which consists in focusing on the statistics of the maxima of blocks of data in link with their frequency (See *e.g.* [31] and references within).

• Other estimators: many variants of the Hill and Pickands estimators have been proposed, using for example the moments [22] or the median of the order statistics, the DRP estimator [69], ...

Finally, several methods have been proposed to deal specifically with the four parameters of stable distributions: quantile estimation [63], maximum likelihood [28, 65], characteristic functions [32, 51]. See also [27, 62, 80] for critical reviews.

3 Framework and methodology

3.1 Log returns and Value-at-Risk

Let us consider the prices of a financial asset $(S_t, t \geq 0)$ where time is measured on days units. We define the daily *log-returns* as being the sequence $(R_t,\ t \in \mathbb{N})$ given by

$$R_t = \ln\left(\frac{S_{t+1}}{S_t}\right) = \ln(S_{t+1}) - \ln(S_t). \tag{2}$$

We call also *log-losses* the values $L_t = -R_t$ for $R_t < 0$.

The statistical principle of risk estimation consists in assuming that the R_t's are independent realizations of a given law.

The *Value-at-Risk* (*VaR*) at a level $\alpha \in (0,1)$, and for the horizon $T = 1$ (one day), of a financial asset $(S_t, t \geq 0)$, is the lowest amount not exceeded by the loss with probability α (usually α is close to 1), i.e.

$$1 - \alpha = \mathbb{P}\left(S_1 - S_0 < -\ VaR_\alpha\right). \tag{3}$$

In this article, we consider only daily VaR. The 1-day VaR_α may be expressed by the daily log-returns as follows: for α close to 1,

$$1 - \alpha = \mathbb{P}(R_1 < q^R_{1-\alpha}) \text{ with } \ VaR_\alpha = S_0(1 - \exp(q^R_{1-\alpha})) \tag{4}$$

or

$$1 - \alpha = \mathbb{P}(L_1 > q^L_\alpha) \text{ with } \ VaR_\alpha = S_0(1 - \exp(-q^L_\alpha)), \tag{5}$$

where q^X_α is the α-quantile of the random variable X.

After having chosen a class of parametric models, the practical computation of the VaR consists in calibrating the parameters for the common distribution of R or L of daily log-returns or log-losses and computing the quantile $q^R_{1-\alpha}$ or q^L_α. We need to know only the tail of the distribution.

3.2 Models

We consider three classes of parametric univariate models for the log-returns or log-losses:

(I) The classical **Gaussian family for the log-returns** defined by its mean μ and standard deviation σ.

(II) The **stable distribution for the log-returns** defined by its characteristic function

$$\phi_X(t) = \begin{cases} \exp[i\mu t - \sigma^\alpha |t|^\alpha (1 - i\beta \operatorname{sign}(t) \tan(\frac{\pi\alpha}{2}))] & \text{if } \alpha \neq 1 \\ \exp[i\mu t - \sigma|t|(1 + \frac{2}{\pi} i\beta \operatorname{sign}(t) \ln|t|)] & \text{if } \alpha = 1, \end{cases} \tag{6}$$

where $\alpha \in (0, 2)$ is the *tail index*, $\beta \in (-1, 1)$ is the *skewness*, $\sigma \geq 0$ is the *scale parameter* and $\mu \in \mathbb{R}$ the *location parameter* (See *e.g.* [34, 68]).

(III) The **Pareto distribution for tails of the log-losses** gives for the distribution function F of the log-losses,

$$\mathbb{P}[L_t > x] = 1 - F(x) = C/x^\alpha \text{ for } x \geq x_0. \tag{7}$$

This model does not make any supplementary assumption on the bulk of the distribution of the log-returns.

Gaussian distributions are a particular case of stable distributions for $\alpha = 2$. However, in this case, α does not correspond to the tail index. When $\alpha < 2$, a stable distribution is in fact a Generalized Pareto distribution with

$$\mathbb{P}[X \leq x] \approx \frac{C_{\text{st}}}{|x|^\alpha} \text{ when } x \to -\infty \text{ with } C_{\text{st}} = \sigma^\alpha \sin\left(\frac{\pi\alpha}{2}\right) \frac{\Gamma(\alpha)}{\pi}(1 - \beta), \tag{8}$$

where Γ denotes the Gamma function [68, Theorem 1.12].

Gaussian and stable distributions arise naturally as universal classes when looking to limit theorems. The micro-economic justification assumes that the prices are fixed by the interactions of a large amounts of small independent prices changes. Stable distributions are fat-tailed. Yet they have an infinite variance which sometimes produces too high extremes events with respect to the observed log-returns (See Section 2).

Pareto distributions offer a wider variety of fat tails by removing the constraint $\alpha < 2$.

3.3 Parameters estimation and computation of the VaR

Before describing the methods we used to estimate the parameters of the chosen distributions and then to compute the quantiles, we should make the following remark: Choosing a model requires that the series of stock prices (S_t) is *stationary* over the time. This is of course not true for prices spanning over several years. Thus, we apply our estimators over a finite window of time, 1 year in our experiments, where the prices are assumed to be stationary.

(I) **Gaussian distribution** Considering n successive daily log-returns $(R_1, \ldots, R_n)$ assumed to follow the Gaussian distribution of mean μ and standard deviation σ, we set

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n R_i, \quad \text{and} \quad \hat{\sigma} = \left(\frac{1}{n-1}\sum_{i=1}^n (R_i - \hat{\mu})^2\right)^{1/2},$$

and the $(1 - \alpha)$-quantile $q^R_{1-\alpha}$ is approximated by $\hat{q}^R_{1-\alpha} = \hat{\mu} + \Phi^{-1}(1 - \alpha)\hat{\sigma}$ where Φ is the standard Gaussian distribution function.

(II) **Stable Distribution** For the stable distribution, we use for the log-returns the McCulloch method as implemented in the `R` library `fBasics` to estimate the four parameters α, β, σ and μ and to compute the quantile through a direct estimation of the distribution function.

(III) **Pareto distribution** We use a slight modification of the Hill estimator. Let $(L_{(1)}, \dots, L_{(n)})$ be the increasing order statistics of n independent log-losses $(L_1, \dots, L_n)$ whose common distribution is assumed to satisfy $\mathbb{P}[L_1 \geq x] = C/x^\alpha$ for $x \geq x_0$.
For i large enough,

$$\ln L_{(i)} = -\gamma \ln \left(\frac{n+1-i}{n+1} \right) + K + \varepsilon_i, \tag{9}$$

where $\gamma = 1/\alpha$, $K = \gamma \ln C$ is a constant and ε_i is a noise. Plotting $\ln L_{(i)}$ as a function of $-\ln((n+1-i)/(n+1))$ gives a *Pareto plot*. The Hill estimator allows to compute the slope of such a graph, using a weighted least squares estimation. For more stability, we use a variant of this estimator by removing the highest values. After fixing an interval $[d_n, u_n]$, an estimator $\hat{\gamma}$ of $\gamma = \alpha^{-1}$ is given by a standard least squares procedure on (9) for $i \in [d_n, u_n]$:

$$\hat{\gamma} = -\frac{\sum_{i=d_n}^{u_n} \ln(L_{(i)}) \cdot \ln\left(\frac{n+1-i}{n+1}\right)}{\sum_{i=d_n}^{u_n} \left(\ln\left(\frac{n+1-i}{n+1}\right)\right)^2}.$$

The constant C in (7) could be estimated as well by $\exp(\hat{K}\hat{\alpha})$ where $\hat{\alpha} = 1/\hat{\gamma}$ and $\hat{K}$ is given by the least square procedure on (9). However, we choose a procedure which is numerically more stable by borrowing ideas from I. Weissman [79]. For $w \in (0,1)$ close to 1, the constant C and the threshold x_0 in (7) are estimated by

$$\hat{C} = L_{\lfloor nw \rfloor}^{\hat{\alpha}}(1-w) \text{ and } \hat{x}_0 = L_{\lfloor nw \rfloor}. \tag{10}$$

The rationale of this approximation is that $L_{\lfloor nw \rfloor}$ is an approximation of the quantile q_w of $(L_1, \dots, L_n)$. The quantile of the log-losses at level $p \geq w$ is then approximated by

$$\hat{q}_p^L = \left(\frac{\hat{C}}{1-p} \right)^{\hat{\gamma}} = L_{\lfloor nw \rfloor} \left(\frac{1-w}{1-p} \right)^{\hat{\gamma}}.$$

This procedure is illustrated over one year (252 data) of real data in Fig. 1. The first two figures represents the evolution of the prices and the log-returns. The third plot is the Pareto plot, where the points used for the statistical estimation of γ are marked in blue, while the one used for the estimation of $\hat{C}$ is marked in magenta. The last two plots represent the empirical densities of the log-returns, with a zoom on the large losses, as well as the fitted models for normal (red), stable (green) and power tail (blue) distributions.

3.4 Choice of the parameters of the tail index estimator

We estimated the power law distribution by setting $d_n = \lfloor 0.95 \times n \rfloor$, $u_n = \lfloor 0.99 \times n \rfloor$ and $w = 0.90$, where n is the number of log-losses.

As the window size is $W = 252$, it has been observed on all the assets that the number of log-losses is around $W/2$. This means that the parameters for extreme log-losses are estimated from very small samples.

3.5 Backtesting

The backtesting procedure consists in comparing the estimated VaR with the real number of extreme losses [15].

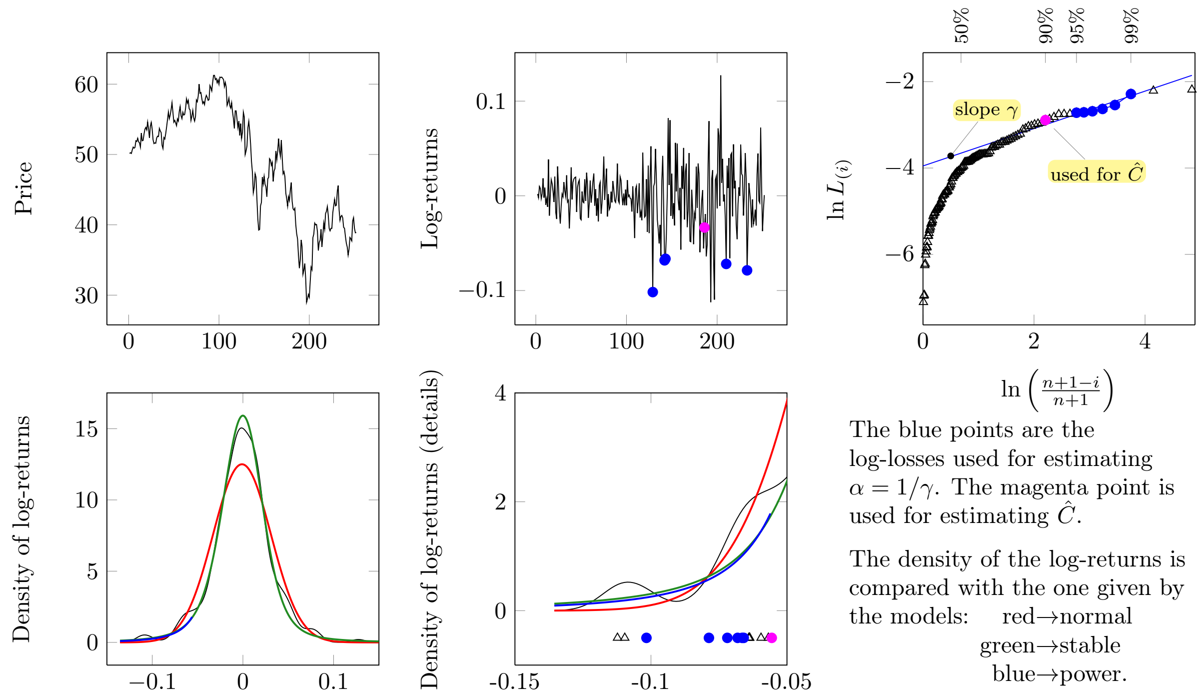


Figure 1: Illustration of the estimation procedure for the parameters of the power law.

The procedure is the following. Given the set of daily prices $(S_0, \dots, S_T)$ of an asset and W a window size, we set $S_{t:t+W} = (S_t, \dots, S_{t+W})$ the series of $W+1$ prices over the time interval $[t, t+W]$. From this, we extract the series of W log-returns $R_{t:t+W}$ as well as the series of log-losses $L_{t:t+W}$. The number of elements in $L_{t:t+W}$ is random. For a fixed level α, the quantiles $q^R_{1-\alpha}$ and q^L_α are then computed from these series $R_{t:t+W}$ and $L_{t:t+W}$ respectively by using the estimation methods described above. The VaR $VaR_{t:t+W}$ at level α is then computed through (4) or (5) and is then compared to the next day's return of the asset.

Hence, we set for $t = 0, \dots, T-W$,

$$J_{t+W+1} = \begin{cases} 1 & \text{if } S_{t+W+1} - S_{t+W} < -\ VaR_{t:t+W}, \\ 0 & \text{otherwise.} \end{cases}$$

As the VaR predicts the threshold for losses that occur with a given probability $1-\alpha$, the mean number of occurrences $\overline{J}$ of 1 in the series $(J_{W+1}, \dots, J_T)$ should be close to $1-\alpha$ for a relevant choice of the model and of the window size W.

If $\overline{J}$ is much bigger than $1-\alpha$, the class of model underestimates the extreme losses. Conversely, if $\overline{J}$ is much smaller than $1-\alpha$, then extreme losses are over-estimated.

One of the difficulties in this procedure comes from the lack of independence of the J_t's. Therefore, it is not easy to give an adequate confidence interval for $\overline{J}$ but in this study, we assume independence of J_t's to compute confidence intervals. A Fisher exact test on the four possibles occurences of (J_{t-1}, J_t) shows up that for most of the assets in our data set, $(J_{W+1}, \dots, J_T)$ cannot be distingushed from a sequence of independent Bernoulli random variables [1].

To construct the confidence interval (CI) for the backtesting, $\sum_{t=0}^{T-W} J_{t+W+1}$ follows a binomial distribution as we have assumed it is the sum of independent random variables.

An exact CI at $100 \times (1-\kappa)\%$ is given by

$$\left[\frac{1}{1+\frac{n-k+1}{k}F_{2(n-k+1),2k}(1-\kappa/2)}, \frac{\frac{k+1}{n-k}F_{2(k+1),2(n-k)}(1-\kappa/2)}{1+\frac{k+1}{n-k}F_{2(k+1),2(n-k)}(1-\kappa/2)}\right],$$

where k is the number of successes, n is the size of the sample, and $F_{\nu_1,\nu_2}(p)$ is the inverse of the quantile at level $p \in [0,1]$ of the F-distribution with degree of freedoms ν_1 and ν_2 [10, 11, 66].

4 Discussion: Empirical results and backtesting analysis

4.1 The dataset: stocks from Euronext Paris

The financial instruments considered in the experiments are stocks exchanged on NYSE Euronext Paris. The market data are provided by eSignal (Interactive Data), and, in the following, the stocks are identified by their eSignal symbol. Out of all the stocks listed on the Euronext Paris exchange in February 2011, more than 600, we selected ones having quotations throughout all the period ranging from January 2001 till February 2011 (more than 11 years). This leads us with a subset of 71 stocks including some of the most liquid stocks making up the CAC40 index. In the following experiments, the time series considered are the log-returns of the end-of-day closing prices of the selected stocks and the parameters of the returns distributions are estimated on a sample made of the last 252 last prices (walk-forward parameter setting). This window length set to one year is a trade-off between the need to have enough data to include recent crises and the increased risk of departure from the stationarity hypothesis with larger data sets. Regarding the stationarity, it is difficult to draw a clear cut conclusion from Dickey-Fuller and Kwiatkowski–Phillips–Schmidt–Shin tests on unit root and stationarity tests [23, 53]. We were not able in our experiments to identify other sample sizes that would consistently outperform one year with regard to the VaR backtesting or stationarity measures.

4.2 Discussion

We applied the methods of estimation of the Value-at-Risk and of backtesting described in Section 3 on the data described above. We plotted for all the assets the asset prices, the historical volatility computed from one-year data over a moving window, and the VaR computed with our three methods (Gaussian, stable and power law) from one year data over a moving window. We then selected 71 assets which sampled all the qualitative behaviors that we could observe in the curves. The list of these assets is found on the horizontal axis of Fig. 2.

For all these 71 assets, we plotted 5 pictures as in Fig. 3, representing:

(a) The asset price and volatility.

(b) The relative VaR, that is the ratio of the VaR and the price, computed with our three methods.

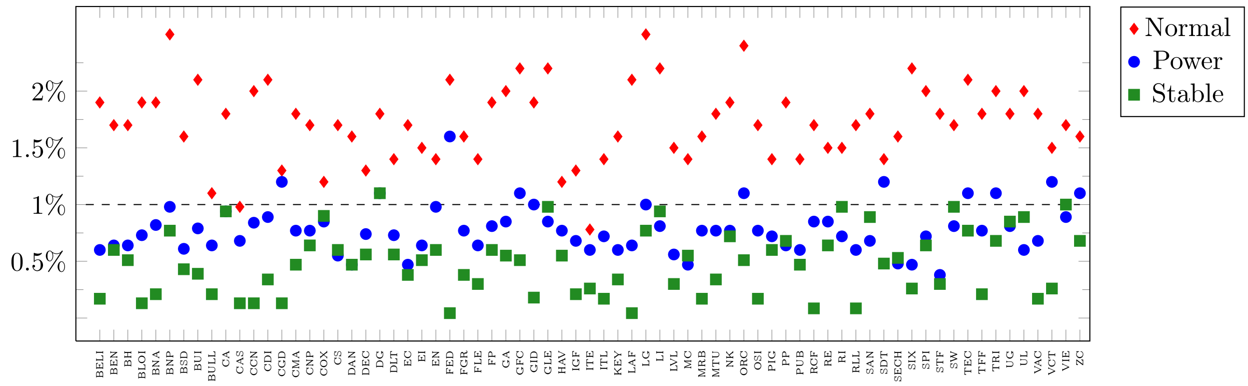
(c) The volatility and the estimated value of the parameter σ of the stable distributions.

(d) The estimated value of the tail index α for the stable and power tail distributions.

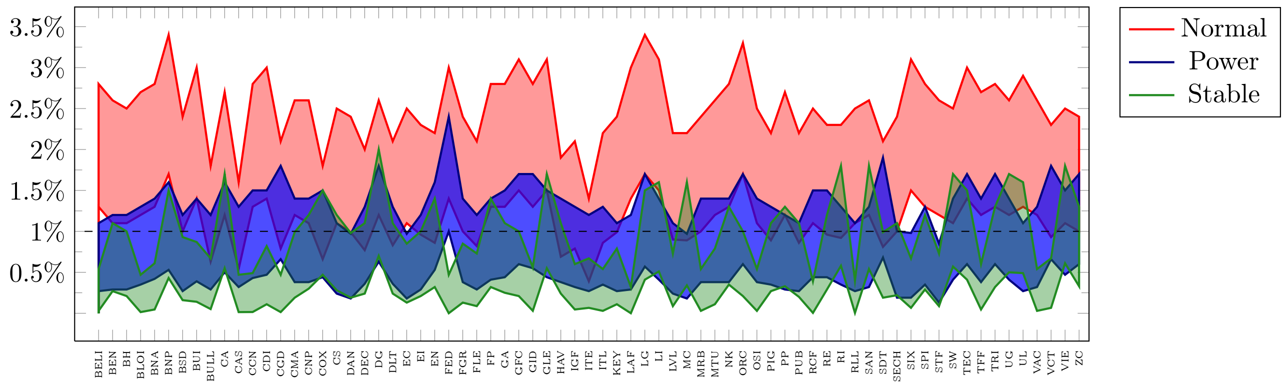
(e) At logarithmic scale, the estimated value of the constant C of the tail distribution of the stable distribution or power tail distribution, given in (8) and (10).

(f) The 99%-CI for each of the models.

All these curves are computed over a one year moving window of data, and represented as a function of the last day of the moving window. We used the following color code: all the curves obtained from an estimation based on the Gaussian distribution are in red, all



(a) Average proportions



(b) 99%-Confidence intervals

Figure 2: Backtesting of the daily VaR over 71 assets on the period 2001-2011 using a moving window of 252 days.

those obtained from the stable distributions are in green, and all those obtained from the power tail distributions are in blue. We made a second selection of 11 curves, shown in Fig. 3, in order to illustrate frequent or less frequent behaviors.

Fig. 2 gives the result of the backtesting for the three methods on the 71 selected assets, as well as confidence intervals. Due to the small amount of data in the backtesting, the confidence intervals are quite large, so it is not possible to give affirmative conclusions for a single asset, but the backtesting results over the 71 assets in Fig. 2 clearly indicate that, in terms of backtesting efficiency, the method of estimation of the VaR based on Gaussian distributions almost always underestimates the VaR: the estimated value seems to be closer to the VaR at 98% than the VaR at 99%. Similarly, the method of estimation of the VaR based on stable distributions often overestimates the VaR. With the power law, 65 out of the 71 99%-CI contain the target value 1%. It is the case for only 37 out of the 71 with the stable model and 23 out of 71 for the Gaussian model. Thus, the method based on power law distributions is thus clearly the one giving the best results in terms of backtesting.

We can interpret this result as follows: the tail distribution in the Gaussian model is too thin, and the tail distribution in the stable law is too fat (since its tail index is always smaller than 2). This is confirmed by the fact that the estimated tail index for stable distribution is always close to 2 for all assets in our dataset. This means that considering stable distributions does not bring so much information about the tail distribution and the extreme events.

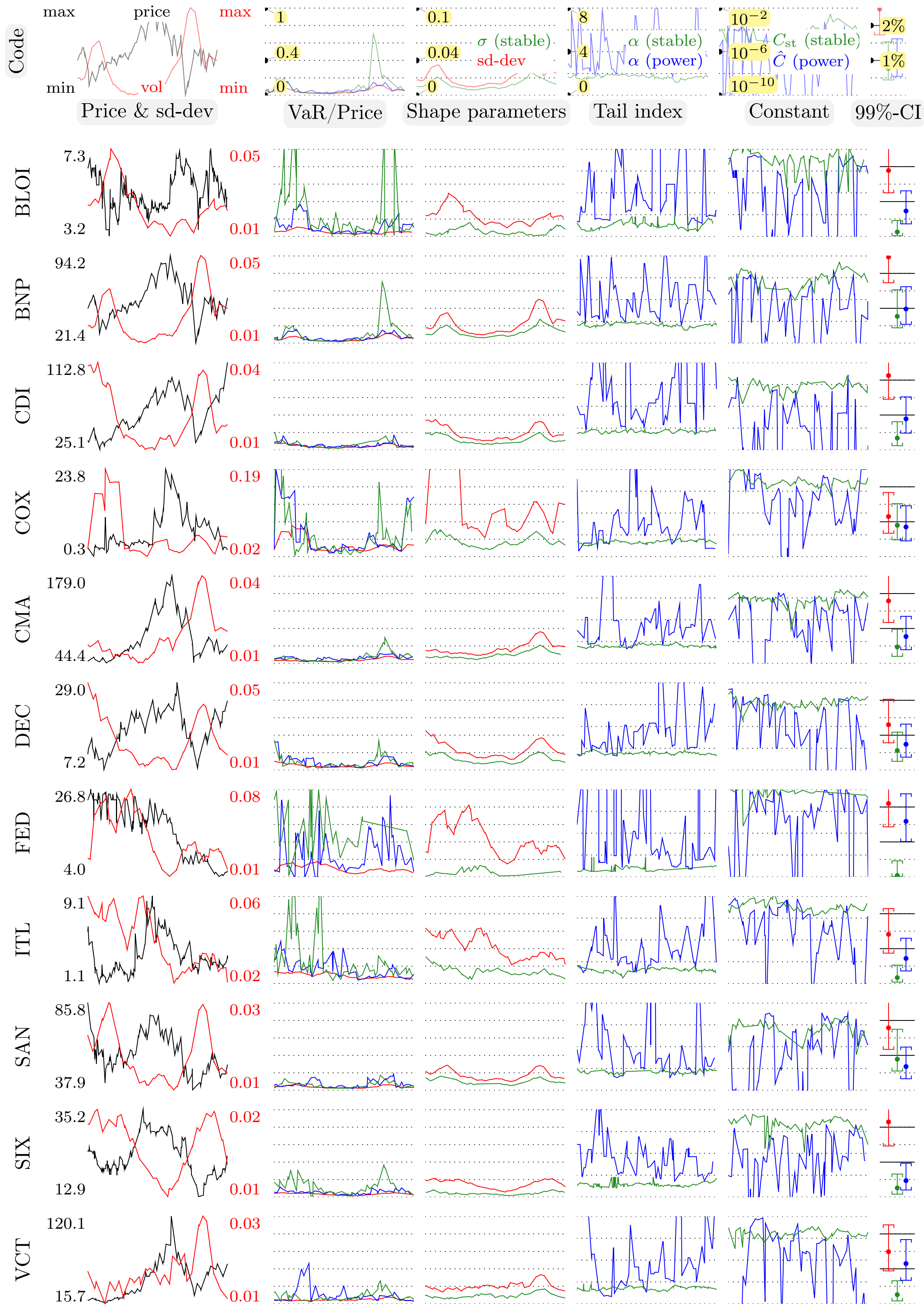


Figure 3: Evolution of the parameters for some of the selected assets.

We can distinguish several classes of assets according to the shape of the volatility curve.

- The most frequent class by far shows two marked volatility peaks around 2002 and 2009, which correspond to the dot-com bubble and the 2008 financial crisis. This is clearly the case for 31 out of 71 assets (see BNP and DEC in Fig. 3).
- Some assets show only one of these peaks: 12 assets, see CMA and BLOI in Fig. 3.
- Several assets have more erratic volatility curves: 5 assets, see FED in Fig. 3.
- Some assets show a decreasing volatility curve: 7 assets, see ITL in Fig. 3.
- Finally, several volatility curves show very small fluctuations and are roughly constant: 16 assets, see SIX in Fig. 3.

The scale parameter σ for the stable distribution plays the role of the volatility for the stable distribution. In all the curves, this parameter always shows a good correlation with the volatility, with a systematic shift of -1% to -2%.

The curves of relative VaR show that the value computed with Gaussian distributions have relatively small random fluctuations, unlike with the two other estimators. This uncertainty comes from the fact that the parameters of the tail distribution in stable laws or power laws are harder to estimate since they only deal with extreme values of the log-losses, and so with a small part of the dataset. In some cases, the VaR computed with one of these methods is clearly unreliable, more often for stable distributions, see *e.g.* BLOI in Fig. 3, sometimes for Power law only, see *e.g.* VCT in Fig. 3, sometimes for both, see *e.g.* FED in Fig. 3. Even for the other assets, we see that a single estimate of the VaR using these methods is not reliable, since the VaR can vary a lot when the window of data is shifted by a few weeks (See *e.g.* BLOI, COX, ITL, ... in Fig. 3). This is particularly true for the estimation method based on stable laws, where we can generally observe a drastic increase in the VaR in periods of higher volatility (See *e.g.* BNP in Fig. 3). Regime changes seem to be faster detected by the Power laws estimators, yet it is hard to be sure that an increase of VaR is due to a regime change rather than a fluctuation (both for stable estimator and power law estimator). Despite this, in mean, the backtesting of these two methods gives better results than for the method based on Gaussian distribution.

Most frequently, the three estimation methods of the VaR give results with small fluctuations: 17 assets out of 71, see DEC and SAN in Fig. 3. In this case, the backtesting is generally good for both power and stable law.

All these arguments seem to favor the method based on power laws for estimating the VaR. However, the curves for the tail index and the constant in the tail distribution, show an extreme uncertainty in the estimation method for these two parameters, compared with the two other methods: the estimated tail index vary between 2 and 8 within a few months for most assets. However, these strong fluctuations are compensated by strong but opposite fluctuations in the constant of the tail distribution, so that the overall result of the VaR and the backtesting is good. The lack of strong fluctuations in the tail index of stable distributions can be explained by the fact that the tail index of stable distributions belongs to a small interval $[0, 2]$, which prevents strong fluctuations as for the power law. However, the constant of the tail distribution is subject to large fluctuations in the stable model.

Note also that the tail index estimator for the power law is rarely below 2 and usually takes much larger values. This confirms that the tail distribution of stable laws is not a good model for the extreme values of the NYSE Euronext Paris financial data, a conclusion already drawn on data from several markets, not only the emergent ones (See *e.g.* [25, 37]). In addition, we observe that the constant of the tail distribution estimated with stable laws is in general greater than the one of the power law. This, combined with the fact that the tail index is almost always smaller for power laws, makes the tail distribution much heavier than the one of the power law, and explains the differences in the backtesting of the two methods.

5 Conclusions and perspectives

Although a large part of these observations seem to indicate that the power law distribution is more suited for VaR estimation, the results of backtesting for stable and power distributions remain contrasted. For many assets, the small sample size leads to an important confidence interval containing the 1% target value both for stable and power tail distributions. The assets may also have very different behaviors. But the stocks' prices from NYSE Euronext Paris could be grouped by similarity of patterns for the price or the volatility. With the exception of some erratic prices or volatility, the power law provides suitable estimates for the assets within a group. This indicates that the power law performs better over a large number of assets.

The finer analysis of the curves associated to the three methods confirms that the VaR estimated by power laws give less excessively small or high values than the other methods. Still, the tail index and constant of the tail distribution show so large fluctuations that one cannot give confidence to a single estimate of the VaR using this method. One must rather look at the curve of estimated VaR to try to detect aberrant estimated values of VaR due to statistical errors.

In any case, the normal distribution is not suitable for dealing with extreme events observed in the markets. Many turbulences and crises arose during the 2001-2011 period, which are clearly seen in prices and volatilities. A difficult question is then to separate "crisis regimes" from "steady-state regime". This requires to have a clear definition of a crisis. A past crisis may have an important impact on the tail index estimators, both for stable and power law. Conversely, the computation of the VaR based on one-year data in a "steady-state regime" does not anticipate a crisis outbreak. Hence, regime change indicators and tests, as well as models on probability of occurrence of crises as the one developed by D. Sornette *et al.* (See *e.g.* [46]), are needed.

We have considered the prices as independent. Correlations and co-movements may happen between stocks. More accurate risk indicators, especially for portfolio management, should use this information. Regarding extreme events, there are several ways to define correlations and links between assets. In a future work, we plan to address this problem, still by focusing on stock prices, while most of the studies have been performed so far on market indices. There is already large literature on this subject, and we refer only to [39, 55, 73, 75] and related content, among many others.

Even without these potential improvements, the results of the experiments suggest to us that the practitioner can already improve there tools to implement sound risk management based on power law, which appear more adapted than stable and normal distributions. Of course, the techniques described in the paper can be refined by considering crisis and non-crisis periods, as well as correlation between stocks.

References

[1] A. Agresti. "A survey of exact inference for contingency tables". In: *Statist. Sci.* 7.1 (1992). With comments and a rejoinder by the author, pp. 131–177.

[2] C. Alexander and E. Sheedy. "Developing a stress testing framework based on market risk models". In: *Journal of Banking & Finance* 32.10 (2008), pp. 2220–2236.

[3] V. Andreev, S. Tinyakov, G. Parahin, and O. Ovchinnikova. "An application of EVT, GPD and POT methods in the Russian stock market (RTS index)". In: *GPD and POT Methods in the Russian Stock Market (RTS Index)(November 17, 2009)* (2009).

[4] A. Assaf. "Extreme observations and risk assessment in the equity markets of MENA region: Tail measures and Value-at-Risk". In: *International Review of Financial Analysis* 18.3 (June 2009), pp. 109–116.

[5] L. Bachelier. "Théorie de la spéculation". In: *Ann. Sci. École Norm. Sup. (3)* 17 (1900), pp. 21–86.

[6] J. Beirlant, Y. Goegebeur, J. Teugels, and J. Segers. *Statistics of extremes.* Wiley Series in Probability and Statistics. Theory and applications, With contributions from Daniel De Waal and Chris Ferro. John Wiley & Sons Ltd., 2004.

[7] J. Berkowitz and J. O'Brien. "How Accurate Are Value-at-Risk Models at Commercial Banks?" In: *The journal of finance* 57.3 (2002), pp. 1093–1111.

[8] F. Black and M. Scholes. "The pricing of options and corporate liabilities". In: *The Journal of Political Economy* (1973), pp. 637–654.

[9] R. C. Blattberg and N. J. Gonedes. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices". In: *The Journal of Business* 47.2 (1974), pp. 244–280.

[10] C. R. Blyth. "Approximate binomial confidence limits". In: *J. Amer. Statist. Assoc.* 81.395 (1986), pp. 843–855.

[11] C. R. Blyth. "Correction: "Approximate binomial confidence limits" [J. Amer. Statist. Assoc. **81** (1986), no. 395, 843–855]". In: *J. Amer. Statist. Assoc.* 84.406 (1989), p. 636.

[12] P. Boothe and D. Glassman. "The Statistical Distribution of Exchange Rates: Empirical Evidence and Economic Implications". In: *Journal of international economics* 22 (1987), pp. 297–320.

[13] B. O. Bradley and M. S. Taqqu. "Financial risk and heavy tails". In: *Handbook of Heavy Tailed Distributions in Finance.* Ed. by S. Rachev. North-Holland, 2003, pp. 35–103.

[14] S. A. Broda, M. Haas, J. Krause, M. S. Paolella, and S. C. Steude. "Stable mixture GARCH models". In: *Journal of Econometrics* 17 (2013), pp. 292–316.

[15] S. D. Campbell. *A review of backtesting and backtesting procedures.* Tech. rep. 2007, pp. 1–17.

[16] P. K. Clark. "A subordinated stochastic process model with finite variance for speculative prices". In: *Econometrica: Journal of the Econometric Society* (1973), pp. 135–155.

[17] R. Cont. "Empirical properties of asset returns: stylized facts and statistical issues". In: *Quantitative Finance* 1 (2001), pp. 223–236.

[18] R. Cont and J.-P. Bouchaud. "Herd behavior and aggregate fluctuations in financial markets". In: *Macroeconomic dynamics* 4.2 (2000), pp. 170–196.

[19] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[20] J. Cotter. "Margin exceedences for European stock index futures using extreme value theory". In: *Journal of Banking & Finance* 25.8 (2001), pp. 1475–1502.

[21] J. D. Curto, J. C. Pinto, and G. N. Tavares. "Modeling stock markets' volatility using GARCH models with Normal, Student's t and stable Paretian distributions". In: *Statistical Papers* 50.2 (July 2007), pp. 311–321.

[22] A. L. M. Dekkers, J. H. J. Einmahl, and L. de Haan. "A moment estimator for the index of an extreme-value distribution". In: *Ann. Statist.* 17.4 (1989), pp. 1833–1855.

[23] D. A. Dickey and W. A. Fuller. "Distribution of the estimators for autoregressive time series with a unit root". In: *J. Amer. Statist. Assoc.* 74.366, part 1 (1979), pp. 427–431.

[24] F. X. Diebold, T. Schuermann, and J. D. Stroughair. "Pitfalls and Opportunities in the Use of Extreme Value Theory in Risk Management". In: *The Journal of Risk Finance* 1.2 (2000), pp. 30–35.

[25] T. Doganoglu, C. Hartz, and S. Mittnik. "Portfolio optimization when risk factors are conditionally varying and heavy tailed". In: *Computational Economics* 29.3-4 (Jan. 2007), pp. 333–354.

[26] H. Drees, L. de Haan, and S. Resnick. "How to make a Hill plot". In: *Ann. Statist.* 28.1 (2000), pp. 254–274.

[27] W. H. DuMouchel. "Estimating the Stable Index α in Order to Measure Tail Thickness: A Critique". In: *the Annals of Statistics* 11.4 (1983), pp. 1019–1031.

[28] W. H. DuMouchel. "On the asymptotic normality of the maximum-likelihood estimate when sampling from a stable distribution". In: *Ann. Statist.* 1 (1973), pp. 948–957.

[29] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events (For insurance and finance)*. Vol. 33. Applications of Mathematics (New York). Berlin: Springer-Verlag, 1997.

[30] E. F. Fama. "Risk, return, and equilibrium". In: *The Journal of Political Economy* 791.1 (1971), pp. 30–55.

[31] D. Faranda, V. Lucarini, G. Turchetti, and S. Vaienti. "Numerical convergence of the block-maxima approach to the Generalized Extreme Value distribution". In: *Journal of statistical physics* 145.5 (2011), pp. 1156–1180.

[32] A. Feuerverger and P. McDunnough. "On efficient inference in symmetric stable laws and processes". In: *Statistics and related topics (Ottawa, Ont., 1980)*. Amsterdam: North-Holland, 1981, pp. 109–122.

[33] J. Fleming and B. S. Paye. "High-frequency returns, jumps and the mixture of normals hypothesis". In: *Journal of Econometrics* 160.1 (Jan. 2011), pp. 119–128.

[34] H. Fofack and J. P Nolan. "Tail behavior, modes and other characteristics of stable distributions". In: *Extremes* 2.1 (1999), pp. 39–58.

[35] X. Gabaix. “Power Laws in Economics and Finance”. In: *Annual Review of Economics* 1 (Sept. 2008), pp. 255–293.

[36] R. Gençay and F. Selçuk. “Extreme value theory and Value-at-Risk: Relative performance in emerging markets”. In: *International Journal of Forecasting* 20.2 (Apr. 2004), pp. 287–303.

[37] M. Gilli and E. Këllezi. “An Application of Extreme Value Theory for Measuring Financial Risk”. In: *Computational Economics* 27.2-3 (2006), pp. 207–228.

[38] A. Gulisashvili. *Analytically tractable stochastic stock price models*. Springer Finance. Heidelberg: Springer, 2012.

[39] P. Hartmann, S. Straetmans, and C. G. de Vries. “Asset market linkages in crisis periods”. In: *Review of Economics and Statistics* 86.1 (2004), pp. 313–326.

[40] B. M. Hill. “A simple general approach to inference about the tail of a distribution”. In: *Ann. Statist.* 3.5 (1975), pp. 1163–1174. JSTOR: 2958379.

[41] L. C. Ho, P. Burridge, J. Cadle, and M. Theobald. “Value-at-risk: Applying the extreme value approach to Asian markets in the recent financial turmoil”. In: *Pacific-Basin Finance Journal* 8.2 (2000), pp. 249–275.

[42] R. Huisman, K. Koedijk, C. Kool, and F. Palm. “Fat tails in small sample”. In: *Available at SSRN 51141* (1997).

[43] D. W. Jansen and C. G. de Vries. “On the frequency of large stock returns: Putting booms and busts into perspective”. In: *The review of economics and statistics* (1991), pp. 18–24.

[44] R. de Jesús and Ortiz. “Risk in Emerging Stock Markets from Brazil and Mexico: Extreme Value Theory and Alternative Value at Risk Models”. In: *Frontiers in Finance and Economics* 8.2 (Oct. 2011), pp. 49–88.

[45] R. de Jesús, E. Ortiz, and A. Cabello. “Long run peso/dollar exchange rates and extreme value behavior: Value at Risk modeling”. In: *North American Journal of Economics and Finance* 24 (Jan. 2013), pp. 139–152.

[46] A. Johansen, D. Sornette, and O. Ledoit. “Predicting financial crashes using discrete scale invariance”. In: *Journal of Risk* 1.4 (1999), pp. 5–32.

[47] P. Jorion. *Value at Risk: The New Benchmark for Managing Financial Risk*. 3rd ed. McGraw-Hill, 2006.

[48] J. Kittiakarasakun and Y. Tse. “Modeling the fat tails in Asian stock markets”. In: *International Review of Economics & Finance* 20.3 (June 2011), pp. 430–440.

[49] K. G. Koedijk, M. Schafgans, and C. G. de Vries. “The tail index of exchange rate returns”. In: *Journal of international economics* (1990).

[50] S. G. Kou. “Chapter 2 Jump-Diffusion Models for Asset Pricing in Financial Engineering”. In: *Handbooks in Operations Research and Management Science*. Ed. by J. R. Birge and V. Linetsky. Elsevier, 2007, pp. 73–116.

[51] I. A. Koutrouvelis. “An iterative procedure for the estimation of the parameters of stable laws”. In: *Comm. Statist. B—Simulation Comput.* 10.1 (1981), pp. 17–28.

[52] K. Kuester, S. Mittnik, and M. S. Paolella. “Value-at-risk prediction: A comparison of alternative strategies”. In: *Journal of Financial Econometrics* 4.1 (2006), pp. 53–89.

[53] D. Kwiatkowski, P. Phillips, and P. Schmidt. “Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root?” In: *Journal of Econometrics* 54 (1992), pp. 159–178.

[54] F. Longin. “The Asymptotic Distribution of Extreme Stock Market Returns”. In: *The Journal of Business* 69.3 (1996), pp. 383–408.

[55] F. Longin and B. Solnik. “Extreme correlation of international equity markets”. In: *The journal of finance* 56.2 (2001), pp. 649–676.

[56] M. Loretan and P. Philips. “Testing the covariance stationarity of heavy-tailed time series: An overview of the theory with applications to several financial datasets”. In: *Journal of Empirical Finance* 1.2 (1994), pp. 211–248.

[57] Y. Malevergne, V. F. Pisarenko, and D. Sornette. “Empirical Distributions of Log-Returns: between the Stretched Exponential and the Power Law?” In: *Quantitative Finance* 5.4 (2005), pp. 379–401.

[58] Y. Malevergne, V. Pisarenko, and D. Sornette. “On the power of generalized extreme value (GEV) and generalized Pareto distribution (GPD) estimators for empirical distributions of stock returns”. In: *Applied Financial Economics* 16.3 (2006), pp. 271–289.

[59] B. Mandelbrot. “The variation of certain speculative prices”. In: *Journal of business* XXXVI (1963), pp. 392–417.

[60] R. Marfè. “A generalized variance gamma process for financial applications”. In: *Quantitative Finance* 12.1 (Jan. 2012), pp. 75–87.

[61] H. Markowitz. “Portfolio Selection”. In: *The journal of finance* 7.1 (1952), pp. 77–91.

[62] J. H. McCulloch. “Measuring tail thickness to estimate the stable index α: a critique”. In: *J. Bus. Econom. Statist.* 15.1 (1997), pp. 74–81.

[63] J. H. McCulloch. “Simple consistent estimators of stable distribution parameters”. In: *Comm. Statist. B—Simulation Comput.* 15.4 (1986), pp. 1109–1136.

[64] J. Mina and J. Y. Xiao. *Return to RiskMetrics: The Evolution of a Standard.* Tech. rep. RiskMetrics, 2001.

[65] S. Mittnik and M. S. Paolella. “A simple estimator for the characteristic exponent of the stable Paretian distribution”. In: *Math. Comput. Modelling* 29.10-12 (1999), pp. 161–176.

[66] J. T. Morisette and S. Khorram. “Exact binomial confidence interval for proportions”. In: *Photogrammetric engineering and remote sensing* 64.4 (1998), pp. 281–282.

[67] Y. Nagahara. “Non-Gaussian distribution for stock returns and related stochastic differential equation”. In: *Financial Engineering and the Japanese Markets* 3.2 (1996), pp. 121–149.

[68] J. Nolan. *Stable Distributions: Models for Heavy Tailed Data.* Boston: Birkhauser, 2013.

[69] V. Paulauskas and M. Vaičiulis. *Once more on comparison of tail index estimators.* Apr. 2011. arXiv: `1104.1242v1`.

[70] J. I. Pickands. “Statistical inference using extreme order statistics”. In: *Ann. Statist.* 3 (1975), pp. 119–131.

[71] S. Rachev, ed. *Handbook of Heavy Tailed Distributions in Finance, Volume 1*. North-Holland, 2003.

[72] C. Scarrott and A. MacDonald. “A review of extreme value thresehold estimation and uncertainty quantication.” In: *REVSTAT–Statistical Journal* 10.1 (2012), pp. 33–60.

[73] R. Serfling. “Quantile functions for multivariate analysis: approaches and applications”. In: *Statistica Neerlandica* 56.2 (2002), pp. 214–232.

[74] W. T. Shaw and M. Schofield. “A model of returns for the post-credit-crunch reality: hybrid Brownian motion with price feedback”. In: *Quantitative Finance* (Jan. 2012), pp. 1–24.

[75] A. Carvalhal da Silva and B. De Melo Mendes. “Value-at-risk and extreme returns in Asian stock markets”. In: *International Journal of Business* 8.1 (2003), pp. 17–40.

[76] A. Sing, D. Allen, and P. Robert. “Extreme Market Risk and Extreme Value Theory”. In: *Mathematics and computers in simulation* (2012).

[77] Supervision, Basel Committee on Banking, ed. *International Convergence of Capital Measurement and Capital Standards – A Revised Framework*. Bank of International Settlements, June 2004.

[78] G. Ünal. “Value-at-risk forecasts: a comparison analysis of extreme-value versus classical approaches”. In: *The Journal of Risk Model Validation* 5.3 (2011), pp. 59–76.

[79] I. Weissman. “Estimation of parameters and large quantiles based on the k largest observations”. In: *J. Amer. Statist. Assoc.* 73.364 (1978), pp. 812–815.

[80] R. Weron. “Levy-stable distributions revisited: tail index > 2 does not exclude the Levy-stable regime”. In: *International Journal of Modern Physics C.* 12.2 (2001), pp. 209–223.