On the null controllability of a 3×3 parabolic system with non-constant coefficients by one or two control forces

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Abstract

This work is concerned with the null controllability of a class of 3×3 linear parabolic systems with non constant coefficients by a single control force or two control forces localized in space. We extend to this class of systems the Kalman rank condition existing for systems with constant or time–dependent coefficients. To prove the result, we construct a solution to the controllability issue using a suitable decomposition. With this decomposition, we are led to study the null controllability of either a non homogeneous system of two equations by one control force acting on the whole domain (in the case of one distributed control force for the initial 3×3 system), or a non homogeneous equation by two forces acting in the whole domain (in the case of two distributed control forces for the 3×3 system).

Keywords: null controllability, observability, parabolic system, Carleman inequality
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Résumé

Ce travail concerne la contrôlabilité à zéro, par une ou deux forces de contrôle localisée(s) en espace, d'une classe de systèmes paraboliques linéaires de trois équations à coefficients non constants. On étend à cette classe de systèmes la condition de Kalman qui existe déjà pour les systèmes à coefficients constants et les systèmes à coefficients ne dépendant que du temps. Pour démontrer ce résultat, on utilise une décomposition adaptée des solutions à contrôler. Cette décomposition permet de transformer le problème de contrôlabilité par une force (*localisée* en espace) en l'étude de la contrôlabilité à zéro d'un système parabolique non homogène de deux équations par l'intermédiaire d'une seule force de contrôle agissant sur *tout* le domaine. De même, le problème de contrôlabilité par deux forces *localisées* en espace se ramène à l'étude de la contrôlabilité à zéro d'une équation parabolique non homogène par l'intermédiaire de deux forces de contrôle agissant sur *tout* le domaine.

1. Statement of the main results and presentation of the method

The starting point of this work is the study of the controllability to trajectories of drug delivery to brain tumors for a distributed parameters model (see (73) and the comments in subsection

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6.3). As we would like to apply the fixed point method described and used in the scalar case by [13] in particular, we are naturally led to investigate the null controllability of a linear 3×3 parabolic system by a single control force localized in space. In the literature devoted to this kind of systems, most of the results on null controllability by one force are proved for systems of two equations (see for instance [2], [17] ,[18] or, more recently [1]). There are very few results concerning the case of systems of n equations, with $n \geq 3$. To the author knowledge, the first characterizations of the null controllability for a linear parabolic system of n equations are proved by Ammar Khodja et al. in [3] for the case of *constant coefficients* and in [4] for the case of *time-dependent coefficients*. For coefficients depending on both variables x and t, we mention the paper of González-Burgos and de Teresa [16] which deals with the case of cascade systems. Recent results obtained by Benabdallah et al. in [7] and [8] for 3×3 systems get round the restrictive hypothesis of cascade systems but assume a geometrical constraint on the boundary of the control domain. For a recent survey on controllability results for parabolic systems, we refer the reader to the paper by Ammar Khodja et al. [5].

The main goal of the present paper is to provide sufficient conditions to control a parabolic system of three equations by one or two forces supported in space (for the boundary controllability of parabolic systems, we refer to [6]). More precisely, we analyze the null controllability of the 3×3 system

$$\left\{ \begin{array}{ll} \partial_t y = \Delta y + Ay + B v 1_\omega & \text{in } Q_T = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with boundary $\partial\Omega$ of class C^2, ω is an arbitrary nonempty open subset of Ω and T is a positive real number. In (1), y denotes a three components vector $y = (y_1, y_2, y_3)^T$, $A = (a_{ij})_{1 \leq i,j \leq 3}$ is a matrix with coefficients $a_{ij} \in L^\infty(Q_T)$ for all $1 \leq i, j \leq 3$, $B = (b_{ij})_{1 \leq i \leq 3, 1 \leq j \leq k}$ is a control operator and v is a searched control belonging to $(L^2(\omega \times (0,T)))^k$ with $k \in \{1,2\}$. We will consider the cases where B equals one of the two matrices

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us introduce the following notation.

Notation 1.

- Set $q_T = \omega \times (0,T)$ and $q_T^0 = \omega^0 \times (0,T)$ for every open subset $\omega^0 \subset \omega$ of Ω.
- $W^{2,1}_\infty(q_T) = W^{1,\infty}(q_T) \cap L^\infty(0,T; W^{2,\infty}(\omega))$, where $W^{p,\infty}(O) = \{f \,/\, D^r f \in L^\infty(O),\ \forall\, 0 \leq r \leq p\}$.
- For a given positive measurable function ρ defined on a subset O of Q_T, let us denote by $L^2(O,\rho)$ the space of functions f such that $f\rho \in L^2(O)$, endowed with the norm $\|f\|_{L^2(O,\rho)} = \|f\rho\|_{L^2(O)}$.
- For any dense subspace U of a Hilbert space H, we define

  $$W(0,T; U, U') = \left\{ \psi \in L^2(0,T;U) \,/\, \partial_t \psi \in L^2\big(0,T;U'\big) \right\},$$

  where U' denotes the dual of U with respect to the pivot space H. The norm of an element $\psi \in W(0,T;U,U')$ is defined by

  $$\|\psi\|_{W(0,T;U,U')} = \left(\|\psi\|^2_{L^2(0,T;U)} + \|\partial_t \psi\|^2_{L^2(0,T;U')} \right)^{1/2}.$$

For simplicity of notation we write, for every open subset O of Ω,

$$W^1_O(0,T) = W\Big(0,T; H^1_0(O), H^{-1}(O)\Big),$$
$$W^2_O(0,T) = W\Big(0,T; H^2(O)\cap H^1_0(O), L^2(O)\Big),$$

where the pivot spaces are $H = L^2(O)$ for $W^1_O(0,T)$ and $H = H^1_0(O)$ for $W^2_O(0,T)$, respectively.

- We use the symbol $\|A\|_\infty$ to denote the norm of A: $\|A\|_\infty = \sum_{i,j=1}^3 \|a_{ij}\|_\infty$, with $\|a_{ij}\|_\infty = \|a_{ij}\|_{L^\infty(Q_T)}$.

We recall below the well–known result of existence and uniqueness for the solutions to the general system

$$\begin{cases} \partial_t y = \Delta y + Ay + f & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \tag{2}$$

Proposition 1. 1. *If $y^0 \in (L^2(\Omega))^3$ and $f \in L^2\Big(0,T;(H^{-1}(\Omega))^3\Big)$, then (2) admits a unique solution $y \in (W^1_\Omega(0,T))^3$ in the distributional sense*

$$\langle \partial_t y(\cdot), z\rangle_{H^{-1},H^1_0} + \langle \nabla y(\cdot), \nabla z\rangle_{L^2} - \langle A(\cdot)y(\cdot), z\rangle_{L^2} = \langle f(\cdot), z\rangle_{H^{-1},H^1_0},$$

for every $z \in (H^1_0(\Omega))^3$. Moreover, y satisfies the estimate

$$\|y\|^2_{(W^1_\Omega(0,T))^3} \le e^{CM_T}\Big(\|y^0\|^2_{(L^2(\Omega))^3} + \|f\|^2_{L^2(0,T;(H^{-1}(\Omega))^3)}\Big),$$

where C is a positive constant which depends neither on y^0, neither on f, nor on y, and M_T is given by

$$M_T = 1 + \|A\|_\infty + T(1 + \|A\|^2_\infty).$$

2. *If $y^0 \in (H^1_0(\Omega))^3$ and $f \in (L^2(Q_T))^3$, then (2) has a classical solution $y \in (W^2_\Omega(0,T))^3$ with the estimate*

$$\|y\|^2_{(W^2_\Omega(0,T))^3} \le e^{CM_T}\Big(\|y^0\|^2_{(H^1_0(\Omega))^3} + \|f\|^2_{(L^2(Q_T))^3}\Big), \tag{3}$$

with M_T and C as above.

For the proof of Proposition 1 we refer to the arguments used in the book of Ladyženskaja, Solonnikov and Ural'ceva ([19, ch. III]). This result can also be obtained by the Galerkin method (see, for instance [10, chap. viii]).

1.1. Main results

The aim of this paper is to prove the following controllability result.

Theorem 2 (Controllability by one force, $B = B_1$)**.** *Let us assume that a_{13}, $a_{23} \in W^{2,1}_\infty(q_T)$ and that there exist two positive constants α and c such that*

$$|a_{23}| \ge \alpha \quad in\ q_T \tag{4}$$

and

$$\frac{\det K}{a_{23}^2} + \partial_t\left(\frac{a_{13}}{a_{23}}\right) \ge c \quad in\ q_T \quad or \quad \frac{\det K}{a_{23}^2} + \partial_t\left(\frac{a_{13}}{a_{23}}\right) \le -c \quad in\ q_T, \tag{5}$$

where $K = (B_1, AB_1, A^2B_1)$ is the Kalman matrix corresponding to the matrix A and the control matrix B_1. Then for every $y^0 \in (L^2(\Omega))^3$, there exists at least one function $v \in L^2(q_T)$ such that the solution y to

$$\begin{cases} \partial_t y = \Delta y + Ay + B_1 v 1_\omega & \textit{in } Q_T, \\ y = 0 & \textit{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \textit{in } \Omega, \end{cases} \tag{6}$$

satisfies

$$y(\cdot, T) = 0 \ \ \textit{in } \Omega.$$

Remark 1.

- We clearly see that if both coefficients a_{13} and a_{23} of the coupling matrix A are identically equal to zero in Q_T, then the first two equations in system (6) are decoupled from the third one, so that we can not expect controllability in this case. A necessary condition for the controllability of (6) is that either a_{13}, or a_{23} does not vanish at a place, namely either $\mathrm{supp}(a_{13}) \neq \emptyset$, or $\mathrm{supp}(a_{23}) \neq \emptyset$. The method presented in this paper supposes that $\mathrm{supp}(a_{23}) = q_T$. This extends to the case $\mathrm{supp}(a_{23}) \cap q_T \neq \emptyset$. Indeed, applying Theorem 2 on a part $\widetilde{q}_T = \widetilde{\omega} \times]0, T[$ contained in $\mathrm{supp}(a_{23}) \cap q_T$, we obtain the controllability of system (6) in $\widetilde{\omega}$ and consequently in $\omega \supset \widetilde{\omega}$. However, the case where $\mathrm{supp}(a_{23})$ and the control domain q_T are disjoint is a difficult open problem and there are only few results concerning this geometrical configuration. We refer, in particular, to [1] for an example of a system of two coupled parabolic equations —with coupling terms depending on the space variable $x \in \Omega$— controlled by one force acting on a region that can be disjoint from the coupling region.

- In the statement of Theorem 2, the coefficients a_{13} and a_{23} play symmetric roles. More precisely, the conclusion of Theorem 2 is still true if we replace a_{23} by a_{13} in condition (4), and if condition (5) is turned into

$$\frac{\det K}{a_{13}^2} - \partial_t\left(\frac{a_{23}}{a_{13}}\right) \geq c \ \text{ in } q_T \quad \text{or} \quad \frac{\det K}{a_{13}^2} - \partial_t\left(\frac{a_{23}}{a_{13}}\right) \leq -c \ \text{ in } q_T.$$

  Note that (5) and the above condition are "equivalent" since we have formally

$$\frac{\det K}{a_{13}^2} - \partial_t\left(\frac{a_{23}}{a_{13}}\right) = \left(\frac{a_{23}}{a_{13}}\right)^2\left(\frac{\det K}{a_{23}^2} + \partial_t\left(\frac{a_{13}}{a_{23}}\right)\right),$$

  and since either a_{13} or a_{23} satisfies (4).

The proof of Theorem 2 is based on a suitable decomposition of the solution y to (6) as

$$y = (1 - \theta)\widehat{y} + \eta\theta Y + F, \tag{7}$$

where

- Y is the solution without control,
- $\widehat{y}$ is a well–chosen controlled solution of (1) associated with three control forces i.e. for $B = I_3$ (see Theorem 14),
- θ and η are two truncation functions satisfying (12), and

- F is to be determined such that y is a controlled solution of (6). Such a F is obtained by the resolution of a null controllability problem for a 2×2 non homogeneous system controlled by only one force acting on the whole domain.

This decomposition was inspired by [17] in the case of a parabolic system of two equations. In fact, in [17], the authors use a similar decomposition to construct, from two controls, a regularized control acting on only one equation.

Remark 2. Theorem 2 generalizes the Kalman rank condition given in [4] for matrices A depending only on time to the case of matrices A depending on space and time. Precisely, in [4], the authors prove that system (1) with $A \in C^2([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in C([0,T];\mathcal{L}(\mathbb{R}^k,\mathbb{R}^n))$ ($n \geq 2$, $k \geq 1$), is null controllable if and only if there exists a dense subset E of $(0,T)$ such that

$$\operatorname{rank} \widetilde{K}_{k,n}(t) = n, \quad \forall t \in E, \tag{8}$$

where $\widetilde{K}_{k,n}(t) = (b_0(t),\cdots,b_{n-1}(t))$, and the sequence $(b_i)_{0\leq i\leq n-1}$ is defined by $b_0(t) = B(t)$ and $b_i(t) = A(t)b_{i-1}(t) - \frac{d}{dt}b_{i-1}(t)$. For $n = 3$, $k = 1$ and $B = B_1$, condition (8) writes

$$\det \widetilde{K}(t) \neq 0, \quad \forall t \in E,$$

where $\widetilde{K}(t) = (B_1, A(t)B_1, A(t)^2B_1 - \frac{d}{dt}A(t)B_1)$ and $\det \widetilde{K}(t) = \det K(t) - a_{13}\frac{d}{dt}a_{23} + a_{23}\frac{d}{dt}a_{13}$, with K defined in Theorem 2. The result of [4] mentioned above ensures the controllability of (6) under this condition. In Theorem 2, the coefficient a_{23} is bounded from both sides, so that (5) is equivalent to

$$\det \widetilde{K}(x,t) \geq \widetilde{c}, \quad \forall (x,t) \in q_T \quad \text{or} \quad \det \widetilde{K}(x,t) \leq -\widetilde{c}, \quad \forall (x,t) \in q_T.$$

However, in the present paper we do not investigate the equivalence between (5) and the controllability of (6), and we only deal with the case of the control matrix with constant coefficients B_1.

We also apply the decomposition (7) to prove the controllability of (1) by two forces ($B = B_2$), which can be stated as follows.

Theorem 3 (Controllability by two forces, $B = B_2$)**.** *Let us assume that a_{12}, $a_{13} \in W^{2,1}_\infty(q_T)$ and that there exists a positive constant c such that*

$$|a_{12}|^2 + |a_{13}|^2 \geq c \ \text{ in } q_T. \tag{9}$$

Then for every $y^0 \in (L^2(\Omega))^3$ there exists a vector $v = (v_1, v_2)^T \in (L^2(q_T))^2$ such that the solution y to

$$\begin{cases} \partial_t y = \Delta y + Ay + B_2 v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \tag{10}$$

satisfies

$$y(\cdot,T) = 0 \ \text{ in } \Omega.$$

1.2. Presentation of the method

In this paragraph, we use hypothesis (4) to transform the controllability problem for system (6) into a controllability problem for a 2×2 non homogeneous system. Let ω^0 be a nonempty open subset of Ω contained in ω. Let $(\widehat{y}, \widehat{v})$ be a solution to

$$\begin{cases} \partial_t \widehat{y} = \Delta \widehat{y} + A\widehat{y} + \widehat{v}\, 1_{\omega^0} & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot, 0) = y^0, \quad \widehat{y}(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{11}$$

$(\widehat{y}, \widehat{v})$ will be suitably chosen in Section 5. Let us consider $p \in \mathbb{N}$ and two truncation functions $\eta \in C^\infty([0, T])$ and $\theta \in C_c^2(\overline{\Omega})$ satisfying

$$\begin{array}{ll} 0 \leq \eta \leq 1, & \\ \eta = 1 \ \text{ in } [0, T/4], & \operatorname{supp}(\theta) \subset \omega, \\ \eta = 0 \ \text{ in } [3T/4, T], & 0 \leq \theta \leq 1, \\ \eta(t) \leq C_\eta (T - t)^{p/2}, \ t \in [0, T], & \theta = 1 \text{ on } \overline{\omega^0}. \end{array} \tag{12}$$

Let Y be the solution to the system without control which is

$$\begin{cases} \partial_t Y = \Delta Y + AY & \text{in } Q_T, \\ Y = 0 & \text{on } \Sigma_T, \\ Y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

We search a solution y to (6) such that $y(\cdot, T) = 0$ in the form

$$y = (1 - \theta)\widehat{y} + \eta\theta Y + F, \tag{13}$$

where $F(x, t)$ is to be determined. Since the researched control forces v are acting on ω, the function F can be chosen with support in $\omega \times [0, T]$. For fixed v, the function y defined by (13) is a solution to (6) satisfying $y(\cdot, T) = 0$ if and only if F satisfies

$$\begin{cases} F(\cdot, 0) = F(\cdot, T) = 0 & \text{in } \omega, \\ \partial_t F - \Delta F - AF - h = (0, 0, v)^T & \text{in } q_T, \end{cases} \tag{14}$$

where

$$h = -2\nabla\theta \cdot \nabla\widehat{y} - (\Delta\theta)\widehat{y} - (\eta'\theta - \eta\Delta\theta)Y + 2\eta\nabla\theta \cdot \nabla Y. \tag{15}$$

Writing $F = (F_1, F_2, F_3)^T$, $F_0 = (F_1, F_2)^T$, $A_0 = (a_{ij})_{1 \leq i,j \leq 2}$, and $B_0 = (a_{13}, a_{23})^T$, we see that there exists a function v and a function F satisfying (14), with support in $\omega \times [0, T]$, if and only if there exists a function F_3, with

$$F_3(\cdot, 0) = F_3(\cdot, T) = 0 \ \text{ in } \omega \tag{16}$$

such that the solution F_0 to

$$\begin{cases} \partial_t F_0 = \Delta F_0 + A_0 F_0 + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + B_0 F_3 & \text{in } q_T, \\ F_0 = 0 & \text{on } \sigma_T := \partial\omega \times (0, T), \\ F_0(\cdot, 0) = 0 & \text{in } \omega, \end{cases} \tag{17}$$

satisfies

$$F_0(\cdot, T) = 0 \ \text{ in } \omega. \tag{18}$$

In this case, the corresponding control v for system (1) is given by

$$v = \partial_t F_3 - \Delta F_3 - a_{31}F_1 - a_{32}F_2 - a_{33}F_3 - h_3. \tag{19}$$

Remark 3. The decomposition (13) enables us to state that controlling system (6) with one force consists in controlling the two equations of (17) with the same control force F_3. In the 2×2 system (17) the control operator is $B_0 = (a_{13}, a_{23})^T$ and then $B_0^*\phi = a_{13}\phi_1 + a_{23}\phi_2$ for $\phi = (\phi_1, \phi_2)^T$. To the author knowledge, the existing techniques used to prove observability inequalities for parabolic systems do not apply for a control operator of this form (even if the control acts on the whole domain).

In view of Remark 3, we apply a change of variables to transform (17) into a 2×2 system where the control force acts only on one equation. Indeed, hypothesis (4) allows us to consider the new variables

$$z = (z_1, z_2)^T,\ z_1 = F_1 - \frac{a_{13}}{a_{23}} F_2,\ z_2 = F_2,\ u = F_3, \tag{20}$$

and to rewrite system (17) as

$$\begin{cases} \partial_t z = \Delta z + \widetilde{A} z + g + \widetilde{B} u & \text{in } q_T, \\ z = 0 & \text{on } \sigma_T, \\ z(\cdot, 0) = 0 & \text{in } \omega, \end{cases} \tag{21}$$

where $\widetilde{B} = (0, a_{23})^T$, $\widetilde{A} = (\widetilde{a}_{ij})_{1\le i,j\le 2}$ with

$$\begin{aligned} \widetilde{a}_{11} &= \tfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{23}}, \\ \widetilde{a}_{12} &= -\tfrac{\det K}{a_{23}^2} + (\Delta - \partial_t)\left(\tfrac{a_{13}}{a_{23}}\right) + 2\nabla\left(\tfrac{a_{13}}{a_{23}}\right).\nabla, \\ \widetilde{a}_{21} &= a_{21}, \\ \widetilde{a}_{22} &= \tfrac{a_{21}a_{13} + a_{22}a_{23}}{a_{23}}, \end{aligned} \tag{22}$$

and

$$g = (g_1, g_2)^T,\ g_1 = h_1 - \frac{a_{13}}{a_{23}} h_2,\ g_2 = h_2. \tag{23}$$

Remark 4. As explained before, z_1 is chosen by the mean of the change of variables (20) so as to have one control force only. Note that —unlike the localized control force v in (6)— this control force u acts on the *whole* domain ω where the solution z to (21) evolves. This will be the key point of the proofs in the following section.

To shorten notation in the sequel, we set

$$\|\widetilde{A}\|_\infty = \|\widetilde{a}_{11}\|_\infty + \|\widetilde{a}_{21}\|_\infty + \|\widetilde{a}_{22}\|_\infty + a_\infty, \tag{24}$$

where

$$a_\infty = \left\|\frac{\det K}{a_{23}^2}\right\|_\infty + \left\|(\Delta + \partial_t)\left(\frac{a_{13}}{a_{23}}\right)\right\|_\infty + \left\|\nabla\left(\frac{a_{13}}{a_{23}}\right)\right\|_\infty.$$

K is the 3×3 Kalman matrix given in Theorem 2 whose determinant is

$$\det K = a_{13}(a_{22}a_{23} + a_{21}a_{13}) - a_{23}(a_{11}a_{13} + a_{12}a_{23}).$$

Note that $\widetilde{a}_{12}$ is not a L^∞ coefficient, but a first order operator in space. We have simplified the adjoint of the control operator of the 2×2 system (17). With the new operator $\widetilde{B}^*\phi = a_{23}\phi_2$ for $\phi = (\phi_1, \phi_2)^T$, we are now able to prove the controllability of (21). Rewriting conditions (16), (18) and (19) with the change of variables (20), we have:

Lemma 4. *If there exists a function $u \in W^2_\omega(0,T)$ satisfying*

$$u(\cdot,0) = u(\cdot,T) = 0, \tag{25}$$

such that the solution z to (21) satisfies $z(\cdot,T) = 0$, then there exists a function $v \in L^2(q_T)$ such that the solution y to (6) satisfies $y(\cdot,T) = 0$.
Moreover, v can be obtained as

$$v = \partial_t u - \Delta u - a_{31}z_1 - \left(a_{32} + \frac{a_{13}a_{31}}{a_{23}}\right)z_2 - a_{33}u - h_3, \tag{26}$$

where h is defined by (15).

Remark 5. For the controllability of system (21) we need that the source term g belongs to an appropriate space. By the definition of g (see (15) and (23)) this implies some constraints on the solution $(\widehat{y}, \widehat{v})$ to (11).

The paper is organized as follows. Sections 2 and 3 concern the proof of Theorem 2. Section 2 is devoted to the proof of an observability inequality for the backward system associated with (21). In Section 3 we follow the method provided by Lemma 4 to prove Theorem 2: first, we prove that, under some hypotheses on the solution $(\widehat{y}, \widehat{v})$ to system (11), the reduced system (21) is null controllable with control forces $u \in L^2(q_T)$, and then, we choose a control u belonging to $W^2_\omega(0,T)$ (so that v defined by (26) belongs to $L^2(q_T)$) and satisfying (25). In section 4, we apply the decomposition (13) to investigate the null controllability of (1) by two forces and to prove Theorem 3. Section 5 is devoted to the proof of the existence of a solution $(\widehat{y}, \widehat{v})$ to (11) satisfying the hypotheses required in sections 3 and 4. Some remarks and further results are discussed in section 6.

2. An observability inequality for the non homogeneous backward system associated with (21)

As it is usual, we state the controllability of system (21) as a consequence of the observability of its adjoint system. Let us consider the following non homogeneous backward system associated with (21):

$$\begin{cases} -\partial_t\phi_1 = \Delta\phi_1 + \widetilde{a}_{11}\phi_1 + \widetilde{a}_{21}\phi_2 + f_1 & \text{in } q_T, \\ -\partial_t\phi_2 = \Delta\phi_2 + \widetilde{a}^*_{12}\phi_1 + \widetilde{a}_{22}\phi_2 + f_2 & \text{in } q_T, \\ \phi_1 = \phi_2 = 0 & \text{on } \sigma_T, \\ \phi_1(\cdot,T) = \phi^0_1,\ \phi_2(\cdot,T) = \phi^0_2 & \text{in } \omega, \end{cases} \tag{27}$$

where $\widetilde{a}^*_{12}$ is the formal adjoint of the operator $\widetilde{a}_{12}$. This section is devoted to the proof of the following observability result for the solutions to (27).

Proposition 5. *Under hypotheses of Theorem 2, for every $p \in \mathbb{N}$, $p \geq 3$, there exists a positive constant $C_0 = C_0(R_0, \|\widetilde{A}\|_\infty, c, \alpha, p, T)$ (where $\|\widetilde{A}\|_\infty$ is defined in (24) and R_0 in (35)) such that for every $\phi^0 = (\phi^0_1, \phi^0_2)^T \in (L^2(\omega))^2$ and every $f = (f_1, f_2)^T \in (L^2(q_T))^2$, the solution $\phi = (\phi_1, \phi_2)^T$ to (27) satisfies*

$$\|\phi(0)\|^2_{(L^2(\omega))^2} + \int_{q_T}(T-t)^p|\phi|^2$$
$$\leq C_0\left(\int_{q_T} t^{p-3}(T-t)^{p-3}(\widetilde{B}^*\phi)^2 + \int_{q_T}(T-t)^p|f|^2\right).$$

The proof of Proposition 5 is decomposed in two steps (see page 11). In the first step we establish a weak observability inequality (38) with an observation on the two components of the solution to system (27). In the second step we remove the first component ϕ_1, as it is estimated by the second one ϕ_2. This second step is the key point of the proof of Proposition 5 and it can be formulated as the following lemma.

Lemma 6. *Under hypotheses of Theorem 2, for every $p \in \mathbb{N}$, $p \geq 3$, there exists a positive constant $C_1 = C_1(\|\widetilde{A}\|_\infty, c, p, T)$ such that, for every $\phi^0 \in (L^2(\omega))^2$ and every $f \in (L^2(q_T))^2$, the solution $\phi = (\phi_1, \phi_2)^T$ to (27) satisfies*

$$\int_{q_T} t^p(T-t)^p {\phi_1}^2 \leq C_1 \left(\int_{q_T} t^{p-3}(T-t)^{p-3} {\phi_2}^2 + \int_{q_T} t^p(T-t)^p |f|^2 \right). \tag{28}$$

Proof. Fix $\delta \in (0,1)$. To simplify notations, let us consider the function φ defined by

$$\varphi(t) = t(T-t)\,,\ t \in [0,T].$$

All along the proof $K = K(\|\widetilde{A}\|_\infty, p, T)$, $K_\epsilon = K_\epsilon(\|\widetilde{A}\|_\infty, p, T, \epsilon)$, and the values of those constants may change from one line to another. Multiplying the second equation of (27) by $\varphi^p\phi_1$ and integrating by parts over q_T, we obtain

$$\begin{aligned} \int_{q_T} \varphi^p(\widetilde{a}_{12}^*\phi_1)\phi_1 \quad &= \quad \int_{q_T}(p\varphi'\varphi^{p-1} - \varphi^p(\widetilde{a}_{11} + \widetilde{a}_{22}))\phi_1\phi_2 + 2\int_{q_T} \varphi^p \nabla\phi_1.\nabla\phi_2 \\ &\quad - \int_{q_T} \varphi^p \widetilde{a}_{21} {\phi_2}^2 - \int_{q_T} \varphi^p(f_2\phi_1 + f_1\phi_2). \end{aligned} \tag{29}$$

Besides, by the definition of $\widetilde{a}_{12}$ and simple computations, we can prove that

$$\begin{aligned} \int_{q_T} \varphi^p(\widetilde{a}_{12}^*\phi_1)\phi_1 \quad &= \quad \int_{q_T} \varphi^p \phi_1(\widetilde{a}_{12}\phi_1) \\ &= \quad -\int_0^T \varphi^p(t) \left[\int_\omega \left(\frac{\det K}{a_{23}^2} + \partial_t \left(\frac{a_{13}}{a_{23}} \right) \right) {\phi_1}^2 \, \mathrm{d}x \right] \mathrm{d}t. \end{aligned} \tag{30}$$

Recalling hypothesis (5), we deduce from (30) that

$$c \int_{q_T} \varphi^p {\phi_1}^2 \leq \left| \int_{q_T} \varphi^p(\widetilde{a}_{12}^*\phi_1)\phi_1 \right|. \tag{31}$$

Now, from (29) and (31) and using $\varphi'(t) = T - 2t \in [-T, T]$ and the inequality $ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$ for every $a, b \in \mathbb{R}^+$ and $\epsilon > 0$, we deduce that

$$\begin{aligned} c\int_{q_T} \varphi^p {\phi_1}^2 \quad &\leq \quad \epsilon K \left(\int_{q_T} \varphi^p {\phi_1}^2 + \int_{q_T} \varphi^{p+1} |\nabla\phi_1|^2 \right) \\ &\quad + K_\epsilon \left(\int_{q_T} \varphi^{p-1} |\nabla\phi_2|^2 + \int_{q_T} \varphi^{p-3} {\phi_2}^2 + \int_{q_T} \varphi^p |f|^2 \right). \end{aligned}$$

Choosing ϵ such that $c - \epsilon K \geq \delta c$, we obtain

$$\begin{aligned} \delta c \int_{q_T} \varphi^p {\phi_1}^2 \quad &\leq \quad \epsilon K \int_{q_T} \varphi^{p+1} |\nabla\phi_1|^2 \\ &\quad + K_\epsilon \left(\int_{q_T} \varphi^{p-1} |\nabla\phi_2|^2 + \int_{q_T} \varphi^{p-3} {\phi_2}^2 + \int_{q_T} \varphi^p |f|^2 \right). \end{aligned} \tag{32}$$

To "control" the variable ϕ_1 by the variable ϕ_2 and the source term f, we have to eliminate the terms in $\nabla\phi_1$ and $\nabla\phi_2$ in the right–hand side of (32). We begin by getting rid of the term in $\nabla\phi_2$. To this end, we multiply the second equation of (27) by $\varphi^{p-1}\phi_2$ and we integrate by parts over q_T:

$$\begin{aligned}\int_{q_T}\varphi^{p-1}|\nabla\phi_2|^2 &= -\int_{q_T}\varphi^{p-1}\phi_2\Delta\phi_2\\ &= \int_{q_T}\varphi^{p-1}\phi_2\left(\partial_t\phi_2+\widetilde{a}^*_{12}\phi_1+\widetilde{a}_{22}\phi_2+f_2\right)\\ &= \int_{q_T}\varphi(\varphi\widetilde{a}_{22}-\frac{p-1}{2}\varphi')\varphi^{p-3}{\phi_2}^2+\int_{q_T}\varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1\\ &\quad+\int_{q_T}\varphi^{p-1}f_2\phi_2. \end{aligned}\tag{33}$$

By the definition of $\widetilde{a}_{12}$ (see (22)) and integration by parts on ω, we can prove that

$$\begin{aligned}\int_{q_T}\varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1 &= -\int_{q_T}\varphi^{p-1}\left(\frac{\det K}{a_{23}^2}+(\Delta+\partial_t)\left(\frac{a_{13}}{a_{23}}\right)\right)\phi_1\phi_2\\ &\quad-2\int_{q_T}\varphi^{p-1}\phi_2\nabla\phi_1.\nabla\left(\frac{a_{13}}{a_{23}}\right).\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{q_T}\varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1\right| &\leq \epsilon^2 a_\infty\left(\int_{q_T}\varphi^p{\phi_1}^2+\int_{q_T}\varphi^{p+1}|\nabla\phi_1|^2\right)\\ &\quad+\frac{a_\infty}{\epsilon^2}\left(\frac{T^2}{4}+1\right)\int_{q_T}\varphi^{p-3}{\phi_2}^2.\end{aligned}$$

Combining this inequality with (33) and (32), we obtain

$$\begin{aligned}\delta c\int_{q_T}\varphi^p{\phi_1}^2 &\leq \epsilon K\left(\int_{q_T}\varphi^p{\phi_1}^2+\int_{q_T}\varphi^{p+1}|\nabla\phi_1|^2\right)\\ &\quad+K_\epsilon\left(\int_{q_T}\varphi^{p-3}{\phi_2}^2+\int_{q_T}\varphi^p|f|^2\right).\end{aligned}$$

Now we choose ϵ such that $\delta c-\epsilon K\geq\delta^2 c$, and we deduce from this inequality that

$$\delta^2 c\int_{q_T}\varphi^p{\phi_1}^2\leq\epsilon K\int_{q_T}\varphi^{p+1}|\nabla\phi_1|^2+K_\epsilon\left(\int_{q_T}\varphi^{p-3}{\phi_2}^2+\int_{q_T}\varphi^p|f|^2\right).\tag{34}$$

To eliminate $\nabla\phi_1$ in the right–hand side of (34), we multiply the first equation of (27) by $\varphi^{p+1}\phi_1$. After integrations by parts in q_T, we obtain

$$\int_{q_T}\varphi^{p+1}|\nabla\phi_1|^2=\int_{q_T}(\varphi\widetilde{a}_{11}-\frac{p+1}{2}\varphi')\varphi^p{\phi_1}^2+\int_{q_T}\varphi^{p+1}\widetilde{a}_{21}\phi_1\phi_2+\int_{q_T}\varphi^{p+1}f_1\phi_1.$$

Hence

$$\int_{q_T}\varphi^{p+1}|\nabla\phi_1|^2\leq K\left(\int_{q_T}\varphi^p{\phi_1}^2+\int_{q_T}\varphi^{p-3}{\phi_2}^2+\int_{q_T}\varphi^p|f|^2\right).$$

Combining this inequality with (34) gives

$$\delta^2 c \int_{q_T} \varphi^p {\phi_1}^2 \leq \epsilon K \int_{q_T} \varphi^p {\phi_1}^2 + K_\epsilon \left(\int_{q_T} \varphi^{p-3} {\phi_2}^2 + \int_{q_T} \varphi^p |f|^2 \right).$$

For ϵ satisfying $\delta^2 c - \epsilon K \geq \delta^3 c$, we deduce from this inequality that

$$\int_{q_T} \varphi^p {\phi_1}^2 \leq \frac{K_\epsilon}{\delta^2 c} \left(\int_{q_T} \varphi^{p-3} {\phi_2}^2 + \int_{q_T} \varphi^p |f|^2 \right).$$

This proves the lemma. □

We also recall the energy inequality satisfied by the solutions to (27).

Lemma 7. *For every $\phi^0 \in (L^2(\omega))^2$ and every $f \in (L^2(q_T))^2$, the solution ϕ to (27) satisfies for all $(t_1, t_2) \in [0, T]^2$ such that $t_1 \leq t_2$*

$$\|\phi(t_1)\|^2_{(L^2(\omega))^2} \leq e^{4R_0(t_2-t_1)} \|\phi(t_2)\|^2_{(L^2(\omega))^2} + e^{4R_0(T-t_1)} \int_{t_1}^{t_2} \|f(t)\|_{(L^2(\omega))^2} dt$$

where

$$\begin{array}{rcl} R_0 & = & \|a_{11}\|_\infty + \|a_{12}\|_\infty + \|a_{21}\|_\infty + \|a_{22}\|_\infty \\ & & + \dfrac{1}{\alpha} \left(\|a_{11}\|^2_\infty + \|a_{13}\|^2_\infty + \|a_{21}\|^2_\infty + \|a_{22}\|^2_\infty \right) \\ & & + \dfrac{\|a_{13}\|^4_\infty}{\alpha^3} + \left\| (\Delta + \partial_t) \left(\dfrac{a_{13}}{a_{23}} \right) \right\|_\infty + \left\| \nabla \left(\dfrac{a_{13}}{a_{23}} \right) \right\|^2_\infty . \end{array} \quad (35)$$

We do not give the proof of Lemma 7 which is standard.

Proof of Proposition 5. Applying Lemma 7, firstly with $t_1 = t \in [0, T/4]$, $t_2 = T/4$, secondly with $t_1 = T/4$, $t_2 = t \in [T/4, T/2]$, we obtain

$$\begin{array}{rcl} \displaystyle\int_0^{T/4} \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t & \leq & \displaystyle e^{2R_0 T} \int_{T/4}^{T/2} \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t \\ & & \displaystyle + \frac{T}{4} e^{4R_0 T} \int_0^{T/2} \|f(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t. \end{array} \quad (36)$$

Since

$$\int_{q_T} (T-t)^p |\phi|^2 \leq T^p \int_0^{T/4} \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t + \frac{2^{2p}}{T^p} \int_{T/4}^T t^p (T-t)^p \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t,$$

we deduce from (36) that

$$\begin{array}{rcl} \displaystyle\int_{q_T} (T-t)^p |\phi|^2 & \leq & \displaystyle T^p e^{2R_0 T} \int_{T/4}^{T/2} \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t \\ & & \displaystyle + \frac{T^{p+1}}{4} e^{4R_0 T} \int_0^{T/2} \|f(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t \\ & & \displaystyle + \frac{2^{2p}}{T^p} \int_{T/4}^T t^p (T-t)^p \|\phi(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t. \end{array} \quad (37)$$

Using the fact that the functions $t^p(T-t)^p$ and $(T-t)^p$ are lower bounded by a positive constant for $t \in [T/4, T/2]$ and $t \in [0, T/2]$ respectively, we obtain from (37)

$$\int_{q_T} (T-t)^p |\phi|^2 \leq C \left(\int_{q_T} t^p (T-t)^p |\phi|^2 + \int_0^{T/2} (T-t)^p \|f(t)\|^2_{(L^2(\omega))^2} \mathrm{d}t \right), \tag{38}$$

where $C = C(R_0, p, T)$. Combining this inequality with (28), we finally obtain

$$\int_{q_T} (T-t)^p |\phi|^2 \leq C \left(\int_{q_T} t^{p-3} (T-t)^{p-3} {\phi_2}^2 + \int_{q_T} (T-t)^p |f|^2 \right),$$

with $C = C(R_0, \|\widetilde{A}\|_\infty, c, p, T)$. This ends the proof of Proposition 5, recalling that $\widetilde{B}^* \phi = a_{23} \phi_2$ with a_{23} satisfying (4). □

Remark 6. In order to deal with the controllability of non linear systems, it is crucial to know the explicit dependence on the parameters T, α and $\|a_{ij}\|_\infty$ $(i, j = 1, 2, 3)$ of the observability constant C_0 in Proposition 5. Analyzing in details the proofs of Lemma 6 and Proposition 5, we can obtain

$$C_0 = \exp(\kappa N_T),$$

where κ is a positive constant which depends only on p and c, and N_T is given by

$$N_T = 1 + M_0 + \left\| \nabla \left(\frac{a_{13}}{a_{23}} \right) \right\|_\infty + \frac{1}{\alpha^2} + T + M_0 T + \frac{1}{T}, \tag{39}$$

where $M_0 = R_0 - \left\| \nabla \left(\frac{a_{13}}{a_{23}} \right) \right\|^2_\infty$, with R_0 as in (35).

3. Controllability of (21) and proof of Theorem 2

For the moment, let us assume that the source term g in (21) satisfies

$$g \in (L^2(q_T, \rho))^2, \tag{40}$$

where

$$\rho(t) = (T-t)^{-p/2}, \quad \forall t \in (0, T). \tag{41}$$

This will be proved in details in section 5.

The aim of the present section is to prove Theorem 2. According to Lemma 4, Theorem 2 will be proved if we construct a regular control u for system (21) which ensures that the control v for system (6) defined by (26) belongs to $L^2(q_T)$. This is the subject of the following result.

Theorem 8. *Let assumptions of Theorem 2 hold. Then there exists at least one function $u \in W^2_\omega(0, T)$ satisfying (25) and such that the solution z to (21) satisfies $z(\cdot, T) = 0$.*

Before proving this theorem, let us recall the well–known result of existence of regular solutions to the following parabolic system

$$\begin{cases} \partial_t z = \Delta z + \widetilde{A} z + f & \text{in } q_T, \\ z = 0 & \text{on } \sigma_T, \\ z(\cdot, 0) = 0 & \text{in } \omega, \end{cases} \tag{42}$$

whose proof can be obtained using the same method as in [19].

Proposition 9. *For every $f \in (L^2(q_T))^2$, system (42) admits a unique solution $z \in (W^2_\omega(0,T))^2$, which satisfies*

$$\|z\|^2_{(W^2_\omega(0,T))^2} \le e^{CR_T}\|f\|^2_{(L^2(q_T))^2},$$

where $C = C(\omega) > 0$ and $R_T = (1+T)(1+R_0)$ (with R_0 given by (35)).

The proof of Theorem 8 follows the idea developed by Fursikov and Imanuvilov in [15] to prove the existence of solutions to parabolic equations which exponentially decrease at $t = T$. This method will also be applied in section 4 for the proof of Lemma 13 and in section 5 for the proof of Theorem 14.

Proof of Theorem 8. Let us introduce the following notation

$$Lz = \partial_t z - \Delta z - \widetilde{A}z, \qquad L^*\phi = -\partial_t\phi - \Delta\phi - \widetilde{A}^*\phi.$$

For $p \in \mathbb{N}$ with $p \ge 8$, we consider the weight functions defined for $t \in (0,T)$ and for $k \in \mathbb{N}^*$ by,

$$\widetilde{\rho}_0(t) = (t(T-t))^{-(p-3)/2}, \qquad \rho_k(t) = (T + \frac{1}{k} - t)^{-p/2}.$$

Then the observability inequality of Proposition 5 may be written as

$$\|\phi(\cdot,0)\|^2_{(L^2(\omega))^2} + \int_{q_T} \rho^{-2}|\phi|^2 \le C_0\left(\int_{q_T} \widetilde{\rho}_0^{-2}(\widetilde{B}^*\phi)^2 + \int_{q_T} \rho^{-2}|L^*\phi|^2\right), \tag{43}$$

where $C_0 = \exp(\kappa N_T)$ with N_T given by (39). All along this proof, C stands for a generic positive constant depending only on ω, p and on the parameter c occurring in hypothesis (5). Let us consider, for each $k \in \mathbb{N}^*$, the following minimization problem

$$\left\{\begin{array}{l} \text{minimize } \widetilde{\mathcal{J}}_k(u) = \frac{1}{2}\int_{q_T} \widetilde{\rho}_0^2 u^2 + \frac{1}{2}\int_{q_T} \rho_k^2 |z_u|^2, \\ u \in L^2(q_T, \widetilde{\rho}_0), \end{array}\right. \tag{44}$$

where z_u stands for the solution to (21) associated with $u \in L^2(q_T,\widetilde{\rho}_0)$. The functional $\widetilde{\mathcal{J}}_k$: $L^2(q_T,\widetilde{\rho}_0) \to \mathbb{R}^+$ is clearly differentiable, coercive and strictly convex on $L^2(q_T,\widetilde{\rho}_0)$. Therefore, following [20], we deduce that the minimization problem (44) admits a unique solution u_k which is characterized by the following optimality conditions

$$Lz_k = g + \widetilde{B}u_k \text{ in } q_T, \quad z_k = 0 \text{ on } \sigma_T, \quad z_k(\cdot,0) = 0, \tag{45}$$

$$L^*\phi_k = \rho_k^2 z_k \text{ in } q_T, \quad \phi_k = 0 \text{ on } \sigma_T, \quad \phi_k(\cdot,T) = 0, \tag{46}$$

$$u_k = -\widetilde{\rho}_0^{-2}\widetilde{B}^*\phi_k. \tag{47}$$

Using (47) and (46), we can write

$$\widetilde{\mathcal{J}}_k(u_k) = \frac{1}{2}\int_{q_T} \widetilde{\rho}_0^{-2}(\widetilde{B}^*\phi_k)^2 + \frac{1}{2}\int_{q_T} \rho_k^{-2}|\rho_k^2 z_k|^2.$$

Since $\rho_k \le \rho$, we can deduce from the observability inequality (43) that

$$\widetilde{\mathcal{J}}_k(u_k) \ge \frac{1}{2C_0}\left(\|\phi_k(\cdot,0)\|^2_{(L^2(\omega))^2} + \int_{q_T} \rho^{-2}|\phi_k|^2\right). \tag{48}$$

Besides, the optimality conditions also imply

$$\begin{aligned}\widetilde{\mathcal{J}}_k(u_k) &= -\frac{1}{2}\int_{q_T}(\widetilde{B}^*\phi_k)u_k - \frac{1}{2}\int_{q_T}(\partial_t\phi_k + \Delta\phi_k + \widetilde{A}^*\phi_k)z_k \\ &= -\frac{1}{2}\int_{q_T}\phi_k\widetilde{B}u_k + \frac{1}{2}\int_{q_T}(\partial_t z_k - \Delta z_k - \widetilde{A}^* z_k)\phi_k \\ &= \frac{1}{2}\int_{q_T}\phi_k g.\end{aligned}$$

By (48), it follows that

$$\widetilde{\mathcal{J}}_k(u_k) \leq \sqrt{\frac{C_0}{2}}\sqrt{\widetilde{\mathcal{J}}_k(u_k)}\,\|g\|_{(L^2(q_T,\rho))^2}.$$

Consequently,

$$\widetilde{\mathcal{J}}_k(u_k) \leq \frac{C_0}{2}\|g\|^2_{(L^2(q_T,\rho))^2}. \tag{49}$$

Besides, from (40) we have $\|g\|^2_{(L^2(q_T))^2} \leq T^p\|g\|^2_{(L^2(q_T,\rho))^2}$. By the definition of $\widetilde{\rho}_0$ and the estimate (49), we also have $\|\widetilde{B}u_k\|^2_{L^2(q_T)} \leq \|a_{23}\|^2_\infty \frac{T^{2p-6}}{4^{p-3}}\int_{q_T}\widetilde{\rho}_0^2|u_k|^2 \leq \|a_{23}\|^2_\infty C_0\|g\|^2_{(L^2(q_T,\rho))^2}$, so that the source term $g + \widetilde{B}u_k$ in (45) belongs to $(L^2(q_T))^2$, with the estimate

$$\|g + \widetilde{B}u_k\|^2_{(L^2(q_T))^2} \leq \|a_{23}\|^2_\infty C_0\|g\|^2_{(L^2(q_T,\rho))^2}.$$

From Proposition 9, it follows that the solution z_k to (45) belongs to $(W^2_\omega(0,T))^2$, with the estimate

$$\|z_k\|^2_{(W^2_\omega(0,T))^2} \leq e^{\kappa S_T}\|g\|^2_{(L^2(q_T,\rho))^2}, \tag{50}$$

where $S_T = R_T + N_T + \|a_{23}\|_\infty$. From (49) and (50), we deduce the existence of subsequences, still denoted z_k and u_k, such that as $k \to \infty$, we have

$$\begin{array}{ll} u_k \rightharpoonup u & \text{in } L^2(q_T,\widetilde{\rho}_0), \\ z_k \rightharpoonup z & \text{in } (W^2_\omega(0,T))^2, \\ \rho_k z_k \rightharpoonup \rho z & \text{in } (L^2(q_T))^2. \end{array}$$

Passing to the weak–limit in (45) as $k \to +\infty$, we see that z is the solution to (21) associated with u. Since $z \in (W^2_\omega(0,T))^2$, we have $z \in C([0,T];(H^1_0(\omega))^2)$. The fact that $\rho z \in (L^2(q_T))^2$ implies that $z(\cdot,T) = 0$, since the weight ρ blows up at $t = T$.

Note that if u belongs to $W^2_\omega(0,T)$, then u necessarily satisfies (25), since $u \in L^2(q_T,\widetilde{\rho}_0)$ and $\widetilde{\rho}_0$ blows up at $t = 0$ and $t = T$. Consequently, the proof of Theorem 8 will be ended if we prove that u belongs to $W^2_\omega(0,T)$. Let us recall that u is given by the weak–limit of u_k in $L^2(q_T,\widetilde{\rho}_0)$, where each u_k satisfies the optimality conditions (45)-(47). Let k be fixed. The weight function ρ_k is bounded on q_T, so that the solution $\phi_k = (\phi_{k,1},\phi_{k,2})$ to (46) belongs to $(W^2_\omega(0,T))^2$. It follows that $u_k \in W^2_\omega(0,T)$, since we have by (47) $u_k = -\widetilde{\rho}_0^{-2}a_{23}\phi_{k,2}$ with $a_{23} \in W^{2,1}_\infty(q_T)$. The idea is to prove that u_k weakly converges in $W^2_\omega(0,T)$. This will be obtained by proving that the norm of u_k in $W^2_\omega(0,T)$ is bounded from above independently of k. The function $\psi_k = \widetilde{\rho}_0^{-2}\phi_k$ is the solution to

$$L^*\psi_k = \widetilde{\rho}_0^{-2}\rho_k^2 z_k - \partial_t(\widetilde{\rho}_0^{-2})\phi_k \ \text{ in } q_T, \qquad \psi_k = 0 \ \text{ on } \sigma_T, \qquad \psi_k(\cdot,T) = 0.$$

Using the facts that $p \geq 6$ and $\int_{q_T} \rho_k^2 |z_k|^2 \leq \frac{C_0}{2} \|g\|^2_{(L^2(q_T,\rho))^2}$ (given by (49)), we can prove that $\int_{q_T} |\widetilde{\rho}_0^{-2} \rho_k^2 z_k|^2 \leq e^{CN_T} \|g\|^2_{(L^2(q_T,\rho))^2}$. The estimate $\int_{q_T} |\partial_t(\widetilde{\rho}_0^{-2})\phi_k|^2 \leq e^{CN_T} \|g\|^2_{(L^2(q_T,\rho))^2}$ follows from the inequality $\int_{q_T} \rho^{-2} |\phi_k|^2 \leq C_0 \|g\|^2_{(L^2(q_T,\rho))^2}$ (obtained by combination of (48) and (49)) and the fact that $p \geq 8$. Therefore, we have $\|L^* \psi_k\|^2_{(L^2(q_T))^2} \leq e^{CN_T} \|g\|^2_{(L^2(q_T,\rho))^2}$. Applying the inequality of Proposition 9 to ψ_k, we obtain

$$\|\psi_k\|^2_{(W^2_\omega(0,T))^2} \leq e^{C(R_T+N_T)} \|g\|^2_{(L^2(q_T,\rho))^2},$$

so that for a subsequence we have $\psi_k \rightharpoonup \psi$ in $(W^2_\omega(0,T))^2$. Then, $\widetilde{B}^* \psi_k$ weakly converges to $\widetilde{B}^* \psi$ in $W^2_\omega(0,T)$, since $a_{23} \in W^{2,1}_\infty(q_T)$. Therefore, $u_k = -\widetilde{\rho}_0^{-2} \widetilde{B}^* \psi_k$ weakly converges to $-\widetilde{\rho}_0^{-2} \widetilde{B}^* \psi$ in $W^2_\omega(0,T)$. By the uniqueness of the weak–limit of u_k in $L^2(q_T)$, we obtain $u = -\widetilde{\rho}_0^{-2} \widetilde{B}^* \psi$, and in particular $u \in W^2_\omega(0,T)$. This ends the proof of Theorem 8. □

4. Application of the method to the controllability by two forces: proof of Theorem 3

In this section we apply the method detailed in section 1.2 to the controllability of system (1) by two forces. The proof is more straightforward in this case than for the controllability by three forces. Indeed, if the decomposition (13) defines a solution y to (10) controlled by two forces $v = (v_1, v_2)^T$, then (14) becomes

$$\begin{cases} F(\cdot,0) = F(\cdot,T) = 0 & \text{in } \omega, \\ \partial_t F - \Delta F - AF - h = (0, v_1, v_2)^T & \text{in } q_T, \end{cases} \tag{51}$$

where h is given by (15). This leads to the controllability of the equation

$$\begin{cases} \partial_t F_1 = \Delta F_1 + a_{11} F_1 + h_1 + a_{21} F_2 + a_{31} F_3 & \text{in } q_T, \\ F_1 = 0 & \text{on } \sigma_T, \\ F_1(\cdot,0) = 0 & \text{in } \omega, \end{cases}$$

by two control forces F_2 and F_3 satisfying $F_2(\cdot,0) = F_2(\cdot,T) = F_3(\cdot,0) = F_3(\cdot,T) = 0$ in ω. The two control forces v_1 and v_2 associated with y are then expressed functions of F_1, F_2 and F_3 thanks to (51). For more readability, we set

$$z = F_1, \quad u_1 = F_2, \quad u_2 = F_3, \quad u = (u_1, u_2)^T, \quad \mathcal{B}u = a_{21} u_1 + a_{31} u_2.$$

The result analogous to Lemma 4 is given below.

Lemma 10. *If there exists $u \in (W^2_\omega(0,T))^2$ satisfying*

$$u(\cdot,0) = u(\cdot,T) = 0, \tag{52}$$

and such that the solution z to

$$\begin{cases} \partial_t z = \Delta z + a_{11} z + h_1 + \mathcal{B}u & \text{in } q_T, \\ z = 0 & \text{on } \sigma_T, \\ z(\cdot,0) = 0 & \text{in } \omega, \end{cases} \tag{53}$$

satisfies $z(\cdot,T) = 0$ in ω, then system (10) is null controllable by two forces v_1 and v_2. Moreover, v_1 and v_2 can be obtained as

$$v_1 = \partial_t u_1 - \Delta u_1 - a_{21} z - a_{22} u_1 - a_{23} u_2 + h_2,$$
$$v_2 = \partial_t u_2 - \Delta u_2 - a_{31} z - a_{32} u_1 - a_{33} u_2 + h_3.$$

The method provided by Lemma 10 to prove Theorem 3 is similar to that provided by Lemma 4 to prove Theorem 2: first, we state that system (53) is controllable by two forces $u = (u_1, u_2)^T \in (L^2(q_T))^2$, then we construct some forces u satisfying (52) and belonging to $(W^2_\omega(0,T))^2$.

Let us consider, for $p \in \mathbb{N}$, $p \geq 1$, the following weight function

$$\rho_0(t) = (t(T-t))^{-(p-1)/2}.$$

Proposition 11. *Let us assume that hypothesis (9) is satisfied. Then there exists $u \in (L^2(q_T, \rho_0))^2$ such that the solution z to (53) satisfies $z(\cdot, T) = 0$ in ω.*

By the definition of $\mathcal{B}$, it is easy to prove that Proposition 11 is true if we replace $u \in (L^2(q_T, \rho_0))^2$ by $u \in (L^2(q_T))^2$, since the source term h_1 belongs to $L^2(q_T)$. But we need to construct control forces u which satisfy (52) and belong to $(W^2_\omega(0,T))^2$. This is the reason why we introduce the weight functions ρ and ρ_0, where ρ is given by (41). It will be proved in section 5 that we can assume h_1 belonging to $L^2(Q_T, \rho)$. As a consequence, Proposition 11 follows by standard duality arguments from the following observability result.

Lemma 12. *Under hypothesis (9), for every $p \in \mathbb{N}$, $p \geq 1$, there exists $\kappa = \kappa(p) > 0$ such that, for every $\phi^0 \in L^2(\omega)$ and every $f \in L^2(q_T)$, the solution ϕ to the backward system*

$$\begin{cases} -\partial_t \phi = \Delta\phi + a_{11}\phi + f & \text{in } q_T, \\ \phi = 0 & \text{on } \sigma_T, \\ \phi(T) = \phi^0 & \text{in } \omega, \end{cases}$$

satisfies

$$\int_{q_T} \rho^{-2}\phi^2 \leq e^{\kappa N_T}\left(\int_{q_T} \rho_0^{-2}|\mathcal{B}^*\phi|^2 + \int_{q_T} \rho^{-2} f^2\right),$$

where $N_T = \frac{1}{T} + (1+T)(1 + \|a_{11}\|_\infty)$.

We do not give the proof of this result which can be obtained by straightforward computations. Note that the proof is simpler than that of Proposition 5, because it concerns only one equation. In general, there is no difficulty to prove the observability inequality when the number of controls in the forward system is greater that the number of equations, a fortiori when the control forces act on the whole domain.

Now, we apply the arguments of the proof of Theorem 8, with $\widetilde{\mathcal{J}}_k$ replaced with

$$\mathcal{J}_k : (L^2(q_T, \rho_0))^2 \to \mathbb{R}, \quad \mathcal{J}_k(u) = \frac{1}{2}\int_{q_T} \rho_0^2 |u|^2 + \frac{1}{2}\int_{q_T} \rho_k^2 {z_u}^2,$$

to obtain the following result.

Lemma 13. *If a_{12} and a_{13} belong to $W^{1,2}_\infty(q_T)$, then there exists $u \in (W^2_\omega(0,T))^2$ satisfying (52) such that the solution z to (53) satisfies $z(\cdot, T) = 0$ in ω.*

In view of Lemma 10, Lemma 13 gives the proof of Theorem 3.

5. Construction of $\widehat{y}$

The aim of this section is to prove that we can construct a solution $(\widehat{y}, \widehat{v})$ to (11) such that the source terms g in (21) and h_1 in (53) respectively belong to $(L^2(q_T, \rho))^2$ and $L^2(q_T, \rho)$, where ρ is defined by (41).

5.1. Statement of the results

The following result states the controllability of (1) with three forces and the existence of solutions which exponentially decrease at $t = T$.

Theorem 14 (Controllability by three forces, $B = I_3$)**.** *If $y^0 \in (L^2(\Omega))^3$ then there exists a function $\widehat{v} \in (L^2(q_T))^3$ such that the solution $\widehat{y}$ to*

$$\begin{cases} \partial_t \widehat{y} = \Delta\widehat{y} + A\widehat{y} + \widehat{v}1_\omega & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \tag{54}$$

satisfies

$$\widehat{y}(\cdot, T) = 0 \text{ in } \Omega,$$

and

$$\widehat{y} \in \left(L^2\left(Q_T, \exp\left(\frac{1}{T-t}\right)\right)\right)^3, \qquad \nabla\widehat{y} \in \left(L^2\left(Q_T, \exp\left(\frac{1}{T-t}\right)\right)\right)^{N\times 3}.$$

Corollary 15. *For every $p \in \mathbb{N}$, the function h defined by (15) belongs to $(L^2(q_T, \rho))^3$, where ρ is given by (41).*

Proof. By the definition of h, we have

$$\begin{aligned} \int_{q_T} \rho^2 |h|^2 \quad \leq \quad & 16\|\theta\|^2_{C^2(\overline{\Omega})} \left(\int_{q_T} \rho^2 (|\nabla\widehat{y}|^2 + |\widehat{y}|^2) + \int_{q_T} \rho^2 |\eta'|^2 |Y|^2 \right. \\ & \left. + \int_{q_T} \rho^2 |\eta|^2 \left(|\nabla Y|^2 + |Y|^2\right) \right). \end{aligned} \tag{55}$$

From Theorem 14 and the definition of ρ, we deduce that $\widehat{y}$ satisfies in particular

$$\int_{q_T} \rho^2 (|\nabla\widehat{y}|^2 + |\widehat{y}|^2) < +\infty.$$

Besides, the definitions of η (see (12)) and ρ imply that

$$\int_{q_T} \rho^2 |\eta|^2 (|\nabla Y|^2 + |Y|^2) \leq C_\eta \int_{q_T} \left(|\nabla Y|^2 + |Y|^2\right) \leq C_\eta \|Y\|^2_{(W^1_\Omega(0,T))^3} < +\infty.$$

Finally, using that $\eta'(t) = 0$ for $t \in (0, T/4)$ and $t \in (3T/4, T)$, we have

$$\begin{aligned} \int_{q_T} \rho^2 |\eta'|^2 |Y|^2 \quad & \leq \quad \|\eta\|^2_{C^1([0,T])} \int_{T/4}^{3T/4} \frac{|Y|^2}{(T-t)^p} \\ & \leq \quad \|\eta\|^2_{C^1([0,T])} \left(\frac{4}{3T}\right)^p \int_{q_T} |Y|^2 < +\infty. \end{aligned}$$

By (55), this implies that $\int_{q_T} \rho^2 |h|^2 < \infty$, which completes the proof of Corollary 15. $\square$

Since $\frac{a_{13}}{a_{23}}$ is bounded in q_T (according to hypothesis (4)), we deduce from Corollary 15 the following result.

Corollary 16. *For every $p \in \mathbb{N}$, the function g defined by (23) belongs to $(L^2(q_T, \rho))^3$.*

To prove Theorem 14 we still apply the method developed by Fursikov and Imanuvilov in [15]. We also refer to [11] for a similar proof. As for the proofs of Theorem 8 and Lemma 13, the main idea is to state an observability inequality for the backward system associated with (54). This is the goal of the next section.

5.2. An observability inequality for the backward system associated with (54)

The main point is to establish a weighted observability estimate without singularity at $t = 0$ in the weights. First, we prove a global Carleman estimate for the solutions to the non homogeneous backward system associated with (54):

$$\begin{cases} -\partial_t \phi = \Delta\phi + A^*\phi + f & \text{in } Q_T, \\ \phi = 0 & \text{on } \Sigma_T, \\ \phi(\cdot, T) = \phi^0 & \text{in } \Omega. \end{cases} \tag{56}$$

Let us first recall the global Carleman inequality satisfied by the solutions to the backward heat equation.

Lemma 17 (Carleman inequality)**.** *There exist a positive function $\beta_0 \in C^2(\overline{\Omega})$, two positive constants $C_0 = C_0(\Omega, \omega)$ and $c_0 = c_0(\Omega, \omega)$ such that for every $\phi^0 \in L^2(\Omega)$, every $f \in L^2(Q_T)$ and every $s \geq s_0 := c_0(T + T^2)$ the solution to*

$$\begin{cases} -\partial_t \phi = \Delta\phi + f & \textit{in } Q_T, \\ \phi = 0 & \textit{on } \Sigma_T, \\ \phi(\cdot, T) = \phi^0 & \textit{in } \Omega, \end{cases}$$

satisfies

$$\int_{Q_T} e^{-2s\beta}\left[(s\gamma)^{-4}\left((\partial_t\phi)^2 + (\Delta\phi)^2\right) + (s\gamma)^{-2}|\nabla\phi|^2 + \phi^2\right] \\ \leq C_0\left(\int_{Q_T} e^{-2s\beta}(s\gamma)^{-3}f^2 + \int_{q_T^0} e^{-2s\beta}\phi^2\right),$$

where β and γ denote the functions $\beta(x,t) = \frac{\beta_0(x)}{t(T-t)}$ (for $(x,t) \in Q_T$) and $\gamma(t) = \frac{1}{t(T-t)}$ (for $t \in (0,T)$).

The proof of Lemma 17 can be found in [14]. However, in [14] the author does not specify the dependence of the parameter s_0 on T. This explicit dependence has been obtained in [12]. Applying Lemma 17 to each equation of system (56) and summing the three Carleman inequalities obtained, we can easily prove the following Carleman inequality for the solutions to (56).

Lemma 18. *Let $\delta \in (0,1)$. For every $\phi^0 \in (L^2(\Omega))^3$ and every $f \in (L^2(Q_T))^3$ the solution ϕ to (56) satisfies*

$$\int_{Q_T} e^{-2s\beta}\left[(s\gamma)^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\gamma)^{-2}|\nabla\phi|^2 + |\phi|^2\right] \\ \leq \frac{4C_0}{\delta^2}\left(\int_{Q_T} e^{-2s\beta}(s\gamma)^{-3}|f|^2 + \int_{q_T} e^{-2s\beta}|\phi|^2\right),$$

for all

$$s \geq s_1 := \max\left(s_0, \frac{T^2}{4}\left(\frac{4C_0}{\delta(1-\delta)}\right)^{1/3} \|A\|_\infty^{2/3}\right), \tag{57}$$

where C_0 and s_0 are given by Lemma 17.

We deduce from the Carleman estimate of Lemma 18 the following weighted observability estimate.

Lemma 19. *Let $s \geq s_1$ (s_1 given by (57)). There exists a positive constant $C = C(\Omega, \omega, T, \|A\|_\infty, s)$ such that for every $\phi^0 \in (L^2(\Omega))^3$ and every $f \in (L^2(Q_T))^3$ the solution ϕ to (56) satisfies*

$$\int_{Q_T} e^{-2s\widetilde{\beta}} \left[(s\widetilde{\gamma})^{-4} \left(|\partial_t \phi|^2 + |\Delta\phi|^2 \right) + (s\widetilde{\gamma})^{-2} |\nabla\phi|^2 + |\phi|^2 \right]$$
$$\leq C \left(\int_{Q_T} e^{-2s\widetilde{\beta}} (s\widetilde{\gamma})^{-3} |f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}} |\phi|^2 \right),$$

where the functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ are defined by

$$\widetilde{\beta}(x,t) = \frac{t}{T}\beta(x,t), \quad \widetilde{\gamma}(t) = t\gamma(t).$$

Remark 7. From Lemma 19 to the end of this section, both parameters s and T are fixed and the constant C in the statement of Lemma 19 depends of these parameters.

Proof. Let η be a function in $C^\infty([0,T])$ such that $\eta = 1$ in $[0, T/4]$, $\eta = 0$ in $[3T/4, T]$ and $0 \leq \eta \leq 1$ and $|\eta'(t)| \leq c_0/T$ in $[0,T]$ (c_0 being a positive constant independent of T). Let ϕ be the solution to (56) associated with ϕ^0 and f. Then the function $\psi(x,t) = \eta(t)\phi(x,t)$ is the solution to

$$\begin{cases} -\partial_t \psi = \Delta\psi + A^*\psi + g & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(\cdot, T) = 0 & \text{in } \Omega, \end{cases}$$

where $g = \eta f - \eta'\phi \in (L^2(Q_T))^3$. In this proof, C stands for a generic positive constant depending only on Ω, ω, T, $\|A\|_\infty$ and s, and κ denotes a positive constant depending only on ω and Ω. The values of those constants may change from one line to another. Applying (3) to ψ and using the definition of g and η, we have

$$\|\psi\|^2_{(W^2_\Omega(0,T))^3} \leq e^{\kappa M_T} \|g\|^2_{(L^2(Q_T))^3} \leq C \left(\int_{Q_{3T/4}} |f|^2 + \int_{T/4}^{3T/4} |\phi|^2 \right).$$

From the definition of β, we have $e^{-2s\beta} \geq e^{-16s\|\beta_0\|_\infty/T^2}$ for $t \in [T/4, 3T/4]$, so that the above inequality implies that

$$\|\psi\|^2_{(W^2_\Omega(0,T))^3} \leq C \left(\int_{Q_{3T/4}} |f|^2 + \int_{Q_T} e^{-2s\beta} |\phi|^2 \right). \tag{58}$$

Since the weights $e^{-2s\widetilde{\beta}}$ and $\widetilde{\gamma}^{-3}$ are lower bounded for $t \in [0, 3T/4]$, we have

$$\int_{Q_{3T/4}} |f|^2 \leq C \int_{Q_{3T/4}} e^{-2s\widetilde{\beta}} (s\widetilde{\gamma})^{-3} |f|^2. \tag{59}$$

The integral term $\int_{Q_T} e^{-2s\beta}|\phi|^2$ in (58) can be estimated by applying Lemma 18:

$$\int_{Q_T} e^{-2s\beta} \left[(s\gamma)^{-4} \left(|\partial_t \phi|^2 + |\Delta\phi|^2 \right) + (s\gamma)^{-2} |\nabla\phi|^2 + |\phi|^2 \right]$$

$$\leq \kappa\left(\int_{Q_T} e^{-2s\beta}(s\gamma)^{-3}|f|^2 + \int_{q_T} e^{-2s\beta}|\phi|^2\right)$$
$$\leq \kappa\left(T^3 \int_{Q_T} e^{-2s\widetilde{\beta}}(s\widetilde{\gamma})^{-3}|f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}}|\phi|^2\right), \tag{60}$$

since $e^{-2s\widetilde{\beta}} = e^{-2s\beta t/T} \geq e^{-2s\beta}$ and $\widetilde{\gamma}^{-1} = \frac{\gamma^{-1}}{t} \geq \frac{\gamma^{-1}}{T}$ for every $t \in [0,T]$. This implies that

$$\int_{Q_T} e^{-2s\beta}|\phi|^2 \leq \kappa\left(T^3 \int_{Q_T} e^{-2s\widetilde{\beta}}(s\widetilde{\gamma})^{-3}|f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}}|\phi|^2\right). \tag{61}$$

Combining (58), (59) and (61), we obtain

$$\|\psi\|^2_{(W^2_\Omega(0,T))^3} \leq C\left(\int_{Q_T} e^{-2s\widetilde{\beta}}(s\widetilde{\gamma})^{-3}|f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}}|\phi|^2\right).$$

On the other hand, by the definitions of $\widetilde{\beta}$ and $\widetilde{\gamma}$, we have

$$\begin{aligned}\|\psi\|^2_{(W^2_\Omega(0,T))^3} &\geq \|\psi\|^2_{(W^2_\Omega(0,T/4))^3}\\ &\geq K\int_{Q_{T/4}} e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right],\end{aligned}$$

where

$$K = \frac{\exp(2s\ \min_{\overline{\Omega}}\beta_0/T^2)}{1+s^2T^2+s^4T^4}.$$

Adding the last two inequalities, we have

$$\int_{Q_{T/4}} e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right]$$
$$\leq C\left(\int_{Q_T} e^{-2s\widetilde{\beta}}(s\widetilde{\gamma})^{-3}|f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}}|\phi|^2\right). \tag{62}$$

For the estimation of $e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right]$ on $[T/4,T]$, remark that for $(x,t) \in \Omega\times[T/4,T]$ we have

$$e^{-2s\widetilde{\beta}(x,t)} \leq e^{8s\|\beta_0\|_\infty/T^2}e^{-2s\beta(x,t)}, \qquad \widetilde{\gamma}(t)^{-1} \leq \frac{4}{T}\gamma(t)^{-1},$$

which implies that

$$\int_{T/4}^{T} e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right]$$
$$\leq C\int_{T/4}^{T} e^{-2s\beta}\left[(s\gamma)^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\gamma)^{-2}|\nabla\phi|^2 + |\phi|^2\right].$$

Combining this inequality with (60), we deduce that

$$\int_{T/4}^{T} e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right]$$
$$\leq C\left(\int_{Q_T} e^{-2s\widetilde{\beta}}(s\widetilde{\gamma})^{-3}|f|^2 + \int_{q_T} e^{-2s\widetilde{\beta}}|\phi|^2\right).$$

Adding this inequality with (62) gives the inequality announced in the statement of Lemma 19. $\square$

From now on, let us choose

$$s = s_0 + \tfrac{T^2}{4}\Big(\tfrac{4C_0}{\delta(1-\delta)}\Big)^{\frac{1}{3}}\|A\|_\infty^{\frac{2}{3}}, \tag{63}$$

so that $s \geq s_1$, with s_0 and s_1 given by Lemma 17 and (57) respectively. In what follows, we will use the notation:

$$\widehat{\rho}(x,t) = \frac{e^{s\widetilde{\beta}(x,t)}}{(T-t)^{3/2}}, \quad \widehat{\rho_0}(x,t) = \widehat{\rho}(x,t)(T-t)^{3/2}.$$

Analyzing the different steps in the proof of Lemma 19, we can obtain the following estimate of the constant C for this particular choice of s:

$$C = e^{\kappa P_T},$$

where $\kappa = \kappa(\omega,\Omega) > 0$ and P_T is defined in the following proposition.

Proposition 20. *There exists $\kappa = \kappa(\omega,\Omega) > 0$ such that, for every $\phi^0 \in (L^2(\Omega))^3$ and every $f \in (L^2(Q_T))^3$, the solution ϕ to (56) satisfies the inequality*

$$\|\phi(\cdot,0)\|^2_{(L^2(\Omega))^3} \leq e^{\kappa P_T}\left(\int_{Q_T}\widehat{\rho}^{\,-2}|f|^2 + \int_{q_T}\widehat{\rho_0}^{\,-2}|\phi|^2\right), \tag{64}$$

with

$$P_T = 1 + \frac{1}{T} + \|A\|_\infty^{2/3} + T\Big(1 + \|A\|_\infty + \|A\|_\infty^{1/3}\Big).$$

Proof. Lemma 19 applied to ϕ with s defined by (63) gives

$$\int_{Q_T} e^{-2s\widetilde{\beta}}\left[(s\widetilde{\gamma})^{-4}\left(|\partial_t\phi|^2 + |\Delta\phi|^2\right) + (s\widetilde{\gamma})^{-2}|\nabla\phi|^2 + |\phi|^2\right]$$
$$\leq e^{\kappa P_T}\left(\int_{Q_T}\widehat{\rho}^{\,-2}|f|^2 + \int_{q_T}\widehat{\rho_0}^{\,-2}|\phi|^2\right). \tag{65}$$

By the definitions of $\widetilde{\beta}$ and $\widetilde{\gamma}$, the weights $e^{-2s\widetilde{\beta}}$ and $(s\widetilde{\gamma})^{-1}$ are uniformly bounded from below on $Q_{T-\delta}$ by a positive constant, so that (65) implies

$$C_\delta \int_{Q_{T-\delta}}\Big(|\partial_t\phi|^2 + |\Delta\phi|^2 + |\nabla\phi|^2 + |\phi|^2\Big) \leq e^{\kappa P_T}\left(\int_{Q_T}\widehat{\rho}^{\,-2}|f|^2 + \int_{q_T}\widehat{\rho_0}^{\,-2}|\phi|^2\right),$$

with $C_\delta = e^{-\frac{2s\|\beta_0\|_\infty}{T\delta}}\min\Big(1, s^{-2}\delta^2, s^{-4}\delta^4\Big)$, for every $\delta \in (0,T)$. The last inequality reads also

$$\|\phi\|^2_{(W^2_\Omega(0,T-\delta))^3} \leq \frac{e^{\kappa P_T}}{C_\delta}\left(\int_{Q_T}\widehat{\rho}^{\,-2}|f|^2 + \int_{q_T}\widehat{\rho_0}^{\,-2}|\phi|^2\right). \tag{66}$$

Using the continuous embedding (see, for instance, [10, chap. VIII])

$$W^2_\Omega(0,T-\delta) \subset C\Big([0,T-\delta];H^1_0(\Omega)\Big),$$

we deduce from (66) the inequality

$$\|\phi(\cdot,0)\|^2_{(H^1_0(\Omega))^3} \leq e^{\kappa P_T}\left(\int_{Q_T}\widehat{\rho}^{\,-2}|f|^2 + \int_{q_T}\widehat{\rho_0}^{\,-2}|\phi|^2\right),$$

which yields (64). $\square$

5.3. Proof of Theorem 14

The construction of $\widehat{v}$ and $\widehat{y}$ can be done in a similar way to that of u and z in Theorem 8, the only difference being in the definition of the functional $\widetilde{\mathcal{J}}_k$. Instead of $\widetilde{\mathcal{J}}_k$, we consider the functional $\mathcal{I}_k : (L^2(q_T, \widehat{\rho_0}))^3 \to \mathbb{R}$, defined by

$$\mathcal{I}_k(v) = \frac{1}{2}\int_{q_T} \widehat{\rho_0}^2 |v|^2 + \int_{Q_T} \widehat{\rho_k}^2 |y_v|^2,$$

where y_v denotes the solution to (54) associated with v and

$$\widehat{\rho_k}(t) = \exp\left(\frac{s\beta(x)}{T(T+1/k-t)}\right)(T+1/k-t)^{-3/2}, \qquad \forall t \in [0,T].$$

Similar arguments to those in the proof of Theorem 8 (in particular the use of observability inequality (64)) give the existence of a function $\widehat{v} \in (L^2(q_T, \widehat{\rho_0}))^3$, with $\|\widehat{v}\|^2_{(L^2(q_T,\widehat{\rho_0}))^3} \le e^{\kappa P_T}\|y^0\|^2_{(L^2(\Omega))^3}$ (P_T defined in Proposition 20), such that the solution $\widehat{y}$ to (54) associated with $\widehat{v}$ satisfies $\widehat{y} \in (W^2_\Omega(0,T))^3$, $\widehat{y} \in (L^2(Q_T, \widehat{\rho}))^3$ with $\|\widehat{y}\|^2_{(L^2(Q_T,\widehat{\rho}))^3} \le e^{\kappa P_T}\|y^0\|^2_{(L^2(\Omega))^3}$ and $\widehat{y}(\cdot,T) = 0$ in $(L^2(\Omega))^3$. In particular, it follows from the definition of $\widehat{\rho}$ (see (41)) that

$$\widehat{y} \in \left(L^2\left(Q_T, \exp\left(\frac{1}{T-t}\right)\right)\right)^3.$$

The only point remaining is to prove that $\nabla\widehat{y} \in \left(L^2\left(Q_T, \exp\left(\frac{1}{T-t}\right)\right)\right)^{N\times 3}$. Multiplying scalarly the equation of $\widehat{y}$ by $\widehat{\rho}(\cdot,t)^2(T-t)^2\widehat{y}(\cdot,t)$ and integrating on $[0,T]$, we obtain

$$\begin{aligned}\int_{Q_T} \widehat{\rho}^2 (T-t)^2\, |\nabla\widehat{y}|^2 &= \int_{Q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}(A\widehat{y}) + \int_{q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}\,\widehat{v} \\ &\quad -\frac{1}{2}\int_{Q_T} \widehat{\rho}^2 (T-t)^2\, \partial_t|\widehat{y}|^2. \end{aligned} \tag{67}$$

From $\widehat{y} \in (L^2(Q_T, \widehat{\rho}))^3$ and the definition of $\widehat{\rho}$, we deduce that $\lim_{t\to T} \widehat{\rho}^2(\cdot,t)(T-t)^2|\widehat{y}(\cdot,t)|^2 = 0$ in $(L^2(\Omega))^3$, so that, integrating by parts on $[0,T]$ in the last term of (67), we obtain

$$\begin{aligned}\int_{Q_T} \widehat{\rho}^2 (T-t)^2\, |\nabla\widehat{y}|^2 &= \int_{Q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}(A\widehat{y}) + \int_{q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}\,\widehat{v} \\ &\quad +\frac{T^2}{2}\int_{\Omega} \widehat{\rho}(x,0)^2\, |y^0(x)|^2 \\ &\quad +\frac{1}{2}\int_{Q_T} \partial_t(\widehat{\rho}^2 (T-t)^2)\, |\widehat{y}|^2. \end{aligned} \tag{68}$$

It is clear that

$$\int_{Q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}(A\widehat{y}) \le T^2\|A\|_\infty \int_{Q_T} \widehat{\rho}^2 |\widehat{y}|^2 \le e^{\kappa P_T}\|y^0\|^2_{(L^2(\Omega))^3}.$$

Besides, by the definition of $\widehat{\rho}$ and $\widehat{\rho_0}$, we also have

$$\begin{aligned}\int_{q_T} \widehat{\rho}^2 (T-t)^2 \widehat{y}\,\widehat{v} &= \int_{q_T} (T-t)^{1/2} \widehat{\rho}\widehat{\rho_0}\, \widehat{y}\,\widehat{v} \\ &\le T^{1/2}\left(\|\widehat{v}\|^2_{(L^2(q_T,\widehat{\rho_0}))^3} + \|\widehat{y}\|^2_{(L^2(Q_T,\widehat{\rho}))^3}\right) \\ &\le e^{\kappa P_T}\|y^0\|^2_{(L^2(\Omega))^3}, \end{aligned} \tag{69}$$

and

$$T^2 \int_\Omega \widehat{\rho}(x,0)^2\, |y^0(x)|^2 \le T^{-1} e^{2s\|\beta_0\|_\infty/T^2} \|y^0\|^2_{(L^2(\Omega))^3} \le e^{\kappa P_T} \|y^0\|^2_{(L^2(\Omega))^3}, \tag{70}$$

with β_0 given in Lemma 17. Consequently, it remains to bound the last term in (68). Since $\partial_t(\widehat{\rho}^2(T-t)^2) = \widehat{\rho}^2\big(T - t + 2s\beta_0(x)/T\big)$, we can write

$$\begin{aligned}
\int_{Q_T} \partial_t(\widehat{\rho}^2(T-t)^2)\, |\widehat{y}|^2 \quad &\le \quad \left(T + \frac{2s\|\beta_0\|_\infty}{T}\right) \|\widehat{y}\|^2_{(L^2(Q_T,\widehat{\rho}))^3} \\
&\le \quad e^{\kappa P_T} \|y^0\|^2_{(L^2(\Omega))^3}.
\end{aligned} \tag{71}$$

Combining (68)–(71), we finally obtain

$$\int_{Q_T} \widehat{\rho}^2 (T-t)^2\, |\nabla \widehat{y}|^2 \le e^{\kappa P_T} \|y^0\|^2_{(L^2(\Omega))^3},$$

which ensures that

$$\nabla \widehat{y} \in \left(L^2\!\left(Q_T, \exp\left(\frac{1}{T-t}\right)\right)\right)^{N\times 3},$$

by the definition of $\widehat{\rho}$. This completes the poof of Theorem 14.

6. Comments and further results

6.1. Controllability of $n \times n$ parabolic systems by one force.

In a forthcoming paper, we will deal with the null controllability of system (1) in the case $n \ge 3$ (that is $A = (a_{ij})_{1\le i,j\le n} \in (L^\infty(Q_T))^{n\times n}$ and $B = (0,0,0,\cdots,0,1)^T \in \mathbb{R}^n$), using the same approach as for the proof of Theorem 2 and working by induction.

6.2. The nonlinear case.

The knowledge of the dependence of the observability constant C_0 (see Proposition 5 and Remark 6) with respect to the coefficients of A is needed to study the controllability to trajectories of systems like

$$\begin{cases} \partial_t y = \Delta y + F(y) + B_1\, v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \tag{72}$$

with a nonlinearity $F : \mathbb{R}^3 \to \mathbb{R}^3$. The next step of our study is to perform a Kakutani fixed-point argument on a linearized system of (72) to deduce a local controllability result for the solutions to (72).

6.3. The case of distinct diffusion coefficients.

The problem of the controllability to trajectories for (72) is derived from the study of the controllability to trajectories of the following system which models the therapy for brain tumors

$$\begin{cases} \partial_t y = D\Delta y + F(y) + B_1\, v 1_\omega & \text{in } Q_T, \\ \partial_\nu y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \tag{73}$$

In this system $\partial_\nu y = (\partial_\nu y_1, \partial_\nu y_2, \partial_\nu y_3)^T$ stands for the normal derivative of $y = (y_1, y_2, y_3)^T$, D is a diagonal matrix given by $D = \mathrm{diag}(d_1, d_2, d_3)$ with $d_i > 0$ ($i = 1, 2, 3$) and the non linearity F is defined by

$$F(y) = \begin{pmatrix} \alpha_{11} y_1 g_1(y_1) - (\alpha_{12} y_2 + \alpha_{13} y_3) y_1 \\ \alpha_{22} y_2 g_2(y_2) - (\alpha_{21} y_1 + \alpha_{23} y_3) y_2 \\ -\alpha_{33} y_3 \end{pmatrix},$$

where either $g_i(y_i) = 1$, either $g_i(y_i) = 1 - y_i/k_i$ or $g_i(y_i) = \ln(k_i/y_i)$, with $k_i > 0$ for $i = 1, 2$ (see [9] for more details). As for system (72), the study of the controllability to trajectories for (73) begins with the study of the null controllability of the following linear system

$$\begin{cases} \partial_t y = D\Delta y + Ay + B_1\, v 1_\omega & \text{in } Q_T, \\ \partial_\nu y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \tag{74}$$

where A is a 3×3 matrix with coefficients a_{ij} belonging to $L^\infty(Q_T)$. Applying the decomposition (13) where Y is the solution to (74) with $v = 0$ and $(\widehat{y}, \widehat{v})$ is a solution to

$$\begin{cases} \partial_t \widehat{y} = D\Delta \widehat{y} + A\widehat{y} + \widehat{v} 1_{\omega_0} & \text{in } Q_T, \\ \partial_\nu \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot, 0) = y^0, \qquad \widehat{y}(\cdot, T) = 0 & \text{in } \Omega, \end{cases}$$

for an open subset $\omega_0 \subset \omega$ of Ω, we obtain, after the change of variables (20), the new system posed in q_T

$$\begin{cases} \partial_t z_1 = d_1 \Delta z_1 + \widetilde{a}_{11} z_1 + \left(c_{12} + 2d_1 \nabla\left(\frac{a_{13}}{a_{23}}\right) \cdot \nabla + (d_1 - d_2)\frac{a_{13}}{a_{23}}\Delta \right) z_2 + g_1, \\ \partial_t z_2 = d_2 \Delta z_2 + \widetilde{a}_{21} z_1 + \widetilde{a}_{22} z_2 + g_2 + a_{23} u, \\ \partial_\nu z_1 = \partial_\nu z_2 = 0 \quad \text{on } \sigma_T, \\ z_1(\cdot, 0) = z_2(\cdot, 0) = 0 \quad \text{in } \omega, \end{cases}$$

with

$$c_{12} = -\frac{\det K}{a_{23}^2} + (d_1 \Delta - \partial_t)\left(\frac{a_{13}}{a_{23}}\right),$$

K being given in Theorem 2. The new coefficient:

$$\widetilde{a}_{12} = c_{12} + 2d_1 \nabla\left(\frac{a_{13}}{a_{23}}\right) \cdot \nabla + (d_1 - d_2)\frac{a_{13}}{a_{23}}\Delta \tag{75}$$

is a second order operator in space (compare with the corresponding coefficient in the case $d_1 = d_2 = 1$ we have considered in the previous sections).

Actually, the starting point of the proof of Lemma 6 is the inequality (31) which is a consequence of the following property: for almost every $t \in]0, T[$, $\widetilde{a}_{12}(\cdot, t)$ satisfies

$$\begin{array}{l} \int_\omega \psi\,(\widetilde{a}_{12}(\cdot, t)\psi)\,\mathrm{d}x = \int_\omega (\widetilde{a}^*_{12}(\cdot, t)\psi)\psi\,\mathrm{d}x \geq c\int_\omega \psi^2\,\mathrm{d}x, \quad \forall \psi \in H^1_0(\omega), \\ \text{or} \\ \int_\omega \psi(\widetilde{a}_{12}(\cdot, t)\psi)\,\mathrm{d}x \leq -c\int_\omega \psi^2\,\mathrm{d}x, \quad \forall \psi \in H^1_0(\omega). \end{array} \tag{76}$$

An inspection of the proof of our main result shows that the assumption (5) was only needed in the proof of Lemma 6 to establish (76) and so (31). Consequently, the assumption (5) in Theorem 2 can be replaced by the assumption (76). Now, taking into account the diffusion coefficients d_i, $i = 1, 2, 3$, in the proof of Lemma 6, we get the following result.

Theorem 21. *Let us assume that a_{13}, $a_{23} \in W^{2,1}_{\infty}(q_T)$ and that there exist two positive constants α and c such that*

$$|a_{23}| \geq \alpha \ \text{ in } q_T$$

and for almost every $t \in]0, T[$, either

$$\int_{\omega} \psi(\widetilde{a}_{12}(\cdot, t)\psi)\, dx \geq c \int_{\omega} \psi^2\, dx, \quad \forall \psi \in H^2(\omega) \cap H^1_0(\omega)\,,$$

or

$$\int_{\omega} \psi(\widetilde{a}_{12}(\cdot, t)\psi)\, dx \leq -c \int_{\omega} \psi^2\, dx, \quad \forall \psi \in H^2(\omega) \cap H^1_0(\omega)\,,$$

where $\widetilde{a}_{12}$ is defined in (75). Then for every $y^0 \in (L^2(\Omega))^3$, there exists at least one function $v \in L^2(q_T)$ such that the solution y to (74) satisfies

$$y(\cdot, T) = 0 \ \text{ in } \Omega.$$

Proof. The proof is similar to that of Theorem 2, the main step being the proof of Lemma 6. Consequently, we only point out the differences that appear in the proof of Lemma 6 when we consider the diffusion coefficients d_i, $i = 1, 2, 3$. First, note that (32) is still true because in (29) we only have to change the integral term $2\int_{q_T} \varphi^p \nabla\phi_1 \cdot \nabla\phi_2$ into $(d_1 + d_2)\int_{q_T} \varphi^p \nabla\phi_1 \cdot \nabla\phi_2$, where (ϕ_1, ϕ_2) is now the solution to the following backward system

$$\begin{cases} -\partial_t\phi_1 = d_1\Delta\phi_1 + \widetilde{a}_{11}\phi_1 + \widetilde{a}_{21}\phi_2 + f_1 & \text{in } q_T, \\ -\partial_t\phi_2 = d_2\Delta\phi_2 + \widetilde{a}^*_{12}\phi_1 + \widetilde{a}_{22}\phi_2 + f_2 & \text{in } q_T, \\ \phi_1 = \phi_2 = 0 & \text{on } \sigma_T, \\ \phi_1(\cdot, T) = \phi^0_1,\ \phi_2(\cdot, T) = \phi^0_2 & \text{in } \omega. \end{cases} \tag{77}$$

The main difference consists in the estimate of $\nabla\phi_2$. Indeed, in (33) the term $\int_{q_T} \varphi^{p-1}|\nabla\phi_2|^2$ has to be changed in $d_2\int_{q_T} \varphi^{p-1}|\nabla\phi_2|^2$ and the term $\int_{q_T} \varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1$ is now given by the formula

$$\begin{aligned} \int_{q_T} \varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1 \quad = \quad & -\int_{q_T} \varphi^{p-1}\left(\frac{\det K}{a_{23}^2} + (\partial_t + d_2\Delta)\left(\frac{a_{13}}{a_{23}}\right)\right)\phi_1\phi_2 \\ & -2d_2 \int_{q_T} \varphi^{p-1}\phi_2\nabla\phi_1 \cdot \nabla\left(\frac{a_{13}}{a_{23}}\right) \\ & +(d_1 - d_2)\int_{q_T} \varphi^{p-1}\frac{a_{13}}{a_{23}}\phi_2\, \Delta\phi_1. \end{aligned} \tag{78}$$

Therefore, we need an estimate on $\Delta\phi_1$ with respect to ϕ_1 and $\nabla\phi_1$. This estimate is obtained by multiplying the first equation of (77) by $\varphi^{p+2}\Delta\phi_1$. In fact, we have

$$\int_{q_T} \varphi^{p+2}(\Delta\phi_1)^2 \leq C\left(\int_{q_T} \varphi^p{\phi_1}^2 + \int_{q_T} \varphi^{p+1}|\nabla\phi_1|^2 + \int_{q_T} \varphi^{p+2}{\phi_2}^2 + \int_{q_T} \varphi^{p+2}{f_1}^2\right).$$

Combining this inequality with (78), we obtain for $\epsilon > 0$ small enough

$$\left|\int_{q_T} \varphi^{p-1}(\widetilde{a}_{12}\phi_2)\phi_1\right| \leq \epsilon C\left(\int_{q_T} \varphi^p{\phi_1}^2 + \int_{q_T} \varphi^{p+1}|\nabla\phi_1|^2\right) + C_\epsilon\left(\int_{q_T} \varphi^{p-4}{\phi_2}^2 + \int_{q_T} \varphi^{p+2}{f_1}^2\right).$$

The new estimate on $\int_{q_T} \varphi^{p-1}|\nabla\phi_2|^2$ follows from the last inequality. Finally, the elimination of $|\nabla\phi_1|^2$ is obtained as in Lemma 6. This leads to an inequality similar to (28) but with the weight $t^{p-4}(T-t)^{p-4}$ instead of $t^{p-3}(T-t)^{p-3}$ in front of ${\phi_2}^2$. $\square$

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