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Due to a restrictive scenario of sequential projective measurements, multi-point temporal quantum correlations were thought to factorize into two-point ones. We show that sequential generalized measurements lead to genuinely multi-point temporal correlations, which enable us to translate typical spatial genuinely multipartite nonlocal phenomena, like Greenberger-Horne-Zeilinger paradox, into the temporal scenario.

Introduction. Temporal quantum correlations have been widely studied since the seminal paper of Leggett and Garg [1]. They proposed Bell-like inequalities involving two-point correlations between measurement outcomes of a single observable measured at different instances in time on an evolving system. The Leggett-Garg inequalities hold whenever the measurement results are assumed to be well-defined before the actual measurement takes place and are not altered by the act of measurement. A related scenario was proposed by Brukner *et al.* [2] in which there are two time instances and one of two different observables is measured at each time. Under these assumptions a CHSH-like temporal inequality holds. This approach was further generalized to several temporal Bell-type inequalities involving many instances of time [3, 4].

The common feature of all these inequalities is that they involve only two-point temporal correlations. It was believed that multi-point temporal quantum correlations always factor into at most two-point cases, even though a proof exists only in the case of projective measurements on qubits [2]. To the best of our knowledge, there are no constructive examples of genuinely multi-point temporal quantum correlations in a scenario of sequential measurements on a single system. A non-constructive example is given by Budroni *et al.* [4], who show that three-point temporal quantum correlations that arise from sequential projective measurements can violate a Bell inequality proposed in [5]. When the three parties are spatially separated such a violation can only occur using super-quantum correlations what shows that sequential projective measurements can simulate super-quantum spatial correlations. However no concrete physical implementation is known. Another approach to demonstrate genuinely multi-point temporal quantum effects has been proposed by Żukowski [6], however in this scenario a sequence of unitary operations rather than measurements is considered.

Here we consider a single system that is measured from time to time in order to establish probabilities for various sequences of outcomes. We show in a constructive way that it is indeed possible to obtain genuinely multi-point temporal quantum correlations by sequential POVM measurements on a single qubit, which overcomes the no-go result of [2]. Our approach is based on the fact that every pure quantum state of finite dimension can be expressed in the form of a Matrix Product State (MPS) and therefore can be generated in a sequential manner [7]. We utilize the sequential procedure to construct a series of POVM measurements on a single qubit giving rise to temporal correlations identical to the spatial correlations of the corresponding MPS state. In this way we obtain a temporal version of the Greenberger-Horne-Zeilinger paradox and temporal Bell inequalities violation of which detects genuinely multi-point temporal correlations.

Temporal quantum correlations – basic concepts and definitions. The most general temporal correlation function for a sequence of N generalized quantum measurements is defined as follows:

$$E(m_1,\ldots,m_N) = \sum_{i_1,\ldots,i_N} i_1\ldots i_N P(i_1,\ldots,i_N|m_1,\ldots,m_N), \tag{1}$$

where

$$P(i_1,\ldots,i_N|m_1,\ldots,m_N) \tag{2}$$

is the probability to observe a particular sequence of outcomes $\{i_1,\ldots,i_N\}$ conditioned on the settings $\{m_1,\ldots,m_N\}$.

Let us now discuss the case of sequential projective measurements. Consider a single qubit prepared in an initial state described by a Bloch vector $\vec{s}$. The qubit is measured at time instances $t_1,\ldots,t_N$ with the corresponding dichotomic observables parameterised by Bloch vectors $\vec{m}_1,\ldots,\vec{m}_N$, i.e. the measurement outcomes $i_1,\ldots,i_N$ are all equal to ±1. Sequential projective measurements on a single qubit form a Markov chain [2, 6], so that the probability of a sequence of outcomes is given by

$$P(i_1,\ldots,i_N|\vec{m}_1,\ldots,\vec{m}_N) = P(i_1|\vec{m}_1)P(i_2|i_1,\vec{m}_1,\vec{m}_2)\times P(i_3|i_2,\vec{m}_2,\vec{m}_3)\ldots P(i_N|i_{N-1},\vec{m}_{N-1},\vec{m}_N), \tag{3}$$

where $P(i_{j+1}|i_j,\vec{m}_j,\vec{m}_{j+1})$ denotes the probability of measuring outcome i_{j+1} on a qubit which collapsed in the previous measurement to the state $i_j\vec{m}_j$, i.e. $P(i_{j+1}|i_j,\vec{m}_j)=\frac{1}{2}(1+i_{j+1}i_j\,\vec{m}_{j+1}\cdot\vec{m}_j)$ with $\cdot$ being the inner product. Straightforward calculation of the temporal correlations reveals that for odd N we have:

$$E(\vec{m}_1,\ldots,\vec{m}_N)=(\vec{m}_1\cdot\vec{s})(\vec{m}_2\cdot\vec{m}_3)\ldots(\vec{m}_{N-1}\cdot\vec{m}_N), \quad (4)$$

whereas for N even:

$$E(\vec{m}_1,\ldots,\vec{m}_N)=(\vec{m}_1\cdot\vec{m}_2)(\vec{m}_3\cdot\vec{m}_4)\ldots(\vec{m}_{N-1}\cdot\vec{m}_N). \quad (5)$$

The temporal correlations between outcomes of multiple projective measurements always factor into correlations between at most two measurements. In this sense a qubit never gives rise to multi-point correlations in time. This property of projective measurements is related to the fact that they can be treated as a preparation of an initial state of the system, which fully describes all future probabilistic predictions [2]. As we will see, in the case of generalized measurements this intuition is no longer valid.

For generalized measurements, the probability $P(i_1,\ldots,i_N|m_1,\ldots,m_N)$ no longer possesses the Markov property, it can only be expressed in the general form

$$P(i_1,\ldots,i_N|m_1,\ldots,m_N)=P(i_1|m_1)P(i_2|i_1,m_1,m_2)\times \\ \ldots P(i_N|i_1,\ldots,i_{N-1},m_1,\ldots,m_N), \quad (6)$$

where m_k is some parametrization of measurement settings of an arbitrary POVM, defined by measurement operators M_k. The conditional probabilities in the above formula are given by

$$P(i_k|i_1,\ldots,i_{k-1},m_1,\ldots,m_k)=\mathrm{Tr}\left(\rho_k M_k^\dagger M_k\right), \quad (7)$$

where the post-measurement state is defined as a recursive formula

$$\rho_k=\frac{M_{k-1}\rho_{k-1}M_{k-1}^\dagger}{\mathrm{Tr}\left(M_{k-1}\rho_{k-1}M_{k-1}^\dagger\right)}. \quad (8)$$

The correlations of a sequence of POVM measurements depend directly on the measurement operators M_k, corresponding to different possible implementations of given POVM elements $E_k=M_k^\dagger M_k$. The dependence comes via the post-measurement states. This is in contrast to the spatially-separated (usual Bell-type) scenario where all necessary information is given by E_k and does not depend on the way we implement them. Therefore, in the case of sequential measurements, the POVM elements E_k have limited physical meaning and more detailed knowledge, about the measurement operators, is required. This point is illustrated in the section about the temporal GHZ paradox.

Sequential generation of quantum states. Many multipartite quantum states can be generated in a sequential manner. For example, if the source emmiting photons is not initialized after each step, then the emmited photons are entangled [8, 9]. If we additionally allow for operations inside the source, depending on these operations, we can obtain different multiqubit states. It was demonstrated by Schön *et al.* [7] that sequentially generated states can be written as Matrix Product States (MPS) and, what is more, any MPS can be generated sequentially.

The Matrix Product State [10–13] representation is an efficient method of describing multipartite quantum states. It is most often used in the context of one-dimensional spin systems with local interactions since MPS states form sets over which Density Matrix Renormalization Group [14] variational method can be applied. The form of an MPS representation of a state depends on boundary conditions. For our purposes we focus on the open boundary conditions for which MPS are described as a sequence of matrices $A^n_{i_n}$

$$|\Psi\rangle=\sum_{i_1\ldots i_n=1}^{d}A^{[1]}_{i_1}\ldots A^{[N]}_{i_N}\,|i_1\ldots i_N\rangle\,. \quad (9)$$

The first and the last matrices are vectors and each matrix $A^n_{i_n}$ has a maximum dimension $D\times D$. D is so called bond dimension and is an important parameter characterising MPS state. Namely, D is the largest rank of the reduced density matrix with respect to every cut. It was shown in [15] that the rank of the reduced density matrix is a measure of entanglement and therefore the parameter D contains information about the entanglement structure of the state.

For sufficiently large bond dimension any state of a Hilbert space can be represented as an MPS [15]. However in practice the most commonly used is the class of MPS with low bond dimension as it describes well low energy states of 1D systems with local interactions. In the context of quantum information theory we can also find number of states playing an important role in information processing that are MPS with small bond dimension. For example multi-qubit GHZ states [16] and W states [17] have bond dimension $D=2$.

To sequentially generate an MPS state of bond dimension D we start from a chain of N initially uncorrelated d-dimensional particles and implement one of two methods. The first is due to Schön *et al.* [7] who use an ancilla system with dimension D and sequentially couple it to the d-dimensional particles performing unitary operations on each party and the ancillary system. In the last step the ancilla decouples from the system enabling us to generate a pure entangled state. The second method [18] is the one we focus on in this paper. In this method we apply unitary operations on m neighbouring particles in a sequential manner and as a result we obtain a state with bond dimension $D=d^{m-1}$. Given MPS form of a state with N matrices $A^{i_k}_k$ we can find $(N-1)$ unitary

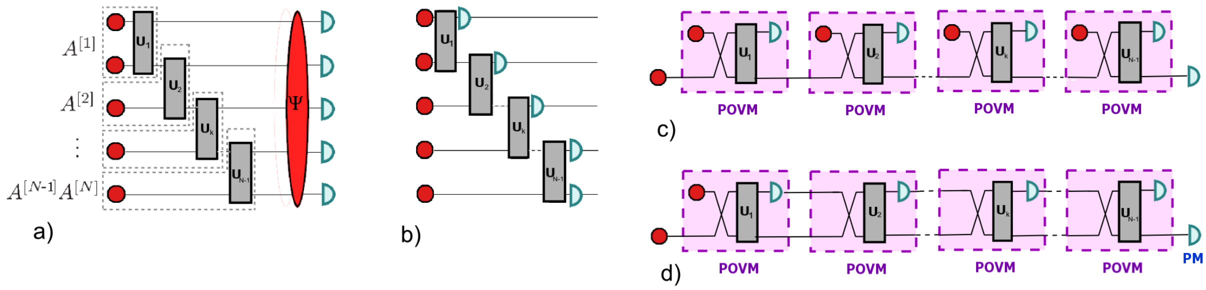

FIG. 1. Sequential generation of a multipartite MPS state with bond dimension two, and its temporal counterpart. a) Two-particle unitary gates U_k (grey rectangles) are sequentially performed on qubits (red circles). After such generation qubits are in the entangled state Ψ (big red shape). In the end projective measurements (blue shapes) can be done on qubits. Dashed lines indicate how unitary gates are related to MPS matrices; b) Measurements can be shifted in time; c) the circuit can be rearranged into a sequence of quantum channels performing POVM measurements; d) We can reduce the required resources to two qubits. After each measurement, one of the qubits is reset and recycled into the remaining protocol.

matrices U_k by use of singular value decomposition as it is explained in [7].

Not only pure states can be generated this way – we can easily generalize the method for a class of mixed states. Group of the particles can be traced out in an effect leaving us with a multipartite mixed state with the bond dimension $D_\rho \leq D_\psi$, where D_ψ describes an original state. The more mixed state we want to obtain the higher dimensional must be the cut bond. So to create n-partite mixed state with bond dimension D that has density matrix of the rank D' we need to sequentially create a purification of this state that contains n-qubits and one particle of dimension D', i.e. $n + \log D'$ qubits.

In particular it is instructive to consider the case of generating pure qubit states with $D = 2$ using two-qubit unitary gates. The scheme is depicted in Fig.1.

Sequential POVM measurements. As described, for any MPS state with bond dimension $D = 2$ we can determine its sequential generation scheme. Now we can utilize this scheme to find a sequence of measurements performed on a *single* qubit such that the correlations between these measurements are exactly the same as correlations measured on the MPS state. Such transition from the spatial to the temporal correlations is depicted in Fig. 1. As illustrated in Fig.1a, at the kth step of the preparation procedure one of the particles prepared in the previous step is being entangled by the gate U_k with the particle that has not been used up to now. This particle is next subject to the gate U_{k+1}, whereas the other particle is left untouched during the rest of the procedure. This important characteristic of the sequential preparation allows one to perform a projective measurement on the first particle right after the gate U_k is applied without disturbing the latter process of preparing the state (see Fig. 1b).

The entire protocol can equivalently be seen as a quantum channel with a single particle input and output that realises a two outcome POVM measurement (Fig. 1c). The corresponding measurement operators $M_i^{(k)}$ can be determined from the relation [19]

$$U_k \left|\psi\right\rangle \otimes \left|0\right\rangle = \sum_{i=\pm 1} \left(M_i^{(k)} \left|\psi\right\rangle\right) \otimes V \left|i\right\rangle, \qquad (10)$$

where V is a unitary rotation from the standard basis to the measurement basis of the k-th observer.

All the above discussion can be summarized in the following proposition being a consequence of equivalence of quantum circuits depicted in Fig. 1.

Proposition 1. *For any N-partite MPS state ψ with bond dimension 2, the arbitrary spatial correlation function for local projective measurements can be simulated by a sequence of $N-1$ consecutive two-outcome POVM measurements followed by a projective measurement in the final step.*

The above construction can be experimentally implemented with only two qubits (Fig. 1d). Such qubit recycling has already been implemented in an experimental realization of the Shor's algorithm [20].

Temporal GHZ paradox. The previous literature [2, 6] suggests, that in the scenario in which a single particle is measured sequentially N times, there is no possibility for genuinely N-partite quantum correlations to arise. However, Proposition 1 shows that this is not the case when we allow generalized quantum measurements at some steps. Especially interesting is the case of N-partite correlations of a GHZ-state

$$\left|\mathrm{GHZ}\right\rangle = \frac{1}{\sqrt{2}} \left(\left|0\dots 0\right\rangle + \left|1\dots 1\right\rangle\right) \qquad (11)$$

which give rise to the so called GHZ paradox. The tripartite version is commonly known [16], whilst the N-partite version [21] is equivalent to the *modulo-4 sum problem* in communication complexity theory [22]. The general form of a correlation function of a GHZ state is given by

$$E(\theta_1, \phi_1, \ldots, \theta_N, \phi_N) = \sin(\theta_1) \cdot \ldots \cdot \sin(\theta_N) \times \cos(\phi_1 + \ldots + \phi_N) \qquad (12)$$

where (θ_k, ϕ_k) denote a Bloch-sphere parametrization of a projective measurement on a k-th qubit.

Since to produce sequentially an N-partite GHZ state one uses a CNOT operation as a unitary entangling gate U_k in eq. (10), it is easy to calculate the POVM measurement operators for this case:

$$M_{-1}^{(k)} = \begin{pmatrix} \cos(\theta_k/2) & 0 \\ 0 & e^{-i\phi_k}\sin(\theta_k/2) \end{pmatrix}$$
$$M_{1}^{(k)} = \begin{pmatrix} e^{i\phi_k}\sin(\theta_k/2) & 0 \\ 0 & -\cos(\theta_k/2) \end{pmatrix} \qquad (13)$$

They give rise to the following POVM elements:

$$E_{-1}^{(k)} = \begin{pmatrix} \cos(\theta_k/2)^2 & 0 \\ 0 & \sin(\theta_k/2)^2 \end{pmatrix}$$
$$E_{1}^{(k)} = \begin{pmatrix} \sin(\theta_k/2)^2 & 0 \\ 0 & \cos(\theta_k/2)^2 \end{pmatrix} \qquad (14)$$

The above implementation of a POVM allows us to obtain the general correlation function for local measurements on a GHZ state by performing sequential POVM measurements on a single qubit followed by a projective one at the last step. Thus we can in fact implement the GHZ paradox in time.

Note that in the above construction, the POVM elements (14) do not depend on the phase ϕ, which enters the non-classical part of the correlation function of the GHZ state, whereas the measurement operators (13) directly depend on this parameter. This shows that the information about the phase is somehow encoded in the way the state collapses at each stage of the sequence of measurements and not in the statistics of individual measurements. This illustrates that knowledge about POVM elements alone is not sufficient to determine temporal correlations.

Multipartite temporal Bell inequalities. The possibility of generating genuinely multi-point temporal correlations enables us to consider multipoint temporal Bell inequalities, a violation of which detects truly n-fold temporal quantum correlations. Let us consider n-partite Bell inequalities with two settings per observer proposed by Seevinck and Svetlichny [23]:

$$\sum_I (-1)^{\frac{t(I)(t(I)+1))}{2}} E(I) \leq 2^{n-1} \qquad (15)$$

where $I = (i_1 = \{0,1\}, \ldots, i_n = \{0,1\})$ denotes the set of measurement settings for each observer, $E(I)$ denotes the correlation function for the measurements and $t(I)$ is the number of occurrence of index 1 in the sequence I. These inequalities are fulfilled by any correlations arising from conditional probabilities of outcomes which are bi-separable. Using quantum mechanical resources one can obtain the maximal violation $\sqrt{2}\cdot 2^{n-1}$ with an n-partite GHZ state [23].

We can treat these inequalities as temporal Bell inequalities in the scenario proposed by Brukner *et al.* [2]. A single system is measured sequentially at n time instances, and at each stage two possible settings are chosen. A violation of the inequalities given in Eq. (15) indicates that the temporal correlation function is not bi-separable. Due to properties (4) and (5) every correlation function arising from a sequence of projective measurements on a qubit is factorisable, therefore a sequence of such measurements cannot violate the above inequalities. However, using our construction, we can violate it up to $\sqrt{2}\cdot 2^{n-1}$ using a sequence of POVM measurements given by (13) and (14), which reconstructs the correlation function (12) of the GHZ state.

We leave it an open question whether there exist a sequence of POVM measurements on qubits which violate temporal Svetlichny inequalities up to the algebraic bound.

Conclusions. We have shown the possibility of obtaining genuinely multi-point temporal quantum correlations in a scenario of sequential generalized quantum measurements on a single particle. In this way we recover in a temporal experiment an arbitrary correlation function of any multipartite quantum state that has a MPS representation with bond dimension 2. Our result sheds a new light on a relation between spatial and temporal quantum correlations, showing that the similarities between these scenarios are much deeper than was presented in previous works. Moreover, a simple modification of existing experiments would allow our construction to be implemented in the laboratory, likely giving a chance of observing the temporal GHZ paradox for higher number of measurements than the number of particles for which this paradox has been observed in space. In analogy to the experiment presented in [20] our construction can be used to implement quantum algorithms which require genuine n-partite correlations in a temporal scenario.

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