Statistical estimation of a class of self-regulating processes

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Abstract

Self-regulating processes are stochastic processes whose local regularity, as measured by the pointwise Hölder exponent, is a function of amplitude. They seem to provide relevant models for various signals arising e.g. in geophysics or biomedicine. We propose in this work an estimator of the self-regulating function (that is, the function relating amplitude and Hölder regularity) of the self-regulating midpoint displacement process introduced in Barrière *et al.* (2012) and study some of its properties. We prove that it is almost surely convergent and obtain a central limit theorem. Numerical simulations show that the estimator behaves well in practice.

Key words: Confidence interval, Pointwise regularity, Strongly consistent estimator, Self-regulating function

1 Introduction and Background

Studying the local regularity of stochastic processes is important in various areas such as stochastic partial differential equations and approximation theory, with applications outside of mathematics in geophysics, signal and image processing, biomedicine or financial modelling. The pointwise Hölder exponent (see next section for definitions) is often used in this connection. For applications, it is useful to have available versatile models allowing one to fit phenomena where the local regularity evolves in time/space, with associated statistical estimators. One popular such model is *multifractional Brownian motion* (mBm) Peltier & Lévy Véhel (1995); Ayache (2002); Benassi *et al.* (1997); Falconer (2003). This is a Gaussian extension of fractional Brownian motion where the Hurst exponent H is replaced by a smooth function ranging h in $(0, 1)$. The pointwise Hölder exponent is then equal to $h(t)$ almost surely for all t. Robust statistical methods have been developed for estimating h Bardet & Surgailis (2013, 2011); Falconer & Fernàndez (2007), permitting meaningful applications to sampled data. Stable counterparts of mBm have also been studied, see, e.g., Stoev & Taqqu (2004, 2005).

In the case of mBm, pointwise regularity is tuned exogenously through the h function. Another paradigm for models with a time varying Hölder exponent is the one of *self-regulating processes*. A process X is called self-regulating if, almost surely, its pointwise Hölder exponent at any point t is equal to $g(X(t))$, where g is a deterministic smooth function. The study of self-regulating processes is motivated by experimental findings: for certain natural signals such as electrocardiograms, temperature records, or natural terrains, there seems to exist a link between the amplitude of the measurements and their pointwise regularity Echelard *et al.* (2010); Echelard & Lévy Véhel (2012). For instance, in young mountains, irregularity typically increases with altitude.

Various classes of self-regulating processes have been constructed in Barrière *et al.* (2012), where some of their probabilistic properties were studied. In view of applications, two issues must be addressed: testing whether the data at hand indeed display self-regulation, and estimating the self-regulating function g. We focus on the second task in this work. More precisely, we cope with the estimation problem for a particular class of such processes, termed *self-regulating midpoint displacement processes*. In this frame, this means obtaining an estimator of the self-regulating function g from sampled data.

The remaining of this article is organised as follows: we recall the definition of the self-regulating midpoint displacement process in the next section. Section 3 presents an estimator of g for this process. Its almost sure convergence is studied in Section 4. Section 5 proves a Central Limit Theorem. Numerical experiments are presented in Section 6. Finally, an appendix gathers some of the more technical or lengthy proofs.

2 Self-regulating midpoint displacement process

We recall the definition and the properties of a self-regulating process constructed in Barrière *et al.* (2012). We will make use of the Schauder basis. Denote

$$\varphi(t) = \begin{cases} 2t & \text{for} \quad t \in [0, 1/2] \\ 2 - 2t & \text{for} \quad t \in (1/2, 1] \\ 0 & \text{for} \quad t \notin [0, 1] \end{cases}$$

and its dilated and translated versions $\varphi_{j,k}(t) = \varphi(2^j t - k)$ (note that we do not use an L^2 normalisation here). It is well-known that $(\varphi_{j,k})_{j\in\mathbb{N},k\in\{0,\ldots,2^j-1\}}$ forms a basis of $C([0, 1])$.

Recall that standard Brownian bridge has the following representation (Bhattacharya & Waymire, 2007, Lemma 7, p. 137):

$$B = \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} 2^{-j/2}\varphi_{j,k} Z_{j,k}, \tag{1}$$

where the $Z_{j,k}$ are independently and identically distributed (i.i.d.) Gaussian random variables $N(0,1)$ (that is with zero mean and unit variance). Heuristically, the fact that Brownian bridge has everywhere almost surely Hölder exponent $\frac{1}{2}$ is related to the scale factor $2^{-j/2}$ in the sum above: the term $2^{j/2}\varphi_{j,k}$ contributes to a variation in amplitude of size $2^{j/2}$ within a time duration of 2^j; thus, variations of time scale $h = 2^j$ are typically of order $h^{1/2}$. A self-regulating process may then obtained by replacing the scale factor $2^{-j/2}$ by an adequate quantity depending on the local amplitude of the process. This is achieved through an iterative construction as follows: set $X_{-1} = 0$. At each step j, $j \in \mathbb{N}$, we add to the process X_{j-1} a "layer" at scale 2^{-j} which is a linear combination of functions $\varphi_{j,k}$. In view of obtaining the self-regulating property, and with inspiration from (1), we weight each $\varphi_{j,k}$ with an expression of the form $2^{-j\alpha_{j,k}}Z_{j,k}$ for a well chosen $\alpha_{j,k}$. Since each $\varphi_{j,k}$ is centred on the point $2^{-j}(k+1/2)$, we take $\alpha_{j,k} = g(X_{j-1}(2^{-j}(k+1/2)))$. Heuristically, this coefficient will yield a regularity equal to the value of $g(X_{j-1})$ at $2^{-j}(k+1/2)$. When j tends to infinity, this will just be $g(X)$.

It turns out that the Gaussian character of the random variables $Z_{j,k}$ is not crucial for our purpose. Rather, we will need the following assumption (which is more restrictive than the one considered in Barrière *et al.* (2012)):
Assumption $\mathcal{A}$:
For any positive c, almost surely, there exists N in $\mathbb{N}$ such that:

$$\forall j \geq N, \max_{k=0..2^j-1} |Z_{j,k}| \leq 2^{jc}.$$

Assumption $\mathcal{A}$ is clearly fulfilled when the $(Z_{j,k})_{j,k}$ follow an $N(0,1)$ law.

For our purpose, a self-regulating random midpoint displacement process is defined as follows:

Theorem and Definition 1. *Let $Z_{j,k}$ be i.i.d. centred random variables with finite variance σ^2 satisfying Assumption $\mathcal{A}$. Let g be a C^1 function from $\mathbb{R}$ to (a,b), where $0 < a < b < 1$. Set $X_{-1} = 0$ and define the sequence of processes $(X_j)_{j\in\mathbb{N}}$ on $[0,1]$:*

$$X_j(t) = X_{j-1}(t) + \sum_{k=0}^{2^j-1} 2^{-jg\left(X_{j-1}\left((k+\frac{1}{2})2^{-j}\right)\right)} Z_{j,k}\varphi_{j,k}(t). \tag{2}$$

Then, almost surely, the sequence $(X_j)_j \in \mathbb{N}$ converges uniformly to a continuous process X called self-regulating midpoint displacement process (abbreviated hereafter srmdp).

The first steps in this construction are illustrated on Figure 1; at step j, the piecewise affine process X_{j-1} is modified as follows: the midpoint of each of its segments is vertically displaced by a random amount depending on the height of this point.

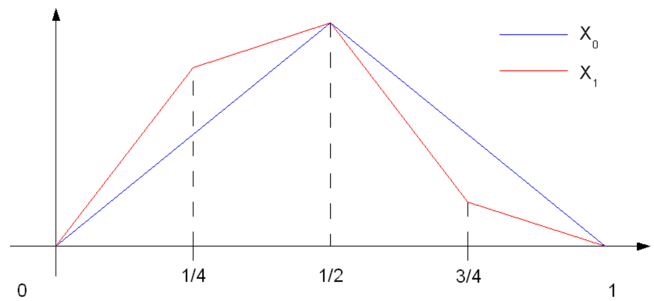

Figure 1: Processes X_0 and X_1.

The proof of the above theorem, as well as of the other results in this section may be found in Barrière *et al.* (2012). It is easy to see that X is a centred process with finite variance (bounded by $\frac{1}{1-2^{-2a}}$).

Note that the limitation on the upper bound on the range of g is solely due to the fact that the Schauder functions are not C^1. A theory very similar to the one presented here may be developed with smooth interpolating splines in lieu of $(\varphi_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$. In this case, g may be taken to range in $(0,n]$ with an arbitrarily large n provided the spline is smooth enough.

Remark 1. *Note that the values $X(k2^{-j})$, $k = 0,\dots,2^j$, are known once the process X_{j-1} has been computed. Indeed, for all $m \geq 0$, and $j \in \mathbb{N}, k \in \{0,\dots,2^j-1\}$, it holds that $X(k2^{-j}) = X_{j+m}(k2^{-j})$. As a consequence, determining X at dyadic points requires only a finite number of iterations of Formula* (2).

Recall the definition of the pointwise Hölder exponent at x_0 of a bounded process f: $\mathbb{R} \to \mathbb{R}$. This is the number α such that :

- $\forall\gamma < \alpha, \lim_{h\to 0} \frac{|f(x_0+h)-P(h)|}{|h|^\gamma} = 0,$
- if $\alpha < +\infty, \forall\gamma > \alpha, \limsup_{h\to 0} \frac{|f(x_0+h)-P(h)|}{|h|^\gamma} = +\infty$

where P is a polynomial of degree not larger than the integer part of α (this definition is valid only if α is not an integer; it has to be adapted otherwise).

We shall denote α_f the *Hölder function* of f, that is, for each x, $\alpha_f(x)$ is the pointwise Hölder exponent of f at x. Clearly, for X a continuous stochastic process, $\alpha_X(x)$ is in general a random variable (with the notable exception of Gaussian processes Ayache & Taqqu (2005)), so that the pointwise Hölder function is also a stochastic process. The main result on the pointwise regularity of X is the following one:

Theorem 1. *Let X be an srmdp. Then, with probability one,*

$$\alpha_X(.) = g(X(.)).$$

In other words, X is indeed self-regulating. Note that this implies that X cannot be a Gaussian process, even when the $(Z_{j,k})$ are Gaussian random variables.

Figure 2 displays realizations of an srmdp with an increasing g function. Note in particular that the larger X is, the more regular it looks, and conversely. However, in contrast with mBm, one cannot say in advance where the process will be smooth and where it will be irregular, since this depends on each realization, as is apparent from the figure. Figure 3 displays another example, yielding processes which are smooth when their amplitude is close to 0, and irregular otherwise (the construction is here slightly modified so that the processes start at -1 and ends at 1 instead of 0)[1].

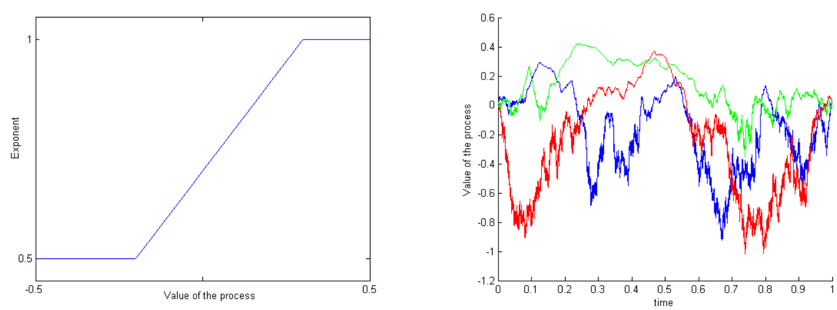

Figure 2: Left: self-regulating function g. Right: three sample paths of X.

3 An estimator of the self-regulating function

We derive in this section the form of our estimator of g. Note that determining the self-regulating function is sufficient to characterize an srmdp. In particular, the Hölder exponent at any point t is easily computed from the relation $\alpha_X(t) = g(X(t))$.

The main difficulty when dealing with an srmdp is that we do not have a closed form expression for X nor its moments (except its mean). Additionally,

[1] The careful reader will have noticed that, in both examples, g is not C^1 but merely Hölder continuous. The theory goes through for such functions, although we shall not consider this extension here

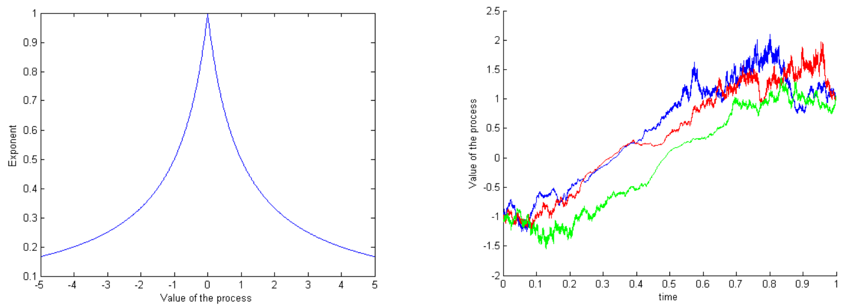

Figure 3: Left: self-regulating function g. Right: three sample paths of X.

neither its marginal laws nor its dependence structure are known. However, the finite scale approximations of X are relatively simple: indeed, conditionally on X_j, X_{j+1} is a sum of independent random variables, as is apparent from (2). This is a the starting point of our estimation scheme.

Hereafter we assume that a sample of $N = 2^j$ regularly spaced points of the process is given. In other words, we have observed a realisation of X_j of Definition 1. Clearly, from a single realization, one cannot hope to estimate the whole of g: only at those x that lie in the observed range of X may one obtain an approximate value of $g(x)$. Fix then x in $(\min(X_j), \max(X_j))$. Since X is continuous, there exists $t \in I$ such that $X(t) = x$. Moreover, given $\varepsilon > 0$, there exists $\eta > 0$ such that $[t-\eta, t+\eta] \subset X^{-1}([x-\varepsilon, x+\varepsilon])$. We will need to estimate the value of η as a function of ε. In that view, we make use of the self-regulating property: the pointwise Hölder exponent of X at t is almost surely $g(X(t)) = g(x)$. This means that, for small enough $\eta > 0$, there exists $C > 0$ such that the following holds:

$$\forall u,\ |t-u| \leq \eta \Rightarrow |X(t) - X(u)| \leq C|t-u|^{\beta'},$$

for all $\beta' < g(x)$. Thus, setting $\eta = \varepsilon^{\frac{1}{\beta}}$ with $\beta < \beta'$, ensures that, for ε small enough, all u inside $[t-\eta, t+\eta]$ are almost surely such that $|X(t) - X(u)| \leq \varepsilon$.

Fix such an ε. Let $s_1, \ldots, s_{n_j}$ denote the real numbers of the form $(k + 1/2)2^{-j}$ such that $X_{j-1}(s_i) \in\ I := [x-\varepsilon, x+\varepsilon]$, and $k_1, \ldots, k_{n_j}$ denote those integers k such that $s_i = (k_i + 1/2)2^{-j}$ (see Figure 4). Note that, by continuity, $n_j > 0$ for j large enough. More precisely,

$$n_j \geq [\eta 2^j] = [\varepsilon^{\frac{1}{\beta}} 2^j], \quad \forall \beta < g(x), \tag{3}$$

where $[z]$ denotes the integer part of z. Both n_j and the integers $k_1, \ldots, k_{n_j}$ depend only on X_{j-1}, and thus on $(Z_{l,k})$ for $l \leq j-1, k = 0, \ldots, 2^l - 1$. In other words, n_j and $k_1, \ldots, k_{n_j}$ are $\mathcal{F}_{j-1}$-measurable, where $\mathcal{F}_{j-1}$ denotes the $\sigma-$algebra generated by the random variables $Z_{l,k}$ for $l \leq j-1, k = 0, \ldots, 2^l - 1$. See again Figure 4. What is crucial for our purpose is that, once the sample

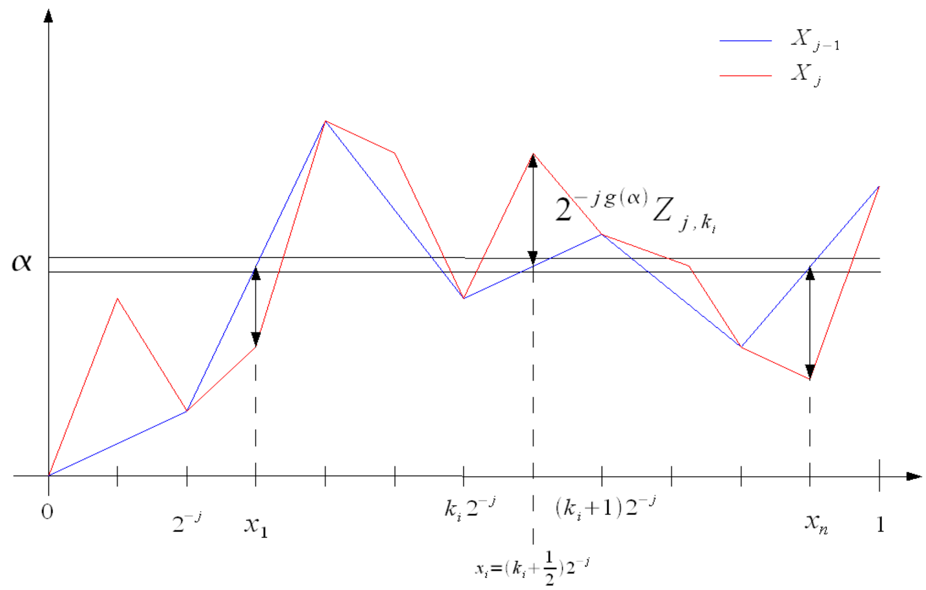

Figure 4: The processes X_{j-1} and X_j.

has been observed and x, ε have been fixed, the value of n_j is known. It is thus enough in the sequel to compute the relevant laws conditional on the value of n_j.

By definition of X_j, the following equality holds for all $i \in [1, \ldots, n_j]$:

$$X_j(s_i) - X_{j-1}(s_i) = 2^{-jg(X_{j-1}(s_i))} Z_{jk_i}. \tag{4}$$

Set $W_{j,n_j} = \sum_{i=1}^{n_j} Z_{jk_i}^2$ and

$$T_{j,n_j} = \sum_{i=1}^{n_j} \left(X_j(s_i) - X_{j-1}(s_i)\right)^2. \tag{5}$$

Since g is C^1, there exists a constant K such that:

$$|g(X_{j-1}(s_i)) - g(x)| \leq K\varepsilon \tag{6}$$

for all $i = 1, ..., n_j$. Squaring (4) and adding over the values of i, we get:

$$2^{-2j(g(x)+K\varepsilon)} W_{j,n_j} \leq T_{j,n_j} \leq 2^{-2j(g(x)-K\varepsilon)} W_{j,n_j}. \tag{7}$$

In order to derive the form of the estimator, it is useful to consider first the particular case where the $Z_{j,k}$ are Gaussian random variables.

Lemma 1. *Let $u_\gamma(k) > 0$ be the γ-quantile of a χ^2 law with k degrees of freedom. Assume that $Z_{1,1}$ is a Gaussian random variable. Then:*

$$P(W_{j,n_j} \in [\sigma^2 u_\gamma(n_j), \sigma^2 u_{1-\gamma}(n_j)]) = 1 - 2\gamma.$$

See Section 7.1 for a proof of this Lemma.

Using (7), Lemma 1 translates into:

$$P\left(g\,(x) \in \left[-K\varepsilon + \frac{1}{2j}\log_2\left(\frac{\sigma^2 u_\gamma(n_j)}{T_{j,n_j}}\right)\ ,\ K\varepsilon + \frac{1}{2j}\log_2\left(\frac{\sigma^2 u_{1-\gamma}(n_j)}{T_{j,n_j}}\right)\right]\right) \geq 1 - 2\gamma. \quad (8)$$

This provides confidence bands when $Z_{1,1}$ is Gaussian. These bands cannot be computed since they depend on the unknown constant K. Nevertheless, the middle of this confidence interval does not depend on K, and thus can be used as a pointwise estimator of $g(x)$. Note furthermore that, if n_j converges almost surely (a.s.) to infinity when j goes to infinity, one has

$$\begin{aligned}\log_2(u_{1-\gamma}(n_j)) &\sim \log_2(\sqrt{2n_j}q(1-\gamma) + n_j) \\ &\sim \log_2(n_j) \quad \text{(a.s.)}\end{aligned} \quad (9)$$

where $q(\gamma)$ denotes the γ-quantile of the standard Gaussian distribution.

These heuristic considerations lead one to consider the following pointwise estimator of $g(x)$ for x in $(\min(X_j), \max(X_j))$:

$$\hat{g}_j(x) := \frac{\log_2(n_j)}{2j} - \frac{\log_2(T_{j,n_j})}{2j} + \frac{\log_2(\sigma^2)}{2j}, \quad (10)$$

where T_{j,n_j} is defined in (5), or when σ^2 is unknown,

$$\widetilde{g}_j(x) := \frac{\log_2(n_j)}{2j} - \frac{\log_2(T_{j,n_j})}{2j} + \frac{\log_2(\hat{\sigma}_j^2)}{2j}, \quad (11)$$

where $\hat{\sigma}_j^2$ is an estimator of σ^2. The problem of estimating σ^2 is dealt with in Section 4, see Formula (15) and Proposition 1.

Remark 2. *Let $\mathcal{D}$ be the support of the law of $Z_{1,1}$. It is easy to see that, for all $j > 0$, the range of X_j contains the interval $[0, 2^{-g(0)}Z_{1,1}]$ (or $[2^{-g(0)}Z_{1,1}, 0]$). As a consequence, for $x \in 2^{-g(0)}\mathcal{D}$, n_j is positive (and thus tends to infinity with n) with positive probability. This entails that $\hat{g}_j(x)$ and $\widetilde{g}_j(x)$ are well defined with positive probability at least on $2^{-g(0)}\mathcal{D}$. For instance, when $Z_{1,1}$ is Gaussian, both estimators are defined with positive probability on $\mathbb{R}$. As a consequence,*

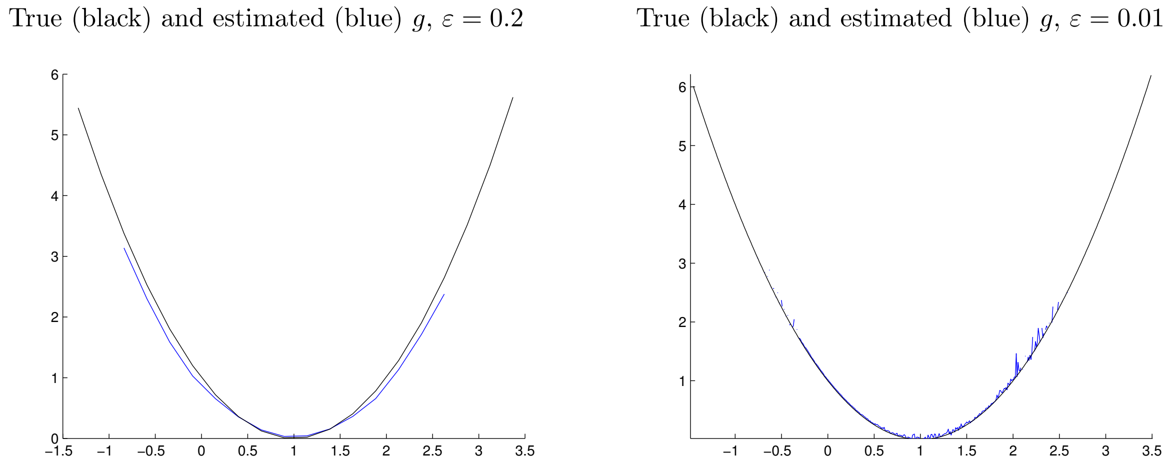

Figure 5: Estimation of the self-regulating function for an srmdp sampled on 65536 points with two values for ε.

Both estimators $\hat{g}_j$ and $\widetilde{g}_j$ depend on ε. Figure 5 shows examples of estimation using (10) on an srmdp with self-regulating function $g(z) = (1-z)^2$. The left graph is obtained with $\varepsilon = 0.2$, while ε is equal to 0.01 on the graph on the right. As may be expected, for smaller ε, the estimated self-regulating function is rough but well centred on the true g, as for larger ε, it is smoother but departs sometimes significantly from g. This is typical of a bias-variance trade-off situation. Indeed, a large ε translates into a large number of samples used to compute the statistics, which decreases the variance. However, it also increases the bias since it essentially assumes that g is constant over an interval. When this is not the case, a large ε means that we are incorporating in the computation points where the exponent may significantly differ from $g(x)$. The choice of ε is discussed in Section 5, where conditions are given ensuring that a Central Limit Theorem holds for the estimators (10) and (11).

4 Almost sure convergence results

In this section, we first propose an estimator of σ^2 and prove that it is almost surely convergent (Proposition 1). Then, we prove almost sure convergence of the estimators of g defined by (10) and (11) (Theorem 2). We begin with a technical lemma.

Lemma 2. *Let X be an srmdp. Define, for $j \geq 2$,*

$$\mathcal{M}_j = \left\{ x : \exists k \in \{0, \dots, 2^j - 1\}, x = X\left(\left(k + \frac{1}{2}\right) 2^{-j}\right)\right\}.$$

Set $m_j = \min(\mathcal{M}_j), M_j = \max(\mathcal{M}_j)$, and $\varepsilon_j = 2^{-j\gamma}$, where $\gamma < a$. Then, almost surely, for all large enough j, there exists an interval I of width at most $4\ \varepsilon_j(M_j - m_j)$ included in $[m_j, M_j]$ such that:

$$\text{card}\left(I \bigcap \mathcal{M}_{j-1}\right) \geq \frac{2^{j(1-\gamma)}}{2} \tag{12}$$

and

$$\text{card}\left(I \bigcap \mathcal{M}_{j}\right) \geq \frac{2^{j(1-\gamma)}}{2}. \tag{13}$$

Proof. To simplify the notations, we assume that, almost surely, card $(\mathcal{M}_{j-1}) = 2^{j-1}$. This is for instance the case if the law of $Z_{1,1}$ has a density. The general case may be handled similarly.

Divide the interval $[m_{j-1}, M_{j-1}]$ into bins of equal size $\varepsilon_j(M_{j-1} - m_{j-1})$. More precisely, let $\Delta = [\frac{1}{\varepsilon_j}]^{-1}(M_{j-1} - m_{j-1})$, and define, for $l = 1, \dots, [\frac{1}{\varepsilon_j}]$,

$$I^l_{j-1} = [m_{j-1} + (l-1)\Delta, m_{j-1} + l\Delta].$$

Inequality (12) is just an application of the Dirichlet drawer principle: the interval $[m_{j-1}, M_{j-1}]$ contains all the points of $\mathcal{M}_{j-1}$. The $[\frac{1}{\varepsilon_j}]$ intervals I^l_{j-1} form a partition of $[m_{j-1}, M_{j-1}]$. Thus, almost surely, at least one of these intervals must contain at least $\frac{2^{j-1}}{[\frac{1}{\varepsilon_j}]} \geq \frac{2^{j(1-\gamma)}}{2}$ points. Let $I^{l^*}_{j-1}$ be one of these intervals, and let I be the interval whose center coincide with the one of $I^{l^*}_{j-1}$ and whose width is twice the one of $I^{l^*}_{j-1}$. Obviously, (12) holds true for such an I, and the width of I is not larger than $4\ 2^{-j\gamma}(M_j - m_j)$. Let K^* denote the set of integers k such that $X\left(\left(k + \frac{1}{2}\right) 2^{-(j-1)}\right)$ belongs to $I^{l^*}_{j-1}$. A sufficient condition for a point of the form $X\left(\left(k + \frac{1}{2}\right) 2^{-j}\right)$ with k in K^* to belong to I is that

$$|Z_{j,k}| < \frac{2^{j(a-\gamma)}}{2}. \tag{14}$$

Assumption $\mathcal{A}$ ensures that (14) is verified almost surely for all k in in K^* provided j is large enough. As a consequence, the inequality card $(I \bigcap \mathcal{M}_j) \geq \frac{2^{j(1-\gamma)}}{2}$ also holds true.

$\square$

In order to obtain a strongly consistent estimator of σ^2, we shall use estimators of g at two consecutive scales.

Fix γ in $(\frac{a}{1+2a}, a)$. Proposition 2 ensures that there exists an interval I of width at most $4\,2^{-j\gamma}(M_j - m_j)$ that contains simultaneously at least $\nu_j = \frac{2^{j(1-\gamma)}}{2}$ points of the form $X\left(\left(k+\frac{1}{2}\right)2^{-(j-1)}\right)$ and $X\left(\left(k+\frac{1}{2}\right)2^{-j}\right)$. Let us denote $t_1, ..., t_{\nu_j}$ the abscissas of these points. They are of the form $\left(k_i''+\frac{1}{2}\right)2^{-(j-1)}$ and $\left(k_i'+\frac{1}{2}\right)2^{-j}$

Denoting C_j the middle of the interval I, we consider the estimators $\hat{g}_j(C_j)$ defined by (10) at scales 2^j and 2^{j-1}, and use the intuitive fact that these estimators should be close to deduce an estimator of σ^2. More precisely, define

$$T_j' = \sum_{i=1}^{\nu_j} (X_j(t_i) - X_{j-1}(t_i))^2 \quad \text{and} \quad T_j'' = \sum_{i=1}^{\nu_j} (X_{j-1}(t_i) - X_{j-2}(t_i))^2.$$

The estimator of σ^2 is obtained by setting

$$\log_2\left(\hat{\sigma}_j^2\right) = j\log_2\left(\frac{T_j''}{\nu_j}\right) - (j-1)\log_2\left(\frac{T_j'}{\nu_j}\right) \tag{15}$$

This estimator depends on γ. The following proposition proves that it is strongly consistent and allows one to adjust γ in order to speed up the rate of almost sure convergence.

Proposition 1. *Let X be an srmdp. Assume that the distribution of $Z_{1,1}$ satisfies the following condition :*

> $\mathbf{H_Z}$ *: there exists $g > 0$ and $T > 0$ with the property that, for all $|t| < T$, $E(e^{tZ_{1,1}}) < e^{g\frac{t^2}{2}}$.*

Then the estimator $\hat{\sigma}_j^2$ defined by (15) converges almost surely to σ^2.

Moreover, the choice $\gamma = \begin{cases} a-\delta & \text{if } a \le 1/3 \\ 1/3 & \text{if } a > 1/3 \end{cases}$, where δ is an arbitrarily small positive real number, ensures that, almost surely, for j large enough,

$$\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) \le Cj^2 2^{-j(\min(a,\frac{1}{3})-\delta)} \tag{16}$$

where C is a positive constant.

Remark 3. *For centred random variables, Condition $\mathbf{H_Z}$ is equivalent to the existence of an exponential moment of arbitrary small order. It is obviously verified for instance for Gaussian random variables.*

Proof.

Let $W'_j = \sum_{i=1}^{\nu_j} Z^2_{j,k'_i}$ and $W''_j = \sum_{i=1}^{\nu_j} Z^2_{j-1,k''_i}$. Using Inequality (7), one gets

$$\left|\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right)\right| \leq j\log_2\left(\frac{W''_j}{\sigma^2\nu_j}\right) - (j-1)\log_2\left(\frac{W'_j}{\sigma^2\nu_j}\right) + 2j(j-1)K\varepsilon'_j$$

where $\varepsilon'_j = 4\ 2^{-j\gamma}(M_j - m_j)$. For j large enough, Proposition 4 (see Section 7.3 in the Appendix) implies that, for all $\beta \in (0,1)$ (C denotes a generic positive constant that may change from line to line),

$$\begin{aligned}\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) &\leq j\log_2(1+\nu_j^{(\beta-1)/2}) - (j-1)\log_2(1-\nu_j^{(\beta-1)/2}) + 2j(j-1)K\varepsilon'_j \\ &\leq Cj\nu_j^{(\beta-1)/2} + 2j(j-1)K\varepsilon'_j \\ &\leq C\left(\log_2(N)N^{-(1-\gamma)(1-\beta)/2} + \log_2(N)^2N^{-\gamma}\right) \qquad (17)\end{aligned}$$

where we recall that $N = 2^j$. As γ and β belong to $(0,1)$, we get

$$\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) \xrightarrow[a.s.]{N\to\infty} 0.$$

A lower bound of $\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right)$ with the same convergence rate is obtained in a similar way.

An optimal value for the upper bound on the right hand side of (17) is obtained by equating the exponents of N in the two terms inside the parenthesis. Given the condition that $\gamma \in (\frac{a}{1+2a}, a)$, and noticing that $\frac{a}{1+2a}$ is always smaller than $\frac{1}{3}$, one easily obtains that the optimal choice for γ is

$$\gamma^* = \begin{cases} a - \delta & \text{if } a \leq 1/3 \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad (18)$$

where δ is an arbitrarily small positive real number. For $\gamma = \gamma^*$, one has

$$\left|\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right)\right| \leq C\begin{cases}\log_2(N)^2N^{-a+\delta} & \text{if } a \leq \frac{1}{3} \\ \log_2(N)N^{-\frac{1}{3}(1-\beta)} & \text{otherwise.}\end{cases} \qquad (19)$$

This conclude the proof. $\square$

Let us now prove that (10) and (11) are strongly consistent estimators of g.

Theorem 2. *Let X be an srmdp and fix $x \in \mathbb{R}$.*

Choose a sequence $(\varepsilon_N)_N$ such that

$$\varepsilon_N = o(1) \text{ and } N^{-a} = o(\varepsilon_N) \qquad (20)$$

as $N = 2^j$ goes to infinity. Then the estimator defined by (10) *converges almost surely on the set* $\{\omega : x \in (\min(X), \max(X))\}$ *as* j *tends to infinity*

$$\hat{g}_j(x) - g(x) \xrightarrow[j\to\infty]{a.s.} 0. \tag{21}$$

Under the additional assumption $\mathbf{H_Z}$ *on the law of* $Z_{1,1}$*, the estimator defined by* (11) *converges almost surely on the set* $\{\omega : x \in (\min(X), \max(X))\}$ *as* j *tends to infinity:*

$$\widetilde{g}_j(x) - g(x) \xrightarrow[j\to\infty]{a.s.} 0. \tag{22}$$

Remark 4. *By Remark 2, when* $Z_{1,1}$ *is Gaussian, the above estimators allow one to evaluate* $g(x)$ *with positive probability for all real* x*.*

Proof. According to Definition 1, $(Z_{j,k}^2)_{j,k\in\mathbb{N}}$ is an array of i.i.d. random variables with finite variance and $E(Z_{1,1}^2) = \sigma^2$. By the Strong Law of Large Numbers for arrays proved in Hu *et al.* (1989), one has

$$\frac{1}{j}\sum_{k=1}^{j} Z_{j,k}^2 = \frac{W_{j,j}}{j} \xrightarrow[j\to\infty]{a.s.} \sigma^2.$$

Furthermore, since $x \in (\min(X), \max(X))$, there exists J such that, for all $j > J$, $x \in (\min(X_j), \max(X_j))$ and thus $n_j > 0$. Inequality (3) applied with $\beta = a$ and Condition (20) then imply that the sequence n_j converges almost surely to infinity as j tend to infinity. Hence $\frac{W_{j,n_j}}{n_j}$ converges almost surely to σ^2 when $j \to \infty$. By definition of $\hat{g}_j$ in (10), one has

$$\hat{g}_j(x) - g(x) = -\frac{1}{2j}\log_2\left(\frac{1}{n_j\sigma^2}T_{j,n_j}2^{2jg(x)}\right).$$

From (7), it is easy to deduce the following inequality

$$|\hat{g}_j(x) - g(x)| \le K\varepsilon_N + \frac{1}{2j}\left|\log_2\left(\frac{W_{j,n_j}}{n_j}\right) - \log_2(\sigma^2)\right|.$$

This concludes the proof of (21).
By definition of $\widetilde{g}_j(x)$ in (11), one has

$$\widetilde{g}_j(x) = \hat{g}_j(x) - \frac{1}{2j}\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right).$$

Therefore, (22) follows immediately from Proposition 1 and (21). □

5 Central Limit theorems

We state in this section Central Limit Theorems for both estimators (10) and (11).

Proposition 2. *Let X be an srmdp and fix $x \in \mathbb{R}$. Choose a sequence $(\varepsilon_N)_N$ of positive real numbers such that*

$$N^{-a} = o(\varepsilon_N) \qquad \text{and} \qquad \sqrt{n_j}\varepsilon_N \log_2(N) = o(1) \tag{23}$$

as $N = 2^j$ goes to infinity. Then, conditionally on $A_x = \{\omega : x \in (\min(X), \max(X))\}$,

$$j\sqrt{2n_j}(\hat{g}_j(x) - g(x)) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1),$$

that is,

$$P\left(j\sqrt{2n_j}(\hat{g}_j(x) - g(x)) \leq t \mid A_x\right) \to \Phi(t)$$

for all t, where Φ is the cumulative distribution function of a standard Gaussian random variable.

A proof of this result is given in the appendix, Section 7.2.

Remark 5. *Remark 2 ensures that, when $Z_{1,1}$ is Gaussian, the event A_x has positive probability for all real x.*

Remark 6. *Inequality (3) entails that $n_j \geq [N\varepsilon_N^{1/a}]$ a.s.. As a consequence, the assumption $N^{-a} = o(\varepsilon_N)$ ensures that n_j will converge almost surely to infinity. Moreover, it is always possible to select a number of points of the order of $N\varepsilon_N^{1/a}$. With this choice, one can find ε_N of the form $N^{-\gamma}$, with $\gamma \in (\frac{a}{2a+1}\,, a)$, such that the condition $\sqrt{n_j}\varepsilon_N \log_2(N) = o(1)$ holds. Now, the rate of convergence entailed by the theorem is of the order of $\frac{1}{j\sqrt{2n_j}}$. With $\varepsilon_N = N^{-\gamma}$, a lower bound on this rate is obtained upon replacing n_j by $\varepsilon_N^{\frac{1}{a}} N = N^{1-\frac{\gamma}{a}}$. Since $\gamma \in (\frac{a}{2a+1}\,, a)$, one sees that it is possible to reach a convergence speed not smaller than $\log_2(N)^{-1}N^{-\frac{a}{2a+1}+c}$, where $c > 0$ is an arbitrary small positive real number. One checks that, as is intuitively clear, the convergence speed is an increasing function of the global smoothness of the process (i.e. of a).*

Remark 7. *The length of the confidence interval in* (8) *is equal to*

$$\frac{\log_2(u_{1-\gamma}(n_j)) - \log_2(u_\gamma(n_j))}{2j} + 2K\varepsilon.$$

The first term in this sum may be interpreted as a variance term (increasing ε_N increases n_j and thus reduces the width of the χ^2 confidence interval), while the second one is a bias term. The choice of ε giving optimal length is the one which makes both terms of the same order. However, in order to obtain a Central Limit Theorem, the assumptions of Proposition 2 are needed, and they do not allow one to reach the optimum. Indeed, Condition (23) *implies that the bias $2K\varepsilon$ is negligible with respect to the variance $\frac{1}{2j}\log_2(u_{1-\gamma}(n_j)) - \log_2(u_\gamma(n_j))$ which is of the order of $\frac{q(1-\gamma)}{j\sqrt{n_j}}$ when $j \to \infty$.*

Let us now state a Central Limit Theorem for the more realistic case where the variance of $Z_{1,1}$ is unknown.

Theorem 3. *Let X be an srmdp such that $Z_{1,1}$ satisfies Assumption $\mathbf{H_Z}$ and fix $x \in \mathbb{R}$. Let $(\varepsilon_N)_N$ be a sequence of positive real numbers such that, for some $\kappa \in (0\,, \min(a, 1/3))$,*

$$N^{-\min(a,\frac{1}{3})+\kappa} = O(\varepsilon_N) \qquad \text{and} \qquad \sqrt{n_j}\varepsilon_N \log_2(N)^2 = o(1) \tag{24}$$

as $N = 2^j$ goes to infinity. Then, conditionally on A_x,

$$j\sqrt{2n_j}\log_2(2)(\widetilde{g}_j(x) - g(x)) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1).$$

Proof. One computes

$$j\sqrt{2n_j}\log_2(2)(\widetilde{g}_j(x) - g(x)) = \\ j\sqrt{2n_j}\log_2(2)(\hat{g}_j(x) - g(x)) - \sqrt{2n_j}\log_2(2)\frac{1}{2}\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right).$$

Since (24) implies (23), Proposition 2 applies and it is sufficient to prove that

$$\sqrt{n_j}\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) \xrightarrow[j\to\infty]{probability} 0$$

Applying (16) with $\delta = \kappa$, one obtains

$$\begin{aligned}\sqrt{n_j}\log_2\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) &\leq C\sqrt{n_j}\log_2(N)^2 N^{-min(a,1/3)+\kappa} \\ &\leq o(1)\varepsilon_n^{-1} N^{-\min(a,1/3)+\kappa} \\ &= o(1),\end{aligned}$$

where we have used Assumption (24) in the last two lines. □

Remark 8. *Selecting* $n_j = N\varepsilon_N^{1/a}$ *points and choosing* ε_N *of the form* $N^{-\gamma}$*, leads to* $\gamma \in (\frac{a}{2a+1}, \min(a, \frac{1}{3}))$*. One reaches to the same lower bound on the convergence rate as the one obtained in Remark 6.*

6 Numerical Experiments

The algorithm for estimating g is a direct application of Formulas (11) and (15). One starts by determining the range $[x_m, x_M]$ of observed values of X. One then fixes ε, and discretize $[x_m, x_M]$ at m regularly spaced points x_i (typically, m is of the order of 2 $\frac{x_M - x_m}{\varepsilon}$). For each x_i, one then finds the points $s_{n_1}, \ldots, s_{n_j}$ and computes T_{j,n_j} with Formula (5).

We display results of experiments in estimating the self-regulation function with (11) on signals sampled on 65536 points with g function equal to $0.2 + \frac{0.5}{(1+z^2)}$, and $Z_{1,1}$ following a centred normal law with variance respectively 0.3 (Figure 6), 1 (Figure 7) and 3 (Figure 8). Confidence intervals given by Theorem 3 are also shown. In order to estimate the variance in a more robust way, we average the estimator given by (15) over all intervals. Note that, since a new realisation is computed for each experiment, the part of the self-regulated function which is estimated differs in each figure.

7 Appendix

7.1 Proof of Lemma 1

It is enough to prove that the conditional law W_{j,n_j} knowing n_j is a χ^2 with n_j degrees of freedom. The random variables $(Z_{j,k_i})_{k_i}$ are independent, thus:

$$E(e^{i\theta W_{j,n_j}}|\mathcal{F}_{j-1}) = \prod_{i=1}^{n_j} E(e^{i\theta Z_{j,k_i}^2}|\mathcal{F}_{j-1}).$$

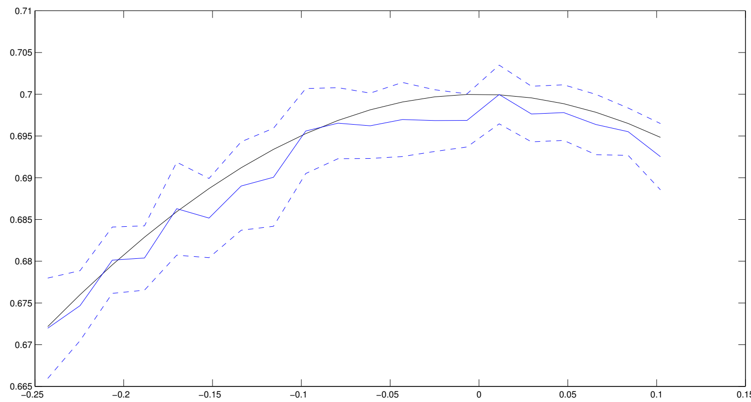

Figure 6: Theoretical g (black), estimated one (blue, solid) and confidence interval (blue, dotted), in the case where $Z_{1,1}$ follows a centred normal law with variance equal to 0.3.

The fact that Z_{j,k_i}^2 is independent of $\mathcal{F}_{j-1}$ entails that $E(e^{i\theta Z_{i,k_i}^2}|\mathcal{F}_{j-1}) = (1-2i\theta)^{-1/2}$. As a consequence,

$$E(e^{i\theta W_{j,n_j}}|\mathcal{F}_{j-1}) = (1-2i\theta)^{-n_j/2},$$

or:

$$E(e^{i\theta W_{j,n_j}}|n_j) = (1-2i\theta)^{-n_j/2}$$

which is the characteristic function of a χ^2 law with n_j degrees of freedom. Therefore we have shown that:

$$P(W_{j,n_j} \in [u_\gamma(n_j), u_{1-\gamma}(n_j)]|n_j) = 1 - 2\gamma.$$

This concludes the proof by taking the expectation. $\square$

7.2 Proof of Proposition 2

From the Definition (10) of $\hat{g}_j$, one deduces

$$\hat{g}_j(x) - g(x) = -\frac{1}{2j}\log_2\left(\frac{1}{\sigma^2 n_j} T_{j,n_j} 2^{2jg(x)}\right)$$

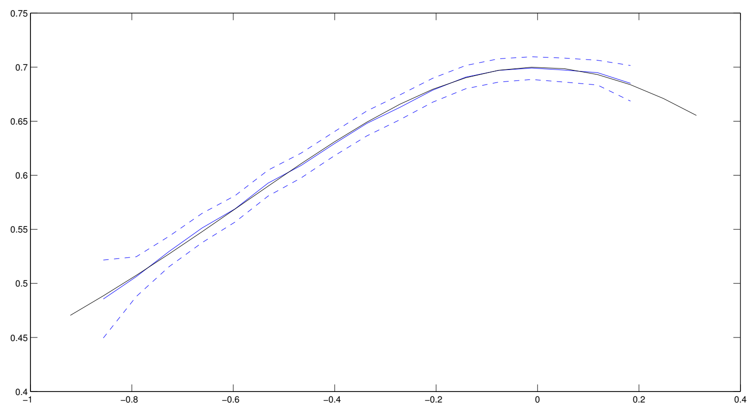

Figure 7: Theoretical g (black), estimated one (blue, solid) and confidence interval (blue, dotted), in the case where $Z_{1,1}$ follows a centred normal law with unit variance.

and

$$\begin{aligned}\frac{1}{\sigma^2 n_j} T_{j,n_j} 2^{2jg(x)} - 1 &= \sum_{i=1}^{n_j} \frac{1}{n_j} 2^{-2j(g(X_{j-1}(s_i))-g(x))} \frac{Z_{j,s_i}^2}{\sigma^2} - 1 \\ &= \sum_{i=1}^{n_j} w_{j,i} \left(\frac{Z_{j,s_i}^2}{\sigma^2} - 1\right) + \sum_{i=1}^{n_j} w_{j,i} - 1 \end{aligned} \tag{25}$$

where

$$w_{j,i} = \frac{1}{n_j} 2^{-2j(g(X_{j-1}(s_i))-g(x))}.$$

Step 1 : we prove that, conditionally on $\mathcal{F}_{j-1}$,

$$\frac{1}{v_j} \sum_{i=1}^{n_j} w_{j,i} \left(\frac{Z_{j,s_i}^2}{\sigma^2} - 1\right) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1) \tag{26}$$

where $v_j^2 = 2\sum_{i=1}^{n_j} w_{j,i}^2$.

Since $(Z_{j,i})_i$ and $(n_j, (w_{j,i})_i) \in \mathcal{F}_{j-1}$ are independent, convergence may be proved using Lindeberg's theorem (see for instance DasGupta (2008), Chapter 5). As a consequence, it suffices to check that almost surely:

$$\max_{i=1,\dots,n_j} \frac{w_{j,i}^2}{v_j^2} \to 0 \quad \text{as } j \to \infty.$$

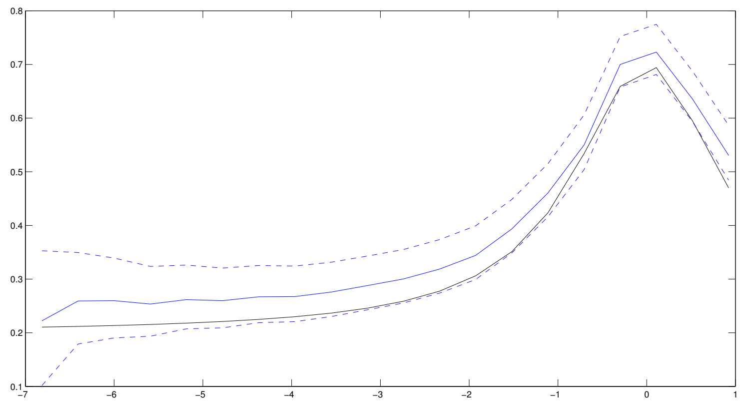

Figure 8: Theoretical g (black), estimated one (blue, solid) and confidence interval (blue, dotted), in the case where $Z_{1,1}$ follows a centred normal law with variance equal to 3.

Using (6), we have that, uniformly in $i = 1, ..., n_j$, $w_{j,i} \in \left[2^{-2j\varepsilon_N K}\frac{1}{n_j}\ ;\ 2^{2j\varepsilon_N K}\frac{1}{n_j}\right]$. It follows immediately that

$$v_j^2 \in \left[\frac{1}{n_j}2^{-4j\varepsilon_N K+1}; \frac{1}{n_j}2^{4j\varepsilon_N K+1}\right] \tag{27}$$

and

$$\max_{i=1,\dots,n_j} \frac{w_{j,i}^2}{v_j^2} \leq 2^{8j\varepsilon_N K}\frac{1}{2n_j} = N^{8K\varepsilon_N}\frac{1}{2n_j} \qquad \text{a.s.}$$

From (3) and (23), one deduces that the sequence n_j tends to infinity almost surely. Then the second condition in (23) implies the almost sure convergence of the sequence $\varepsilon_N \log_2(N)$ to zero as $N \to \infty$. Therefore the right-hand side goes almost surely to zero and (26) holds according to Lindeberg's theorem.

Step 2 : Let us now show that

$$\frac{1}{v_j}\left(\sum_{i=1}^{n_j} w_{j,i} - 1\right) \xrightarrow[j\to\infty]{a.s.} 0.$$

One has

$$\begin{aligned}\frac{1}{v_j}\left(\sum_{i=1}^{n_j} w_{j,i} - 1\right) &\leq (2^{2j\varepsilon_N K} - 1)\sqrt{n_j}2^{2j\varepsilon_N K - 1/2} \\ &\leq \sqrt{\frac{n_j}{2}}(e^{2K\varepsilon_N \log_2(N)} - 1)e^{2K\varepsilon_N \log_2(N)}.\end{aligned}$$

Since $\varepsilon_N \log_2(N)$ goes to zero as $N \to \infty$, there exists a constant C such that

$$\frac{1}{v_j}(\sum_{i=1}^{n_j} w_{j,i} - 1) \leq C\sqrt{n_j}\log_2(N)\varepsilon_N.$$

Condition (23) then entails that the right hand side of the above inequality tends almost surely to zero.
Step 3 Steps 1 and 2 with (25) ensure that, conditionally on $\mathcal{F}_{j-1}$,

$$\frac{1}{v_j}(\frac{1}{n_j\sigma^2}T_{j,n_j}2^{2jg(x)} - 1) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1).$$

According to (27), we have $\frac{n_j}{2}v_j^2 \in \left[N^{-4\varepsilon_N K}; N^{4\varepsilon_N K}\right]$. Since $\log_2(N)\varepsilon_N \to 0$, the sequence $v_j\sqrt{n_j/2}$ converges almost surely to 1 and thus

$$\sqrt{\frac{n_j}{2}}(\frac{1}{n_j\sigma^2}T_{j,n_j}2^{2jg(x)} - 1) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1), \quad \text{conditionally on } \mathcal{F}_{j-1}.$$

As shown in Step 1, the sequence $(n_j)_j$ tends almost surely to infinity. So the delta method can be applied, and one gets that, conditionally on $\mathcal{F}_{j-1}$,

$$\sqrt{\frac{n_j}{2}}2j(\hat{g}_j(x) - g(x)) \xrightarrow[j\to\infty]{law} \mathcal{N}(0,1).$$

This implies that $\sqrt{2n_j}j(\hat{g}_j(x) - g(x))$ converges in distribution to the standard Gaussian law. □

7.3 Large deviation inequalities

Let us first state a simple large deviation result, whose proof is included only for completeness.

Proposition 3. *Let $(U_j)_{j\in\mathbb{N}}$ be i.i.d. centred random variables such that there exists $g > 0$ and $T > 0$ with the property that, for all $|t| < T$, $E(e^{tU_1}) < e^{g\frac{t^2}{2}}$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers tending to 0. Then*

$$P\left(\left|\sum_{j=1}^{n} U_j\right| \geq n\rho_n\sqrt{\frac{g}{2}}\right) \leq e^{-\frac{n\rho_n^2}{4}}.$$

Proof. This is a direct application of (Petrov, 1995, Theorem 2.6), which asserts that random variables as the $(U_j)_{j\in\mathbb{N}}$ verify

$$P\left(\sum_{j=1}^{n} U_j \geq x\right) \leq e^{-\frac{x^2}{2ng}}$$

and

$$P\left(\sum_{j=1}^{n} U_j \leq -x\right) \leq e^{-\frac{x^2}{2ng}}$$

for $0 \leq x \leq ngT$. Just apply these inequalities with $x = n\rho_n\sqrt{\frac{g}{2}}$ which is indeed smaller than ngT for n large enough since ρ_n tends to 0. $\square$

The following proposition is used in the proof of Proposition 1:

Proposition 4. *Let $(V_j)_{j\in\mathbb{N}}$ be i.i.d. centred random variables with variance σ^2 such that $U_j = V_j^2 - \sigma^2$ verifies the exponential moment condition of Proposition 3. Then, almost surely, for n large enough,*

$$\sigma^2(1 - n^{\frac{\beta-1}{2}}) \leq \sum_{j=1}^{n} \frac{V_j^2}{n} \leq \sigma^2(1 + n^{\frac{\beta-1}{2}})$$

for all $\beta \in (0, 1)$.

Proof. Set $\rho_n = n^{\frac{\beta-1}{2}}$, and note that ρ_n tends to 0 as n tends to infinity since $\beta < 1$. One computes:

$$\begin{aligned}
P\left(\sum_{j=1}^{n} \frac{V_j^2}{n} \geq \sigma^2(1+\rho_n)\right) &= P\left(\sum_{j=1}^{n} U_j + n\sigma^2 \geq n\sigma^2(1+\rho_n)\right) \\
&= P\left(\sum_{j=1}^{n} U_j \geq n\sigma^2\rho_n)\right) \\
&\leq e^{-\frac{n\rho_n^2}{4}} \\
&= e^{-\frac{n^\beta}{4}}.
\end{aligned}$$

The Borel-Cantelli lemma entails that, almost surely, for n large enough, $\sum_{j=1}^{n} \frac{V_j^2}{n} \leq \sigma^2(1+\rho_n)$.

The lower bound is obtained in a similar way.

$\square$

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