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Abstract

Estimation of the occurrence of extreme events actually is that of risk premiums interest in actuarial Sciences, Insurance and Finance. Heavy-tailed distributions are used to model large claims and losses. In this paper we deal with the empirical estimation of the distortion risk premiums for heavy tailed losses by using the extreme value statistics. This approach can produce a potential bias in the estimation. Thus we look at this framework here and propose a reduced-bias approach of the classical estimators already suggested in the literature. A finite sample behavior is investigated, both for simulated data and real insurance data, in order to illustrate the efficiency of our approach.

Keyswords: Risk premiums · Distortion · Heavy-tailed distribution · Tail index · Extreme quantiles · Bias reduction

1 Introduction

Risk premiums are used to quantify insurance losses and financial assessments. For various examples and properties of such principles, we refer to Goovaerts et al. (1984), Denuit and Charpentier (2004), Young (2004) and references therein. One of the most commonly used one is *the net premium* defined for a non-negative loss random variable X with tail distribution function $\overline{F} := 1 - F$ as

$$\pi = \mathbb{E}(X) = \int_0^\infty \overline{F}(x)dx.$$

In general, premiums are required to be greater than or equal to the net premium $\mathbb{E}(X)$ in order to avoid that the insurer loses money on average. One way to achieve this goal consists in introducing an increasing, concave function g that maps $[0, 1]$ onto $[0, 1]$, such that $g(0) = 0$ and $g(1) = 1$ and to define the following distortion risk premium introduced by Wang (1996):

$$\pi(g) = \int_0^\infty g(\overline{F}(x))dx,$$

Note that the distortion Risk premiums can be seen as the expectation with respect to *distorted probabilities*. The function g is called *distortion function* and is in general parametrized by a one-dimentional parameter called the *distortion parameter*. This parameter controls the amount of the risk loading included in the premium for given riskiness of the loss variable X. The concavity of g makes the corresponding distortion premiums $\pi(g)$ coherent (Artzner et al., 1999; Wirch and Hardy, 1999). It is assumed throughout the present paper that F is a continous loss distribution. Let Q be the quantile function corresponding to F and defined by $Q(t) = \inf\{x : F(x) \geq t\}$, for every $t \in [0, 1)$. By a change of variables and integration by parts, the distortion risk premium $\pi(g)$ can be rewritten in terms of the quantile function Q as follows:

$$\pi(g) = \int_0^1 Q(1-s)g'(s)ds, \tag{1}$$

where g' denotes the lebesgues derivative of g. The quantile function Q plays a pivotal role in defining numerous risk measures, and is a well known risk measure itself, called the Value-at-Risk (VaR_t) at a level t.

To this frequent use will be made of extreme value statistics in the context when the distribution functions F of the risk are heavy-tailed as we will discussed in more detail further on. We start by assuming that the distortion functions g is such that $t \to g(t)$ is regularly varying at zero with index $\frac{1}{\beta} \in (0, 1]$ that is

$$g(t) = t^{1/\beta}\ell_g(t), \tag{2}$$

where $\ell_g(\cdot)$ is a slowly varying function at zero satisfying $\ell_g(\lambda t)/\ell_g(t) \to 1$ as $t \to 0$, for $\lambda > 0$. Before we comment in more details statistical inference for distortion risk premiums, we mention some examples of usual distortion functions g satisfying the condition (2).

- Net premium principle

$$g(t) = t \rightsquigarrow \begin{cases} \beta = 1, \\ \ell_g(t) = 1. \end{cases}$$

- Tail Value-at-Risk (TVaR) principle, $0 < \alpha < 1$

$$g(t) = \min\left(\frac{t}{\alpha}, 1\right) \rightsquigarrow \begin{cases} \beta = 1, \\ \ell_g(t) = 1 \ \text{ if } \ t \leq \alpha. \end{cases}$$

Since the cdf F is cotinuous, the (TVaR) coincides with the Conditional Tail Expection (CTE) which is the average amount of loss given that the loss exceeds a specified quantile (the Value-at-Risk) and difined by $\mathrm{CTE}(\alpha) = \mathbb{E}(X|X > \mathrm{VaR}_\alpha)$.

- Proportional Hazard Transform principle (PHT), $\varrho \geq 1$

$$g(t)=t^{1/\varrho}\rightsquigarrow\begin{cases}\beta=\varrho,\\ \ell_g(t)=1.\end{cases}$$

- Dual-power function principle, $\alpha>1$

$$g(t)=1-(1-t)^{\alpha}=t\left\{\alpha-\frac{\alpha(\alpha-1)}{2}t+o(t)\right\}\rightsquigarrow\begin{cases}\beta=1,\\ \ell_g(t)=\alpha-\dfrac{\alpha(\alpha-1)}{2}t+o(t)\quad\text{as}\quad t\downarrow 0.\end{cases}$$

- Gini principle, $0<\alpha\leq 1$

$$g(t)=(1+\alpha)t-\alpha t^2=t\left\{1+\alpha-\alpha t\right\}\rightsquigarrow\begin{cases}\beta=1,\\ \ell_g(t)=1+\alpha-\alpha t.\end{cases}$$

- Beta-distortion risk premium, $a\leq 1\leq b$, (e.g., Wirch and Hardy, 1999)

$$g(t)=\int_0^t\frac{s^{a-1}(1-s)^{b-1}}{\beta(a,b)}ds=t^a\left\{\frac{1}{a\beta(a,b)}\right\}\rightsquigarrow\begin{cases}\beta=\frac{1}{a},\\ \ell_g(t)=\dfrac{1}{a\beta(a,b)},\end{cases}$$

where $\beta(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}ds.$

- MINMAXVAR2 risk premium, $\mu>0,\ \nu>0$, (e.g., Madan and Schoutens, 2010)

$$g(t)=1-(1-x^{\frac{1}{1+\mu}})^{1+\nu}=t^{\frac{1+\mu}{1+\nu}}\left\{t^{\frac{1+\nu}{1+\mu}}-\left(t^{\frac{1}{1+\mu}}-1\right)^{1+\nu}\right\}\rightsquigarrow\begin{cases}\beta=\frac{1+\nu}{1+\mu},\\ \ell_g(t)=t^{\frac{1+\mu}{1+\nu}}-\left(t^{\frac{1}{1+\mu}}-1\right)^{1+\nu}.\end{cases}$$

For more details about the risk premiums, we refer e.g. to Wang (1998, 2000), Denuit et al. (2005) and references therein. A discussion of their empirical estimation is given by Jone and Zitikis (2003).

Suppose now that $\pi(g)$ is to be estimated from an independent and identically distributed (i.i.d) one-dimentional observations $X_1,...,X_n$, whose the common distribution function is that of the risk X and with let $X_{1,n}\leq...\leq X_{n,n}$ be the corresponding order statistics. The empirical distribution F_n of the sample and its corresponding empirical quantile function are respectively difined by:

$$F_n(x)=n^{-1}\sum_{j=1}^n\mathbb{1}(X_i\leq x),\ \text{for any } x\in\mathbb{R}$$

and

$$Q_n(s)=\inf\{t>0,F_n(t)\geq s\},\quad s\in(0,1).$$

One natural candidate for the empirical estimate of $\pi(g)$ in (1) is obtained by replacing the true quantile Q with the sample quantiles Q_n to yield a linear combination of order statistics called L-statistics:

$$\widehat{\pi}_n(g):=\sum_{j=1}^n a_{j,n}^{(g)}X_{n-j+1,n},\tag{3}$$

where the coefficients $a_{j,n}^{(g)}$ are $a_{j,n}^{(g)}=g\left(\frac{j}{n}\right)-g\left(\frac{j-1}{n}\right),\ j=1,..,n.$

For some statistical inference for distortion risk premiums, we refer to Peng *et al* (2001) Jones and

Zitikis (2003), Necir and Boukhetala (2004), Centeno and Andrade (2005), Necir *et al.* (2007), Jones and Zitikis (2007), Brazauskas *et al.* (2008), Furman and Zitikis (2008a, 2008b), Greselin *et al* (2009), Necir *et al.* (2009), Necir and Meraghni (2009, 2010), Brahimi *et al.* (2011, 2012), Necir and Zitikis (2012), Peng *et al* (2012), Rassoul (2012), Deme *et al* (2013a, 2013b) and the references therein.

Using the asymptotic theory for L-statistics (e.g., Shorack and Wellner, 1986), Jones and Zitikis (2007) prove that, for underlying distributions with a sufficient number of finite moments and under certain regularity conditions on the distortion function g, the following asymptotic-normality result holds:

$$n^{1/2}\left(\widehat{\pi}_n(g)-\pi(g)\right)\overset{\mathcal{D}}{\rightarrow}\mathcal{N}\left(0,\sigma_g^2\right) \tag{4}$$

provided that the variance

$$\sigma_g^2=\int_0^1\int_0^1(\min(s,t)-st)g'(s)g'(t)dQ(1-s)dQ(1-t)<\infty. \tag{5}$$

Hence, another approach is used and based on extreme values statistics for deriving statistical inferential results in the case of such distributions, and we shall do so next.

The remainder of the paper is organized as follows. In Section 2, we give a short review about Extreme Value methodology on estimating the underline class of distortion risk premiums for heavy-tailed losses. In Section 3, we give the asymptotic normality of the estimator under study by illustrating the fact that this last one can exhibit severe bias in many situations. To overcome this problem a reduced-bias approach is also proposed. The efficiency of our method is shown by a simple simulation study and a real dataset in Section 4.1. Section 5 is devoted on the proofs.

2 Extreme Value Methodology

Acturial and financial applications emphasis often lie on the modeling of rare events, i.e. events with low frequency, but with a hight and often disastrous impact. Analysing of such extreme events can be performed using extreme values methodology. where the tail behavior of distribution function is carracterized mainly by its extreme value index denoted by γ. This real-valued parameter helps to indicate the size and frequency of certains extreme events under given probability distribution: the heavier the tail. In this paper, we concentrate on the estimation of the extreme value index and derived quantiles in case of heavy-tailed distributions ($\gamma>0$).

2.1 First Order Regularity Variation

Extreme Value Theory (EVT) establishes the asymptotic behavior of the largest observations in a sample. It provides methods for extending the empirical distribution functions beyond the observed data. It is thus possible to estimate quantities related to the tail of a distribution such as small exceedance probabilities or extreme quantiles. We assume that the properly centred and normed sample maxima $X_{n,n}$ converge to a non degenerate limit. Then, the limit distribution G necessary is of generalized extreme value type (Fisher and Tippet, 1928). More specifically, when there existe sequence of constants $(a_n > 0)$ and $(b_n \in \mathbb{R})$ such that

$$\lim_{n\to\infty} \mathbb{P}(X_{n,n} \leq a_n x + b_n) = G(x)$$

for all continuity points of G necessarilly has to be of the form

$$G_\gamma(x) = \exp\left(-(1+\gamma x)_+\right)^{-1/\gamma}$$

where $y_+ = \max(y, 0)$. Here, the real-valued parameter is reffered to as the extreme value index γ of F, which in turn is said to belong to the maximum domain of attraction of G_γ, denoted by $F \in \mathcal{DM}(G_\gamma)$. We refer to Galambos (1978), Resnick (1987), Embrechts *et al.* (1997), de Haan and Ferreira (2006) for general accounts on extreme-value theory.

Most common continuous distribution functions satisfy this weak condition quite naturally. Distributions for which $\gamma > 0$ are called heavy-tailed distributions, as their typically decays as a power function, i.e.

$$\overline{F}(x) = x^{-1/\gamma}\ell_F(x), \text{ for any } x > 0. \tag{6}$$

where ℓ_F is a slowly varying function at infinity satisfying $\ell_F(\lambda x)/\ell_F(x) \to 1$ as $x \to \infty$ for $\lambda > 0$. Clearly the parameter γ governs the tail behavior, with larger values indicating heavier tails. The present model is now often restated as the assumption of regular variation at infinity of $1 - F$ with index $-1/\gamma$ (see e.g. Bingham *et al.*, 1987). In terms of the quantile function $Q(1-\cdot)$, the equation (6) is equivalent to $Q(1-\cdot)$ is regular variation zero with index $-\gamma$ i.e.

$$Q(1-s) = s^{-\gamma}\ell_Q(s), \text{ for any } s \in (0,1),$$

where ℓ_Q is also a slowly varying function at zero satisfying $\ell_Q(\lambda s)/\ell_Q(s) \to 1$ as $s \to 0$ for $\lambda > 0$. The class of heavy-tailed distributions includes distributions such as Pareto, Burr, Student, Lévy-stable, and log-gamma, which are known to be appropriate models for fitting large insurance claims, large fluctuations of prices, log-returns, etc. (see, e.g., Rolski et al., 1999; Beirlant *et al.*, 2001; Reiss and Thomas, 2007).

We focus our paper on the case $\gamma \in (\frac{1}{2}, 1)$ and $\beta \in [1, \frac{1}{\gamma})$ in order to ensure that the distortion risk premiums is finite and since in that case the results in (4) cannot be applied while $\sigma_g^2 = \infty$.

The estimation of γ has been extensively studied in the litterature and the most famous estimator is the Hill (1975) estimator defined by

$$\gamma^{\mathrm{H}}_{n,k_n} = k_n^{-1}\sum_{j=1}^{k_n} j\left(\log X_{n-j+1,n} - \log X_{n-j,n}\right)$$

for an intermediate sequence k_n i.e a sequence such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. More generally, Cosörgő *et al.* (1985) extended the Hill estimator into a kernel class of estimators

$$\widehat{\gamma}_{n,k_n}^{K} = \frac{1}{k_n} \sum_{j=1}^{k_n} K\left(\frac{j}{k_n+1}\right) Z_{jk},$$

wher K is a kernel function integrating to one and $Z_{jk} = j\left(\log X_{n-j+1,n} - \log X_{n-j,n}\right)$. Note that the Hill estimator coresponds to the particular case where $K = \underline{K} := \mathbb{1}_{(0,1)}$.

2.2 Estimating $\pi(g)$ when $F \in \mathcal{DM}(G_\gamma),\ \gamma > 0$

Extreme Quantile Estimation: As above mentioned, the use of empirical quantiles to estimate risk premiums $\pi(g)$ does not garantee the asymptotic normality when losses folow have a heavy-tailed distribution. Therefore, it is necessary to adopt another approach based on extreme quantiles. A quantile of level $0 < t < 1$ of df F is the point $q_t = Q(1-t)$. High quantiles correspond to situations where t is very small, more specifically and since we use asymptotic theory, the number s must depend on the sample size n, i.e. $t = t_n$, in such a way that as $n \to \infty$, $t_n \downarrow 0$ and $nt_n \to c > 0$. The estimation of extreme quantiles for heavy-tailed distributions has been of must interest in the literature. For details, we refer to Weissman (1978), de Haan and Rootzén (1983), Dekkers and de Haan (1989), Matthys and Beirlant (2003), Gomes *et al* (2005), and references therein. In this paper, we suggest Weissman's estimator (see Weissman, 1978) for $Q(1-t)$:

$$Q_n^{\mathrm{W}}(1-t) = (nt/k_n)^{-\gamma_{n,k_n}^{\mathrm{H}}}\, X_{n-k_n,n}, \quad t \downarrow 0.$$

Estimating the Distortion Risk Premiums: Transforming $\pi(g)$ and integrating by parts yield

$$\begin{aligned}\pi(g) &= \left\{\int_{k_n/n}^{1} g'(s)Q(1-s)ds\right\} + \left\{g(k_n/n)Q(1-k_n/n) - \int_0^{k_n/n} g(s)dQ(1-s)\right\}, \\ &= \pi_n^{(1)}(g) + \pi_n^{(2)}(g).\end{aligned}$$

Remark that $X_{n-k_n,n}$ is the simple estimator of $Q(1-k_n/n)$. Hence, coming back to the quantile $Q(1-s)$, we estimate it by using the empirical estimator $Q_n(1-s)$ when $s \in (k_n/n, 1)$ and by using the Weissman's estimator $Q_n^{\mathrm{W}}(1-s)$ when $s \in (0, k_n/n)$.
Thus, as an estimator of $\pi_n^{(1)}(g)$ we take the sample one that is

$$\widetilde{\pi}_n^{(1)}(g) = \sum_{j=k_n+1}^{n} a_{j,n}^{(g)} X_{n-j+1,n},$$

where the coefficients $a_{j,n}^{(g)}$ are those of the L-statistic $\widehat{\pi}_n(g)$ defined in (3). Since the distortion functions g satisfy the condition (2), with $\boldsymbol{\beta} \in [1, \frac{1}{\gamma})$ and since $\gamma_{n,k_n}^{\mathrm{H}}$ is a consistent estimator of γ

(see Masson (1982)) then we have for all large values of n, $\mathbb{P}(\gamma^{\mathrm{H}}_{n,k_n} > \frac{1}{\beta}) = o(1)$ and

$$\begin{aligned} -\int_0^{k_n/n} g(s)dQ_n^{\mathrm{W}}(1-s) &= \gamma^{\mathrm{H}}_{n,k_n}\left(\frac{k_n}{n}\right)^{\gamma^{\mathrm{H}}_{n,k_n}} X_{n-k_n,n}\int_0^{k_n/n} s^{-1-\gamma^{\mathrm{H}}_{n,k_n}} g(s)ds \\ &= \frac{\gamma^{\mathrm{H}}_{n,k_n}}{\frac{1}{\boldsymbol{\beta}}-\gamma^{\mathrm{H}}_{n,k_n}} g(k_n/n)X_{n-k_n,n}(1+o(1)). \end{aligned}$$

Hence we may estimate $\pi_n^{(2)}(g)$ by

$$\widetilde{\pi}_n^{(2)}(g) = g(k_n/n)X_{n-k_n} + \frac{\gamma^{\mathrm{H}}_{n,k_n}}{\frac{1}{\boldsymbol{\beta}}-\gamma^{\mathrm{H}}_{n,k_n}} g(k_n/n)X_{n-k_n,n} = \frac{g(k_n/n)}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}} X_{n-k_n,n}.$$

Thus, the final form of the estimator of $\pi(g)$ is

$$\widetilde{\pi}_n(g) = \sum_{j=k_n+1}^{n} a_{j,n}^{(g)} X_{n-j+1,n} + \frac{g(k_n/n)}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}} X_{n-k_n,n}. \tag{7}$$

A universal estimator of the distortion risk premiums $\pi(\rho)$ may be summarized by $\widehat{\pi}_n^*(g) = \widetilde{\pi}_n(g)\mathbb{1}_{\{\sigma_g^2=\infty\}} + \widehat{\pi}_n(g)\mathbb{1}_{\{\sigma_g^2<\infty\}}$, where $\widehat{\pi}_n(g)$ is as in (3). More precisely

$$\widehat{\pi}_n^*(g) = \widetilde{\pi}_n(g)\mathbb{1}_{\left\{S(\gamma,\beta)\right\}} + \widehat{\pi}_n(g)\mathbb{1}_{\left\{\overline{S}(\gamma,\beta)\right\}},$$

where $S(\gamma,\beta) = \left\{(\gamma,\beta)\in(0,\infty)\times[1,\infty),\ \ \gamma\in(\frac{1}{2},1)\ \text{ and }\ \beta<\frac{1}{\gamma}\right\}$ and $\overline{S}(\gamma,\beta)$ is its complementary in $(0,\infty)\times[1,\infty)$.

A number of special cases that are covered by statistical inferential theory for distortion risk premiums have been investigated in the literature within the heavy-tailed framework by making use the extreme values theory. One can refer to Peng *et al* (2001), Necir and Boukhetala (2004), Necir *et al.* (2007), Necir *et al.* (2009), Necir and Meraghni (2009, 2010), Brahimi *et al.* (2011, 2012), Necir and Zitikis (2012), Peng *et al* (2012), Rassoul (2012) and Deme *et al* (2013a, 2013b). Note that, in the special case where the distortion function g is a power function i.e $g(t) = t^{1/\beta}$ (which corresponds to the PHT premiums) its corresponding estimator $\widetilde{\pi}_n(g)$ is those proposed by Necir and Meraghni (2009). Necir *et al.*, (2009) introduced an estimator of the conditional tail espectation for heavy-tailed losses which is another special case of $\widetilde{\pi}_n(g)$ where $g(t) = \min(\frac{t}{\alpha},1)$, $0<\alpha\leq 1$. The estimator $\widetilde{\pi}_n(g)$ is also used by Necir and Zitikis (2012) in order to introduce an estimator of a coupled risk premiums for heavy-tailed losses.

However, the use of extreme values approach in the case of heavy-tailed losses still has a problem due to the fact that, it is based on the estimation of extreme quantile of $Q(1-s)$ known to be largely biased. In the statistic of extreme values, many reduced estimators are proposed in the literature as an alternative to extreme quantiles, see, for instance, Feuerverger and Hall, (1999), Beirlant et al. (2002), Gomes and Martins, (2002), Matthys and Beirlant (2003), Caeiro *et al.* (2004), Gomes and Martins, (2004), Matthys *et al.* (2004), Peng and Qi, (2004), Gomes and Figueiredo, (2006), Gomes and Pestana, (2007), Beirlant *et al.* (2008), Caeiro *et al.* (2009) and Li *et al.* (2010).

Recently, many estimators with reduced biases are proposed in the literature as an alternative special cases of distortion risk premiums in the context of heavy-tailed distributions. Brahimi *et al.* (2012a) proposed a bias reduction estimator for the mean (which correspond to the net premium case) by using the estimation of extreme quantiles developped by Li *et al.* (2010). Brahimi *et al.* (2012b) give a bias reduction estimator of the distortion premiums based on the extreme quantiles estimators introduced by Matthys and Beirlant (2003) and Deme *et al.* (2013a, 2013b) used the estimation of extreme quantiles proposed by Feuerverger and Hall, (1999), Beirlant *et al.* (2002) and Matthys *et al.* (2004) to introduce respectively a bias-reduced estimators of the Proportional Hazard Transform principle and the Conditional Tail Expectation. This present paper generalizes the frameworks proposed by the last authors for estimating the distortion risk premiums.

2.3 Second Order Regularity Variation

Note that, the asymptoctic normality of $\widetilde{\pi}_n(g)$ is related to that of Hill's estimator $\gamma^{\mathrm{H}}_{n,k_n}$ which is established under suitable assumptions. To prove such a result, a second order regularity variation condition is required in order to specify the bias-term. This assumption can be expressed in terms of the quantile function $Q(1-\cdot)$ as follows:

Sesond order condition $(\mathcal{R}_{A,\gamma,\rho})$. There exist a function $A(x) \to 0$ of constant sign for large values of x and a second order parameter $\rho < 0$ [1] such that, for every $x > 0$,

$$\lim_{t\to 0} \frac{1}{A(1/t)}\left(\frac{Q(1-tx)}{Q(1-t)} - x^{-\gamma}\right) = x^{-\gamma}\frac{x^{-\rho}-1}{\rho}. \tag{8}$$

Let us remark that the condition $(\mathcal{R}_{A,\gamma,\rho})$ implies that $|A|$ is regularly varying at infinity with index ρ (see, e.g. Geluck and de Haan, 1987). The condition $(\mathcal{R}_{A,\gamma,\rho})$ is not too restrictive; for instance, the important Hall class of Pareto-type models (Hall and Welsh, 1985) for which the tail quantile function is of the form

$$Q(1-t) = ct^{-\gamma}(1 + dt^{-\rho} + o(t^{-\rho})), \qquad (t \to \infty),$$

with some constants $c > 0$ and $d \neq 0$ satisfies the condition condition $(\mathcal{R}_{A,\gamma,\rho})$ with $A(1/t) = \rho d t^{-\rho}$. Most common heavy-tailed distributions can see to satisfy the above assumptions. Through these conditions, we now obtain the asymptotic normality and bias of the Hill's estimator and subsequently also of the distortion risk premiums.

3 Main results

In the next theorem, we establish the asymptotic normality of the estimator $\widetilde{\pi}_n(g)$. As it exhibits some bias, we propose an unbiased estimator whose asymptotic normality is also obtained.

[1] In a general setup of the second order condition, it is possible to have a second order index ρ equals to zero (see e.g. Haan and Ferreira (2006)). Nevertheless, for bias correction studies, it is usually assumed that $\rho < 0$. The parameter ρ determine the rate of convergence of $Q(1-tx)/Q(1-t)$ to its limit $x^{-\gamma}$, as $t \to 0$.

Asymptotic result for the Distortion Risk Premium estimator.

Theorem 1. *Assume that F satisfies $(\mathcal{R}_{A,\gamma,\rho})$ with $\gamma \in (\frac{1}{2}, 1)$ and its corresponding quantile function $Q(\cdot)$ is continuously differentiable on $[0,1)$. For any differentiable distortion function g satisfying the condition (2) with $\beta \in [1, \frac{1}{\gamma})$, and for any sequence of integer k_n satisfying $k_n \to \infty$, $k_n/n \to 0$ and $k_n^{1/2} A(n/k_n) \to \lambda \in \mathbb{R}$, as $n \to \infty$,one has*

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\Big(\widetilde{\pi}_n(g) - \pi(g)\Big) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(\lambda \mathcal{AB}(\gamma,\rho,\beta), \mathcal{AV}(\gamma,\beta)\Big),$$

where

$$\mathcal{AB}(\gamma,\rho,\beta) = \frac{\beta\rho(\gamma\beta+\beta-1)}{(1-\rho)(\gamma\beta+\rho\beta-1)(1-\gamma\beta)^2} \quad \text{and} \quad \mathcal{AV}(\gamma,\beta) = \frac{\beta\gamma^2(\gamma\beta+\beta-1)^2}{(2\gamma\beta+\beta-2)(1-\beta\gamma)^4}.$$

Thus Theorem 1 generalizes Theorem 2 and 3.1 in Necir and Meraghni (2009) and Necir *et al.* (2009) in the case $\lambda \neq 0$ when we use a general regularly varying distortion function g.

Reduced-bias method with the Least Squared approach: from Theorem 1, it is clear that the estimator $\widetilde{\pi}_n(g)$ exhibit a bias due to the fact that we use in its construction the Weissman's estimator which is known to have such a problem. To solve this issue, we propose to use the exponential regression model introduced in Feuerverger and Hall (1999) and Beirlant *et al.* (1999) to construct a reduced-bias estimator. More precisely, using $(\mathcal{R}_{A,\gamma,\rho})$, Feuerverger and Hall (1999) and Beirlant et al. (1999, 2002) proposed the following exponential regression model for the log-spacings of order statistics:

$$Z_{jk} = j\log\left(\frac{X_{n-j+k_n,n}}{X_{n-k_n,n}}\right) \sim \left(\gamma + A(n/k_n)\left(\frac{j}{k_n+1}\right)^{-\rho}\right) + \varepsilon_{jk},\ 1 \le j \le k_n, \tag{9}$$

where ε_{jk} are zero-centered error terms. If we ignore the term $A(n/k_n)$ in (9), we get the Hill-type estimator $\hat{\gamma}^{\text{H}}_{n,k_n}$ by taking the mean of the right-hand side of (9). By using a least-square approach, the equation (9) can be further exploited to propose a reduced-bias estimator for γ in which ρ is substituted by a consistent estimator $\widehat{\rho} = \widehat{\rho}_{n,k_n}$ (see for instance Beirlant et al, 2002) or by a canonical choice, such as $\widehat{\rho} = -1$ (see e.g. Feuerverger and Hall (1999) or Beirlant et al (1999)). The least squares estimators for γ and $A(n/k_n)$ are then given respectively by

$$\hat{\gamma}^{\text{LS}}_{n,k_n}(\widehat{\rho}) = \frac{1}{k_n}\sum_{j=1}^{k_n} Z_{jk} - \frac{\hat{A}^{\text{LS}}_{n,k_n}(\widehat{\rho})}{1-\widehat{\rho}} = \hat{\gamma}^{\text{H}}_{n,k_n} - \frac{\hat{A}^{\text{LS}}_{n,k_n}(\widehat{\rho})}{1-\widehat{\rho}}, \tag{10}$$

and

$$\hat{A}^{\text{LS}}_{n,k_n}(\widehat{\rho}) = \frac{(1-2\widehat{\rho})(1-\widehat{\rho})^2}{\widehat{\rho}^2}\frac{1}{k_n}\sum_{j=1}^{k_n}\left(\left(\frac{j}{k_n+1}\right)^{-\widehat{\rho}} - \frac{1}{1-\widehat{\rho}}\right) Z_{jk}. \tag{11}$$

The asymptotic normalities of $\hat{\gamma}^{\text{LS}}_{n,k_n}(\widehat{\rho})$ and $\hat{A}^{\text{LS}}_{n,k_n}(\widehat{\rho})$ are stablished in Beirlant et al. (2002, Theorem 3.2). Note that $\hat{\gamma}^{\text{LS}}_{n,k_n}(\widehat{\rho})$ can be viewed as a kernel estimator

$$\widehat{\gamma}^{\text{LS}}_{n,k_n}(\widehat{\rho}) = \frac{1}{k_n}\sum_{j=1}^{k_n} K_{\widehat{\rho}}\left(\frac{j}{k_n+1}\right) Z_{jk},$$

where

$$K_{\rho}(s) = \frac{1-\rho}{\rho}\underline{K}(s) + \left(1 - \frac{1-\rho}{\rho}\right)\underline{K}_{\rho}(s), \quad \text{for} \quad 0 < s \leq 1,$$

with $\underline{K} = 1\!\!1_{\{0<s<1\}}$ and $\underline{K}_{\rho}(s) = ((1-\rho)/\rho)(s^{-\rho}-1)1\!\!1_{\{0<s<1\}}$.

Now, we are going to propose an adaptive asymptotically unbiased estimation procedure for $\widetilde{\pi}_n(g)$ that is based on the following unbiased Weissman's estimator of the extreme quantile $Q(1-s)$ for $s \downarrow 0$,

$$\widehat{Q}_n^{\mathrm{LS}}(1-s,\widehat{\rho}) = \left(\frac{k_n}{ns}\right)^{\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})} \left\{1 - \hat{A}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})\frac{1-(k_n/n)^{\widehat{\rho}}\, s^{-\widehat{\rho}}}{\widehat{\rho}}\right\} X_{n-k_n,n},$$

see e.g. Matthys *et al.* (2004).

Thus, in the spirit of (7), we arrive at the following asymptotically unbiased estimators of the $\pi(g)$:

$$\begin{aligned} \widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho}) &= \sum_{j=k_n+1}^{n} a_{j,n}^{(g)} X_{n-j+1,n} + \frac{g(k_n/n)}{1-\beta\hat{\gamma}^{\mathrm{LS}}_{n,k}(\widehat{\rho})}\left(1 - \frac{\hat{A}^{\mathrm{LS}}_{n,k}(\widehat{\rho})}{\hat{\gamma}^{\mathrm{LS}}_{n,k}(\widehat{\rho}) + \widehat{\rho} - \frac{1}{\beta}}\right) X_{n-k,n}, \qquad (12) \\ &= \widetilde{\pi}_n^{(1)}(g) + \widetilde{\pi}_n^{(3)}(g). \end{aligned}$$

Our next goal is to establish, under suitable asumptions, the asymptotic normality of $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$. This is done in the following theorem.

Theorem 2. *Under the assumptions of Theorem 1, if $\widehat{\rho}$ is consistent of ρ, then*

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\left(\widetilde{\pi}_n^{LS}(g,\widehat{\rho}) - \pi(g)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \widetilde{\mathcal{AV}}(\gamma,\beta,\rho)\right)$$

where

$$\widetilde{\mathcal{AV}}(\gamma,\beta,\rho) = \frac{\beta\gamma^2(\beta\gamma+\beta-1)^2(\beta\gamma+\beta-\beta\rho-1)^2}{(2\beta\gamma+\beta-2)(\beta\gamma+\beta\rho-1)^2(1-\beta\gamma)^4}.$$

4 Finite sample behavior

In order to illustrate the efficiency of the proposed statistical methods, their finite sample behavior is investigated, both for simulated data and real insurance data.

4.1 Simulated data

In this section, the biased estimator $\widetilde{\pi}_n(g)$ and the reduced-bias one $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ (with the canonical choice $\widehat{\rho} = -1$) are compared on a small simulation study. To this aim, 500 samples of size 500 are simulated from a Burr distribution defined as: $\overline{F}(x) = (1+x^{-\frac{3}{2}\rho})^{1/\rho}$. The associated extreme-value index is $\gamma = 2/3$ and ρ is the second order parameter. Different values of $\rho \in \{-0.75, -1, -1.5\}$ are considered to assess its impact. Concerning the premium calculation principles, we have restricted ourselves to the case of the net and the dual-power premium principle discussed in Section 1. In the dual-power premium principle, we have set the loading parameter α at 1.366, as in Wang

(1996). The median and median squared error (MSE) of these estimators are estimated over the 500 replications. The results are displayed on Figure 1 and Figure 2. It appears on Figure 1 that the closer ρ is to 0, the more important is the bias of $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ whatever the value of α is. The effect of the bias correction on the MSE is illustrated on Figure 2. We can observe that the MSE of the reduced-bias estimator $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ is almost constant with respect to k, especially when the bias of $\widetilde{\pi}_n(g)$ is strong, $i.e$ when ρ is close to 0.

4.2 Real insurance data

Our real dataset concerns a Norwegian fire insurance portfolio from 1972 until 1992. Together with the year of occurrence, the data contain the value ($\times 1\,000$ Krone) of the claims. A priority of 500 units was in force. These data were of some concern in that the number of claims had risen systematically with a maximum in 1988 as illustrated in Figure 3(a). We concentrate here on the year 1976 where the average claim size per year reached a peak as was the case in 1988. The sample size is $n = 207$. The data were corrected, among others, for inflation. As argued in Beirlant *et al.* (2004), the Pareto model seems to form a good fit to the tail of the claim size observations, suggesting that the data originate from a heavy-tailed distribution. Figure 3(b) shows the histogram corresponding to this year 1976. From Figure 3(c) we can observe the difficulty to find a stable part in the plot of the Hill estimator $\widehat{\gamma}_{n,k_n}^{\mathrm{H}}$ as a function of k, due to the bias of this estimator. We can apply our methodology to this real dataset as the extreme value index (or at least its estimator) is in the interval $(1/2,1)$ whatever the value of k is. Figure 3(d) and (e) shows the biased estimator $\widetilde{\pi}_n(g)$ (dashed line) and the reduced-bias one $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ (full line) for the net and the dual-power premium principle the loading parameter $\alpha = 1.366$. The reduced-bias estimator $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ is almost constant for a large range of values of k which makes the choice of k easier in practice.

5 Proof

We will use in this section the Csörgő *et al.* (1985) approach. We construct a probability space $(\Omega,\mathbb{A},\mathbb{P})$, carrying a sequence $\xi_1,\xi_2...$ of independent random variables uniformly distributed on (0,1) and a sequence of Bronian bridges $\mathbb{B}_n(s)$, $0 \le s \le 1$, $n = 1,2,...$ such that for every $0 \le \nu \le 1/2$ and for all n

$$\sup_{1/n \le s \le 1-1/n} \frac{|\beta_n(s) - \mathbb{B}(s)|}{(s(1-s))^{1/2-\nu}} = O\left(n^{-\nu}\right), \tag{13}$$

where the resulting empirical quantile $\beta_n(\cdot)$ is defined by

$$\beta_n(t) = n^{1/2}(t - \mathbb{V}_n(t)),\ (0 \le t \le 1), \tag{14}$$

with $\mathbb{V}_n(s) = \xi_{j,n},\ (j-1)/n < s \leq j/n,\ j = 1, ..., n$, and $\mathbb{V}_n(0) = \mathbb{V}_n(0+)$. The two sequences of order statistics $X_{1,n} \leq ... \leq X_{n,n}$ and $\xi_{1,n} \leq ... \leq \xi_{n,n}$ are linked via the following equality in distribution

$$X_{j,n} \stackrel{\mathcal{D}}{=} Q(1 - \xi_{n-j+1,n}),\ j = 1, ..., n.$$

5.1 Preliminary results

The following preliminary results will be instrumental for our needs. Their proofs are postponed to Section 7. The next two lemmas establish the asymptotic expansions of the two random terms appearing in (7).

Lemma 1. *Under the assumptions of Theorem 1, we have*

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\left(\widetilde{\pi}_n^{(1)}(g) - \pi_n^{(1)}(g)\right) \stackrel{\mathcal{D}}{=} \mathbb{W}_{n,1} + o_{\mathbb{P}}(1),$$

with

$$\mathbb{W}_{n,1} = -\sqrt{\frac{k_n}{n}}\ \frac{\int_{k_n/n}^{1} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds}{g(k_n/n)Q(1-k_n/n)}.$$

The Lemma 1 generalizes the statement (11) in Necir and Meraghni (2009), in the case where a regularly varying distortion functions g is used.

Lemma 2. *Under the assumptions of Theorem 1, we have*

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\left(\widetilde{\pi}_n^{(2)}(g) - \pi_n^{(2)}(g)\right) \stackrel{\mathcal{D}}{=} \lambda\mathcal{AB}(\gamma,\rho) + \mathbb{W}_{n,2} + \mathbb{W}_{n,3} + o_{\mathbb{P}}(1),$$

where

$$\begin{cases} \mathbb{W}_{n,2} := -\dfrac{\gamma}{1-\gamma\beta}\sqrt{\dfrac{n}{k_n}}\mathbb{B}_n\left(1-k_n/n\right), \\ \\ \mathbb{W}_{n,3} := \dfrac{\gamma\beta}{(1-\gamma\beta)^2}\sqrt{\dfrac{n}{k_n}}\displaystyle\int_0^1 s^{-1}\mathbb{B}_n(1-sk_n/n)d(s\underline{K}(s)), \end{cases}$$

with $\underline{K}(s) = \mathbb{1}_{\{0<s<1\}}$.

The following lemma establishes the asymptotic expansion of the random terms appearing in (12).

Lemma 3. *Under the assumptions of Theorem 1, if $\widehat{\rho}$ is consistent of ρ, then*

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\left(\widetilde{\pi}_n^{LS}(g) - \pi_n(g)\right) \stackrel{\mathcal{D}}{=} \mathbb{W}_{n,1} + \mathbb{W}_{n,2} + \mathbb{W}_{n,4} + \mathbb{W}_{n,5} + o_{\mathbb{P}}(1),$$

where

$$\begin{cases} \mathbb{W}_{n,4} := \dfrac{(1-\rho)(1-\gamma\beta)}{1-\gamma\beta-\rho\beta}\mathbb{W}_{n,3}, \\ \\ \mathbb{W}_{n,5} := -\dfrac{\beta\rho\gamma(\gamma\beta+\beta-1)}{(1-\gamma\beta-\rho\beta)(1-\gamma\beta)^2}\displaystyle\int_0^1 s^{-1}\mathbb{B}\left(1-\frac{sk}{n}\right)d(sK_\rho(s)). \end{cases}$$

Last lemma is a direct consequence of Karamata's Theorem (see Propositions 1.5.8 in Bingham *et al.*, 1987).

Lemma 4. *Let ℓ be a slowly varying function at 0. Then for all $\alpha > 1$*

$$\lim_{s\to 0} \frac{1}{s^{1-\alpha}\ell(s)} \int_s^1 t^{-\alpha} dt = \frac{1}{\alpha - 1}.$$

6 Proofs of main results

Proof of Theorem 1. Combining Lemmas 1 and 2, we get

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)}\left(\widetilde{\pi}_n(g) - \pi(g)\right) \stackrel{\mathcal{D}}{=} \lambda\mathcal{AB}(\gamma,\rho) + \mathbb{W}_{n,1} + \mathbb{W}_{n,2} + \mathbb{W}_{n,3} + o_{\mathbb{P}}(1).$$

The limiting process is a Gaussian random with mean zero and asymptotic variance given by

$$\mathcal{AV}(\gamma,\beta) = \lim_{n\to\infty} \mathbb{E}\left((\mathbb{W}_{n,1} + \mathbb{W}_{n,2} + \mathbb{W}_{n,3})^2\right).$$

The computations are tedious but quite direct. We only give below the main arguments, i.e.

$$\begin{aligned}
\mathbb{E}\left(\mathbb{W}_{n,1}^2\right) &= \frac{2\int_{k_n/n}^1 sg'(s)Q'(1-s)\left(\int_s^1 (1-t)g'(t)Q'(1-t)dt\right)ds}{(n/k_n)g^2(k_n/n)Q^2(1-k_n/n)}, \\
&= \frac{2\int_{k_n/n}^1 sg'(s)Q'(1-s)\left(\int_s^1 g'(t)Q'(1-t)dt\right)ds}{(n/k_n)g^2(k_n/n)Q^2(1-k_n/n)} \\
&\quad -\frac{2\int_{k_n/n}^1 sg'(s)Q'(1-s)\left(\int_s^1 tg'(t)Q'(1-t)dt\right)ds}{(n/k_n)g^2(k_n/n)Q^2(1-k_n/n)} \\
&:= Q_{1,n} + Q_{2,n}.
\end{aligned}$$

By remarking that $d\left(\int_s^1 g'(t)Q'(1-t)dt\right) = -g'(s)Q'(1-s)ds$, $s \in (0,1)$, we obtain

$$\begin{aligned}
Q_{1,n} &= \frac{\int_{k_n/n}^1 \left(\int_s^1 g'(t)Q'(1-t)dt\right)^2 ds}{(n/k_n)g^2(k_n/n)Q^2(1-k_n/n)} + \frac{k_n}{n}\left(\frac{\int_{k_n/n}^1 g'(t)Q'(1-t)dt}{(n/k_n)^{1/2}g(k_n/n)Q(1-k_n/n)}\right)^2, \\
&:= Q_{1,n}^{(1)} + Q_{1,n}^{(2)}.
\end{aligned}$$

Since $g(\cdot)$ and $Q(1-\cdot)$ are both regularly varying functions at zero with index respectively $\frac{1}{\beta} > 0$ and $-\gamma < 0$ and with $g(0) = 0$, then by using the 11th assertion of Proposition B.1.9 (page 367) in de Haan and Ferreira (2006), yields that for $s \downarrow 0$

$$Q'(1-s) = \gamma(1+o(1))s^{-1}Q(1-s) \;\text{ and }\; g'(s) = \frac{1}{\beta}(1+o(1))s^{-1}g(s). \tag{15}$$

It follows that $g'(\cdot)$ and $Q'(1-\cdot)$ are both regularly varying at zero with index respectively $\frac{1}{\beta} - 1$ and $-\gamma - 1$. Hence, there exists two slowly varying functions at zero $\bar{\ell}_g(s)$ and $\bar{\ell}_Q(s)$ such that $g'(s) = s^{\frac{1}{\beta}-1}\bar{\ell}_g(s)$ and $Q'(1-s) = s^{-\gamma-1}\bar{\ell}_Q(s)$. Let $\bar{\ell}(\cdot) = \bar{\ell}_g(\cdot)\bar{\ell}_Q(\cdot)$, we have

$$g'(s)Q'(1-s) = s^{\frac{1}{\beta}-\gamma-2}\;\bar{\ell}(s). \tag{16}$$

It is clear that, $s \mapsto \bar{\ell}(s)$ is a slowly varying function at zero. From (15), we also have

$$g(s)Q(1-s) \sim \frac{\beta}{\gamma} s^2 g'(s) Q'(1-s) = \frac{\beta}{\gamma} s^{\frac{1}{\beta}-\gamma}\ \bar{\ell}(s). \tag{17}$$

$$\begin{aligned}
Q_{1,n}^{(1)} &= \frac{\int_{k_n/n}^1 \left[\int_s^1 Q'(1-t)g'(t)dt\right]^2 ds}{(n/k_n)\, g^2(k_n/n) Q^2(1-k_n/n)} \\
&= \frac{\int_{k_n/n}^1 \left[\int_s^1 t^{\frac{1}{\beta}-\gamma-2}\ \bar{\ell}(t)dt\right]^2 ds}{(n/k_n)\, g^2(k_n/n) Q^2(1-k_n/n)} \qquad \text{(from 16)} \\
&\sim \frac{\gamma^2}{\beta^2} \frac{\int_{k_n/n}^1 \left[\int_s^1 t^{\frac{1}{\beta}-\gamma-2}\ \bar{\ell}(t)dt\right]^2 ds}{(n/k_n)\left[(k_n/n)^{\frac{1}{\beta}-\gamma}\ \bar{\ell}(k_n/n)\right]^2} \qquad \text{(from 17)} \\
&= \frac{\gamma^2}{\beta^2} \frac{\int_{k_n/n}^1 \left[\int_s^1 t^{\frac{1}{\beta}-\gamma-2}\ \bar{\ell}(t)dt\right]^2 ds}{(k_n/n)\left[\int_{k_n/n}^1 t^{\frac{1}{\beta}-\gamma-2}\ \bar{\ell}(t)dt\right]^2} \left[\frac{\int_{k_n/n}^1 t^{\frac{1}{\beta}-\gamma-2}\ \bar{\ell}(t)dt}{(k_n/n)^{\frac{1}{\beta}-\gamma-1}\ \bar{\ell}(k_n/n)}\right]^2.
\end{aligned}$$

Since $\gamma \in (\frac{1}{2}, 1)$ and $\beta \in [1, \frac{1}{\gamma})$, then by Lemma 4,

$$Q_{1,n}^{(1)} \quad \to \quad \frac{\gamma^2}{(2\gamma\beta+\beta-2)\,(\gamma\beta+\beta-1)^2}.$$

Similary, we also have

$$Q_{1,n}^{(2)} \to \frac{\gamma^2}{(\gamma\beta+\beta-1)^2}.$$

Hence,

$$Q_{1,n} \to \frac{2\gamma^2}{(2\gamma\beta+\beta-2)\,(\gamma\beta+\beta-1)}.$$

Next, we have

$$\begin{aligned}
Q_{2,n} &= \left[\frac{\int_{k_n/n}^1 tQ'(1-t)g'(t)dt}{(n/k_n)^{1/2}\, g(k_n/n)Q(1-k_n/n)}\right]^2 \\
&\sim \frac{\gamma^2}{\beta^2}\frac{k_n}{n}\left[\frac{\int_{k_n/n}^1 t^{\frac{1}{\beta}-\gamma-1}\ \bar{\ell}(t)dt}{(k_n/n)^{\frac{1}{\beta}-\gamma}\ \bar{\ell}(k_n/n)}\right]^2, \\
&= o(1).
\end{aligned} \tag{18}$$

This last result coming from the fact that, according to Proposition 1.3.6 in Bingham et al. (1987)

$$\begin{aligned}
\forall\varepsilon > 0 \quad x^{-\varepsilon}\bar{\ell}(x) &\to \infty \text{ as } x \to 0 \\
\forall\delta > 0 \quad x^{\delta}\bar{\ell}(x) &\to 0 \text{ as } x \to 0.
\end{aligned}$$

Thus, by choosing $0 < \delta < \frac{1}{\beta} - \gamma$ and $0 < \varepsilon < \gamma - \frac{1}{\beta} + \frac{1}{2}$, entails

$$0 \le s\left[\frac{\int_s^1 t^{\frac{1}{\beta}-\gamma-1}\bar{\ell}(t)dt}{s^{\frac{1}{\beta}-\gamma}\bar{\ell}(s)}\right]^2 \le s\left[Cs^{\gamma-\frac{1}{\beta}-\varepsilon}\right]^2 = O\left(s^{1+2[\gamma-\frac{1}{\beta}-\varepsilon]}\right) = o(1)$$

under our assumptions, where C is a suitable constant.
Finally $\mathbb{E}\left(\mathbb{W}_{1,n}^2\right) \to \dfrac{2\gamma^2}{(2\gamma\beta+\beta-2)\,(\gamma\beta+\beta-1)}$. Direct computations now lead to

$$\begin{aligned}
\mathbb{E}\left(\mathbb{W}_{2,n}^2\right) &\to \frac{\gamma^2}{(1-\beta\gamma)^2} \\
\mathbb{E}\left(\mathbb{W}_{3,n}^2\right) &\to \frac{\gamma^2\beta^2}{(1-\beta\gamma)^4} \quad \text{by Corollary 1 in Deme } et\ al. \text{ (2013)} \\
\mathbb{E}\left(\mathbb{W}_{n,1}\mathbb{W}_{2,n}\right) &\to \frac{\gamma^2\beta^2}{(\gamma\beta+\beta-1)(1-\beta\gamma)} \quad \text{by using the same method that allowed to set } \mathbb{E}\left(\mathbb{W}_{1,n}^2\right) \\
\mathbb{E}\left(\mathbb{W}_{n,1}\mathbb{W}_{3,n}\right) &\to 0 \quad \text{by (18)} \\
\mathbb{E}\left(\mathbb{W}_{n,2}\mathbb{W}_{3,n}\right) &= 0.
\end{aligned}$$

Combining all these results, Theorem 1 follows. ■

Proof of Theorem 2. From Lemma 3, we only have to compute the asymptotic variance of the limiting process. As Theorem 1, the computations are quite direct and lead to the desired asymptotic variance. This ends the proof of the Theorem 2. ■

7 Proofs of auxiliary results

Proof of Lemma 1: We have

$$\begin{aligned}
\widetilde{\pi}_n^{(1)}(g) - \pi_n^{(1)}(g) &= \int_{k_n/n}^{1-1/n} g'(s)\left(Q_n(1-s) - Q(1-s)\right) ds \\
&\quad + \int_{1-1/n}^{1} g'(s)\left(Q_n(1-s) - Q(1-s)\right) ds \\
&= A_{n,1} + A_{n,2}.
\end{aligned}$$

We first show that

$$\frac{k_n^{1/2} A_{n,2}}{g(k_n/n)Q(1-k_n/n)} = \frac{n^{1/2}A_{n,2}}{(k_n/n)^{-1/2}g(k_n/n)Q(1-k_n/n)} = o_{\mathbb{P}}(1). \tag{19}$$

Note that $Q_n(1-s) = X_{1,n}$ when $1-1/n \le s < 1$ and $X_{1,n} = O_{\mathbb{P}}(1)$. Since $g(1) = 1$, we get

$$A_{n,2} = (1 - g(1-1/n))X_{1,n} - \int_{1-1/n}^{1} g'(s)Q(1-s)ds.$$

Since g is continous on [0,1], then $(1-1/n)^{-\frac{1}{\beta}}g(1-1/n) \to 1$. Hence, for all larges values of n we get

$$\begin{aligned}
n^{1/2}[1 - g(1-1/n)] &\sim n^{1/2}\left[1 - (1-1/n)^{\frac{1}{\beta}}\right] \\
&\sim n^{\frac{1}{2}-\frac{1}{\beta}},
\end{aligned}$$

which tends to zero as $n \to \infty$ because $\frac{1}{2} - \frac{1}{\beta} < 0$. Consequently, $n^{1/2} \int_{1-1/n}^{1} g'(s) Q_n(1-s) ds = o_{\mathbb{P}}(1)$. On the other hand, applying the mean-value Theorem to the function

$$\varphi(x) = \int_{\delta}^{x} g\prime(s) Q(1-s) ds \quad \text{for} \quad 0 < \delta \leq 1 - 1/n \leq x \leq 1,$$

we get $\lim\limits_{n\to\infty} n^{1/2} \displaystyle\int_{1-1/n}^{1} g'(s) Q(1-s) ds = 0$. This prove statement (21) because

$$(k_n/n)^{-1/2} g(k_n/n) Q(1 - k_n/n) = (k_n/n)^{-(\gamma - \frac{1}{\beta} + \frac{1}{2})} \ell_g(k_n/n) \ell_Q(k_n/n) \to \infty, \tag{20}$$

since $\gamma - \frac{1}{\beta} + \frac{1}{2} > 0$ and $\ell_g(\cdot)\ell_Q(\cdot)$ is a slowly varying function at zero.

Next, we investigate the asymptotic behaviour of

$$\frac{k_n^{1/2} A_{n,1}}{g(k_n/n) Q(1 - k_n/n)} = \frac{n^{1/2} A_{n,1}}{(k_n/n)^{-1/2} g(k_n/n) Q(1 - k_n/n)}.$$

Note that

$$\{Q_n(1-s),\ 0 < s < 1\}_{n\in\mathbb{N}^*} \overset{\mathcal{D}}{=} \{Q(\mathbb{V}_n(1-s)),\ \ 0 < s < 1\}_{n\in\mathbb{N}^*}.$$

By the differentiability of Q, we have

$$\begin{aligned}
n^{1/2}(Q_n(1-s) - Q(1-s)) &\overset{\mathcal{D}}{=} n^{1/2}(Q(\mathbb{V}_n(1-s)) - Q(1-s)) \\
&= n^{1/2} Q'(1 - \vartheta_n(s))(\mathbb{V}_n(1-s) - 1 + s), \quad \text{(by Taylor expansion)} \\
&= -\beta_n(1-s) Q'(1 - \vartheta_n(s))
\end{aligned}$$

where $\{\vartheta_n(s),\ \ 0 < s < 1\}_{n\in\mathbb{N}^*}$ is a sequence of ramdom variables with values in the open interval of endpoints $s \in (0,1)$ and $1 - \mathbb{V}(1-s)$ and $\beta_n(s)$ is given in (14). It follows that

$$\begin{aligned}
n^{1/2} A_{n,1} &= -\int_{k_n/n}^{1-1/n} g'(s) \beta_n(1-s) Q'(1 - \vartheta_n(s)) ds \\
&= -\int_{k_n/n}^{1-1/n} g'(s) \beta_n(1-s) Q'(1-s) ds \\
&\quad + \int_{k_n/n}^{1-1/n} g'(s) \beta_n(1-s) (Q'(1 - \vartheta_n(s)) - Q(-s) ds \\
&= A_{n,1}^{(1)} + A_{n,1}^{(2)}
\end{aligned}$$

and each term is studied separately.

<u>Term $A_{n,1}^{(1)}$</u>. We have

$$\begin{aligned}
A_{n,1}^{(1)} &= -\int_{k_n/n}^{1-1/n} g'(s) Q'(1-s) \mathbb{B}_n(1-s) ds \\
&\quad -\int_{k_n/n}^{1-1/n} g'(s) Q'(1-s) \left(\beta_n(1-s) - \mathbb{B}_n(1-s)\right) ds \\
&= A_{n,1}^{(1,1)} + A_{n,1}^{(1,2)}
\end{aligned}$$

Let $0 < \nu < 1/2$, we have

$$\begin{aligned}
\left|A_{n,1}^{(1,2)}\right| &= \left|\int_{k_n/n}^{1-1/n} g'(s)Q'(1-s)\left(\beta_n(1-s) - \mathbb{B}_n(1-s)\right) ds\right| \\
&\leq \int_{k_n/n}^{1-1/n} |g'(s)Q'(1-s)|\,|\beta_n(1-s) - \mathbb{B}_n(1-s)|\, ds \\
&\leq \sup_{1/n \leq s \leq 1-1/n} \frac{|\beta_n(s) - \mathbb{B}_n(s)|}{(s(1-s))^{1/2-\nu}} \int_{k_n/n}^{1} s^{1/2-\nu}|g'(s)Q'(1-s)|ds \\
&= O_{\mathbb{P}}(n^{-\nu}) \int_{k_n/n}^{1} s^{1/2-\nu}|g'(s)Q'(1-s)|ds, \quad \text{by Csörgő } \textit{et al.} \text{ (1986)}.
\end{aligned}$$

The functions $-Q(1-\cdot)$ and $g(\cdot)$ are increasing and differentiable on (0,1), then $-Q'(1-s) \geq 0$ and $g'(s) \geq 0$ for any $s \in (0,1)$. It follows that $|Q'(1-s)| = -Q'(1-s)$ for any $s \in (0,1)$. Hence, for all large values of n

$$\left|A_{n,1}^{(1,2)}\right| \leq -O_{\mathbb{P}}(n^{-\nu}) \int_{k_n/n}^{1} s^{1/2-\nu} g'(s)Q'(1-s)ds = -O_{\mathbb{P}}(1)n^{-\nu} \int_{k_n/n}^{1} s^{1/2-\nu} g'(s)Q'(1-s)ds.$$

Further, from(16) and (17), we have

$$\begin{aligned}
\frac{n^{-\nu}\int_{k_n/n}^{1} s^{1/2-\nu} g'(s)Q'(1-s)ds}{(k_n/n)^{-1/2} g(k_n/n)Q(1-k_n/n)} &= \frac{n^{-\nu}\int_{k_n/n}^{1} s^{\frac{1}{\beta}-\gamma-\frac{3}{2}-\nu}\bar{\ell}(s)ds}{(k_n/n)^{-1/2} g(k_n/n)Q(1-k_n/n)} \\
&\sim \frac{\gamma}{\beta}\frac{k_n^{-\nu}\int_{k_n/n}^{1} s^{\frac{1}{\beta}-\gamma-\frac{3}{2}-\nu}\bar{\ell}(s)ds}{(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}-\nu}\bar{\ell}(k_n/n)}.
\end{aligned}$$

Then by taking $\alpha = -\left(\beta - \gamma - \frac{3}{2} - \nu\right) > 1$, we get from Lemma 4,

$$\frac{\gamma}{\beta}\frac{k_n^{-\nu}\int_{k_n/n}^{1} s^{\frac{1}{\beta}-\gamma-\frac{3}{2}-\nu}\bar{\ell}(s)ds}{(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}-\nu}\bar{\ell}(k_n/n)} = \frac{\gamma}{\beta}\frac{k_n^{-\nu}\int_{k_n/n}^{1} s^{-\alpha}\bar{\ell}(s)ds}{(k_n/n)^{1-\alpha}\bar{\ell}(k_n/n)} = \frac{\gamma}{\beta(\alpha-1)} k_n^{-\nu}(1-o(1)), \quad \text{as} \quad n \to \infty.$$

Hence, $A_{n,1}^{(1,2)} = o_{\mathbb{P}}(1)$, $n \to \infty$. Therefore, by using again (20), we get

$$\begin{aligned}
\frac{A_{n,1}^{(1)}}{(k_n/n)^{-1/2} g(k_n/n)Q(1-k_n/n)} &= \frac{A_{n,1}^{(1,1)}}{(k_n/n)^{-1/2} g(k_n/n)Q(1-k_n/n)} + o_{\mathbb{P}}(1) \\
&= -\sqrt{\frac{k_n}{n}}\frac{\int_{k_n/n}^{1-1/n} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds}{g(k_n/n)Q(1-k_n/n)} + o_{\mathbb{P}}(1).
\end{aligned}$$

N'ext, we are going to prove that

$$\sqrt{\frac{k_n}{n}}\frac{\int_{1-1/n}^{1} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds}{g(k_n/n)Q(1-k_n/n)} = o_{\mathbb{P}}(1).$$

Note that $\mathbb{E}\left(\mathbb{B}_n(s))^2\right) = s(1-s)$, for $0 \leq s \leq 1$, then by using the the Cauchy-Schwarz inequality, for each $n \in \mathbb{N}^*$ we get

$$\mathbb{E}\left(|\mathbb{B}_n(1-s)|\right) \leq \left(\mathbb{E}\left(\mathbb{B}_n^2(s)\right)^{1/2} \leq s^{1/2} \leq 1, \quad \text{for any} \quad 0 \leq s \leq 1.$$

Since $Q'(1-s)$ is continuous on $(0,1]$

$$\begin{aligned}\mathbb{E}\left(\left|\int_{1-1/n}^{1} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds\right|\right) &\le -\int_{1-1/n}^{1} g'(s)Q'(1-s)\mathbb{E}\left(|\mathbb{B}_n(1-s)|\right)ds\\ &\le \int_{1-1/n}^{1} g'(s)|Q'(1-s)|ds\\ &\le \sup_{1-1/n\le s\le 1}|Q'(1-s)|\Big(1-g(1-1/n)\Big)\\ &\le \sup_{0<s\le 1}|Q'(1-s)|\Big(1-g(1-1/n)\Big).\end{aligned}$$

Since the distortion function g is continuous and $g(1)=1$, then $g(1-1/n)$ tends to 1 as $n\to\infty$. It follows that

$$\left|\int_{1-1/n}^{1} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds\right| = o_{\mathbb{P}}(1).$$

This prove the result by using (20). And finally, this implies that

$$\frac{A_{n,1}^{(1)}}{(k_n/n)^{-1/2}g(k_n/n)Q(1-k_n/n)} = -\sqrt{\frac{k_n}{n}}\frac{\int_{k_n/n}^{1} g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds}{g(k_n/n)Q(1-k_n/n)} + o_{\mathbb{P}}(1). \tag{21}$$

Term $A_{n,1}^{(2)}$. Let $0<\varepsilon<1$ be small enough but fixed, we have

$$\begin{aligned}A_{n,1}^{(2)} &= \int_{k_n/n}^{\varepsilon}\left\{1-\frac{Q'(1-\vartheta_n(s))}{Q'(1-s)}\right\}g'(s)Q'(1-s)\beta_n(1-s)ds\\ &\quad+\int_{\varepsilon}^{1-1/n} g'(s)\left\{Q'(1-s)-Q'(1-\vartheta_n(s))\right\}\beta_n(1-s)ds\\ &= A_{n,1}^{(2,1)}+A_{n,1}^{(2,2)}.\end{aligned}$$

We have

$$\mathbb{E}(|A_{n,1}^{(2,1)}|) \le -\sup_{k_n/n<s<\varepsilon}\left|\frac{Q'(1-\vartheta_n(s))}{Q'(1-s)}-1\right|\int_{k_n/n}^{\varepsilon} g'(s)Q'(1-s)\mathbb{E}(|\beta_n(1-s)|)ds.$$

Fixe $0<\nu<1/2$, and write

$$\beta_n(1-s) = \frac{\beta_n(1-s)-\mathbb{B}_n(1-s)}{(s(1-s))^{1/2-\nu}}(s(1-s))^{1/2-\nu}+\mathbb{B}_n(1-s).$$

Then, for any $s\in[k_n/n,\varepsilon]$,

$$|\beta_n(1-s)| \le \sup_{1/n\le s\le 1-1/n}\frac{|\beta_n(1-s)-\mathbb{B}_n(1-s)|}{(s(1-s))^{1/2-\nu}}s^{1/2}+|\mathbb{B}_n(1-s)|.$$

Since $\mathbb{E}(|\mathbb{B}_n(1-s)|)\le s^{1/2}$, then by Csörgő *et al.* (1986), we get for all large values of n

$$\mathbb{E}(|\beta_n(1-s)|) \le \left(1+O\left(n^{-\nu}\right)\right)s^{1/2} < (1+\varepsilon)s^{1/2}, \quad \text{for any } \; k_n<s<\varepsilon.$$

Hence,

$$\begin{aligned}\mathbb{E}(|A_{n,1}^{(2,1)}|) &\le -(1+\varepsilon)\sup_{k_n/n<s<\varepsilon}\left|\frac{Q'(1-\vartheta_n(s))}{Q'(1-s)}-1\right|\int_{k_n/n}^{\varepsilon} s^{1/2}g'(s)Q'(1-s)ds\\ &\le -(1+\varepsilon)\sup_{k_n/n<s<\varepsilon}\left|\frac{Q'(1-\vartheta_n(s))}{Q'(1-s)}-1\right|\int_{k_n/n}^{1} s^{1/2}g'(s)Q'(1-s)ds.\end{aligned}$$

From Lemma 3 in Necir and Meraghni (2009),

$$\sup_{k_n/n<s<\varepsilon}\left|\frac{Q'(1-\vartheta_n(s))}{Q'(1-s)}-1\right|=o_{\mathbb{P}}(1).$$

Since $\gamma\in(\frac{1}{2},1)$ and $\beta\in[1,\frac{1}{\gamma})$, then by using again (16) and (17), we get

$$\begin{aligned}\int_{k_n/n}^{1}s^{1/2}g'(s)Q'(1-s)ds &= \int_{k_n/n}^{1}s^{\frac{1}{\beta}-\gamma-\frac{3}{2}}\bar{\ell}(s)ds\\ &= \frac{\int_{k_n/n}^{1}s^{\frac{1}{\beta}-\gamma-\frac{3}{2}}\bar{\ell}(s)ds}{(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}}\bar{\ell}(k_n/n)}(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}}\bar{\ell}(k_n/n)\\ &= (k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}}\bar{\ell}(k_n/n)\frac{1}{\gamma-\frac{1}{\beta}+\frac{1}{2}}(1+(o(1)).\end{aligned}$$

Therfore, for all large values of n

$$\frac{\mathbb{E}(|A_{n,1}^{(2,1)}|)}{(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}}\bar{\ell}(k_n/n)}=o\left(1\right).$$

Hence, by remarking that $(k_n/n)^{-1/2}g(k_n/n)Q(1-k_n/n)=(k_n/n)^{\frac{1}{\beta}-\gamma-\frac{1}{2}}\bar{\ell}(k_n/n)$, we obtain for all large values of n,

$$\frac{A_{n,1}^{(2,1)}}{(k_n/n)^{-1/2}g(k_n/n)Q(1-k_n/n)}=o_{\mathbb{P}}(1).$$

We now consider $A_{n,1}^{(2,2)}$, we have

$$|A_{n,1}^{(2,1)}|\leq\sup_{\varepsilon\leq s\leq 1}|Q'(1-\vartheta_n(s))-Q'(1-s)|\int_{\varepsilon}^{1}g'(s)|\beta_n(1-s)|ds.$$

Since $\mathbb{E}(|\beta_n(1-s)|)<(1-\varepsilon)s^{1/2}<(1-\varepsilon)$ and $g(1)=1$, it follows that for all large n

$$\mathbb{E}\left(\int_{\varepsilon}^{1}g'(s)|\beta_n(1-s)|ds\right)<(1+\varepsilon)(1-g(\varepsilon)).$$

From Lemma 3 in Necir and Meraghni (2009), we have $\sup_{\varepsilon\leq s\leq 1}|Q'(1-\vartheta_n(s))-Q'(1-s)|=o(1)$. Hence, in view of (20), we get

$$\frac{A_{n,1}^{(2,2)}}{(k_n/n)^{-1/2}g(k_n/n)Q(1-k_n/n)}=o_{\mathbb{P}}\left(1\right).$$

Therefore, for all large values of n

$$\frac{k_n^{1/2}\left(\widetilde{\pi}_n^{(1)}(g)-\pi_n^{(1)}(g)\right)}{g(k_n/n)Q(1-k_n/n)}=-\sqrt{\frac{k_n}{n}}\frac{\int_{k_n/n}^{1-1/n}g'(s)Q'(1-s)\mathbb{B}_n(1-s)ds}{g(k_n/n)Q(1-k_n/n)}+o_{\mathbb{P}}(1).$$

■

Proof of Lemma 2: Note that from the equality $X_{j,n}\overset{\mathcal{D}}{=}Q(1-\xi_{n-j+1,n})$, $\widetilde{\pi}_n^{(2)}(g)$ can be rewritten as follows

$$\widetilde{\pi}_n^{(2)}(g)\overset{\mathcal{D}}{=}\frac{g\left(k_n/n\right)}{1-\beta\gamma_{n,k_n}^{\mathrm{H}}}Q\left(1-\xi_{k_n+1,n}\right).$$

As a consequence, the following expansion holds:

$$
\begin{aligned}
\frac{k_n^{1/2}\left(\widetilde{\pi}_n^{(2)}(g)-\pi_n^{(2)}(g)\right)}{g(k_n/n)Q(1-k_n/n)} &\overset{\mathcal{D}}{=} k_n^{1/2}\left[\frac{1}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}}\times\frac{Q\left(1-\xi_{k_n+1,n}\right)}{Q(1-k_n/n)}-\frac{\int_0^{k_n/n}g'(s)Q(1-s)ds}{g(k_n/n)Q(1-k_n/n)}\right],\\
&:= T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4},
\end{aligned}
$$

with

$$
\begin{aligned}
T_{n,1} &= \frac{k_n^{1/2}}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}}\left[\frac{Q\left(1-\xi_{k_n+1,n}\right)}{Q\left(1-k_n/n\right)}-\left(\frac{n}{k_n}(1-\xi_{k_n+1,n})\right)^{-\gamma}\right],\\
T_{n,2} &= \frac{k_n^{1/2}}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}}\left[\left(\frac{n}{k_n}(1-\xi_{k_n+1,n})\right)^{-\gamma}-1\right],\\
T_{n,3} &= \frac{\beta}{(1-\beta\gamma^{\mathrm{H}}_{n,k_n})(1-\beta\gamma)}k_n^{1/2}\left[\gamma^{\mathrm{H}}_{n,k_n}-\gamma\right],\\
T_{n,4} &= k_n^{1/2}\left[\frac{1}{1-\beta\gamma}-\frac{\int_0^{k_n/n}g'(s)Q(1-s)ds}{g(k_n/n)Q(1-k_n/n)}\right].
\end{aligned}
$$

We study each term separately.

Term $T_{n,1}$. From Deme *et al* (2012), Theorem 1, since $k_n^{1/2}A(n/k_n)\to\lambda\in\mathbb{R}$ as $n\to\infty$, we have

$$
k_n^{1/2}\left(\gamma^{\mathrm{H}}_{n,k_n}-\gamma\right) \overset{\mathcal{D}}{=} \frac{\lambda}{1-\rho}+\gamma\sqrt{\frac{n}{k_n}}\int_0^1 s^{-1}\mathbb{B}_n(1-sk_n/n)d(s\underline{K}(s))+o_{\mathbb{P}}(1), \tag{22}
$$

where $\underline{K}(s)=\mathbb{1}_{\{0<s<1\}}$. In particular, $\gamma^{\mathrm{H}}_{n,k_n}$ is a consistent estimator of γ. Hence, we get

$$
\frac{1}{1-\beta\gamma^{\mathrm{H}}_{n,k_n}}\overset{\mathbb{P}}{\to}\frac{1}{1-\beta\gamma}\quad\text{as}\quad n\to\infty.
$$

Next, according to de Haan and Ferreira (2006, p. 60 and Theorem 2.3.9, p. 48), for any $\delta>0$, we get

$$
\begin{aligned}
\frac{Q\left(1-\xi_{k_n+1,n}\right)}{Q(1-k_n/n)}-\left(\frac{n}{k_n}\xi_{k_n+1,n}\right)^{-\gamma} &= A_0\left(\frac{n}{k_n}\right)\left(\frac{n}{k_n}\xi_{k_n+1,n}\right)^{-\gamma}\frac{\left(\frac{n}{k_n}\xi_{k_n+1,n}\right)^{-\rho}-1}{\rho}\\
&+ A_0\left(\frac{n}{k_n}\right)o_{\mathbb{P}}(1)\left(\frac{n}{k_n}\xi_{k_n+1,n}\right)^{-\gamma-\rho\pm\delta},
\end{aligned}
$$

where $A_0(t)\sim A(t)$ as $t\to\infty$. Thus, since $\frac{n}{k_n}\xi_{k_n+1,n}=1+o_{\mathbb{P}}(1)$ and $k_n^{1/2}A(n/k_n)\to\lambda\in\mathbb{R}$, we have

$$
k_n^{1/2}\left[\frac{Q\left(1-\xi_{k_n+1,n}\right)}{Q(1-k_n/n)}-\left(\frac{n}{k_n}\xi_{k_n+1,n}\right)^{-\gamma}\right]=o_{\mathbb{P}}(1). \tag{23}
$$

Hence,

$$
T_{n,1}=o_{\mathbb{P}}(1). \tag{24}
$$

Term $T_{n,2}$. We have

$$\begin{aligned}\sqrt{k}\left(\left(\frac{n}{k}\xi_{k_n+1,n}\right)^{-\gamma}-1\right) &= -\gamma\sqrt{k}\left(\frac{n}{k}(1-\xi_{n-k,n})-1\right)(1+o_{\mathbb{P}}(1)) \quad \text{by a Taylor expansion}\\ &= \gamma\sqrt{\frac{n}{k}}\beta_n\left(1-\frac{k}{n}\right)(1+o_{\mathbb{P}}(1))\\ &= -\gamma\sqrt{\frac{n}{k}}\left(\mathbb{B}_n\left(1-\frac{k}{n}\right)+O_{\mathbb{P}}(n^{-\nu})\left(\frac{k}{n}\right)^{1/2-\nu}\right)(1+o_{\mathbb{P}}(1)),\end{aligned}$$

for $0 \le \nu < 1/2$, by Csörgő *et al.* (1986). Thus

$$T_{n,2} \stackrel{\mathcal{D}}{=} -\frac{\gamma}{1-\beta\gamma}\sqrt{\frac{n}{k}}\mathbb{B}_n\left(1-\frac{k}{n}\right)(1+o_{\mathbb{P}}(1)) = \mathbb{W}_{n,2}+o_{\mathbb{P}}(1). \tag{25}$$

Term $T_{n,3}$. In view of statement (22), we get

$$\begin{aligned}T_{3,n} &\stackrel{\mathcal{D}}{=} \frac{\lambda\beta}{(1-\rho)(1-\beta\gamma)^2}+\frac{\gamma\beta}{(1-\beta\gamma)^2}\sqrt{\frac{n}{k_n}}\int_0^1 s^{-1}\mathbb{B}_n(1-sk_n/n)d(s\underline{K}(s))+o_{\mathbb{P}}(1)\\ &= \frac{\lambda\beta}{(1-\rho)(1-\beta\gamma)^2}+\mathbb{W}_{n,3}+o_{\mathbb{P}}(1).\end{aligned} \tag{26}$$

Term $T_{n,4}$. A change of variables and the computations in Section 6 yield

$$\begin{aligned}T_{n,4} &= k_n^{1/2}\left[\frac{1}{1-\beta\gamma}-\frac{k_n}{n}\int_0^1\frac{g'(sk_n/n)}{g(k_n/n)}\frac{Q(1-sk_n/n)}{Q(1-k_n/n)}ds\right].\\ &= k_n^{1/2}\left[\frac{1}{1-\beta\gamma}-\frac{k_n}{n}\int_0^1 s^{-\gamma}\frac{g'(sk_n/n)}{g(k_n/n)}ds-\frac{k_n}{n}\int_0^1\frac{g'(sk_n/n)}{g(k_n/n)}\left(\frac{Q(1-ks/n)}{Q(1-k_n/n)}-s^{-\gamma}\right)ds\right].\end{aligned}$$

Since $\gamma \in (\frac{1}{2},1)$ and $\beta \in [1,\frac{1}{\gamma})$, we get from (15),

$$\begin{aligned}\frac{k_n}{n}\int_0^1 s^{-\gamma}\frac{g'(sk_n/n)}{g(k_n/n)}ds &= \left(\frac{k_n}{n}\right)^{\frac{1}{\beta}}\int_0^1 s^{\frac{1}{\beta}-\gamma-1}\frac{\bar{\ell}_g(sk_n/n)}{g(k_n/n)}ds,\\ &\sim \int_0^1 s^{\frac{1}{\beta}-\gamma-1}\frac{\bar{\ell}_g(sk_n/n)}{\bar{\ell}_g(k_n/n)}ds\\ &\to \frac{1}{1-\beta\gamma}, \quad \text{as } n\to\infty, \quad \text{by the uniform convergence.}\end{aligned} \tag{27}$$

Finally, we get for all large value of n

$$T_{n,4} \sim -\,k_n^{1/2}\left(\frac{k_n}{n}\right)\int_0^1\frac{g'(sk_n/n)}{g(k_n/n)}\left(\frac{Q(1-ks/n)}{Q(1-k_n/n)}-s^{-\gamma}\right)ds.$$

Thus, Theorem 2.3.9 in de Haan and Ferreira (2006, p. 48) entails that for a possibly different function A_0, with $A_0(x) \sim A(x)$, $tx \to \infty$, and for any $\delta > 0$, that there exists a threshold $s_\delta \in (0,1)$ such that for all t, $ts \le s_\delta$,

$$\left|\frac{1}{A_0(1/t)}\left(\frac{Q(1-ts)}{Q(1-t)}-s^{-\gamma}\right)-s^{-\gamma}\frac{s^{-\rho}-1}{\rho}\right| \le \delta s^{-\gamma-\rho}\max(s^{\delta},s^{-\delta}). \tag{28}$$

Since g is increasing and differentiable then $g'(s) \geq 0$. Hence, by using the inequality (28) with $t = k_n/n \to 0$ and $s \in (0,1)$, we get

$$\left| \frac{1}{A_0(n/k_n)} \int_0^1 \frac{g'(sk_n/n)}{g(k_n/n)} \left[\frac{Q(1-k_n s/n)}{Q(1-k_n/n)} - s^{-\gamma} \right] ds - \frac{1}{\rho} \int_0^1 s^{-\gamma}(s^{-\rho}-1) \frac{g'(sk_n/n)}{g(k_n/n)} ds \right|$$
$$\leq \delta \int_0^1 s^{-\gamma-\rho-\delta} \frac{g'(sk_n/n)}{g(k_n/n)} ds.$$

By using the same computations as to prove (27), we get

$$\int_0^1 s^{-\gamma-\rho-\delta} \frac{g'(sk_n/n)}{g(k_n/n)} ds = O(1),$$

and

$$\begin{aligned} T_{4,n} &= \frac{1}{\rho} k_n^{1/2} A_0(n/k_n) \left\{ \int_0^1 s^{-\gamma}(s^{-\rho}-1) \frac{g'(sk_n/n)}{g(k_n/n)} ds \right\} (1+o(1)), \\ &= k_n^{1/2} A_0(n/k_n) \frac{\beta}{(\gamma\beta+\rho\beta-1)(1-\beta\gamma)} (1+o(1)) \end{aligned}$$

Hence, since $k_n^{1/2} A(n/k_n) \to \lambda \in \mathbb{R}$, we get

$$T_{4,n} = \frac{\lambda\beta}{(\gamma\beta+\rho\beta-1)(1-\beta\gamma)} (1+o(1)). \tag{29}$$

Combining (24), (25), (26) and (29), Lemma 2 follows. ■

Proof of Lemma 3. We have

$$\frac{k_n^{1/2}\left(\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho}) - \pi(g)\right)}{g(k_n/n)Q(1-k_n/n)} = \frac{k_n^{1/2}\left(\widetilde{\pi}_n^{(1)}(g) - \pi_n^{(1)}(g)\right)}{g(k_n/n)Q(1-k_n/n)} + \frac{k_n^{1/2}\left(\widetilde{\pi}_n^{(3)}(g) - \pi_n^{(2)}(g)\right)}{g(k_n/n)Q(1-k_n/n)},$$

with

$$\begin{aligned} \widetilde{\pi}_n^{(3)}(g) &= \frac{\beta}{\beta - \hat{\gamma}_{n,k}^{\mathrm{LS}}(\widehat{\rho})} \left(1 - \frac{\hat{A}_{n,k}^{\mathrm{LS}}(\widehat{\rho})}{\hat{\gamma}_{n,k}^{\mathrm{LS}}(\widehat{\rho}) + \widehat{\rho} - \beta} \right) g(k_n/n)\ X_{n-k,n} \\ &\overset{\mathcal{D}}{=} \frac{\beta}{\beta - \hat{\gamma}_{n,k}^{\mathrm{LS}}(\widehat{\rho})} \left(1 - \frac{\hat{A}_{n,k}^{\mathrm{LS}}(\widehat{\rho})}{\hat{\gamma}_{n,k}^{\mathrm{LS}}(\widehat{\rho}) + \widehat{\rho} - \beta} \right) g(k_n/n)\ Q\left(1 - \xi_{k_n+1,n}\right). \end{aligned}$$

According to Lemma 1, we have

$$\frac{k_n^{1/2}}{g(k_n/n)Q(1-k_n/n)} \left(\widetilde{\pi}_n^{(1)}(g) - \pi_n^{(1)}(g)\right) \overset{\mathcal{D}}{=} \mathbb{W}_{n,1} + o_{\mathbb{P}}(1).$$

Now, we are going to established the limiting process of $\widetilde{\pi}_n^{(3)}(g) - \pi_n^{(2)}(g)$. As in the proof of Lemma 2, we have

$$\frac{k_n^{1/2}\left(\widetilde{\pi}_n^{(3)}(g) - \pi_n^{(2)}(g)\right)}{g(k_n/n)Q(1-k_n/n)} \overset{\mathcal{D}}{=} S_{n,1} + S_{n,2} + S_{n,3} + S_{n,4} + S_{n,5},$$

with

$$\begin{aligned}
S_{n,1} &= \frac{1}{1-\beta\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})}\left(1-\frac{\hat{A}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})}{\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})+\widehat{\rho}-\beta}\right)k_n^{1/2}\left[\frac{Q\left(1-\xi_{k_n+1,n}\right)}{Q\left(1-k_n/n\right)}-\left(\frac{n}{k_n}(1-\xi_{k_n+1,n})\right)^{-\gamma}\right],\\
S_{n,2} &= \frac{1}{1-\beta\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})}\left(1-\frac{\hat{A}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})}{\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})+\widehat{\rho}-\beta}\right)k_n^{1/2}\left[\left(\frac{n}{k_n}(1-\xi_{k_n+1,n})\right)^{-\gamma}-1\right],\\
S_{n,3} &= \frac{\beta}{(1-\beta\gamma)(1-\beta\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho}))}k_n^{1/2}\left(\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})-\gamma\right),\\
S_{n,4} &= \beta\, k_n^{1/2}\left[\frac{A(n/k_n)}{(1-\beta\gamma)(\beta\gamma+\beta\rho-1)}-\frac{\hat{A}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})}{\left(1-\beta\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})\right)\left(\beta\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})+\beta\widehat{\rho}-1\right)}\right],\\
S_{n,5} &= k_n^{1/2}\left[\frac{1}{1-\beta\gamma}\left(1-\frac{\beta A(n/k_n)}{\beta\gamma+\beta\rho-1}\right)-\frac{k_n}{n}\int_0^{k_n/n}\frac{g'(sk_n/n)}{g(k_n/n)}\frac{Q(1-sk_n/n)}{Q(1-k_n/n)}ds\right].
\end{aligned}$$

From Deme *et al* (2012), Lemma 5, since $k_n^{1/2}A(n/k_n)\to\lambda\in\mathbb{R}$ as $n\to\infty$ and since $\widehat{\rho}$ is a consistent estimator of ρ, we get

$$\sqrt{k}\left(\hat{\gamma}^{\mathrm{LS}}_{n,k}(\widehat{\rho})-\gamma\right)\overset{\mathcal{D}}{=}\gamma\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}\left(1-\frac{sk}{n}\right)d(sK_\rho(s))+o_{\mathbb{P}}(1), \tag{30}$$

and

$$\sqrt{k}\left(\widehat{A}^{\mathrm{LS}}_{n,k}(\widehat{\rho})-A(n/k)\right)\overset{\mathcal{D}}{=}\gamma(1-\rho)\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}\left(1-\frac{sk}{n}\right)d(s(\underline{K}(s)-K_\rho(s)))+o_{\mathbb{P}}(1), \tag{31}$$

where

$$K_\rho(s)=\frac{1-\rho}{\rho}\underline{K}(s)+\left(1-\frac{1-\rho}{\rho}\right)\underline{K}_\rho(s),\quad\text{for}\quad 0<s\leq 1,$$

and with $\underline{K}(s)=\mathbb{1}_{\{0<s<1\}}$ and $\underline{K}_\rho(s)=((1-\rho)/\rho)(s^{-\rho}-1)\mathbb{1}_{\{0<s<1\}}$.

Term $S_{n,1}$. In view (30) and (31), we obtain $\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})=\gamma+o_{\mathbb{P}}(1)$ and $\hat{A}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})=o_{\mathbb{P}}(1)$. Hence, by using (23) we get

$$S_{n,1}=o_{\mathbb{P}}(1). \tag{32}$$

Term $S_{n,2}$. By using the same arguments as to prove $T_{2,n}$, we get

$$S_{n,2}\overset{\mathcal{D}}{=}-\frac{\gamma}{1-\beta\gamma}\sqrt{\frac{k_n}{n}}\,\mathbb{B}_n(1-k_n/n)+o_{\mathbb{P}}(1)=\mathbb{W}_{n,2}+o_{\mathbb{P}}(1). \tag{33}$$

Term $S_{n,3}$. We have

$$S_{n,3}\overset{\mathcal{D}}{=}\frac{\beta}{(1-\beta\gamma)^2}k_n^{1/2}(\hat{\gamma}^{\mathrm{LS}}_{n,k_n}(\widehat{\rho})-\gamma)+o_{\mathbb{P}}(1).$$

In view of statement (30), we get

$$S_{n,3}\overset{\mathcal{D}}{=}\frac{\beta\gamma}{(1-\beta\gamma)^2}\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}\left(1-\frac{sk}{n}\right)d(sK_\rho(s))+o_{\mathbb{P}}(1).$$

Term $S_{n,4}$. We have

$$S_{n,4} = k_n^{1/2}A(n/k_n)\left[\frac{\beta}{(1-\beta\gamma)(\beta\gamma+\beta\rho-1)} - \frac{\beta}{\left(1-\beta\hat{\gamma}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})\right)\left(\beta\hat{\gamma}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})+\beta\widehat{\rho}-1\right)}\right] - \frac{\beta}{\left(1-\beta\hat{\gamma}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})\right)\left(\beta\hat{\gamma}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})+\beta\widehat{\rho}-1\right)}k_n^{1/2}\left(\hat{A}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})-A(n/k_n)\right).$$

Since $k_n^{1/2}A(n/k_n) \to \lambda \in \mathbb{R}$ and $\widehat{\rho}$ is consistent to ρ then by using the statement (31) and the consistence of $\hat{\gamma}_{n,k_n}^{\mathrm{LS}}(\widehat{\rho})$ to γ, we obtain

$$S_{n,4} = -\frac{\beta\gamma(1-\rho)}{(1-\beta\gamma)(\beta\gamma+\beta\rho-1)}\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}\left(1-\frac{sk}{n}\right)d\left(s(\underline{K}(s)-K_\rho(s))\right)+o_{\mathbb{P}}(1).$$

It is easy to see that

$$S_{n,3}+S_{n,4} = \mathbb{W}_{n,4}+\mathbb{W}_{n,5}+_{\mathbb{P}}(1). \tag{34}$$

Term $S_{n,5}$. By using the same arguments as to prove $T_{n,5}$, we get

$$S_{n,5} = o_{\mathbb{P}}(1). \tag{35}$$

Combining (32), (33), (34) and (35), the Lemma 3 follows. ■

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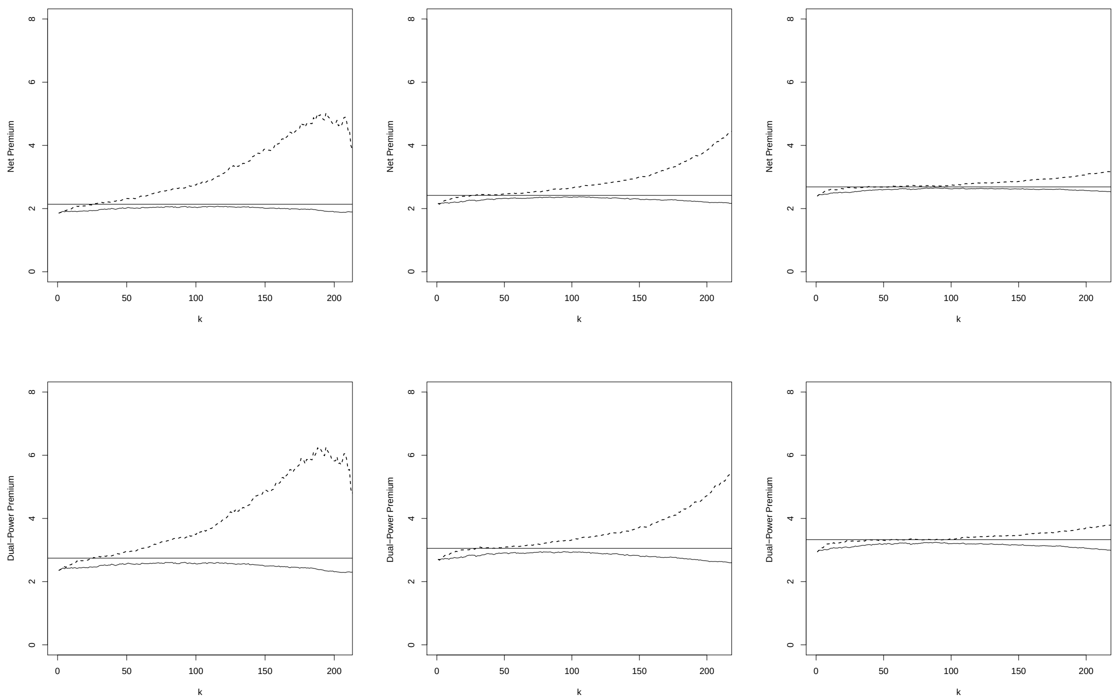

Figure 1: Median of $\widetilde{\pi}_n(g)$ (dotted line) and $\widetilde{\pi}_n^{\mathrm{LS}}(g, \widehat{\rho})$ (full line) as a function of k based on 500 samples of size 500 for Net Premium (top) and Dual-Power premium with its loading parameter $\alpha = 1.366$ (bottom) from a Burr distribution defined as $\overline{F}(x) = (1 + x^{-\frac{3\rho}{2}})^{1/\rho}$. From the left to the right: $\rho = -0.75$, $\rho = -1$ and $\rho = -1.5$. The horizontal line represents the true value of the premium $\pi(g)$.

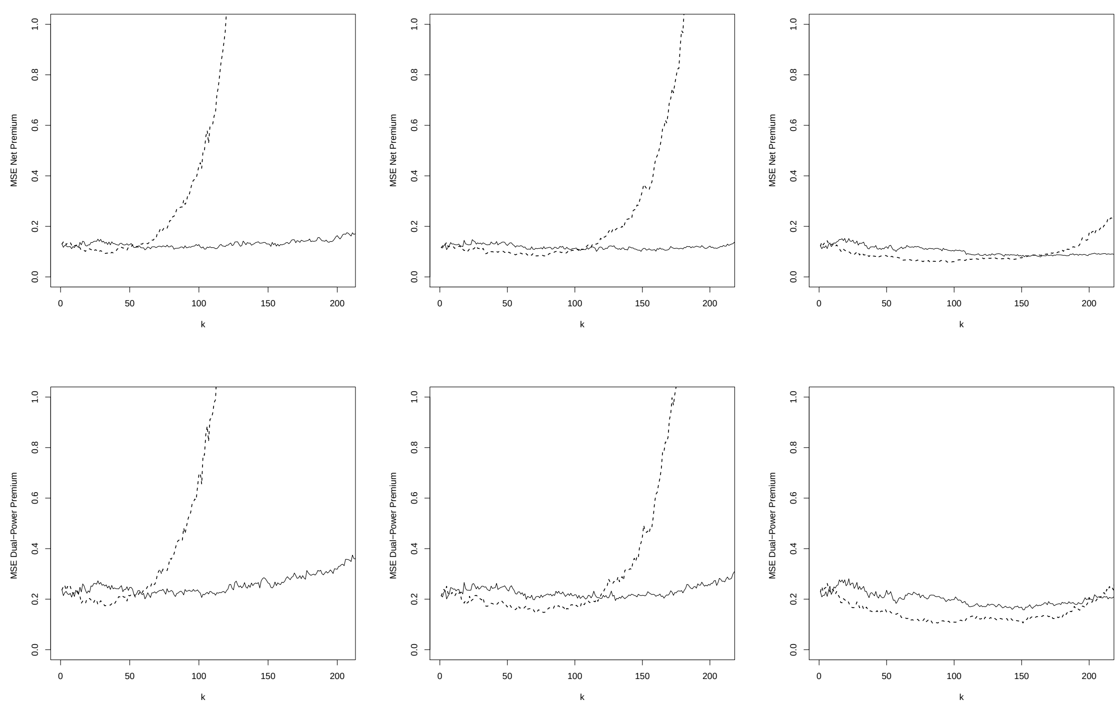

Figure 2: MSE of $\widetilde{\pi}_n(g)$ (dotted line) and $\widetilde{\pi}_n^{\mathrm{LS}}(g,\widehat{\rho})$ (full line) as a function of k based on 500 samples of size 500 for Net Premium (top) and Dual-Power premium with its loading parameter $\alpha = 1.366$ (bottom) from a Burr distribution defined as $\overline{F}(x) = (1 + x^{-\frac{3\rho}{2}})^{1/\rho}$. From the left to the right: $\rho = -0.75$, $\rho = -1$ and $\rho = -1.5$.

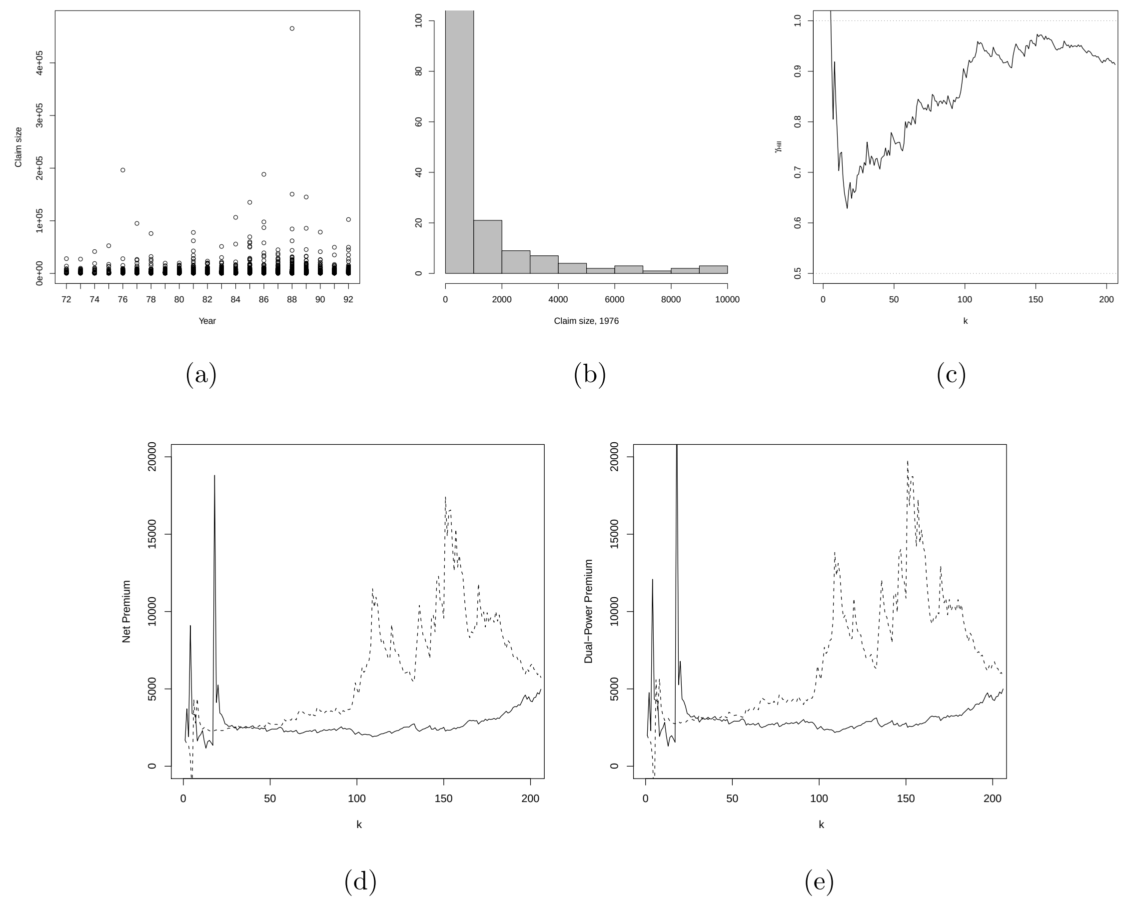

Figure 3: (a) Time plot for the Norwegian fire insurance data; (b) Histogram of the claim size for the year 1976; (c) Hill estimator as a function of k for the year 1976; Biased estimator $\widetilde{\pi}_n(g)$ (dotted line) and reduced-bias one $\widetilde{\pi}_n^{\mathrm{LS}}(g, \widehat{\rho})$ (full line) as a function of k for the net premium principle (d) and the dual-power premium principle with its loading parameter $\alpha = 1.366$ (e).