A Configuration Model for the Line Planning Problem

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Abstract

We propose a novel extended formulation for the line planning problem in public transport. It is based on a new concept of *frequency configurations* that account for all possible options to provide a required transportation capacity on an infrastructure edge. We show that this model yields a strong LP relaxation. It implies, in particular, general classes of facet defining inequalities for the standard model.

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1 Introduction

Line planning is an important strategic planning problem in public transport. The task is to find a set of lines and frequencies such that a given demand can be transported. There are usually two main objectives: minimizing the travel times of the passengers and minimizing the line operating costs.

Since the late nineteen-nineties, the line planning literature has developed a variety of integer programming approaches that capture different aspects, see Schöbel [15] for an overview. Bussieck, Kreuzer, and Zimmermann [8] (see also the thesis of Bussieck [7]) propose an integer programming model to maximize the number of direct travelers. Operating costs are discussed in the articles of Claessens, van Dijk, and Zwaneveld [9] and Goossens, van Hoesel, and Kroon [11, 12]. Schöbel and Scholl [16] and Borndörfer and Karbstein [3] focus on the number of transfers and the number of direct travelers, respectively, and further integrate line planning and passenger routing in their models. Borndörfer, Grötschel, and Pfetsch [2] also propose an integrated line planning and passenger routing model that allows to generate lines dynamically.

All these models employ some type of *capacity* or *frequency demand constraints* in order to cover a given demand. In this paper we propose a concept to strengthen such constraints by means of a novel extended formulation. The idea is to enumerate the set of possible *configurations* of line frequencies for each capacity constraint. We show that such an extended formulation implies general facet defining inequalities for the standard model. We remark that configuration models have also been used successfully in railway vehicle rotation planning [4] and railway track allocation applications [5].

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2 Problem Description

We consider the following basic *line planning problem*. We have an undirected graph $G = (V, E)$ representing the transportation network, and a set $\mathcal{L} = \{l_1, \ldots, l_n\}$, $n \in \mathbb{N}$, of lines, where every line l_i is a path in G. Denote by $\mathcal{L}(e) := \{l \in \mathcal{L} : e \in l\}$ the set of lines on edge $e \in E$. Furthermore, we are given an ordered set of frequencies $\mathcal{F} = \{f_1, \ldots, f_k\} \subseteq \mathbb{N}$, such that $0 < f_1 < \ldots < f_k$, $k \in \mathbb{N}$, and costs $c_{l,f}$ for operating line $l \in \mathcal{L}$ at frequency $f \in \mathcal{F}$. Finally, each edge $e \in E$ in the network bears a positive frequency demand $F(e)$ giving the number of line operations that are necessary to cover the demand on this edge.

A *line plan* $(\bar{\mathcal{L}}, \bar{f})$ consists of a subset $\bar{\mathcal{L}} \subseteq \mathcal{L}$ of lines and an assignment $\bar{f} : \bar{\mathcal{L}} \to \mathcal{F}$ of frequencies to these lines. A line plan is *feasible* if the frequencies of the lines satisfy the given frequency demand $F(e)$ for each edge $e \in E$, i.e., if

$$\sum_{l \in \bar{\mathcal{L}}(e)} \bar{f}(l) \geq F(e) \text{ for all } e \in E. \tag{1}$$

We define the cost of a line plan $(\bar{\mathcal{L}}, \bar{f})$ as $c(\bar{\mathcal{L}}, \bar{f}) = \sum_{l \in \bar{\mathcal{L}}} c_{l,\bar{f}(l)}$. The *line planning problem* is to find a feasible line plan of minimal cost.

2.1 Standard Model

The common way to formulate the line planning problem uses binary variables $x_{l,f}$ indicating whether line $l \in \mathcal{L}$ is operated at frequency $f \in \mathcal{F}$, cf. the references listed in the introduction. In our case, this results in the following *standard model*:

$$\begin{array}{llrlr}
\text{(SLP)} & \min & \displaystyle\sum_{l \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{l,f} x_{l,f} & & \\
& \text{s.t.} & \displaystyle\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{l,f} \geq F(e) & \forall e \in E & (2) \\
& & \displaystyle\sum_{f \in \mathcal{F}} x_{l,f} \leq 1 & \forall l \in \mathcal{L} & (3) \\
& & x_{l,f} \in \{0, 1\} & \forall l \in \mathcal{L}, \forall f \in \mathcal{F}. & (4)
\end{array}$$

Model (SLP) minimizes the cost of a line plan. The *frequency demand constraints* (2) ensure that the frequency demand is covered. The *assignment constraints* (3) ensure that every line operates at only one frequency. Hence, the solutions of (SLP) correspond to the feasible line plans.

2.2 Extended or Configuration Model

In the following, we give an extended formulation for (SLP) in order to tighten the LP-relaxation. The formulation is based on the observation that the frequency demand for an edge $e \in E$ can also be expressed by specifying the numbers q_f of lines that are operated at frequency f, $f \in \mathcal{F}$, on edge e. We explain the idea using the example in Figure 1. The transportation network consists of two edges and three lines. Each line can be operated at frequency 2 or 8. The frequency demand on edge $\{u, v\}$ is 9. To cover this demand using at most three lines we need at least two lines with frequency 8 or one line with frequency 2 and one line with frequency 8. We call these feasible frequency combinations *configurations*. In this case the set of all possible configurations is $\bar{Q}(\{u, v\}) = \{(0, 2), (0, 3), (1, 1), (1, 2), (2, 1)\}$, where the first coordinate gives the number of lines with frequency 2 and the second coordinate gives the number of lines with frequency 8. The set of minimal configurations is $Q(\{u,v\}) = \{(0,2),(1,1)\}$, i.e., any line plan that matches one configuration covers the frequency demand of the edge $\{u,v\}$. Similarly, the minimal configurations for edge $\{v,w\}$ are $Q(\{v,w\}) = \{(0,1),(1,0)\}$. Any line plan that matches one configuration for each edge is a feasible line plan. A formal description is as follows.

▶ **Definition 1.** For $e \in E$ denote by

$$\bar{\mathcal{Q}}(e) := \Big\{ q = (q_{f_1},\dots,q_{f_k}) \in \mathbb{N}_0^k : \sum_{f\in\mathcal{F}} q_f \le |\mathcal{L}(e)|, \sum_{f\in\mathcal{F}} f\cdot q_f \ge F(e) \Big\}$$

the set of *(feasible) (frequency) configurations of* e and by

$$\mathcal{Q}(e) := \big\{ q \in \bar{\mathcal{Q}}(e) : (q_{f_1},\dots,q_{f_i}-1,\dots,q_{f_k}) \notin \bar{\mathcal{Q}}(e) \quad \forall i = 1,\dots,k \big\}$$

the set of *minimal configurations of* e.

Let $q = \big(q(e)\big)_{e\in E}$ be some vector of minimal configurations, i.e., $q(e) \in \mathcal{Q}(e)$ for all $e \in E$, and $(\bar{\mathcal{L}},\bar{f})$ a line plan. If $(\bar{\mathcal{L}},\bar{f})$ satisfies the inequality $|\{l \in \bar{\mathcal{L}}(e) : \bar{f}(l) = f\}| \ge q(e)_f$ for all $e \in E$, $f \in \mathcal{F}$, then $(\bar{\mathcal{L}},\bar{f})$ is feasible. Conversely, if $(\bar{\mathcal{L}},\bar{f})$ is a feasible line plan, then $q_f = |l \in \bar{\mathcal{L}}(e) : \bar{f}(l) = f|$, $f \in \mathcal{F}$, is a feasible frequency configuration for each edge e. In other words, satisfying the frequency demand is equivalent to choosing a feasible configuration for each edge.

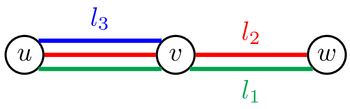
We extend (SLP) using binary variables $y_{e,q}$ that indicate for each edge $e \in E$ which configuration $q \in \mathcal{Q}$ is chosen. This results in the following extended formulation:

$$
\begin{array}{lllll}
\text{(QLP)} & \min & \displaystyle\sum_{l\in\mathcal{L}}\sum_{f\in\mathcal{F}} c_{l,f}x_{l,f} & & \\
& \text{s.t.} & \displaystyle\sum_{l\in\mathcal{L}(e)} x_{l,f} \ge \sum_{q\in\mathcal{Q}(e)} q_f\cdot y_{e,q} & \forall e\in E, \forall f\in\mathcal{F} & (5)\\
& & \displaystyle\sum_{q\in\mathcal{Q}(e)} y_{e,q} = 1 & \forall e\in E & (6)\\
& & \displaystyle\sum_{f\in\mathcal{F}} x_{l,f} \le 1 & \forall l\in\mathcal{L} & (7)\\
& & x_{l,f} \in \{0,1\} & \forall l\in\mathcal{L}, \forall f\in\mathcal{F} & (8)\\
& & y_{e,q} \in \{0,1\} & \forall e\in E, \forall q\in\mathcal{Q}(e). & (9)
\end{array}
$$

The *(extended) configuration model* (QLP) also minimizes the cost of a line plan. The *configuration assignment constraints* (6) ensure that exactly one configuration for each edge is chosen while the *coupling constraints* (5) guarantee that sufficient numbers of lines are operated at the frequencies of the chosen configurations.

▶ **Example 2.** Consider the line planning problem in Figure 1. We define the costs as $c_{l_1,f} = c_{l_2,f} = 2\cdot f$ and $c_{l_3,f} = f$. The standard model for this example reads as follows:

$$
\begin{array}{lllllllll}
\text{(SLP)} & \min & 4x_{l_1,2}+16x_{l_1,8} & +4x_{l_2,2}+16x_{l_2,8} & +2x_{l_3,2}+8x_{l_3,8} & \\
& \text{s.t.} & 2x_{l_1,2}+8x_{l_1,8} & +2x_{l_2,2}+8x_{l_2,8} & +2x_{l_3,2}+8x_{l_3,8} & \ge 9\\
& & 2x_{l_1,2}+8x_{l_1,8} & +2x_{l_2,2}+8x_{l_2,8} & & \ge 1\\
& & x_{l_1,2}\ +x_{l_1,8} & & & \le 1\\
& & & +x_{l_2,2}\ +x_{l_2,8} & & \le 1\\
& & & & +x_{l_3,2}\ +x_{l_3,8} & \le 1\\
& & & & x_{l_i,f} & \in \{0,1\}.
\end{array}
$$



$\mathcal{F} = \{2, 8\}$

e	$e_1 = \{u, v\}$	$e_2 = \{v, w\}$
$F(e)$	9	1
$\mathcal{L}(e)$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2\}$
$\mathcal{Q}(e)$	$\{(0, 2), (1, 1)\}$	$\{(0, 1), (1, 0)\}$

Figure 1 An instance of the line planning problem. *Left*: Transportation network consisting of two edges and three lines. *Right*: The given set of frequencies, frequency demands, and the minimal frequency configurations.

The configuration model for this example is:

$$
\begin{array}{lllllllllllll}
\text{(QLP)} & \min & \multicolumn{11}{l}{4x_{l_1,2}+16x_{l_1,8}+4x_{l_2,2}+16x_{l_2,8}+2x_{l_3,2}+8x_{l_3,8}} \\
 & \text{s.t.} & x_{l_1,2} & & +x_{l_2,2} & & +x_{l_3,2} & & & -y_{e_1,q_2} & & & \geq 0 \\
 & & & +x_{l_1,8} & & +x_{l_2,8} & & +x_{l_3,8} & -2y_{e_1,q_1} & -y_{e_1,q_2} & & & \geq 0 \\
 & & x_{l_1,2} & & +x_{l_2,2} & & & & & & & -y_{e_2,q_2} & \geq 0 \\
 & & & +x_{l_1,8} & & +x_{l_2,8} & & & & & -y_{e_2,q_1} & & \geq 0 \\
 & & x_{l_1,2} & +x_{l_1,8} & & & & & & & & & \leq 1 \\
 & & & & +x_{l_2,2} & +x_{l_2,8} & & & & & & & \leq 1 \\
 & & & & & & +x_{l_3,2} & +x_{l_3,8} & & & & & \leq 1 \\
 & & & & & & & & +y_{e_1,q_1} & +y_{e_1,q_2} & & & = 1 \\
 & & & & & & & & & & +y_{e_2,q_1} & +y_{e_2,q_2} & = 1 \\
 & & & & & & & & & & & x_{l_i,f} & \in \{0,1\} \\
 & & & & & & & & & & & y_{e,q} & \in \{0,1\}.
\end{array}
$$

3 Comparison of the Models

In this section we compare the standard and the extended configuration model for the line planning problem. We need some further notation. For an integer program (IP) $= \min\{c^T x : Ax \geq b, x \in \mathbb{Z}^n\}$ we denote by $\mathrm{P_{IP}}$ (IP) the polyhedron defined by the convex hull of all feasible solutions of (IP) and by $\mathrm{P_{LP}}$ (IP) the set of feasible solutions of the LP relaxation of (IP), i.e., $\mathrm{P_{IP}}\text{ (IP)} = \operatorname{conv}\{x \in \mathbb{Z}^n : Ax \geq b\}$ and $\mathrm{P_{LP}}\text{ (IP)} = \{x \in \mathbb{R}^n : Ax \geq b\}$. For a polyhedron $P = \{(x, y) \in \mathbb{R}^{n+m} : Ax + By \geq b\}$ denote by $P|_x := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ s.t. } (x, y) \in P\}$ the projection of P onto the space of x-variables.

Using this notation, we can state that solving (QLP) is equivalent to solving (SLP):

▶ **Lemma 3.** (QLP) *provides an extended formulation for* (SLP)*, i.e.,*

$$\mathrm{P_{IP}}(\mathrm{QLP})\,|_x = \mathrm{P_{IP}}(\mathrm{SLP})\,.$$

For the LP relaxations, however, the following holds:

▶ **Theorem 4.** *The LP relaxation of* $\mathrm{P_{IP}}(\mathrm{QLP})\,|_x$ *is tighter than the LP relaxation of* $\mathrm{P_{IP}}(\mathrm{SLP})$*, i.e.,*

$$\mathrm{P_{LP}}(\mathrm{QLP})\,|_x \subseteq \mathrm{P_{LP}}(\mathrm{SLP})\,.$$

Proof. Let $(\bar{x}, \bar{y}) \in \mathrm{P_{LP}}(\mathrm{QLP})$. Obviously, $\bar{x}$ satisfies (3) and (4). We further get

$$\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot \bar{x}_{l,f} \overset{(5)}{\geq} \sum_{f \in \mathcal{F}} \Big(f \cdot \sum_{q \in \mathcal{Q}(e)} q_f \cdot \bar{y}_{e,q} \Big) = \underbrace{\sum_{q \in \mathcal{Q}(e)} \bar{y}_{e,q}}_{\overset{(6)}{=} 1} \cdot \underbrace{\sum_{f \in \mathcal{F}} f \cdot q_f}_{\geq F(e) \forall q \in \mathcal{Q}(e)} \geq F(e).$$

Hence, $\bar{x}$ satisfies (2) as well and is contained in $\mathrm{P_{LP}}(\mathrm{SLP})$. ◀

The converse, i.e., $P_{LP}(SLP) \subseteq P_{LP}(QLP)\,|_x$, does not hold in general, indeed, the ratio of the optimal objectives of the two LP relaxations can be arbitrarily large.

▶ **Example 5.** Consider an instance of the line planning problem involving only one edge $E = \{e\}$, one line $\mathcal{L}(e) = \{l\}$, a frequency demand $F(e) = 6$, and one frequency $\mathcal{F} = \{M\}$ such that $M > 6$ with cost function $c_{l,M} = M$. The only minimal configuration for e is $q = (1)$.

$$\begin{aligned} \text{QLP}_{\text{LP}}:\quad \min\quad & M \cdot x_{l,M} \\ \text{s.t.}\quad x_{l,M} - y_q &\geq 0 \\ y_q &= 1 \\ y_q, x_{l,M} &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{SLP}_{\text{LP}}:\quad \min\quad & M \cdot x_{l,M} \\ \text{s.t.}\quad M \cdot x_{l,M} &\geq 6 \\ \\ x_{l,M} &\geq 0 \end{aligned}$$

Obviously, $x_{l,M} = 1$ is the only and hence optimal solution to QLP_{LP} with objective value M and $x_{l,M} = \frac{6}{M}$ is an optimal solution to SLP_{LP} with objective value 6.

In the following subsections we show that the LP relaxation of the configuration model implies general classes of facet defining inequalities for the line planning polytope $P_{IP}(SLP)$ that are discussed in the literature.

3.1 Band Inequalities

In this section we analyze band inequalities, which were introduced by Stoer and Dahl [19] and are closely related to the knapsack cover inequalities, see Wolsey [21].

▶ **Definition 6.** Let $e \in E$.

- A *band* $f_{\mathcal{B}} : \mathcal{L}(e) \to \mathcal{F} \cup \{0\}$ assigns to each line containing e a frequency or 0. We call $f_{\mathcal{B}}$ a *valid band of* e if

$$\sum_{l \in \mathcal{L}(e)} f_{\mathcal{B}}(l) < F(e).$$

- We call the band $f_{\mathcal{B}}$ *maximal* if $f_{\mathcal{B}}$ is valid and there is no valid band $f_{\mathcal{B}'}$ with $f_{\mathcal{B}}(l) \leq f_{\mathcal{B}'}(l)$ for every line $l \in \mathcal{L}(e)$ and $f_{\mathcal{B}}(l) < f_{\mathcal{B}'}(l)$ for at least one line $l \in \mathcal{L}(e)$.
- We call the band $f_{\mathcal{B}}$ *symmetric* if $f_{\mathcal{B}}(l) = f$ for all $l \in \mathcal{L}(e)$ and for some $f \in \mathcal{F}$.

Applying the results of Stoer and Dahl [19] yields

▶ **Proposition 7.** *Let $f_{\mathcal{B}}$ be a valid band of $e \in E$, then*

$$\sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > f_{\mathcal{B}}(l)}} x_{l,f} \geq 1 \tag{10}$$

is a valid inequality for $P_{IP}(SLP)$.

The simplest example is the case $f_{\mathcal{B}}(l) \equiv 0$, which states that one must operate at least one line on every edge, i.e., one has to cover the demand.

▶ **Proposition 8.** *The* set cover *inequality*

$$\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} x_{l,f} \geq 1 \tag{11}$$

is valid for $P_{IP}(SLP)$ *for all* $e \in E$.

The set cover inequalities (11) do not hold in general for the LP relaxation of the standard model, compare with Example 5. Note that they are symmetric band inequalities.

Maximal band inequalities often define facets of the single edge relaxation of the line planning polytope [13]. The symmetric ones are implied by the configuration model.

▶ **Theorem 9.** *The LP relaxation of the configuration model implies all band inequalities* (10) *that are induced by a valid symmetric band.*

Proof. Assume $f_{\mathcal{B}}$ is a valid symmetric band of some edge e with $f_{\mathcal{B}}(l) = \tilde{f}$ for all $l \in \mathcal{L}(e)$ and for some $\tilde{f} \in \mathcal{F}$, $\tilde{f} < f_k$. Thus $\sum_{l \in \mathcal{L}(e)} f_{\mathcal{B}}(l) = |\mathcal{L}(e)| \cdot \tilde{f} < F(e)$. Hence, in every minimal configuration $q \in \mathcal{Q}(e)$ there is a frequency $f > \tilde{f}$ such that $q_f \geq 1$. Starting from (5), we get:

$$
\begin{aligned}
& & \sum_{l \in \mathcal{L}(e)} x_{l,f} &\geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q & \forall f \in \mathcal{F} \\
\Rightarrow & & \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} \sum_{l \in \mathcal{L}(e)} x_{l,f} &\geq \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q & \\
\Leftrightarrow & & \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} x_{l,f} &\geq \sum_{q \in \mathcal{Q}(e)} y_q \cdot \underbrace{\sum_{\substack{f \in \mathcal{F} \\ f > \tilde{f}}} q_f}_{\geq 1} & \\
& & &\geq \sum_{q \in \mathcal{Q}(e)} y_q = 1. &
\end{aligned}
$$

◀

The same does not hold for the standard model as the following example shows.

▶ **Example 10** (Example 2 continued)**.** A valid symmetric band for edge e_1 in Figure 1 is given by $f_{\mathcal{B}}(l) = 2$ for all $l \in \mathcal{L}(e_1)$. The corresponding band inequality

$$x_{l_1,8} + x_{l_2,8} + x_{l_3,8} \geq 1 \tag{12}$$

is violated by $\tilde{x} \in \mathrm{P_{LP}(SLP)}$, where $\tilde{x}_{l_2,8} = \frac{7}{8}$, $\tilde{x}_{l_3,2} = 1$, and $\tilde{x}_{l,f} = 0$ otherwise. One can show that (12) is facet-defining for $\mathrm{P_{IP}(SLP)}$ in this example.

3.2 MIR Inequalities

We study in this section the mixed integer rounding (MIR) inequalities and their connection to the configuration model. MIR inequalities can be derived from the basic MIR inequality as defined by Wolsey [22], see also Raack [14].

▶ **Lemma 11** (Wolsey [22])**.** *Let* $Q_I := \{(x, y) \in \mathbb{Z} \times \mathbb{R} : x + y \geq \beta, y \geq 0\}$*. The basic MIR inequality*

$$rx + y \geq r\lceil \beta \rceil$$

with $r := r(\beta) = \beta - \lfloor \beta \rfloor$ *is valid for* Q_I *and defines a facet of* $\mathrm{conv}(Q_I)$ *if* $r > 0$*.*

We use mixed integer rounding to strengthen the demand inequalities (2).

▶ **Proposition 12.** *Let $\lambda \in \mathbb{R}_+$, $e \in E$, and define $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$. The MIR inequality*

$$\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \lfloor \lambda f \rfloor + \min(r_f, r)) x_{l,f} \geq r \lceil \lambda F(e) \rceil \tag{13}$$

induced by the demand inequality (2) *scaled by λ is valid for* (SLP)*.*

Proof. Scaling inequality (2) by $\lambda > 0$ yields

$$\begin{aligned}
\lambda \cdot F(e) \leq \lambda \cdot \sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} f \cdot x_{l,f} &= \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} \lambda \cdot f \cdot x_{l,f} + \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} \lambda \cdot f \cdot x_{l,f} \\
&\leq \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} (\lfloor \lambda \cdot f \rfloor + r_f) \cdot x_{l,f} + \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} (\lfloor \lambda \cdot f \rfloor + 1) \cdot x_{l,f} \\
&= \underbrace{\sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot x_{l,f}}_{\geq 0} + \underbrace{\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot x_{l,f} + \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} x_{l,f}}_{\in \mathbb{Z}}.
\end{aligned}$$

Applying Lemma 11 yields

$$\begin{aligned}
r \cdot \lceil \lambda \cdot F(e) \rceil &\leq \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot x_{l,f} + r \cdot \Big(\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot x_{l,f} + \sum_{l \in \mathcal{L}(e)} \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} x_{l,f} \Big) \\
&= \sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) \cdot x_{l,f}.
\end{aligned}$$

◀

Notice that $\lambda \in \mathbb{R}_+$ only produces a non-trivial MIR inequality (13) if $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor \neq 0$. Dash, Günlük and Lodi [10] analyze for which λ the MIR inequality (13) is non-redundant.

▶ **Proposition 13** (Dash, Günlük and Lodi [10])**.** *Each non-redundant MIR inequality* (13) *is defined by $\lambda \in (0, 1)$, where λ is a rational number with denominator equal to some $f \in \mathcal{F}$.*

Again, we can show that these inequalities are implied by the LP relaxation of the configuration model. The proof is based on the following lemma, a configuration version of Proposition 12.

▶ **Lemma 14.** *For $e \in E$, $q \in \mathcal{Q}(e)$, and $\lambda \in (0, 1)$, it holds*

$$\sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) q_f \geq r \cdot \lceil \lambda \cdot F(e) \rceil,$$

where $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$.

Proof. $q \in \mathcal{Q}(e)$ implies $\sum\limits_{f \in \mathcal{F}} f \cdot q_f \geq F(e)$ and hence we get for $\lambda \in (0,1)$

$$\begin{aligned}
\lambda \cdot F(e) \leq \lambda \cdot \sum_{f \in \mathcal{F}} f \cdot q_f &= \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} \lambda \cdot f \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} \lambda \cdot f \cdot q_f \\
&\leq \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} (\lfloor \lambda \cdot f \rfloor + r_f) \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} (\lfloor \lambda \cdot f \rfloor + 1) \cdot q_f \\
&= \underbrace{\sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot q_f}_{\geq 0} + \underbrace{\sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} q_f}_{\in \mathbb{Z}}.
\end{aligned}$$

Applying Lemma 11 yields

$$\begin{aligned}
r \cdot \lceil \lambda \cdot F(e) \rceil &\leq \sum_{\substack{f \in \mathcal{F} \\ r_f < r}} r_f \cdot q_f + r \cdot \Big(\sum_{f \in \mathcal{F}} \lfloor \lambda \cdot f \rfloor \cdot q_f + \sum_{\substack{f \in \mathcal{F} \\ r_f \geq r}} q_f \Big) \\
&= \sum_{f \in \mathcal{F}} (r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)) \cdot q_f.
\end{aligned}$$

◀

▶ **Theorem 15.** *Let $\lambda \in (0,1)$, $e \in E$, $r = \lambda F(e) - \lfloor \lambda F(e) \rfloor$ and $r_f = \lambda f - \lfloor \lambda f \rfloor$. Then the MIR inequality*

$$\sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} (r \lfloor \lambda f \rfloor + \min(r_f, r)) x_{l,f} \geq r \lceil \lambda F(e) \rceil$$

is implied by the LP relaxation of the configuration model, i.e., the MIR inequalities (13) *are valid for* $\mathrm{P}_{\mathrm{LP}}(\mathrm{QLP})\,|_x$.

Proof. Let $(x,y) \in \mathrm{P}_{\mathrm{LP}}(\mathrm{QLP})$. Then by (5)

$$\sum_{l \in \mathcal{L}(e)} x_{l,f} \geq \sum_{q \in \mathcal{Q}(e)} q_f \cdot y_q \qquad \forall f \in \mathcal{F}.$$

Scaling this inequality by $\lambda_f^r := r \cdot \lfloor \lambda f \rfloor + \min(r_f, r)$ yields

$$\begin{aligned}
& \sum_{l \in \mathcal{L}(e)} \lambda_f^r \cdot x_{l,f} \geq \sum_{q \in \mathcal{Q}(e)} \lambda_f^r \cdot q_f \cdot y_q \qquad \forall f \in \mathcal{F} \\
\Rightarrow & \sum_{f \in \mathcal{F}} \sum_{l \in \mathcal{L}(e)} \lambda_f^r \cdot x_{l,f} \geq \sum_{f \in \mathcal{F}} \sum_{q \in \mathcal{Q}(e)} \lambda_f^r \cdot q_f \cdot y_q \\
\Leftrightarrow & \sum_{l \in \mathcal{L}(e)} \sum_{f \in \mathcal{F}} \lambda_f^r \cdot x_{l,f} \geq \sum_{q \in \mathcal{Q}(e)} \sum_{f \in \mathcal{F}} \lambda_f^r \cdot q_f \cdot y_q \\
& \qquad \overset{(*)}{\geq} \sum_{q \in \mathcal{Q}(e)} r \cdot \lceil \lambda \cdot F(e) \rceil \cdot y_q \\
& \qquad = r \cdot \lceil \lambda \cdot F(e) \rceil \cdot \sum_{q \in \mathcal{Q}(e)} y_q \\
& \qquad \overset{(6)}{=} r \cdot \lceil \lambda \cdot F(e) \rceil.
\end{aligned}$$

(∗) apply Lemma 14 here. ◀

Table 1 Statistics on the line planning instances. The columns list the instance name, the number of edges of the preprocessed transportation network, the number of lines, the number of variables for lines and frequencies, and the number of configuration variables in the configuration model and in the mixed model.

			(SLP)/(SLP$^+$)		(SLP^Q)		(QLP)	
name	$\|E\|$	$\|\mathcal{L}\|$	#vars	#cons	#vars	#cons	#vars	#cons
China1	27	474	2 793	499 / 620	3 732	654	41 196	661
China2	27	4 871	29 170	4 896 / 5 016	36 757	5 058	67 575	5 058
China3	27	19 355	116 074	19 380 /19 500	145 736	19 542	154 479	19 542
Dutch1	30	402	1 544	424 / 502	1 760	580	1 760	580
Dutch2	30	2 679	11 779	2 701 / 2 779	11 997	2 859	11 997	2 859
Dutch3	30	7 302	33 988	7 324 / 7 402	34 206	7 482	34 206	7 482
SiouxFalls1	37	866	5 188	902 / 1 113	6 680	1 117	753 840	1 124
SiouxFalls2	37	9 397	56 374	9 433 / 9 644	73 531	9 648	902 703	9 655
SiouxFalls3	37	15 365	92 182	15 401 /15 612	117 711	15 616	938 511	15 623
Potsdam1998b	351	1 907	10 765	1 998 / 2 679	13 795	3 969	38 637	4 114
Potsdam1998c	351	4342	25 306	4 431 / 5 112	32 037	6 484	53 184	6 549
Potsdam2010	517	3433	9 535	3 109 / 3 584	11 524	4 986	11 524	4 986
Chicago	1 028	23 109	131 915	24 066 /28 297	165 083	30 229	2 503 163	30 285

Again, we can give an example where a MIR inequality is not valid for the LP relaxation of the standard model.

▶ **Example 16** (Example 2 continued)**.** Let $\lambda = \frac{1}{8}$, then the MIR inequality for edge e_1

$$x_{l_1,2} + x_{l_1,8} + x_{l_2,2} + x_{l_2,8} + x_{l_3,2} + x_{l_3,8} \geq 2 \tag{14}$$

is violated by $\tilde{x} \in \mathrm{P_{LP}(SLP)}$, where $\tilde{x}_{l_2,8} = \frac{7}{8}$, $\tilde{x}_{l_3,2} = 1$, and $\tilde{x}_{l,f} = 0$ otherwise. It can be verified that (14) is even facet-defining for $\mathrm{P_{IP}(SLP)}$ in this example.

4 Computational Results

We have implemented the configuration approach to provide a computational evaluation of the strength of the extended formulation (QLP). We compare it with the standard model (SLP) and with two additional models (SLP$^+$) and (SLP^Q). Model (SLP$^+$) is obtained by adding the set cover, symmetric band, and MIR inequalities for all edges to the standard model (SLP). Model (SLP^Q) has been developed to cut down on the number of configuration variables, which can explode for large instances. This model is situated between (SLP$^+$) and (QLP) and constructed as follows. We order the edges with respect to an increasing number of minimal configurations and generate the configuration variables and the associated constraints iteratively as long as the number of generated configuration variables does not exceed 25% of the number of variables for lines and frequencies. For the remaining edges we use the set cover, symmetric band, and MIR inequalities.

Our test set consists of five transportation networks that we denote as China, Dutch, SiouxFalls, Chicago, and Potsdam. The instances SiouxFalls and Chicago use the graph and the demand of the street network with the same name from the Transportation Network Test Problems Library of Bar-Gera [20]. Instances China, Dutch, and Potsdam correspond to public transportation networks. The Dutch network was introduced by Bussieck in the context of line planning [6]. The China instance is artificial; we constructed it as a showcase example, connecting the twenty biggest cities in China by the 2009 high speed

Table 2 Statistics on the computations for the models (SLP), (SLP^+), (SLP^Q), and (QLP). The columns list the instance name, model, computation time, number of branching nodes, the integrality gap, the primal bound, the dual bound, and the dual bound after solving the root node.

name	model	time	nodes	gap	primal	dual	root dual
China1	(SLP)	1h	1524169	1.22 %	236631,2	233772.3	233566.3
	(SLP^+)	1h	808186	0.37 %	235873.4	235006.5	234772.3
	(SLP^Q)	1h	1147588	0.21 %	235531.2	235038.6	234828.6
	(QLP)	1h	31204	0.49 %	236149.0	235005.2	234878.3
China2	(SLP)	1h	154009	2.47 %	238187.4	232436.5	232294.8
	(SLP^+)	1h	24751	1.42 %	237333.4	234011.1	233860.8
	(SLP^Q)	1h	21388	0.50 %	235249.0	234076.8	233890.2
	(QLP)	1h	13872	0.63 %	235549.0	234071.3	233891.2
China3	(SLP)	1h	21078	3.78 %	241046.0	232271.9	232203.5
	(SLP^+)	1h	2214	0.99 %	236067.0	233760.1	233735.5
	(SLP^Q)	1h	3880	0.88 %	235925.8	233862.9	233778.1
	(QLP)	1h	3914	1.20 %	236639.6	233844.8	233778.1
Dutch1	(SLP)	1h	7427826	1.03 %	59000.0	58400.2	58227.4
	(SLP^+)	3.81s	1301	0.00 %	59000.0	59000.0	58841.7
	(SLP^Q)	0.98s	23	0.00 %	59000.0	59000.0	58868.6
	(QLP)	0.99s	23	0.00 %	59000.0	59000.0	58868.6
Dutch2	(SLP)	1h	609931	12.76 %	59300.0	52587.5	52492.3
	(SLP^+)	1934.67s	352128	0.00 %	58600.0	58600.0	58392.2
	(SLP^Q)	45.62s	6407	0.00 %	58600.0	58600.0	58435.7
	(QLP)	45.62s	6407	0.00 %	58600.0	58600.0	58435.7
Dutch3	(SLP)	1h	87746	14.64 %	59700.0	52075.0	52022.2
	(SLP^+)	1h	168915	0.38 %	58600.0	58376.6	58356.3
	(SLP^Q)	77.15s	1915	0.00 %	58500.0	58500.0	58372.9
	(QLP)	76.64s	1915	0.00 %	58500.0	58500.0	58372.9
SiouxFalls1	(SLP)	1029.24s	1115540	0.00 %	2409.8	2409.8	2352.6
	(SLP^+)	270.45s	125157	0.00 %	2409.8	2409.8	2365.0
	(SLP^Q)	177.8s	51099	0.00 %	2409.8	2409.8	2357.2
	(QLP)	1h	0	infinite	-	-	-
SiouxFalls2	(SLP)	1h	11664	26.07 %	1815.3	1439.9	1439.9
	(SLP^+)	1h	44565	3.48 %	1704.2	1647.0	1647.0
	(SLP^Q)	1h	19324	3.48 %	1704.2	1647.0	1647.0
	(QLP)	1h	0	infinite	-	-	-
SiouxFalls3	(SLP)	1h	27994	23.89 %	1527.8	1233.2	1233.2
	(SLP^+)	1h	6452	4.13 %	1420.7	1364.4	1363.9
	(SLP^Q)	1h	7569	3.83 %	1416.3	1364.1	1363.9
	(QLP)	1h	0	infinite	-	-	-
Potsdam1998b	(SLP)	1h	233518	3.74 %	36688.3	35365.0	35124.2
	(SLP^+)	1h	123701	0.77 %	36167.3	35891.0	35735.2
	(SLP^Q)	1h	237661	0.36 %	36067.0	35936.1	35770.6
	(QLP)	1h	124082	0.14 %	36067.0	36018.0	35850.4
Potsdam1998c	(SLP)	1h	105062	4.47 %	36617.1	35051.8	34896.9
	(SLP^+)	1h	38634	1.69 %	36243.5	35641.9	35510.1
	(SLP^Q)	1h	63336	0.56 %	35891.9	35690.7	35575.8
	(QLP)	1h	11681	7.49 %	38345.8	35675.3	35521.8
Potsdam2010	(SLP)	2.47s	1	0.00 %	11066.6	11066.6	11066.6
	(SLP^+)	4.93s	8	0.00 %	11066.6	11066.6	11011.8
	(SLP^Q)	6.31s	7	0.00 %	11066.6	11066.6	11046.9
	(QLP)	6.25s	7	0.00 %	11066.6	11066.6	11046.9
Chicago	(SLP)	1h	2002	5.88 %	22990.6	21713.3	21666.6
	(SLP^+)	1h	553	2.79 %	22327.2	21722.2	21685.3
	(SLP^Q)	1h	319	5.73 %	22948.1	21705.0	21689.4
	(QLP)	1h	0	infinite	-	-	-

train network. The Potsdam instances are real multi-modal public transportation networks for 1998 and 2009.

We constructed a line pool by generating for each pair of terminals all lines that satisfy a certain length restriction. To be more precise, the number of edges of a line between two terminals s and t must be less than or equal to k times the number of edges of the shortest path between s and t. For each network, we increased k in three steps to produce three instances with different line pool sizes. For Dutch and China instance number 3 contains all lines, i.e., all paths that are possible in the network. The Potsdam2010 instance arose within a project with the Verkehr in Potsdam GmbH (ViP) [18] to optimize the 2010 line plan [1]. The line pool contains all possible lines that fulfill the ViP requirements.

For all instances the lines can be operated at frequencies 3, 6, 9, 18, 36, and 72. This corresponds to a cycle time of 60, 30, 20, 10, 5, and 2.5 minutes in a time horizon of 3 hours. We set the line cost to be proportional to the line length and the frequency plus a fixed cost term that is used to reduce the number of lines. The costs and the capacities of the lines depend on the mode of transportation (e.g., bus, tram). In the instances each edge is associated with exactly one mode, i.e., all lines on an edge have the same capacity, see Karbstein [13] for more details. Hence, we can express capacities in terms of frequency demands. Table 1 lists some statistics about the test instances. The second and third columns give the number of edges and lines in the transportation network. The remaining columns list the number of variables and constraints for the four models after preprocessing. The preprocessing eliminates for instance dominated constraints and dominated and infeasible frequency assignments. For example a frequency f is dominated for line l if $f > \max_{e\in l}\{F(e)\}$.

The instances were solved using the constraint integer programming framework SCIP version 3.0.1 [17] with CPLEX 12.5 as LP-solver. We set a time limit of 1 hour for all instances and used the default settings of SCIP, apart from the primal heuristic "shiftandpropagate" which we turned off. All computations were done on an Intel(R) Xeon(R) CPU E3-1290, 3.7 GHz computer (in 64 bit mode) with 13 MB cache, running Linux and 16 GB of memory. The results are shown in Table 2.

The computations show that the set cover, symmetric band, and MIR cuts indeed improve the standard model. The superiority of model (QLP) does not always show up, because its root LP cannot be solved within one hour for those instances where the number of configuration variables is more than 10 times higher than the number of line and frequency variables. For all other instances, the dual bounds after solving the root node for ($\mathrm{SLP}^{\mathrm{Q}}$) and (QLP) are better than those for (SLP^{+}). Model ($\mathrm{SLP}^{\mathrm{Q}}$) is performing best on nearly all instances. Except for Chicago it has a better dual bound after terminating the computations than models (SLP) and (SLP^{+}). Hence, model ($\mathrm{SLP}^{\mathrm{Q}}$) is a good compromise between improving the formulation with configuration variables and keeping the size of the formulation small.

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