Existence results for Hughes' model for pedestrian flows

Debora Amadori, Paola Goatin, Massimiliano D. Rosini

Existence results for Hughes' model for pedestrian flows

Debora Amadori*

DISIM, Università degli Studi dell'Aquila, via Vetoio 1, 67010 L'Aquila, Italy

Paola Goatin

INRIA Sophia Antipolis - Méditerranée, 2004 route des Lucioles - BP 93, 06902 Sophia Antipolis Cedex, France

Massimiliano D. Rosini

ICM, University of Warsaw, ul. Prosta 69, Warsaw, Poland

Abstract

In this paper we prove two global existence results for Hughes' model for pedestrian flows under assumptions that ensure that the traces of the solutions along the turning curve are zero for all positive times.

Key words: crowd dynamics, conservation laws, eikonal equation, Hughes' model for pedestrian flows, wave-front tracking
2010 MSC: 35L65, 35F21, 90B20

1. Introduction

In this paper we study the one dimensional version of Hughes' model [16] for pedestrian flows

$$\partial_t \rho - \partial_x \left[\rho\, v(\rho)\, \frac{\partial_x \varphi}{|\partial_x \varphi|} \right] = 0 \ , \qquad\qquad |\partial_x \varphi| = c(\rho) \ , \tag{1}$$

in the spatial domain $\Omega =]-1,1[$, together with homogeneous Dirichlet boundary conditions

$$\rho(t,-1) = \rho(t,1) = 0 \ , \qquad \varphi(t,-1) = \varphi(t,1) = 0 \ , \qquad t > 0 \tag{2a}$$

*Corresponding author
Email addresses: amadori@univaq.it (Debora Amadori), paola.goatin@inria.fr (Paola Goatin), mrosini@icm.edu.pl (Massimiliano D. Rosini)

Preprint submitted to Elsevier *March 6, 2014*

and initial datum

$$\rho(0,x) = \bar{\rho}(x)\,, \qquad\qquad x \in \left]-1,1\right[\ . \tag{2b}$$

Here $x \in \Omega$ is the space variable, $t \geq 0$ is the time, $\rho = \rho(t,x) \in [0,1]$ is the (normalized) crowd density, while

$$v(\rho) = 1-\rho\,, \qquad c(\rho) = 1/v(\rho)$$

are respectively the mean (normalized) velocity and the running cost. We denote

$$f(\rho) = \rho\, v(\rho) = \rho\,(1-\rho)\,.$$

The initial datum $\bar{\rho}$ is assumed to be in $\mathbf{L}^\infty(\Omega;\mathbb{R})$ with $\|\bar{\rho}\|_\infty < 1$. This assumption, together with the maximum principle proved in [13], will ensure that the cost (which is singular for $\rho = 1$) computed along any solution of (1), (2) is well defined.

As already observed in [2, 13], the system (1) can be rewritten as

$$\partial_t \rho + \partial_x F(t,x,\rho) = 0\ , \tag{3a}$$

$$F(t,x,\rho) = \operatorname{sgn}\left(x-\xi(t)\right) f(\rho)\,,$$

$$\int_{-1}^{\xi(t)} c\left(\rho(t,x)\right) \mathrm{d}x = \int_{\xi(t)}^{1} c\left(\rho(t,x)\right) \mathrm{d}x\ . \tag{3b}$$

Indeed, in order to have a unique viscosity solution to the Dirichlet problem for φ, the derivative $\partial_x\varphi$ can change its sign just once from positive to negative. It is therefore defined the so called *turning curve* $x = \xi(t)$, where $\varphi(t,\cdot)$ reaches its maximum point. After integration of the second equation in (1), the relation (3b) states the continuity of $\varphi(t,\cdot)$ at $x = \xi(t)$, and defines it implicitly. Notice that the flux F is possibly discontinuous along $x = \xi(t)$.

Definition 1. *[13] A map $(t,x) \mapsto \rho(t,x)$ is an entropy weak solution of the initial-boundary value problem (2), (3) if is in $\mathbf{C^0}\left([0,+\infty[;\mathbf{L^1}(\Omega;[0,1])\right)$ and for any $\kappa \in [0,1]$ and any test function $\psi \in \mathbf{C^\infty_c}\left(\mathbb{R}^2;[0,+\infty[\right)$ it satisfies*

$$\int_0^{+\infty}\int_{-1}^{1} \left[|\rho-\kappa|\,\partial_t\psi + \mathcal{F}(t,x,\rho,\kappa)\,\partial_x\psi\right] \mathrm{d}x\,\mathrm{d}t\ + \int_{-1}^{1} |\bar{\rho}(x)-\kappa|\,\psi(0,x)\,\mathrm{d}x \tag{4a}$$

$$+\int_0^{+\infty} \left[f\left(\rho(t,-1+)\right) - f(\kappa)\right]\psi(t,-1)\,\mathrm{d}t\ + \int_0^{+\infty} \left[f\left(\rho(t,1-)\right) - f(\kappa)\right]\psi(t,1)\,\mathrm{d}t \tag{4b}$$

$$+\,2\int_0^{+\infty} f(\kappa)\,\psi\left(t,\xi(t)\right)\,\mathrm{d}t \geq 0 \tag{4c}$$

where

$$\mathcal{F}(t,x,\rho,\kappa) = \operatorname{sgn}(\rho-\kappa)\left[F(t,x,\rho) - F(t,x,\kappa)\right].$$

The first line (4a) originates from the Kružkov definition of entropy weak solution in the case of a Cauchy problem, [17]. Line (4b) comes from the boundary condition introduced by Bardos et al. in [6], see also [4, 8, 9, 18]. The latter line (4c) accounts for the discontinuity of the flux along the turning curve, see [1, 3, 5, 14, 20].

Observe that the strong traces of the solution at the boundary points exist due to the genuine non-linearity of the flux ([19, 21]) and must satisfy

$$\begin{aligned} &f\left(\rho(t,-1+)\right) \geq f(\kappa) \quad \text{for all } \kappa \in [0, \rho(t,-1+)], \\ &f\left(\rho(t,1-)\right) \geq f(\kappa) \quad \text{for all } \kappa \in [0, \rho(t,1-)]. \end{aligned}$$

This in particular implies that

$$\rho(t,-1+) \leq 1/2 \quad \text{and} \quad \rho(t,1-) \leq 1/2\,. \tag{5}$$

In the perspective of studying the well-posedness for problem (1), (2) a first result was proved in [12], where the eikonal equation in (1) is replaced by an elliptic approximation with small fixed parameter.

In the present article, motivated by the availability of a local Riemann solver for (1) provided in [2], we follow the wave-front tracking approach [10, 7]: we construct a sequence of piecewise constant approximate solutions to the Cauchy problem for (3) by solving locally the Riemann problems arising at each jump discontinuity, and prove their convergence by providing the uniform boundedness of their total variation.

Differently from the classical case of scalar conservation laws, we have to face two major problems when applying the wave-front tracking method to (1): either when two wave-fronts interact or a wave-front interacts with the turning curve, it may occur that several new fronts arise at the turning curve; also, the total variation of the solution may generically increase.

In this paper we give sufficient conditions on the initial datum that prevent these situations from occurring, therefore leading to the existence of a sequence of approximate solutions with uniformly bounded total variation. Even in this somewhat simplified situation, the wave-front tracking approach is very useful when studying the variation in time of the turning curve $\xi(t)$, implicitly defined by (3b). A convenient choice of the speed of approximate rarefactions is used, see (10c).

For numerical purposes, the wave-front tracking algorithm for (1) was analyzed in [15].

As a first existence result, we give the following theorem for the "symmetric" case. Let us denote by $\mathcal{S}$ the space of functions $\rho \in \mathbf{L}^\infty(\Omega;[0,1[)$ with $\|\rho\|_\infty < 1$ that are even, namely $\rho(x) = \rho(-x)$ for a.e. $x \in \Omega$.

Theorem 2. *For any initial datum $\bar\rho$ in $\mathcal{S}$ there exists a unique entropy weak solution ρ of* (2), (3) *such that $\rho(t,\cdot) \in \mathcal{S}$ for all $t > 0$.*

Proof. If ρ is an entropy weak solution of (2), (3) and $\rho(t,\cdot) \in \mathcal{S}$ for all $t > 0$, then necessarily $\xi \equiv 0$. Indeed, by (3b) we have that for any $t > 0$

$$\int_{\xi(t)}^{1} c\left(\rho(t,x)\right) \mathrm{d}x = \int_{-1}^{\xi(t)} c\left(\rho(t,x)\right) \mathrm{d}x = \int_{-\xi(t)}^{1} c\left(\rho(t,-x)\right) \mathrm{d}x = \int_{-\xi(t)}^{1} c\left(\rho(t,x)\right) \mathrm{d}x$$

and therefore $\xi(t) = 0$ because by definition $c(\rho) \geq 1$. As a consequence of the Rankine-Hugoniot condition along the turning curve, we have that $f(\rho(t,0+)) + f(\rho(t,0-)) = 0$, namely $\rho(t,0\pm) = 0$. Thus, for any fixed initial datum $\bar{\rho}$ in $\mathcal{S}$ the unique candidate in $\mathcal{S}$ to be the entropy weak solution of (2), (3) is the function $\rho: \ [0,+\infty[\times\Omega \to [0, \|\bar{\rho}\|_\infty]$ defined for $x \in [0,1]$ as the entropy weak solution to the initial-boundary value problem

$$\begin{aligned}
&\partial_t \rho + \partial_x f(\rho) = 0 && t > 0\,,\ x \in\,]0,1[\ ,\\
&\rho(t,0) = \rho(t,1) = 0 && t > 0\,,\\
&\rho(0,x) = \bar{\rho}(x) && x \in\,]0,1[\ ,
\end{aligned}$$

namely, for any $\kappa \in [0,1]$ and any test function $\psi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}\times[0,1]; [0,+\infty[)$

$$\begin{aligned}
&\int_0^{+\infty}\int_0^1 \left\{|\rho-\kappa|\,\partial_t\psi + \operatorname{sgn}(\rho-\kappa)\,[f(\rho)-f(\kappa)]\ \partial_x\psi\right\}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^1 |\bar{\rho}(x)-\kappa|\,\psi(0,x)\,\mathrm{d}x + \int_0^{+\infty}\left[f(\rho(t,1-)) - f(\kappa)\right]\ \psi(t,1)\,\mathrm{d}t\\
&-\int_0^{+\infty}\left[f(\rho(t,0+)) - f(\kappa)\right]\ \psi(t,0)\,\mathrm{d}t \geq 0\,.
\end{aligned}$$

From the maximum principle one has that $\|\rho(t,\cdot)\|_\infty \leq \|\bar{\rho}\|_\infty < 1$ for any $t > 0$. By standard generalized characteristic analysis, we have that $\rho \equiv 0$ in a region that contains $\{(t,x):\ t > 0\,,\ x \in \Omega\,,\ |x| < v(\|\bar{\rho}\|_\infty)\,t\}$. It is therefore immediate to prove that ρ is an entropy weak solution of (2), (3) in the sense of Definition 1. $\square$

In the next theorem we treat a different case. Let $[x]_+ = \max\{x,0\}$, $x \in \mathbb{R}$.

Theorem 3. *If the initial datum $\bar{\rho} \in \mathbf{BV}(\Omega;[0,1[)$ satisfies*

$$3\|\bar{\rho}\|_\infty + \mathrm{TV}\left(c\left(\bar{\rho}\right)\right) + \left[c\left(\bar{\rho}(-1+)\right) - c(1/2)\right]_+ + \left[c\left(\bar{\rho}(1-)\right) - c(1/2)\right]_+ < 2\,, \tag{6}$$

then there exists an entropy weak solution of (2), (3) *defined globally in time.*

The proof is based on the wave-front tracking algorithm. We use the results achieved in [2, 13], where the solutions of the Riemann–type problems associated to (2), (3) are constructed and studied. We underline that (6) implies that $\bar{\rho}(\pm1\mp) < (7-\sqrt{13})/6 \sim 0.565741$.

The paper is organized as follows. In Section 2 we describe the wave-front tracking algorithm used to construct the approximate solutions to (2), (3). In Section 3, provided that (6) holds on the initial data, we prove that the approximate solutions exist globally and finally prove Theorem 3. Some technical lemmas are deferred to the last section.

2. The approximate solution

For any fixed integer $n \geq 1$, consider the approximation parameter $\varepsilon = 2^{-n} > 0$ and the ε-grid $\mathcal{G}^\varepsilon = \{i\,\varepsilon : i = 0, \ldots, \varepsilon^{-1}\}$. Consider the piecewise linear function f^ε that interpolates linearly the points $(\rho_i, f(\rho_i))$, $\rho_i \in \mathcal{G}^\varepsilon$. Therefore f^ε coincides with f on $\mathcal{G}^\varepsilon$, is increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$.

Let $\bar\rho^\varepsilon \in \mathbf{BV}(\Omega; \mathcal{G}^\varepsilon)$ be piecewise constant and define $\bar\xi^\varepsilon$ as the unique solution of the equation

$$\int_{-1}^{\bar\xi^\varepsilon} c\left(\bar\rho^\varepsilon(x)\right) \mathrm{d}x = \int_{\bar\xi^\varepsilon}^{1} c\left(\bar\rho^\varepsilon(x)\right) \mathrm{d}x\ . \tag{7}$$

We call shock waves (respectively, rarefaction waves) the decreasing discontinuities on the left of the turning curve and the increasing discontinuities on the right of the turning curve (respectively, the increasing discontinuities on the left of the turning curve and the decreasing discontinuities on the right of the turning curve) of the solution to (2), (3) with $\bar\rho^\varepsilon$ instead of $\bar\rho$, f^ε instead of f and constructed with the classical Riemann solver $\mathcal{R}_c$. Introduce the simplified Riemann solver $\mathcal{R}_s$, that replaces any rarefaction wave given by $\mathcal{R}_c$ with a rarefaction front (we shall state more precise assumptions below, see (10c)). Apply then $\mathcal{R}_s$ to solve each Riemann problem associated to the boundary $\{-1, 1\}$ and to the jumps of discontinuity of $\bar\rho^\varepsilon$ away from $x = \bar\xi^\varepsilon$. Denote by ρ^ε_L, respectively ρ^ε_R, the juxtaposition of the piecewise constant functions obtained by solving with $\mathcal{R}_s$ the Riemann problems on the left of $x = \bar\xi^\varepsilon$, respectively on the right of $x = \bar\xi^\varepsilon$.

Observe that ρ^ε_L and ρ^ε_R are well defined for sufficiently small times. By applying Theorem 6 given below, we can construct a piecewise constant function ρ^ε_ξ such that if ρ^ε is the juxtaposition of ρ^ε_L, ρ^ε_ξ and ρ^ε_R, then the corresponding turning curve $x^\varepsilon_0 \equiv \xi^\varepsilon$ defined by

$$\int_{-1}^{\xi^\varepsilon(t)} c\left(\rho^\varepsilon(t,x)\right)\, \mathrm{d}x = \int_{\xi^\varepsilon(t)}^{1} c\left(\rho^\varepsilon(t,x)\right)\, \mathrm{d}x \tag{8}$$

satisfies the Rankine-Hugoniot condition (10a) below.

In Theorem 6 we will describe how to construct the local solution ρ^ε_ξ both for $t = 0$ and for any positive interaction time. Before doing this, we need some notation.

Let $t > 0$ be a time at which no interactions occur. Denote by $x^\varepsilon_i(t)$, $i = -h-1, \ldots, -1, 1, \ldots, k+1$, with $-1 = x^\varepsilon_{-h-1} < x^\varepsilon_i(t) < x^\varepsilon_{i+1}(t) < x^\varepsilon_{k+1} = 1$, the discontinuity lines of ρ^ε away from $x^\varepsilon_0 \equiv \xi^\varepsilon$, which we call *fronts*. Then the approximate solution reads as

$$\rho^\varepsilon(t,x) = \sum_{i=-h-1}^{k} \rho^\varepsilon_{i+1/2}\, \chi_{[x^\varepsilon_i(t),\, x^\varepsilon_{i+1}(t)[}(x)$$

where $x^\varepsilon_0(t) = \xi^\varepsilon(t)$ is the turning curve and

$$\rho^\varepsilon_{i-1/2} \neq \rho^\varepsilon_{i+1/2} \text{ if } i \neq 0\,, \qquad \rho^\varepsilon_{-1/2} = \rho^\varepsilon_{1/2} \text{ if and only if } \rho^\varepsilon_{\pm 1/2} = 0\,, \tag{9a}$$

$$\rho^\varepsilon_{-h-1/2} \leq 1/2\,, \qquad \rho^\varepsilon_{k+1/2} \leq 1/2\,. \tag{9b}$$

The ε-approximate solution ρ^ε is prolonged beyond any interaction time (namely, when one front reaches the boundary, or two (or more) fronts approach, or one front reaches the turning curve) by applying $\mathcal{R}_s$ where the interaction takes place away from $x = \xi^\varepsilon$ and by applying then Theorem 6. Observe that, as a result of any interaction, new fronts may originate from the turning curve. However, the resulting ε-approximate solution ρ^ε keeps the structure described above. Therefore, after each interaction time, we can use the same notation introduced before by rearranging the indices and by considering h and k as piecewise constant functions of time. Finally, the turning curve is prolonged by applying (8) as long as ρ^ε is well defined.

In the sequel we refer to upward jumps on the left of $x = \xi^\varepsilon$ and to downward jumps on the right of $x = \xi^\varepsilon$ as *rarefaction fronts*, while the remaining jumps away from $x = \xi^\varepsilon$ are called *shock fronts*. The size of the jumps is defined by

$$\sigma_i(t) = \operatorname{sgn}(i) \left[\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}\right] .$$

By construction ρ^ε satisfies the Rankine-Hugoniot jump conditions along the turning curve and the shock fronts, respectively

$$\left[\rho^\varepsilon_{1/2} - \rho^\varepsilon_{-1/2}\right] \dot{\xi}^\varepsilon = f\left(\rho^\varepsilon_{1/2}\right) + f\left(\rho^\varepsilon_{-1/2}\right) , \tag{10a}$$

$$\dot{x}^\varepsilon_i = \operatorname{sgn}(i) \frac{f\left(\rho^\varepsilon_{i+1/2}\right) - f\left(\rho^\varepsilon_{i-1/2}\right)}{\rho^\varepsilon_{i+1/2} - \rho^\varepsilon_{i-1/2}} \qquad \text{if } i \neq 0 \text{ and } \sigma_i < 0 . \tag{10b}$$

On the other hand, we impose that any rarefaction front x^ε_i travels with speed

$$\dot{x}^\varepsilon_i = \operatorname{sgn}(i) \frac{q\left(\rho^\varepsilon_{i+1/2}\right) - q\left(\rho^\varepsilon_{i-1/2}\right)}{c\left(\rho^\varepsilon_{i+1/2}\right) - c\left(\rho^\varepsilon_{i-1/2}\right)} \qquad \text{if } i \neq 0 \text{ and } \sigma_i > 0 , \tag{10c}$$

where q is the entropy flux associated to c. Indeed, since the cost function c is convex in $[0, 1[$, we can consider it as an entropy for (3) with entropy flux q on $x > \xi^\varepsilon(t)$ (respectively, $-q$ on $x < \xi^\varepsilon(t)$) defined by

$$q(\rho) = -c(\rho) + 2 \log c(\rho) . \tag{11}$$

The function q satisfies $q' = c' f'$, $q(0) = -1$ and $(1 - 2\rho)\, q'(\rho) > 0$ for $\rho \neq 1/2$.

The above choice for the speed of propagation of the rarefaction fronts allows us to simplify the terms appearing in the representation for $\dot{\xi}^\varepsilon$, see next formula (12). Indeed, since entropy conditions hold with an equality along any classical rarefaction, the corresponding terms on the right hand side of (12) reduce to the difference of the entropy fluxes.

In the next proposition we obtain a necessary condition on $\dot{\xi}^\varepsilon$ which is obtained by the derivation of equation (8).

Proposition 4 (*A condition on* $\dot{\xi}^\varepsilon$). *If* ρ^ε *satisfies* (9), *then*

$$\left[c\left(\rho^\varepsilon_{-1/2}\right)+c\left(\rho^\varepsilon_{1/2}\right)\right]\dot{\xi}^\varepsilon \;=\; \sum_{i\neq 0}\operatorname{sgn}(i)\left[c\left(\rho^\varepsilon_{i-1/2}\right)-c\left(\rho^\varepsilon_{i+1/2}\right)\right]\dot{x}^\varepsilon_i\,. \tag{12}$$

Proof. To simplify the notation, we omit the dependence on ε and write $c\left(\rho^\varepsilon_i\right)=c_i$. By (9) we have

$$\int_{-1}^{\xi(t)} c\left(\rho(t,x)\right)\,\mathrm{d}x=\sum_{i=1}^{h+1}\left[x_{1-i}(t)-x_{-i}(t)\right]c_{1/2-i}\,,$$
$$\int_{\xi(t)}^{1} c\left(\rho(t,x)\right)\,\mathrm{d}x=\sum_{i=1}^{k+1}\left[x_{i}(t)-x_{i-1}(t)\right]c_{i-1/2}\,,$$

and by rearranging the indexes we have

$$\int_{-1}^{\xi(t)} c\left(\rho(t,x)\right)\,\mathrm{d}x=c_{-h-1/2}+c_{-1/2}\,\xi(t)+\sum_{i=1}^{h}\left[c_{-i-1/2}-c_{1/2-i}\right]x_{-i}(t)\,,$$
$$\int_{\xi(t)}^{1} c\left(\rho(t,x)\right)\,\mathrm{d}x=c_{k+1/2}-c_{1/2}\,\xi(t)+\sum_{i=1}^{k}\left[c_{i-1/2}-c_{i+1/2}\right]x_{i}(t)\,.$$

By taking the derivative with respect to t we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-1}^{\xi(t)} c\left(\rho(t,x)\right)\,\mathrm{d}x\right]=c_{-1/2}\,\dot{\xi}(t)+\sum_{i=1}^{h}\left[c_{-i-1/2}-c_{1/2-i}\right]\dot{x}_{-i}(t)\,, \tag{13a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\xi(t)}^{1} c\left(\rho(t,x)\right)\,\mathrm{d}x\right]=-c_{1/2}\,\dot{\xi}(t)+\sum_{i=1}^{k}\left[c_{i-1/2}-c_{i+1/2}\right]\dot{x}_{i}(t)\,. \tag{13b}$$

By (8) the above quantities are equal and therefore we deduce (12). □

In the following definition we specify the properties of the ε-approximate solution.

Definition 5. *The piecewise constant function* $(t,x)\mapsto\rho^\varepsilon(t,x)$ *as in* (9) *is an* ε*–admissible approximate solution of* (3) *if it consists of a finite number of fronts traveling according to* (10), *satisfies* (12) *and* $\sup_i\sigma_i\leq\varepsilon$.

Let us underline that the above definition does not require that ρ^ε takes values in $\mathcal{G}^\varepsilon$. This is motivated by the possible appearance of states along the turning curve that do not belong to $\mathcal{G}^\varepsilon$. We defer to the end of this Section for further comments.

For convenience we introduce the following notation,

$$\Psi\left[\rho^{\varepsilon}\right]=\sum_{i\neq 0}\Phi_i\left[\rho^{\varepsilon}\right], \qquad \Phi_i\left[\rho^{\varepsilon}\right]=\begin{cases}\left[c\left(\rho^{\varepsilon}_{-1/2}\right)+c\left(\rho^{\varepsilon}_{1/2}\right)\right]\dot{\xi}^{\varepsilon} & \text{if } i=0\,,\\ \operatorname{sgn}(i)\left[c\left(\rho^{\varepsilon}_{i-1/2}\right)-c\left(\rho^{\varepsilon}_{i+1/2}\right)\right]\dot{x}^{\varepsilon}_i & \text{if } i\neq 0\,,\end{cases} \tag{14}$$

so that (12) rewrites as

$$\Phi_0\left[\rho^{\varepsilon}\right] \;=\; \Psi\left[\rho^{\varepsilon}\right].$$

For later use, for any $\alpha,\beta\in[0,1[$ with $\alpha\neq\beta$, introduce the quantities

$$\lambda_\xi\left(\alpha,\beta\right)=\frac{f\left(\beta\right)+f\left(\alpha\right)}{\beta-\alpha}\,, \qquad \Phi_\xi\left(\alpha,\beta\right)=\left[c\left(\alpha\right)+c\left(\beta\right)\right]\lambda_\xi\left(\alpha,\beta\right), \tag{15a}$$

$$\lambda_s\left(\alpha,\beta\right)=\frac{f\left(\beta\right)-f\left(\alpha\right)}{\beta-\alpha}\,, \qquad \Phi_s\left(\alpha,\beta\right)=\left[c\left(\alpha\right)-c\left(\beta\right)\right]\lambda_s\left(\alpha,\beta\right), \tag{15b}$$

$$\lambda_r\left(\alpha,\beta\right)=\frac{q\left(\beta\right)-q\left(\alpha\right)}{c\left(\beta\right)-c\left(\alpha\right)}\,, \qquad \Phi_r\left(\alpha,\beta\right)=q\left(\alpha\right)-q\left(\beta\right). \tag{15c}$$

Clearly, if x^{ε}_i is a shock front, respectively a rarefaction front, then $\Phi_i\left[\rho^{\varepsilon}\right]=\Phi_s\left(\rho^{\varepsilon}_{i-1/2},\rho^{\varepsilon}_{i+1/2}\right)$, respectively $\Phi_i\left[\rho^{\varepsilon}\right]=\Phi_r\left(\rho^{\varepsilon}_{i-1/2},\rho^{\varepsilon}_{i+1/2}\right)$. Moreover, if $\rho^{\varepsilon}_{-1/2}\neq\rho^{\varepsilon}_{1/2}$, then $\Phi_0\left[\rho^{\varepsilon}\right]=\Phi_\xi\left(\rho^{\varepsilon}_{-1/2},\rho^{\varepsilon}_{1/2}\right)$, otherwise $\rho^{\varepsilon}_{\pm 1/2}=0$, $\Phi_\xi\left(\rho^{\varepsilon}_{-1/2},\rho^{\varepsilon}_{1/2}\right)$ is not well defined and $\Phi_0\left[\rho^{\varepsilon}\right]=2\dot{\xi}^{\varepsilon}=\Psi\left[\rho^{\varepsilon}\right]$.

Some of the introduced quantities have a clear geometric interpretation in the (ρ,f)-plane. Indeed $\lambda_\xi(\alpha,\beta)$ represents the slope of the segment between $(\alpha,-f(\alpha))$ and $(\beta,f(\beta))$, $\lambda_s(\alpha,\beta)$ is the slope of the segment between $(\alpha,f(\alpha))$ and $(\beta,f(\beta))$, $v(\alpha)$ is the slope of the segment between $(0,0)$ and $(\alpha,f(\alpha))$, see Figure 1, left.

Also, $\lambda_r(\alpha,\beta)$ coincides with the slope of $\rho\mapsto f(\rho)$ at $(\eta,f(\eta))$ for some η between α and β. This fact can be easily proved by the mean value theorem applied to the function $\tilde{q}(c)=-c+2\log c$ (see the definition of q, (11)), and using the identity $-1+2/c(\rho)=f'(\rho)$.

As a consequence, see Figure 1, center and right,

$$\lambda_s(\alpha,\gamma)<\lambda_s(0,\gamma)=v(\gamma)=\lambda_\xi(0,\gamma)<\lambda_\xi(\beta,\gamma) \qquad \text{for all } \alpha,\beta<\gamma, \tag{16}$$

$$\lambda_\xi(\alpha,\beta)>\lambda_s(\beta,\gamma) \qquad \text{for all } \alpha<\beta<\gamma. \tag{17}$$

By using the definition of Φ_ξ and with simple calculations, one can show that $\Phi_\xi(\alpha,\beta)\geq v(\alpha)+v(\beta)$ for all $\alpha<\beta$, with the equality that holds if and only if $\alpha=0$. Further properties of the just introduced functions are collected in Section 4.

For any interaction time $t=t_I$ or for $t=0$, introduce also the quantity

$$\Psi_*\left[\rho^{\varepsilon}\right]\left(t_I+\right)=\sum_{\substack{i\neq 0\\ x^{\varepsilon}_i(t_I)\neq x^{\varepsilon}_0(t_I)}}\Phi_i\left[\rho^{\varepsilon}\right]\left(t_I+\right), \tag{18}$$

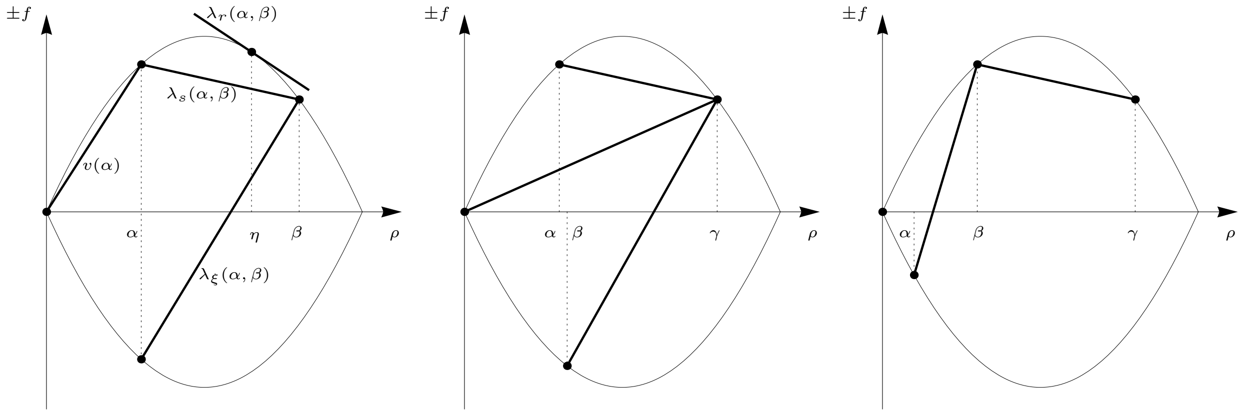

Figure 1: Left: Geometrical interpretation of the quantities introduced in (15). Center and right: Geometrical interpretation of the estimates, respectively, (16) and (17).

where the sum counts the fronts of $\rho^\varepsilon(t_I+)$ that do not start from the turning curve at time $t = t_I$, namely, the fronts of ρ_L^ε and ρ_R^ε.

The next theorem, which upgrades [2, Theorem 1], shows how to construct ρ_ξ^ε once we have ρ_L^ε and ρ_R^ε. In particular the structure of ρ_ξ^ε depends on the three parameters $\rho_{-1/2}(t_I) = \rho_L^\varepsilon(t_I, \xi^\varepsilon(t_I))$, $\rho_{1/2}(t_I) = \rho_R^\varepsilon(t_I, \xi^\varepsilon(t_I))$ and $\Psi_*[\rho^\varepsilon](t_I+)$. For simplicity we omit the dependence on ε.

Theorem 6. *There exists ρ_ξ such that ρ, the juxtaposition of ρ_L, ρ_ξ and ρ_R, is an ε–admissible approximate solution of* (3) *in the sense of Definition 5. In particular we can distinguish the following cases.*

1. *ρ_ξ consists of rarefaction fronts on the right of the turning curve if and only if*

$$0 \le \rho_{1/2}(t_I) < \rho_{-1/2}(t_I) \qquad \textit{and} \qquad \Psi_*[\rho](t_I+) < \Phi_\xi\left(\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right). \tag{19}$$

In this case the rarefaction fronts of ρ_ξ are

$$\begin{aligned} x_i^\xi(t) &= \xi(t_I) + \lambda_r\left(\rho_{i-1/2}^\xi, \rho_{i+1/2}^\xi\right)[t - t_I], & i = 1, \ldots, m-1, \\ x_m^\xi(t) &= \xi(t_I) + \lambda_r\left(\rho_{m-1/2}^\xi, \rho_{1/2}(t_I)\right)[t - t_I], \end{aligned}$$

where $\rho_{1/2}^\xi \in \left]\rho_{1/2}(t_I), \rho_{-1/2}(t_I)\right[$ is the unique ρ–solution of

$$\Phi_\xi\left(\rho_{-1/2}(t_I), \rho\right) - q(\rho) = \Psi_*[\rho](t_I+) - q\left(\rho_{1/2}(t_I)\right),$$

and $\left]\rho_{1/2}(t_I), \rho_{1/2}^\xi\right[\cap \mathcal{G} = \left\{\rho_{3/2}^\xi, \ldots, \rho_{m-1/2}^\xi\right\}$ with $\rho_{i-1/2}^\xi > \rho_{i+1/2}^\xi$. Observe that $\rho_{m+1/2}(t_I+) = \rho_{1/2}(t_I) < \rho_{m-1/2}(t_I+) = \rho_{m-1/2}^\xi < \ldots < \rho_{1/2}(t_I+) = \rho_{1/2}^\xi < \rho_{-1/2}(t_I+) = \rho_{-1/2}(t_I)$.

2. ρ_ξ *consists of a shock front on the right of the turning curve if and only if either*

$$0 = \rho_{-1/2}(t_I) < \rho_{1/2}(t_I) \qquad \text{and} \qquad \Psi_*[\rho](t_I+) < v\left(\rho_{-1/2}(t_I)\right) + v\left(\rho_{1/2}(t_I)\right), \tag{20}$$

or

$$0 < \rho_{-1/2}(t_I) < \rho_{1/2}(t_I) \qquad \text{and} \qquad \Psi_*[\rho](t_I+) \leq -v\left(\rho_{-1/2}(t_I)\right) - v\left(\rho_{1/2}(t_I)\right), \tag{21}$$

or

$$\begin{aligned}&0 < \rho_{1/2}(t_I) < \rho_{-1/2}(t_I) \text{ and}\\ &\Phi_\xi\left(\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right) < \Psi_*[\rho](t_I+) \leq -v\left(\rho_{-1/2}(t_I)\right) - v\left(\rho_{1/2}(t_I)\right).\end{aligned} \tag{22}$$

In the case (20) *the shock front of* ρ_ξ *is*

$$x_1^\xi(t) = \xi(t_I) + v\left(\rho_{1/2}(t_I)\right)[t - t_I],$$

and $\rho_{\pm 1/2}(t_I+) = 0 < \rho_{3/2}(t_I+) = \rho_{1/2}(t_I)$. *In the cases* (21), (22) *the shock front of* ρ_ξ *is*

$$x_1^\xi(t) = \xi(t_I) + \lambda_s\left(\rho_{1/2}^\xi, \rho_{1/2}(t_I)\right)[t - t_I],$$

where $\rho_{1/2}^\xi \in \left[0, \min\left\{\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right\}\right[$ *is the unique* ρ*–solution of*

$$\Psi_*[\rho](t_I+) = \Phi_\xi\left(\rho_{-1/2}(t_I), \rho\right) - \Phi_s\left(\rho, \rho_{1/2}(t_I)\right),$$

and $\rho_{1/2}(t_I+) = \rho_{1/2}^\xi < \min\left\{\rho_{-1/2}(t_I+) = \rho_{-1/2}(t_I),\ \rho_{3/2}(t_I+) = \rho_{1/2}(t_I)\right\}$ *with* $\rho_{1/2}(t_I+) = 0$ *if and only if* $\Psi_*[\rho](t_I+) = -v\left(\rho_{-1/2}(t_I)\right) - v\left(\rho_{1/2}(t_I)\right)$.

3. ρ_ξ *consists of two shock fronts, one on each side of the turning curve, if and only if*

$$\rho_{\pm 1/2}(t_I) \neq 0 \qquad \text{and} \qquad \left|\Psi_*[\rho](t_I+)\right| < v\left(\rho_{-1/2}(t_I)\right) + v\left(\rho_{1/2}(t_I)\right). \tag{23}$$

In this case the shock fronts of ρ_ξ *are*

$$\begin{aligned}x_{-1}^\xi(t) &= \xi(t_I) - v\left(\rho_{-1/2}(t_I)\right)[t - t_I],\\ x_1^\xi(t) &= \xi(t_I) + v\left(\rho_{1/2}(t_I)\right)[t - t_I],\end{aligned}$$

$\rho_{\pm 1/2}(t_I+) = 0$ *and* $\rho_{\pm 3/2}(t_I+) = \rho_{\pm 1/2}(t_I)$.

4. ρ_ξ *consists of a shock front on the left of the turning curve if and only if either*

$$0 = \rho_{1/2}(t_I) < \rho_{-1/2}(t_I) \qquad \text{and} \qquad \Psi_*\left[\rho\right](t_I+) > -v\left(\rho_{-1/2}(t_I)\right) - v\left(\rho_{1/2}(t_I)\right), \tag{24}$$

or

$$0 < \rho_{1/2}(t_I) < \rho_{-1/2}(t_I) \qquad \text{and} \qquad \Psi_*\left[\rho\right](t_I+) \geq v\left(\rho_{-1/2}(t_I)\right) + v\left(\rho_{1/2}(t_I)\right), \tag{25}$$

or

$$\begin{aligned}&0 < \rho_{-1/2}(t_I) < \rho_{1/2}(t_I) \text{ and}\\ &v\left(\rho_{-1/2}(t_I)\right) + v\left(\rho_{1/2}(t_I)\right) \leq \Psi_*\left[\rho\right](t_I+) < \Phi_\xi\left(\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right).\end{aligned} \tag{26}$$

In the case (24) *the shock front of* ρ_ξ *is*

$$x^\xi_{-1}(t) = \xi(t_I) - v\left(\rho_{-1/2}(t_I)\right)[t - t_I],$$

and $\rho_{\pm 1/2}(t_I+) = 0 < \rho_{-3/2}(t_I+) = \rho_{-1/2}(t_I)$. *In the cases* (25), (26) *the shock front of* ρ_ξ *is*

$$x^\xi_{-1}(t) = \xi(t_I) - \lambda_s\left(\rho_{-1/2}(t_I), \rho^\xi_{1/2}\right)[t - t_I],$$

where $\rho^\xi_{1/2} \in \left[0, \min\left\{\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right\}\right[$ *is the unique* ρ*–solution of*

$$\Psi_*\left[\rho\right](t_I+) = \Phi_\xi\left(\rho, \rho_{1/2}(t_I)\right) - \Phi_s\left(\rho_{-1/2}(t_I), \rho\right),$$

and $\rho_{-1/2}(t_I+) = \rho^\xi_{1/2} < \min\left\{\rho_{-3/2}(t_I+) = \rho_{-1/2}(t_I), \rho_{1/2}(t_I+) = \rho_{1/2}(t_I)\right\}$ *with* $\rho_{-1/2}(t_I+) = 0$ *if and only if* $\Psi_*\left[\rho\right](t_I+) = v\left(\rho_{-1/2}(t_I)\right) + v\left(\rho_{1/2}(t_I)\right)$.

5. ρ_ξ *consists of rarefaction fronts on the left of the turning curve if and only if*

$$0 \leq \rho_{-1/2}(t_I) < \rho_{1/2}(t_I) \qquad \text{and} \qquad \Psi_*\left[\rho\right](t_I+) > \Phi_\xi\left(\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right). \tag{27}$$

In this case the rarefaction fronts of ρ_ξ *are*

$$\begin{aligned}x^\xi_{-i}(t) &= \xi(t_I) - \lambda_r\left(\rho^\xi_{-i-1/2}, \rho^\xi_{-i+1/2}\right)[t - t_I], && i = 1, \ldots, m-1,\\ x^\xi_{-m}(t) &= \xi(t_I) - \lambda_r\left(\rho_{-1/2}(t_I), \rho^\xi_{-m+1/2}\right)[t - t_I],\end{aligned}$$

where $\rho^{\xi}_{-1/2} \in \left]\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right[$ *is the unique* ρ*–solution of*

$$\Phi_{\xi}\left(\rho, \rho_{1/2}(t_I)\right) + q\left(\rho\right) = \Psi_*\left[\rho\right](t_I+) + q\left(\rho_{-1/2}(t_I)\right),$$

and $\left]\rho_{-1/2}(t_I), \rho^{\xi}_{-1/2}\right[\cap \mathcal{G} = \left\{\rho^{\xi}_{-3/2}, \ldots, \rho^{\xi}_{-m+1/2}\right\}$ *with* $\rho^{\xi}_{i-1/2} > \rho^{\xi}_{i+1/2}$.
Observe that $\rho_{-m-1/2}(t_I+) = \rho_{-1/2}(t_I) < \rho_{-m+1/2}(t_I+) = \rho^{\xi}_{-m+1/2} < \ldots < \rho_{-1/2}(t_I+) = \rho^{\xi}_{-1/2} < \rho_{1/2}(t_I+) = \rho_{1/2}(t_I)$.

6. ρ_{ξ} *consists of the turning curve alone if and only if*

$$\rho_{\pm 1/2}(t_I) = 0\,, \tag{28}$$

or

$$\rho_{-1/2}(t_I) \neq \rho_{1/2}(t_I) \qquad \text{and} \qquad \Psi_*\left[\rho\right](t_I+) = \Phi_{\xi}\left(\rho_{-1/2}(t_I), \rho_{1/2}(t_I)\right). \tag{29}$$

The 6 cases above cover all possible values of $\rho_{\pm 1/2}(t_I)$, $\Psi_*\left[\rho\right](t_I+)$ *and are mutually exclusive.*

Proof. For notational convenience, we write $\rho_{\pm 1/2} = \rho_{\pm 1/2}(t_I)$ and denote $g\left(\rho_i\right) = g_i$, $g(\rho^{\xi}_i) = g^{\xi}_i$ for any function $g\colon [0,1] \to \mathbb{R}$.
(1,"$\Rightarrow$") If ρ_{ξ} consists of rarefaction fronts x^{ξ}_i, $i = 1, \ldots, m$, between the states $\rho^{\xi}_{1/2}$, …, $\rho^{\xi}_{m-1/2}$, $\rho_{1/2}$, on the right of the turning curve, then

$$\rho_{1/2} < \rho^{\xi}_{m-1/2} < \ldots < \rho^{\xi}_{1/2} \qquad \text{and} \qquad \dot{\xi}(t_I+) < \dot{x}^{\xi}_1(t_I+) < \ldots < \dot{x}^{\xi}_m(t_I+)\,,$$

where $\dot{x}^{\xi}_i(t_I+) = \lambda_r\left(\rho^{\xi}_{i-1/2}, \rho^{\xi}_{i+1/2}\right)$, $i = 1, \ldots, m-1$, $\dot{x}^{\xi}_m(t_I+) = \lambda_r\left(\rho^{\xi}_{m-1/2}, \rho_{1/2}\right)$. In particular $\rho^{\xi}_{1/2} \neq 0$ and therefore, by (10a), we have $\rho^{\xi}_{1/2} \neq \rho_{-1/2}$ and $\dot{\xi}(t_I+) = \lambda_{\xi}\left(\rho_{-1/2}, \rho^{\xi}_{1/2}\right)$. Moreover, we have $\rho^{\xi}_{1/2}(t_I) < \rho_{-1/2}$ because, by Lemma 8 and (16), for all $\rho < \rho^{\xi}_{1/2}$

$$\dot{x}^{\xi}_1(t_I+) = \lambda_r\left(\rho^{\xi}_{1/2}, \rho^{\xi}_{3/2}\right) < v^{\xi}_{1/2} < \lambda_{\xi}\left(\rho, \rho^{\xi}_{1/2}\right).$$

As a consequence, the first condition in (19) holds. We then observe that $\Phi_0\left[\rho\right](t_I+) = \Phi_{\xi}\left(\rho_{-1/2}, \rho^{\xi}_{1/2}\right)$ and

$$\begin{aligned}\Psi\left[\rho\right](t_I+) &= \Psi_*\left[\rho\right](t_I+) + \sum_{i=1}^{m-1} \Phi_r\left(\rho^{\xi}_{i-1/2}, \rho^{\xi}_{i+1/2}\right) + \Phi_r\left(\rho^{\xi}_{m-1/2}, \rho_{1/2}\right) \\ &= \Psi_*\left[\rho\right](t_I+) + q^{\xi}_{1/2} - q_{1/2}\,.\end{aligned}$$

By (12) we have that

$$\Phi_\xi\left(\rho_{-1/2},\rho^\xi_{1/2}\right)=\Psi_*\left[\rho\right](t_I+)+q^\xi_{1/2}-q_{1/2}$$

and therefore by Lemma 10

$$\Psi_*\left[\rho\right](t_I+)=\Phi_\xi\left(\rho_{-1/2},\rho^\xi_{1/2}\right)-q^\xi_{1/2}+q_{1/2}<\Phi_\xi\left(\rho_{-1/2},\rho_{1/2}\right).$$

Thus also the second condition in (19) is satisfied.
(2,"$\Rightarrow$") If ρ_ξ consists of a shock front x^ξ_1 on the right of the turning curve between the states $\rho^\xi_{1/2}$ and $\rho_{1/2}$, then

$$\rho_{1/2}>\rho^\xi_{1/2}\,, \qquad\qquad \dot\xi(t_I+)<\dot x^\xi_1(t_I+)=\lambda_s\left(\rho^\xi_{1/2},\rho_{1/2}\right)$$

and

$$\Phi_0\left[\rho\right](t_I+)=\left[c_{-1/2}+c^\xi_{1/2}\right]\dot\xi(t_I+)\,, \qquad\qquad \Psi\left[\rho\right](t_I+)=\Psi_*\left[\rho\right](t_I+)+\Phi_s\left(\rho^\xi_{1/2},\rho_{1/2}\right).$$

By (12) we have

$$\Psi_*\left[\rho\right](t_I+)=\left[c_{-1/2}+c^\xi_{1/2}\right]\dot\xi(t_I+)-\Phi_s\left(\rho^\xi_{1/2},\rho_{1/2}\right).$$

By (17) it must be $\rho_{-1/2}\geq\rho^\xi_{1/2}$. In order to proceed with the proof, we have to distinguish the following cases:

- If $\rho_{-1/2}=\rho^\xi_{1/2}$, then by (10a) we have $\rho_{-1/2}=\rho^\xi_{1/2}=0$. As a consequence

$$\dot\xi(t_I+)<\lambda_s\left(\rho^\xi_{1/2},\rho_{1/2}\right)=v_{1/2}, \qquad\qquad \Phi_s\left(\rho^\xi_{1/2},\rho_{1/2}\right)=v_{1/2}-1$$

and

$$\Psi_*\left[\rho\right](t_I+)<2v_{1/2}-\left[v_{1/2}-1\right]=v_{-1/2}+v_{1/2}\,.$$

- If $\rho_{-1/2}\neq\rho^\xi_{1/2}$, then $\dot\xi(t_I+)=\lambda_\xi\left(\rho_{-1/2},\rho^\xi_{1/2}\right)$ and therefore $\Psi_*\left[\rho\right](t_I+)=\psi\left(\rho^\xi_{1/2}\right)$, where

$$\psi\left(\rho^\xi_{1/2}\right)=-\frac{2\rho^\xi_{1/2}}{\rho_{-1/2}-\rho^\xi_{1/2}}\left[v_{-1/2}c^\xi_{1/2}+1\right]-\left[v_{-1/2}+v_{1/2}\right]c^\xi_{1/2}-\left[c_{-1/2}+c_{1/2}\right]\rho^\xi_{1/2}\,.$$

Clearly the map $\rho\mapsto\psi(\rho)$ is decreasing and

$$\psi(0)=-v_{-1/2}-v_{1/2}, \qquad \lim_{\rho\uparrow\rho_{-1/2}}\psi(\rho)=-\infty\,, \qquad \lim_{\rho\uparrow\rho_{1/2}}\psi(\rho)=\Phi_\xi\left(\rho_{-1/2}(t_I),\rho_{1/2}(t_I)\right).$$

Recalling that $\rho^{\xi}_{1/2} < \min\left\{\rho_{-1/2},\ \rho_{1/2}\right\}$, we deduce (21) and (22).
(3,"⇒") If ρ_ξ consists of a shock front x^{ξ}_{-1} on the left of the turning curve between the states $\rho_{-1/2}$ and $\rho^{\xi}_{-1/2}$, and a shock front x^{ξ}_{1} on the right of the turning curve between the states $\rho^{\xi}_{1/2}$ and $\rho_{1/2}$, then

$$\begin{aligned}
&\rho^{\xi}_{-1/2} < \rho_{-1/2}\,, \qquad \rho^{\xi}_{1/2} < \rho_{1/2}\,,\\
&\dot{x}^{\xi}_{-1}(t_I+) = -\lambda_s\left(\rho_{-1/2},\rho^{\xi}_{-1/2}\right) < \dot{\xi}(t_I+) < \dot{x}^{\xi}_{1}(t_I+) = \lambda_s\left(\rho^{\xi}_{1/2},\rho_{1/2}\right).
\end{aligned}$$

By (17) we have $\lambda_\xi\left(\rho,\rho^{\xi}_{1/2}\right) > \lambda_s\left(\rho^{\xi}_{1/2},\rho_{1/2}\right)$ for all $\rho < \rho^{\xi}_{1/2}$, and therefore $\rho^{\xi}_{1/2} \le \rho^{\xi}_{-1/2}$. Analogously, we have $-\lambda_s\left(\rho_{-1/2},\rho^{\xi}_{-1/2}\right) > \lambda_\xi\left(\rho^{\xi}_{-1/2},\rho\right)$ for all $\rho < \rho^{\xi}_{-1/2}$, and therefore $\rho^{\xi}_{1/2} \ge \rho^{\xi}_{-1/2}$. In conclusion we proved that $\rho^{\xi}_{1/2} = \rho^{\xi}_{-1/2}$ and this, by (10a), implies that $\rho^{\xi}_{\pm 1/2} = 0$. Hence

$$\begin{aligned}
&\Phi_0\left[\rho\right](t_I+) = 2\dot{\xi}(t_I+)\,,\\
&\Psi\left[\rho\right](t_I+) = \Psi_*\left[\rho\right](t_I+) + \Phi_s\left(\rho_{-1/2},0\right) + \Phi_s\left(0,\rho_{1/2}\right) = \Psi_*\left[\rho\right](t_I+) - v_{-1/2} + v_{1/2}
\end{aligned}$$

and by (12) we have

$$\Psi_*\left[\rho\right](t_I+) = 2\dot{\xi}(t_I+) + v_{-1/2} - v_{1/2}\,.$$

Thus (23) holds true because $-v_{-1/2} < \dot{\xi}(t_I+) < v_{1/2}$.
(4,"⇒") & (5,"⇒") In accordance to the symmetry of the problem, the proofs are analogous to that for the cases (2) and (1), respectively.
(6,"⇒") If ρ_ξ is given by the turning curve only, then (28) and (29) follow from (10a) and (12).
("⇐") The converse is obvious by the above construction of the solutions. Indeed, for any given $\rho_{-1/2}$, $\rho_{1/2}$, $\Psi_*\left[\rho\right](t_I+)$, only one of the conditions (19)-(29) is satisfied because $\Phi_\xi\left(\alpha,\beta\right) > v\left(\alpha\right) + v\left(\beta\right)$ for any $0 < \alpha < \beta$ and $\Phi_\xi\left(0,\beta\right) = 1 + v\left(\beta\right)$. Furthermore, ρ^ξ never consists of rarefaction fronts on both sides of the turning curve. Indeed, as already proved, the presence of rarefaction fronts on the left of the turning curve implies that $\dot{\xi}(t_I+) > 0$, as well as the presence of rarefaction fronts on the right of the turning curve implies that $\dot{\xi}(t_I+) < 0$. Moreover, ρ^ξ never consists of rarefaction fronts on one side of the turning curve and a shock front on the other side. Indeed, for instance, the presence of rarefaction fronts on the left of the turning curve implies that $\rho_{-1/2} > 0$ and $\dot{\xi}(t_I+) > 0$, while the presence of a shock front on the right of the turning curve with $\dot{\xi}(t_I+) > 0$ implies that $\rho_{\pm 1/2} = 0$ and this gives a contradiction.

The proof of Theorem 6 is then complete. □

Before the end of the section, we show how to apply the above theorem to construct an ε–admissible approximate solution ρ^ε. Let $t_I \ge 0$, $\rho^\varepsilon(t_I,x)$ be piecewise constant and $\xi^\varepsilon(t_I)$ uniquely defined by (8) for $t = t_I$.

- **Step 1.** Construct ρ_L^ε and ρ_R^ε, as described at the beginning of this Section. For any $t > t_I$, with $t - t_I$ sufficiently small, ρ_L^ε and ρ_R^ε have the following form

$$\begin{aligned}
\rho_L^\varepsilon(t,x) &= \sum_{i=-h-1}^{-2} \rho_{i+1/2}^\varepsilon \, \chi_{[x_i^\varepsilon(t),\, x_{i+1}^\varepsilon(t)[}(x) \; + \; \rho_{-1/2}^\varepsilon \, \chi_{[x_{-1}^\varepsilon(t),\, 1[}(x)\,, \\
\rho_R^\varepsilon(t,x) &= \rho_{1/2}^\varepsilon \, \chi_{[-1,\, x_1^\varepsilon(t)[}(x) \; + \; \sum_{i=1}^{k} \rho_{i+1/2}^\varepsilon \, \chi_{[x_i^\varepsilon(t),\, x_{i+1}^\varepsilon(t)[}(x)\,.
\end{aligned}$$

- **Step 2.** Compute the quantity $\Psi_* [\rho^\varepsilon](t_I+)$ defined by (18), which writes

$$\Psi_* [\rho^\varepsilon](t_I+) = \sum_{i \neq 0} \operatorname{sgn}(i) \left[c\left(\rho_{i-1/2}^\varepsilon\right) - c\left(\rho_{i+1/2}^\varepsilon\right) \right] \dot{x}_i^\varepsilon(t_I+)\,.$$

- **Step 3.** Check which one of the conditions given in Theorem 6 is satisfied and construct the corresponding ρ_ξ^ε.

If for instance condition (19) is satisfied, then there exists a unique $\rho_{1/2}^\xi \in \left]\rho_{1/2}^\varepsilon, \rho_{-1/2}^\varepsilon\right[$ solution to

$$\Phi_\xi \left(\rho_{-1/2}^\varepsilon, \rho_{1/2}^\xi\right) - q\left(\rho_{1/2}^\xi\right) = \Psi_* [\rho^\varepsilon](t_I+) - q\left(\rho_{1/2}^\varepsilon\right).$$

Observe that in general $\rho_{1/2}^\xi$ does not belong to the ε-grid $\mathcal{G}^\varepsilon$. However, even in this case, if $\left]\rho_{1/2}^\varepsilon, \rho_{1/2}^\xi\right[\cap$

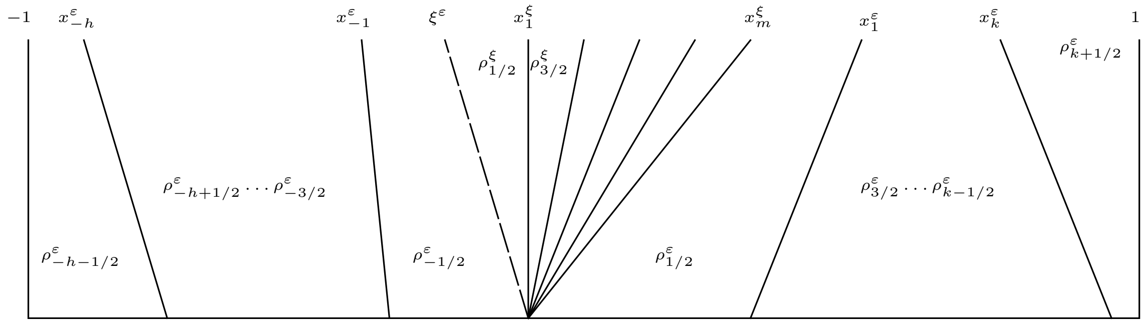

Figure 2: How to prolong an ε–admissible solution of (3) beyond $t = t_I$ in the case described in Theorem 6, case 1.

$\mathcal{G}^\varepsilon = \left\{\rho^\xi_{3/2}, \ldots, \rho^\xi_{m-1/2}\right\}$ with $\rho^\xi_{i-1/2} > \rho^\xi_{i+1/2}$, then for any $t > t_I$ sufficiently small (see Figure 2)

$$\begin{aligned}
\rho^\varepsilon(t,x) = & \sum_{i=-h-1}^{-2} \rho^\varepsilon_{i+1/2}\, \chi_{[x^\varepsilon_i(t),\, x^\varepsilon_{i+1}(t)[}(x) + \rho^\varepsilon_{-1/2}\, \chi_{[x^\varepsilon_{-1}(t),\, \xi^\varepsilon(t)[}(x) \\
& + \rho^\xi_{1/2}\, \chi_{[\xi^\varepsilon(t),\, x^\xi_1(t)[}(x) + \sum_{i=1}^{m-1} \rho^\xi_{i+1/2}\, \chi_{[x^\xi_i(t),\, x^\xi_{i+1}(t)[}(x) \\
& + \rho^\varepsilon_{1/2}\, \chi_{[x^\xi_m(t),\, x^\varepsilon_1(t)[}(x) + \sum_{i=1}^{k} \rho^\varepsilon_{i+1/2}\, \chi_{[x^\varepsilon_i(t),\, x^\varepsilon_{i+1}(t)[}(x)\,,
\end{aligned}$$

with

$$\begin{aligned}
\xi^\varepsilon(t) &= \xi^\varepsilon(t_I) + \lambda_\xi\left(\rho^\varepsilon_{-1/2}, \rho^\xi_{1/2}\right)(t - t_I)\,, & \\
x^\xi_i(t) &= \xi^\varepsilon(t_I) + \lambda_r\left(\rho^\xi_{i-1/2}, \rho^\xi_{i+1/2}\right)(t - t_I)\,, & i = 1, \ldots, m-1, \\
x^\xi_m(t) &= \xi^\varepsilon(t_I) + \lambda_r\left(\rho^\xi_{m-1/2}, \rho^\varepsilon_{1/2}\right)(t - t_I)\,, &
\end{aligned}$$

is an ε–admissible approximate solution of (3) in the sense of Definition 5.

3. Proof of Theorem 3

Let $\bar\rho \in \mathbf{BV}(\Omega; [0,1[)$ satisfy (6). Let n be an integer ≥ 1 and set $\varepsilon = 1/2^n$. We can introduce a piecewise constant function $\bar\rho^\varepsilon \in \mathbf{BV}(\Omega; \mathcal{G}^\varepsilon)$ such that

$$|c(\bar\rho^\varepsilon(\pm 1 \mp)) - c(\bar\rho(\pm 1 \mp))| \leq \varepsilon\,, \qquad \mathrm{TV}(c(\bar\rho^\varepsilon)) \leq \mathrm{TV}(c(\bar\rho)) + \varepsilon\{\sup c'\}\,, \tag{30a}$$

$$\|\bar\rho^\varepsilon\|_\infty \leq \|\bar\rho\|_\infty\,, \qquad \lim_{\varepsilon\to 0+} \|\bar\rho - \bar\rho^\varepsilon\|_1 = 0\,. \tag{30b}$$

As seen in Theorem 6, in general new fronts may start from the turning curve, so that ρ^ε may well take values outside the grid $\mathcal{G}^\varepsilon$. However, as we will see in the next theorem, condition (6) ensures that the ε-approximate solution takes values always in $\mathcal{G}^\varepsilon$. Here we will denote $\bar\rho_\infty = \|\bar\rho\|_\infty$.

Theorem 7. *For $\varepsilon = 1/2^n > 0$ sufficiently small, if the initial datum satisfies the condition* (6)*, then there exists an ε–admissible approximate solution of* (2), (3) *in the sense of Definition 5 with values in $\mathcal{G}^\varepsilon$ and defined globally in time.*

PROOF. Consider $\bar\xi^\varepsilon$ given by (7), $\rho^\varepsilon_L(t,\cdot)$ and $\rho^\varepsilon_R(t,\cdot)$ as in Section 2.

Recalling the definition of the speeds (10b) and (10c), we notice that $|\dot x^\varepsilon_i| \leq 1$ for $i \neq 0$. Then, recalling the definition of the Φ_i in (14), we find that

$$|\Phi_i[\rho^\varepsilon](t)| \;\leq\; \left|c\left(\rho^\varepsilon_{i-1/2}\right) - c\left(\rho^\varepsilon_{i+1/2}\right)\right|, \qquad i \neq 0\,.$$

Since $v(\rho) = 1 - \rho$, by (30a) and the assumption (6), we estimate the quantity Ψ_* in (18) as follows:

$$
\begin{aligned}
|\Psi_*[\rho^\varepsilon](0+)| &\leq \mathrm{TV}\left(c\left(\rho^\varepsilon_L(0+)\right)\right) + \mathrm{TV}\left(c\left(\rho^\varepsilon_R(0+)\right)\right)\\
&\leq \mathrm{TV}\left(c\left(\bar\rho\right)\right) + \left[c\left(\bar\rho(-1+)\right) - c(1/2)\right]_+ + \left[c\left(\bar\rho(1-)\right) - c(1/2)\right]_+ + C_1\varepsilon\\
&< 2 - 3\bar\rho_\infty \;\leq\; 2v(\bar\rho_\infty)\\
&\leq v\left(\bar\rho^\varepsilon_{-1/2}\right) + v\left(\bar\rho^\varepsilon_{1/2}\right)
\end{aligned}
$$

with $C_1 = 2 + \{\sup c'\}$ and ε small enough. Above, the terms $[c(\bar\rho(\pm1\mp) - c(1/2)]_+$ account for the possible rarefactions arising at the boundaries at time $t = 0$.

Thanks to the above bound on Ψ_*, by Theorem 6 we deduce that only cases (20), (23), (24) and (28) can hold true. In all of these cases we have $\rho^\varepsilon_{\pm1/2}(0+) = 0$.

Thus, for $t > 0$ sufficiently small we have $\rho^\varepsilon \equiv 0$ in a region which is bounded by the fronts $x^\varepsilon_{\pm1}$. Necessarily these fronts are shocks, and their speeds are given by $\dot x^\varepsilon_{\pm1}(t) = \pm v\left(\rho^\varepsilon_{\pm3/2}(t)\right)$.

Because of the maximum principle one has $\|\rho^\varepsilon(t)\|_\infty \leq \|\bar\rho^\varepsilon\|_\infty \leq \bar\rho_\infty < 1$, therefore $|\dot x^\varepsilon_{\pm1}| \geq v(\bar\rho_\infty) > 0$: the region with $\rho^\varepsilon \equiv 0$ contains a non-empty cone.

At each interaction we apply the algorithm and Theorem 6 to extend ρ^ε in time as described in Section 2. We want to prove that the condition (28) always holds true by showing that no front can reach the turning curve.

Assume that, for some $\tilde t > 0$, one has $\rho^\varepsilon\left(t, \xi^\varepsilon(t)\pm\right) = 0$ for all $t \in \left]0, \tilde t\right[$. Now we consider $\dot\xi^\varepsilon(\tilde t-)$, given by

$$
2\dot\xi^\varepsilon\left(\tilde t-\right) \;=\; \sum_{i\neq0}\Phi_i[\rho^\varepsilon](\tilde t-)\,,
$$

and write the sum $\sum_{i\neq0}$ as $\sum_{i=\pm1} + \sum_{|i|\geq2}$. Recalling that $c(\rho)\cdot v(\rho) = 1$, we can write

$$
\begin{aligned}
\Phi_{-1}[\rho^\varepsilon]\left(\tilde t-\right) + \Phi_1[\rho^\varepsilon]\left(\tilde t-\right) = &\left[c\left(\rho^\varepsilon_{-3/2}\left(\tilde t-\right)\right) - 1\right] v\left(\rho^\varepsilon_{-3/2}\left(\tilde t-\right)\right)\\
&+ \left[1 - c\left(\rho^\varepsilon_{3/2}\left(\tilde t-\right)\right)\right] v\left(\rho^\varepsilon_{3/2}\left(\tilde t-\right)\right) = \rho^\varepsilon_{-3/2} - \rho^\varepsilon_{3/2}\,.
\end{aligned}
$$

Thus

$$
\left|\Phi_{-1}[\rho^\varepsilon]\left(\tilde t-\right) + \Phi_1[\rho^\varepsilon]\left(\tilde t-\right)\right| \leq \bar\rho_\infty\,. \tag{31}
$$

Moreover we find that, for the same constant C_1 as above,

$$
\begin{aligned}
\sum_{|i|\geq2}\left|\Phi_i[\rho^\varepsilon](\tilde t-)\right| &\leq \sum_{|i|\geq2}\left|c\left(\rho^\varepsilon_{i+\frac12}\right) - c\left(\rho^\varepsilon_{i-\frac12}\right)\right|\\
&\leq \mathrm{TV}\left(c(\bar\rho)\right) + \left[c\left(\bar\rho(-1+)\right) - c(1/2)\right]_+ + \left[c\left(\bar\rho(1-)\right) - c(1/2)\right]_+ + C_1\varepsilon\,.
\end{aligned} \tag{32}
$$

Indeed, recall that the indexes are possibly rearranged at each interaction; the above sum decreases in all possible interactions, that is: when a front leaves the domain Ω; when two or more fronts interact, with all indexes different from ± 1; when the interaction involves a ± 1 front, in which case the resulting front will inherit the ± 1 index.

By virtue of (31), (32) and (6), for ε small enough we have that

$$\begin{aligned} 2\left|\dot{\xi}^\varepsilon\left(\tilde{t}-\right)\right| &\leq \bar{\rho}_\infty + \mathrm{TV}\left(c\left(\bar{\rho}\right)\right) + \left[c\left(\bar{\rho}(-1+)\right) - c(1/2)\right]_+ + \left[c\left(\bar{\rho}(1-)\right) - c(1/2)\right]_+ + C_1\varepsilon \\ &< 2\left[1-\bar{\rho}_\infty\right] \leq 2\left|\dot{x}^\varepsilon_{\pm 1}\left(\tilde{t}-\right)\right| \end{aligned}$$

and therefore the turning curve does not reach $x^\varepsilon_{\pm 1}$ at time $t = \tilde{t}$. As a consequence, no new front starts from the turning curve and we can apply the standard theory of wave-front tracking to prolong ρ^ε after any interaction and to ensure its global existence. $\square$

Finally we prove that, up to a subsequence, as ε goes to zero ρ^ε converges to a solution of (2), (3). By construction, see the proof of Theorem 7, we immediately have that for all $t > 0$

$$\|\rho^\varepsilon(t)\|_\infty \leq \|\bar{\rho}^\varepsilon\|_\infty \leq \bar{\rho}_\infty\,,$$

and

$$\begin{aligned} \mathrm{TV}\left(\rho^\varepsilon(t)\right) &\leq \mathrm{TV}\left(\rho^\varepsilon(0+)\right) \\ &\leq \mathrm{TV}(\bar{\rho}^\varepsilon) + [\bar{\rho}^\varepsilon(-1+) - 1/2]_+ + [\bar{\rho}^\varepsilon(1-) - 1/2]_+ + 2\bar{\rho}_\infty \\ &\leq \mathrm{TV}\left(c(\bar{\rho})\right) + \left[c(\bar{\rho}(-1+)) - c(1/2)\right]_+ + \left[c(\bar{\rho}(1-)) - c(1/2)\right]_+ + C_1\varepsilon + 2\bar{\rho}_\infty < L \end{aligned} \tag{33}$$

for ε small enough, where $L = 2 - \bar{\rho}_\infty$; here we used that $c' \geq 1$ and assumption (6). Finally observe that

$$\int_{-1}^{1} |\rho^\varepsilon(t,x) - \rho^\varepsilon(s,x)|\,\mathrm{d}x \leq L\,|t-s|\,.$$

Indeed, if no interaction occurs for times between t and s, then

$$\begin{aligned} \int_{-1}^{1} |\rho^\varepsilon(t,x) - \rho^\varepsilon(s,x)|\,\mathrm{d}x &\leq \sum_{i\neq 0} \left|(t-s)\,\dot{x}^\varepsilon_i(t)\left(\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}\right)\right| \\ &\leq |t-s|\,\mathrm{TV}\left(\rho^\varepsilon(t)\right) \leq L\,|t-s|\,. \end{aligned}$$

The case when one or more interactions take place for times between t and s is similar, because by (10) the map $t \mapsto \rho^\varepsilon(t)$ is $\mathbf{L^1}$-continuous across interaction times.

Thus, by applying Helly's Theorem in the form [7, Theorem 2.4], there exists a function $\rho \in \mathbf{L^1_{loc}}\left([0,+\infty[\, \times \Omega; [0,1]\right)$ and a subsequence, still denoted ρ^ε, such that

$$\begin{aligned} &\rho^\varepsilon \to \rho \text{ in } \mathbf{L^1_{loc}}([0,+\infty[\,\times\Omega;\mathbb{R}) \text{ as } \varepsilon \downarrow 0\,, && \mathrm{TV}\left(\rho(t)\right) \leq L\,, \\ &\|\rho(t) - \rho(s)\|_1 \leq L\,|t-s| \text{ for all } t,s \geq 0\,, && \|\rho(t)\|_\infty \leq \|\bar{\rho}\|_\infty\,. \end{aligned}$$

Regarding ξ^ε, we observe that the associated sequence is bounded and uniformly equicontinuous, because of the uniform Lipschitz constant. Therefore, by Ascoli-Arzelà theorem, we can extract a subsequence uniformly converging to some $\xi \in W^{1,1}([0,T];\Omega)$, for any $T>0$, with the same Lipschitz constant $(1-\bar\rho_\infty)$. In particular, ξ evolves in a cone where ρ is zero, and it is straightforward to show that (3b) holds.

We want to prove that ρ is an entropy weak solution of the initial-boundary value problem (2), (3) in the sense of Definition 1. For any fixed $\kappa \in [0,1]$ and $\psi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^2;[0,+\infty[)$, we have to prove that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \Bigg[& \int_0^{+\infty} \int_{-1}^1 \left\{ |\rho^\varepsilon - \kappa|\, \partial_t \psi + \mathcal{F}(t,x,\rho^\varepsilon,\kappa)\, \partial_x \psi \right\} \mathrm{d}x\, \mathrm{d}t \\
& + \int_{-1}^1 |\bar\rho^\varepsilon(x) - \kappa|\, \psi(0,x)\, \mathrm{d}x + \int_0^{+\infty} \left\{ f\left(\rho^\varepsilon(t,-1+)\right) - f(\kappa) \right\} \psi(t,-1)\, \mathrm{d}t \\
& + \int_0^{+\infty} \left\{ f\left(\rho^\varepsilon(t,1-)\right) - f(\kappa) \right\} \psi(t,1)\, \mathrm{d}t + 2 \int_0^{+\infty} f(\kappa)\, \psi\left(t,\xi(t)\right)\, \mathrm{d}t \Bigg] \geq 0\,.
\end{aligned} \tag{34}$$

Fix $\psi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^2;[0,+\infty[)$ and choose $T>0$ such that $\psi(t,x)=0$ whenever $t \geq T$. For any $\varepsilon>0$, we can divide the strip $[0,T]\times[-1,1]$ into finitely many regions Γ_i, delimited by the wave fronts, where ρ^ε takes a constant value denoted by $\tilde\rho_i$. The indexes are chosen so that

$$\Gamma_0 = \left\{ (t,x) \in [0,T]\times[-1,1] \,:\, x^\varepsilon_{-1}(t) \leq x \leq x^\varepsilon_1(t) \right\}.$$

Observe that $\tilde\rho_0 = 0$. Divide Γ_0 in $\Gamma_0^\pm = \{(t,x)\in\Gamma_0 : \ \pm[x-\xi(t)] \geq 0\}$. Then

$$\begin{aligned}
I = & \int_0^{+\infty} \int_{-1}^1 \left[|\rho^\varepsilon - \kappa|\, \partial_t \psi + \mathcal{F}(t,x,\rho^\varepsilon,\kappa)\, \partial_x \psi \right] \mathrm{d}x\, \mathrm{d}t \\
= & \sum_{i<0} \int\!\!\int_{\Gamma_i} \left\{ |\tilde\rho_i - \kappa|\, \partial_t\psi - \mathrm{sgn}\,(\tilde\rho_i - \kappa) \left[f(\tilde\rho_i) - f(\kappa) \right] \partial_x \psi \right\} \mathrm{d}x\, \mathrm{d}t \\
& + \int\!\!\int_{\Gamma_0^-} \left[\kappa\, \partial_t \psi - f(\kappa)\, \partial_x \psi \right] \mathrm{d}x\, \mathrm{d}t + \int\!\!\int_{\Gamma_0^+} \left[\kappa\, \partial_t \psi + f(\kappa)\, \partial_x \psi \right] \mathrm{d}x\, \mathrm{d}t \\
& + \sum_{i>0} \int\!\!\int_{\Gamma_i} \left\{ |\tilde\rho_i - \kappa|\, \partial_t\psi + \mathrm{sgn}\,(\tilde\rho_i - \kappa) \left[f(\tilde\rho_i) - f(\kappa) \right] \partial_x \psi \right\} \mathrm{d}x\, \mathrm{d}t\,.
\end{aligned}$$

We apply the divergence theorem on each Γ_i. The contributions to I coming from the boundaries of Γ_i can be listed as follows:

- $I_0 = -\int_{-1}^1 |\bar\rho^\varepsilon(x) - \kappa|\, \psi(0,x)\, \mathrm{d}x$ along $t=0$;
- $I_{\pm 1} = \int_0^T \mathrm{sgn}\,(\rho^\varepsilon(t,\pm1\mp) - \kappa) \left[f\left(\rho^\varepsilon(t,\pm1\mp)\right) - f(\kappa) \right] \psi(t,\pm1)\, \mathrm{d}t$ along $x=\pm1$;
- $I_\xi = -2f(\kappa) \int_0^T \psi\left(t,\xi(t)\right) \mathrm{d}t$ along ξ;

• $\mathcal{I}_i = \int_{t_i'}^{t_i''} \Theta_i^\varepsilon(t;\kappa)\, \psi(t, x_i^\varepsilon)\, \mathrm{d}t$ along x_i^ε, for some $t_i' < t_i''$, where

$$\begin{aligned}
&\Theta_i^\varepsilon(t;\kappa) = \operatorname{sgn}(i)\operatorname{sgn}\left(\rho_{i-1/2}^\varepsilon - \kappa\right)\left[f\left(\rho_{i-1/2}^\varepsilon\right) - f(\kappa)\right]\\
&+ \left[\left|\rho_{i+1/2}^\varepsilon - \kappa\right| - \left|\rho_{i-1/2}^\varepsilon - \kappa\right|\right]\dot{x}_i^\varepsilon - \operatorname{sgn}(i)\operatorname{sgn}\left(\rho_{i+1/2}^\varepsilon - \kappa\right)\left[f\left(\rho_{i+1/2}^\varepsilon\right) - f(\kappa)\right].
\end{aligned}$$

Therefore we obtain that $I = I_0 + I_{-1} + I_1 + I_\xi + \sum_i \mathcal{I}_i$ and that the expression within the square brackets in (34) is equal to

$$\begin{aligned}
&\int_0^T \left[1 + \operatorname{sgn}\left(\rho^\varepsilon(t,-1+) - \kappa\right)\right]\left[f\left(\rho^\varepsilon(t,-1+)\right) - f(\kappa)\right]\psi(t,-1)\,\mathrm{d}t\\
&+ \int_0^T \left[1 + \operatorname{sgn}\left(\rho^\varepsilon(t,1-) - \kappa\right)\right]\left[f\left(\rho^\varepsilon(t,1-)\right) - f(\kappa)\right]\psi(t,1)\,\mathrm{d}t\\
&+ \sum_i \int_{t_i'}^{t_i''} \Theta_i^\varepsilon(t;\kappa)\, \psi(t, x_i^\varepsilon)\, \mathrm{d}t\,.
\end{aligned}$$

Recalling that $\rho^\varepsilon(t, \pm 1\mp) \le 1/2$, we easily obtain that

$$\int_0^T \left[1 + \operatorname{sgn}\left(\rho^\varepsilon(t,\pm1\mp) - \kappa\right)\right]\left[f\left(\rho^\varepsilon(t,\pm1\mp)\right) - f(\kappa)\right]\psi(t,\pm1)\,\mathrm{d}t \ge 0\,.$$

If x_i^ε is a shock front, the term Θ_i^ε has the correct sign ≥ 0, since the shock speed is the exact one.

Hence, to complete the proof, it is sufficient to prove that for the rarefactions one has

$$\left|\Theta_i^\varepsilon(t;\kappa)\right| \le \varepsilon\left|\rho_{i+1/2}^\varepsilon - \rho_{i-1/2}^\varepsilon\right|. \tag{35}$$

Indeed, if x_i^ε is a rarefaction front, then

$$\begin{aligned}
&\operatorname{sgn}(i)\,\Theta_i^\varepsilon(t;\kappa) = \left[\left|\rho_{i+1/2}^\varepsilon - \kappa\right| - \left|\rho_{i-1/2}^\varepsilon - \kappa\right|\right]\lambda_r\left(\rho_{i-1/2}^\varepsilon, \rho_{i+1/2}^\varepsilon\right)\\
&+ \operatorname{sgn}\left(\rho_{i-1/2}^\varepsilon - \kappa\right)\left[f\left(\rho_{i-1/2}^\varepsilon\right) - f(\kappa)\right] - \operatorname{sgn}\left(\rho_{i+1/2}^\varepsilon - \kappa\right)\left[f\left(\rho_{i+1/2}^\varepsilon\right) - f(\kappa)\right].
\end{aligned}$$

If $\kappa < \min\left\{\rho_{i-1/2}^\varepsilon, \rho_{i+1/2}^\varepsilon\right\}$ or $\kappa > \max\left\{\rho_{i-1/2}^\varepsilon, \rho_{i+1/2}^\varepsilon\right\}$, then by Lemma 9

$$\begin{aligned}
\left|\Theta_i^\varepsilon(t;\kappa)\right| &= \left|\rho_{i+1/2}^\varepsilon - \rho_{i-1/2}^\varepsilon\right|\left|\lambda_r\left(\rho_{i-1/2}^\varepsilon, \rho_{i+1/2}^\varepsilon\right) - \lambda_s\left(\rho_{i-1/2}^\varepsilon, \rho_{i+1/2}^\varepsilon\right)\right|\\
&\le \left[\rho_{i+1/2}^\varepsilon - \rho_{i-1/2}^\varepsilon\right]^2 \le \varepsilon\left|\rho_{i+1/2}^\varepsilon - \rho_{i-1/2}^\varepsilon\right|.
\end{aligned}$$

If $\kappa = \alpha \rho^\varepsilon_{i-1/2} + (1-\alpha)\, \rho^\varepsilon_{i+1/2}$ for an $\alpha \in [0,1]$, then again by Lemma 9

$$\begin{aligned}
&\left|\Theta^\varepsilon_i\,(t;\kappa)\right| = \\
=&\left|\left[2\kappa - \rho^\varepsilon_{i+1/2} - \rho^\varepsilon_{i-1/2}\right] \lambda_r\left(\rho^\varepsilon_{i-1/2}, \rho^\varepsilon_{i+1/2}\right) + f\left(\rho^\varepsilon_{i+1/2}\right) + f\left(\rho^\varepsilon_{i-1/2}\right) - 2f(\kappa)\right| \\
=&\left[\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}\right]^2 \left|(2\alpha-1)\, \frac{\lambda_r\left(\rho^\varepsilon_{i-1/2}, \rho^\varepsilon_{i+1/2}\right) - \lambda_s\left(\rho^\varepsilon_{i-1/2}, \rho^\varepsilon_{i+1/2}\right)}{\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}} + 2\alpha\,(\alpha-1)\right| \\
\leq&\left[\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}\right]^2 \left[|2\alpha-1| + 2\alpha\,(1-\alpha)\right] \leq \left[\rho^\varepsilon_{i-1/2} - \rho^\varepsilon_{i+1/2}\right]^2 \leq \varepsilon \left|\rho^\varepsilon_{i+1/2} - \rho^\varepsilon_{i-1/2}\right|.
\end{aligned}$$

In conclusion we proved (35). As a consequence, thanks to (33), we get

$$\begin{aligned}
\left|\sum_{\substack{i \\ x^\varepsilon_i \text{raref.}}} \int_{t'_i}^{t''_i} \Theta^\varepsilon_i\,(t;\kappa)\ \psi\,(t, x^\varepsilon_i)\ \mathrm{d}t\right| &\leq \varepsilon\, \|\psi\|_{\mathbf{C^0}} \sum_i \int_{t'_i}^{t''_i} \left|\rho^\varepsilon_{i+1/2} - \rho^\varepsilon_{i-1/2}\right| \mathrm{d}t \\
&\leq \varepsilon\, \|\psi\|_{\mathbf{C^0}}\, T \sup_{]0,T[} \mathrm{TV}\,(\rho^\varepsilon(t)) \\
&\leq \varepsilon\, \|\psi\|_{\mathbf{C^0}}\, L\, T
\end{aligned}$$

and therefore (34) holds true.

4. Technical section

In this section we estimate the quantities introduced in (15).

Lemma 8 (Estimate for λ_r). *For any $\alpha, \beta \in [0,1]$ with $\beta < \alpha$ we have*

$$\lambda_r\,(\alpha, \beta) < v\,(\alpha)\,.$$

PROOF. Observe that for all $x \geq 1$

$$\log(x) \leq \frac{x^2-1}{2x}\,.$$

Indeed the above estimate holds for $x = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2-1}{2x} - \log(x)\right] = \frac{(x-1)^2}{2x^2} \geq 0$. Therefore

$$\lambda_r\left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right) < \lambda_r\left(1 - \frac{1}{x}, 0\right) = \frac{2\log(x)}{x-1} - 1 \leq \frac{1}{x} = v\left(1 - \frac{1}{x}\right)\,.$$

Then, it is sufficient to take $x = c(\alpha)$ and $y = c(\beta)$ to complete the proof. □

Lemma 9 (Estimate for $|\lambda_r - \lambda_s|$). *For any $\alpha, \beta \in [0,1[$ with $\alpha \neq \beta$ we have*

$$|\lambda_r(\alpha,\beta) - \lambda_s(\alpha,\beta)| \leq |\alpha - \beta| .$$

Proof. Introduce the function $C \in \mathbf{C^0}([0,+\infty[;\mathbb{R})$ defined by

$$C(x) = \begin{cases} -1 & \text{if } x = 0, \\ \frac{x+1}{x-1} - \frac{2x}{(x-1)^2} \log x & \text{if } x \in]0,+\infty[\setminus \{1\}, \\ 0 & \text{if } x = 1. \end{cases}$$

Since $x \mapsto C(x)$ is strictly increasing and $\lim_{x\to+\infty} C(x) = 1$, we have that $\|C\|_{\mathbf{C^0}([0,+\infty[;\mathbb{R})} = 1$. Moreover for any $x, y \geq 1$ with $x \neq y$ we have

$$C\left(\frac{y}{x}\right) = \left[\lambda_r\left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right) - \lambda_s\left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right)\right]\left[\frac{1}{y} - \frac{1}{x}\right]^{-1} .$$

Thus, it is sufficient to take $x = c(\alpha)$ and $y = c(\beta)$ to complete the proof. □

Lemma 10 (Estimates for Φ_ξ). *For any $\alpha \in [0,1]$ we have*

(1) $\beta \mapsto \Phi_\xi(\alpha,\beta) - q(\beta)$ is decreasing in $[0,\alpha[$;

(2) $\beta \mapsto \Phi_\xi(\beta,\alpha) + q(\beta)$ is increasing in $[0,\alpha[$.

Proof. We first observe that

$$\Phi_\xi\left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right) - q\left(1 - \frac{1}{y}\right) = [x + y]\left[\frac{1}{y} - \frac{1}{x} + \frac{x + y - 2}{y - x}\right] + y - 2\log y .$$

and that the derivative with respect to y of the above function is

$$(x,y) \mapsto -\frac{\left(x^2 - y^2 + 2xy\right)\left(x^2 - y^2 + 2xy(y-1)\right)}{x(x-y)^2 y^2}$$

and is negative for all $x > y \geq 1$. Then, it is sufficient to take $x = c(\alpha)$ and $y = c(\beta)$ to complete the proof of (1). The proof of (2) follows from the previous one by observing that Φ_ξ is anti-symmetric. □

Acknowledgment

This research was partially supported by the European Research Council under the European Union's Seventh Framework Program (FP/2007-2013) / ERC Grant Agreement n. 257661 and by Polonium 2011 (French-Polish cooperation program) under the project "CROwd Motion Modeling and Management". The last author was partially supported by ICM, University of Warsaw, and by Narodowe Centrum Nauki, grant 4140.

References

[1] Adimurthi, R. Dutta, S.S. Ghoshal, G.D. Veerappa Gowda, Existence and nonexistence of TV bounds for scalar conservation laws with discontinuous flux, Comm. Pure Appl. Math. 64 (2011) 84–115

[2] D. Amadori, M. Di Francesco, The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions, Acta Math. Sci. Ser. B Engl. Ed. 32 (2012) 259–280

[3] B. Andreianov, K.H. Karlsen, N.H. Risebro, A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux, Arch. Ration. Mech. Anal. 201 (2011) 27–86

[4] B. Andreianov, K. Sbihi, Well-posedness of general boundary-value problems for scalar conservation laws, Trans. AMS (2014) to appear.

[5] E. Audusse, B. Perthame, Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 253–265

[6] C. Bardos, A.Y. le Roux, J.C. Nédélec, First order quasilinear equations with boundary conditions, Comm. Partial Differential Equations 4 (1979) 1017–1034

[7] A. Bressan, Hyperbolic systems of conservation laws, vol. 20 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2000

[8] J. Carrillo, Conservation laws with discontinuous flux functions and boundary condition, J. Evol. Equations 3 (2003) 283–301

[9] R.M. Colombo, M.D. Rosini, Well posedness of balance laws with boundary, J. Math. Anal. Appl. 311 (2005) 683–702

[10] C.M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, J. Math. Anal. Appl. 38 (1972) 33–41

[11] C.M. Dafermos, Hyperbolic conservation laws in continuum physics, volume 325 of *Grundlehren der Mathematischen Wissenschaften*, third ed., Springer-Verlag, Berlin, 2010

[12] M. Di Francesco, P.A. Markowich, J.F. Pietschmann, M.T. Wolfram, On the Hughes' model for pedestrian flow: the one-dimensional case, J. Differential Equations 250 (2011) 1334–1362

[13] N. El-Khatib, P. Goatin, M. Rosini, On entropy weak solutions of Hughes' model for pedestrian motion, Z. Angew. Math. Phys. 64 (2013) 223–251

[14] M. Garavello, R. Natalini, B. Piccoli, A. Terracina, Conservation laws with discontinuous flux, Netw. Heterog. Media 2 (2007) 159–179

[15] P. Goatin, M. Mimault, The wave-front tracking algorithm for Hughes' model of pedestrian motion, SIAM J. Sci. Comput. 35 (2013) B606–B622

[16] R.L. Hughes, A continuum theory for the flow of pedestrians, Transportation Research Part B 36 (2002) 507–535

[17] S.N. Kružhkov, First order quasilinear equations with several independent variables, Mat. Sb. (N.S.) 81 (123) (1970) 228–255

[18] F. Otto, Initial-boundary value problem for a scalar conservation law, C. R. Acad. Sci. Paris Sér. I Math. 322 (1996) 729–734

[19] E.Y. Panov, Existence of strong traces for quasi-solutions of multidimensional conservation laws, J. Hyperbolic Differ. Equ. 4 (2007) 729–770

[20] E.Y. Panov, On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux, J. Hyperbolic Differ. Equ. 06 (2009) 525–548

[21] A. Vasseur, Strong traces for solutions of multidimensional scalar conservation laws, Arch. Ration. Mech. Anal. 160 (2001) 181–193