Dispersion analysis of improved time discretization for simply supported prestressed Timoshenko systems. Application to the stiff piano string.

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Abstract

We study the implicit time discretization of Timoshenko prestressed beams. This model features two types of waves: flexural and shear waves, that propagate with very different velocities. We present a novel implicit time discretization adapted to the physical phenomena occuring at the continuous level. After analyzing the continuous system and the two branches of eigenfrequencies associated with the standing modes, the classical θ-scheme is studied. A dispersion analysis recalls that $\theta = 1/12$ reduces the numerical dispersion, but yields a severely constrained stability condition for our application. Therefore we propose a new θ-like scheme based on two parameters adapted to each wave velocity, which reduces the numerical dispersion while relaxing this stability condition. Numerical experiments successfully illustrate the theoretical results on the specific cas of a realistic piano string. This motivates the extension of the proposed approach for more challenging physics.

Introduction

Piano strings can be modeled as simply supported Timoshenko prestressed beams. This model introduced in [2] accounts for inharmonicity of the transversal displacement, via a coupling with a shear angle resulting in the propagation of flexural and shear waves with very different speeds. Our concern in this work is to develop a new implicit time discretization, which will be associated with finite element methods in space, in order to reduce the numerical dispersion of flexural waves while allowing the use of a large time step in spite of the high shear velocity (compared to the maximal time step allowed with the explicit leap-frog scheme).

1 Continuous system

The prestressed Timoshenko model considers two unknowns (u,φ) which stand respectively for the transversal displacement and the shear angle of the cross section of the the string. We assume that the physical parameters (see [1] for definition) are positive and that $ES > T_0$ (which is true in practice for piano strings). We consider "simply supported" boundary conditions (zero displacement and zero torque). It reads:
Find (u,φ) such that $\forall\, x \in]0,L[, \quad \forall\, t>0$,

$$\left\{\begin{aligned} &\rho S\frac{\partial^2 u}{\partial t^2} - T_0\frac{\partial^2 u}{\partial x^2} + SG\kappa\frac{\partial}{\partial x}\big(\varphi - \frac{\partial u}{\partial x}\big) = \sigma,\\ &\rho I\frac{\partial^2 \varphi}{\partial t^2} - EI\frac{\partial^2 \varphi}{\partial x^2} + SG\kappa\big(\varphi - \frac{\partial u}{\partial x}\big) = 0,\end{aligned}\right. \tag{1}$$

with boundary conditions

$$\begin{aligned} u(x=0,t)&=0, & u(x=L,t)&=0,\\ \partial_x\varphi(x=0,t)&=0, & \partial_x\varphi(x=L,t)&=0,\end{aligned} \tag{2}$$

where σ stands for a source term. Standard energy techniques for systems of wave equations can be used to show a priori estimates on this system thanks to the following energy identity:

$$\frac{d\mathcal{E}}{dt} = \int_0^L \rho S\,\sigma\cdot\partial_t u, \qquad \text{with} \tag{3}$$

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \rho\, S\,|\partial_t u|^2 + \frac{1}{2}\int_0^L \rho\, I\,|\partial_t\varphi|^2 + \frac{1}{2}\int_0^L T_0\,|\partial_x u|^2$$
$$+\frac{1}{2}\int_0^L EI\,|\partial_x\varphi|^2 + \frac{1}{2}\int_0^L SG\kappa\,|\varphi - \partial_x u|^2. \tag{4}$$

If we seek a solution of the form $\,^t(u,\varphi)(x,t) = V(x)e^{-2i\pi f\,t}$, then there exists (see [3]) ℓ such that $f = f_\ell^\pm$, where

$$\left\{\begin{aligned} &f_\ell^- = \ell f_0^-\left(1+\epsilon\,\ell^2\right) + \mathcal{O}(\ell^5), &\text{(5a)}\\ &\text{where}\ \ f_0^- = \frac{1}{2L}\sqrt{\frac{T_0}{\rho S}},\ \epsilon = \frac{\pi^2}{2L^2}\frac{EI}{T_0}\Big[1-\frac{T_0}{ES}\Big]. &\\ &f_\ell^+ = f_0^+\left(1+\eta\,\ell^2\right) + \mathcal{O}(\ell^4), &\text{(5b)}\\ &\text{where}\ \ f_0^+ = \frac{1}{2\pi}\sqrt{\frac{SG\kappa}{\rho I}},\ \eta = \frac{\pi^2}{2L^2}\frac{EI+IG\kappa}{SG\kappa}. &\end{aligned}\right.$$

2 Discretisation

Space discretisation is done with high order finite elements on a mesh of size h. After this process, we get the following equation:

$$\frac{d^2}{dt^2} M_h U_h + K_h U_h = M_h \mathbf{\Sigma}_h \tag{6}$$

where M_h is symmetric positive definite and K_h is positive semi-definite.

2.1 Time discretisation with a classical θ-scheme

Using a classical θ-scheme leads to :

$$\begin{aligned} &M_h \frac{U_h^{n+1} - 2U_h^n + U_h^{n-1}}{\Delta t^2} \\ &+ K_h \left(\theta U_h^{n+1} + (1-2\theta) U_h^n + \theta U_h^{n-1}\right) = M_h \mathbf{\Sigma}_h \end{aligned} \tag{7}$$

Stability of this numerical scheme and a priori estimates can be shown with energy techniques. First, any numerical solution is shown to satisfy an energy identity. If $\theta \geq 1/4$, this discrete energy is always positive, while if $\theta < 1/4$, the time step Δt must be lower than a maximal value Δt^θ. Then, the scheme is shown to be stable if the energy is positive. Orignial proofs of stability are proposed in [1].

We also remind that if we seek a solution of the form $U_h^n = V_h^0 e^{2i\pi\, f_h\, n\Delta t}$, then there exists ℓ such that $f_h = f_{h,\ell}$, where

$$f_{h,\ell} = f_\ell + \frac{f_\ell^3}{2} \left(\frac{1}{12} - \theta\right) \Delta t^2 + \mathcal{O}(\Delta t^4 + h^4) \tag{8}$$

where $f_\ell = f_\ell^\pm$ is one of the eigenfrequencies of the continuous problem given in (5). Choosing $\theta = 1/12$ reaches fourth order of accuracy. This value being lower than $1/4$, it leads to a conditionally stable scheme. This condition will be very severe, because of the large velocity of the shear waves of the system of a realistic piano string.

Our goal is to construct a numerical scheme with a small numerical dispersion on the flexural wave, without undergoing the time step restriction coming from the shear waves.

2.2 Time discretisation with a new θ-scheme.

The idea is to separate the matrix K_h into the sum of two matrices $\overline{K}_h$ and $\underline{K}_h$, respectively inducing the energy terms $T_0 \left|\partial_x u\right|^2$ and $EI \left|\partial_x \varphi\right|^2 + SG\kappa \left|\varphi - \partial_x u\right|^2$. We consider the following scheme, with $(\theta, \overline{\theta}) \in [0,\ 1/2]^2$:

$$\begin{aligned} &M_h \frac{\mathbf{U}_h^{n+1} - 2\mathbf{U}_h^n + \mathbf{U}_h^{n-1}}{\Delta t^2} \\ &\qquad + \underline{K}_h \left\{\mathbf{U}_h\right\}_\theta^n + \overline{K}_h \left\{\mathbf{U}_h\right\}_{\overline{\theta}}^n = M_h \mathbf{\Sigma}_h^n \end{aligned} \tag{9}$$

where the θ-approximation of $\mathbf{U}_h(t^n)$ is the weighted average on three time steps:

$$\left\{\mathbf{U}_h\right\}_\theta^n = \theta\, \mathbf{U}_h^{n+1} + (1-2\theta)\, \mathbf{U}_h^n + \theta\, \mathbf{U}_h^{n-1}, \tag{10}$$

Stability of this scheme and a priori estimates can be shown via energy techniques. Sufficient conditions of stability can be given according to the values of $(\theta, \overline{\theta})$ (see [1]).

We show that if we seek a solution of the form $U_h^n = V_h^0 e^{2i\pi\, f_h\, n\Delta t}$, then there exists ℓ such that $f_h = f_{h,\ell}$, where

$$\begin{cases} f_{h,\ell}^- = \ell\, f_0^- \left(1 + \epsilon_{\Delta t}\, \ell^2\right) + \mathcal{O}(\ell^5 + \Delta t^4 + h^4), & \text{(11a)} \\ f_{h,\ell}^+ = f_{0,\Delta t}^+ \left(1 + \eta_{\Delta t}\, \ell^2\right) + \mathcal{O}(\ell^3 + \Delta t^4 + h^4), & \text{(11b)} \end{cases}$$

with (f_0^-, f_0^+, ϵ and η were defined in (5))

$$\begin{cases} \epsilon_{\Delta t} = \epsilon + 2\pi^2\, \Delta t^2 \left(\dfrac{1}{12} - \overline{\theta}\right) \left(f_0^-\right)^2, & \text{(12a)} \\[2ex] f_{0,\Delta t}^+ = f_0^+ \left[1 + (2\pi f_0^+)^2 \left(\dfrac{1}{12} - \theta\right) \Delta t^2\right], & \text{(12b)} \\[2ex] \eta_{\Delta t} = \eta + \dfrac{\pi^2\,(E + G\kappa)}{2\rho L^2} \left(\theta - \dfrac{1}{12}\right) \Delta t^2. & \text{(12c)} \end{cases}$$

We note that the value $\overline{\theta} = 1/12$ provides fourth order accuracy for the approximation $f_{h,\ell}^-$ of the flexural eigenfrequencies given by (5a), for small ℓ.

The main interest of this scheme is to choose, for the slow wave, a value of $\overline{\theta}$ that diminishes numerical dispersion, and for the fast wave, a value of θ that ensures stability under acceptable conditions, typically $(\overline{\theta}, \theta) = (1/12, 1/4)$.

Numerical illustrations that show the interest of this scheme for piano strings will be displayed.

References

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