Numerical Analysis of a reduced formulation of an elasto-acoustic scattering problem

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Abstract

We present a reduced formulation for the elasto-acoustic scattering problem. The modeling is based on the On Surface Radiation Condition method. It leads to a discrete system which solution is assessed when using Discontinuous Finite Elements.

Introduction

The numerical reconstruction of the shape of a solid immersed into a fluid is an interesting issue both from a mathematical and a practical point of view. Indeed, if engineers have solved this problem from a while in relatively simple configurations, it continues to deserve attention because its solution requires to invert a sparse linear system composed of two discretized Helmholtz equations which are known to be very sensitive to the values of the frequency [1]. The computational costs are then very high and become quickly prohibitive in particular in 3D. Solution methodologies that are able to decrease the computational burden are thus welcome. Obviously, they can be minimized by choosing suitable finite elements [2], [**?**]. In this work, we propose to investigate a different approach by using a reduced problem that can be solved with lower computational costs. For the construction of the reduced problem, we propose to play with the boundary condition that is used to limit the fluid domain. This is an absorbing boundary condition that is set on a surface which can be more or less far from the solid. Now in the simplest case of a sound-soft or hard obstacle, Kriegsmann *et al.* [4], have shown that it is possible to compute an approximate solution by setting the ABC directly on the surface of the scatterer. The corresponding reduced problem is then given by an equation called On Surface Radiation Condition (OSRC) that is set on the boundary of the scatterer and that requires less computations than the initial problem. The interest of OSRC methodology has been demonstrated later for solving the scattering problem of a penetrable object immersed into a fluid [5] and to the best of our knowledge, it has not been investigated in the case of an elasto-acoustic problem. The accuracy of the OSRC method depends on the geometry of the scatterer and on the frequency regime and it can not provide an accurate solution in all the situations. Nevertheless, it can certainly quickly deliver a solution that could be used as an initial guess for the solution of the inverse problem.

1 Problem setting

Let Ω_s be a bounded domain representing the solid and let Γ be the boundary of Ω_s. We denote by n the normal vector defined on Γ and outwardly directed to Ω_s. The solid is immersed into a fluid Ω_f which is limited by a surface Σ that has been introduced for numerical reasons. We then consider the mixed boundary value problem:

$$\nabla\cdot\sigma(u)+\omega^2\rho_s\,u=0 \qquad \text{in } \Omega_s \tag{1}$$
$$\Delta p+(\omega^2/c_f^2)\,p=0 \qquad \text{in } \Omega_f \tag{2}$$
$$\omega^2\rho_f u\cdot n=\partial_n p+\partial_n p^{inc} \qquad \text{on } \Gamma \tag{3}$$
$$\sigma(u)n=-pn-p^{inc}n \qquad \text{on } \Gamma \tag{4}$$
$$\partial_n p+\alpha p-\beta\Delta_\Sigma p=0 \qquad \text{on } \Sigma \tag{5}$$

to model the behavior of the pair (u,p) representing the displacement u into the solid and the pressure p in the fluid. ρ_f and ρ_s are the density moduli of the fluid and the solid, c_f is the propagation velocity in the fluid and the positive constant ω is the pulsation. In the following, we set $k_f=\omega/c_f$. The stress tensor $\sigma(u)$ is defined by $\sigma(u)=C\varepsilon(u)=\frac{1}{2}C\left(\nabla u+\nabla^t u\right)$, where C is the elasticity tensor. The solid S is illuminated by the incident wave p^{inc} propagating into the fluid and impinging the surface Γ according to the transmission conditions. Now let x be a generic point in $I\!R^2$ and γ_δ be the level set defined by $\gamma_\delta=\{x:=\tau+\delta n\}$ where $\tau:=\tau(s)$ denotes the orthogonal projection of x onto γ_δ and s is the curvilinear abscissa. We then have $\gamma_0=\Gamma$ and we choose $\Sigma=\gamma_R$ for a given $R>0$. By this way, Γ and Σ are parallel and the parameter δ measures the distance between the two surfaces. The boundary condition on Σ is an ABC which involves the Laplace-Beltrami operator Δ_Σ and the coefficient α incorporates the geometry of Γ and depends on δ. For instance, following the same ideas than in [6], we get :

$$\alpha = (\xi - \kappa_\delta/2 + ik_f)^{-1}(\xi(\kappa_\delta + ik_f) - k_f^2) \qquad (6)$$

where κ_δ denotes the curvature of Σ defined by:

$$\kappa_\delta = \frac{\kappa}{1+\delta\kappa}$$

with κ the curvature of Γ. Regarding β, we have:

$$\beta = (\xi - \kappa_\delta + ik_f)^{-1}. \qquad (7)$$

In the above definitions, ξ is a positive parameter which is determined empirically.

2 Reduced problem

The construction of a reduced formulation begins with assuming that γ and Σ are close enough to be merged. It is well-known that if p and $\partial_n p$ can be computed on Γ, it is possible to reconstruct the pressure field p in the fluid by using an integral representation. Based on this remark, we propose to replace $\partial_n p$ on Γ in the transmission condition (3) by using the ABC (5). By this way, we get the reduced problem composed of Eq.(1) combined with a modified version of (3) given by

$$\omega^2 \rho_f u \cdot n = -\alpha p + \beta \Delta_\Gamma p + \partial_n p^{inc}, \qquad (8)$$

and Eq.(4). The solution to the reduced problem is then given by the pair $(u, p_{|\Gamma})$ and the computation are performed in Ω_s only.

To compute the solution to the elasto-acoustic scattering problem, we thus propose to solve a discrete system related to Eq. (1) set in the bounded domain Ω_s combined with the boundary conditions (8) and (4) at first. The pressure field can next be reconstructed by computing $\partial_n p_{|\Gamma}$ thanks to (5) set on Γ, which enables to get the expression of p in Ω_f thanks to the Kirchhoff integral formulation. While (4) is included in the variational formulation of (1), Condition (8) is weakly taken into account by a surfacic variational formulation on Γ.

To get an approximate solution of this problem, we propose to use an Interior Penalty Discontinuous Galerkin (IPDG) method. We denote by Ω_h (resp. Γ_h) a triangulation of Ω (resp. Γ) and we consider the spaces

$$V_h = \left\{ u \in L^2(\Omega) \,|\, u_{|T} \in P^r(T),\, \forall T \in \Omega_h \right\} \text{ and}$$

$$W_h = \left\{ u \in L^2(\Gamma) \,|\, p_{|\Sigma} \in P^r(\Sigma),\, \forall \Sigma \in \Gamma_h \right\},$$

where $P^r(T)$ denotes the space of polynomials of degree r on T. We denote by $\Phi = (\phi_i)_{i=1..N_s}$ (resp. $\Psi = (\psi_i)_{i=1..N_f}$) a basis of V_h (resp. W_h). It is worth noting that N_f is very small compared to N_s since we only compute an approximation of p on the boundary Γ. The linear system to be solved reads as

$$\begin{array}{ccccc} (M_s+K_s)U & + & B_{sf}P & = & F_1, \\ B_{fs}U & + & (M_f+K_f)P & = & F_2, \end{array} \qquad (9)$$

where U and P are two vectors of size N_s and N_f containing the components of the approximation of u and p in the basis Φ and Ψ. M_s and M_f are block-diagonal mass matrices of size $N_s \times N_s$ and $N_f \times N_f$ defined by

$$(M_s)_{i,j} = \int_\Omega \phi_i \cdot \phi_j d\Omega \text{ and } (M_f)_{i,j} = \int_\Gamma \alpha\psi_i\,\psi_j d\Gamma.$$

K_s and K_f are stiffness matrices of size $N_s \times N_s$ and $N_f \times N_f$ defined by

$$(K_s)_{i,j} = a(\phi_i\,,\,\phi_j) \text{ and } (K_f)_{i,j} = b(\psi_i\,,\,\psi_j),$$

where $a(.,.)$ and $b(.,.)$ are the bilinear forms obtained by the IPDG discretization of the operators $\nabla\cdot\sigma(u)$ and $-\beta\Delta_\Gamma$. Finally, B_{sf} and B_{fs} are two coupling matrices of size $N_s \times N_f$ and $N_f \times N_s$ defined by

$$(B_{sf})_{i,j} = -\int_\Gamma \phi_i \cdot \psi_j n d\Gamma \text{ and } (B_{fs})_{i,j} = \omega^2\rho_f (B_{sf})_{j,i}.$$

We present numerical results that illustrate the performance of the reduced model. We also assess the impact of parameter δ on the accuracy of the solution.

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