Sensitivity of rough differential equations: an approach through the Omega lemma

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Abstract

The Itô map gives the solution of a Rough Differential Equation, a generalization of an Ordinary Differential Equation driven by an irregular path, when existence and uniqueness hold. By studying how a path is transformed through the vector field which is integrated, we prove that the Itô map is Hölder or Lipschitz continuous with respect to all its parameters. This result unifies and weakens the hypotheses of the regularity results already established in the literature.

Keywords: rough paths; rough differential equations; Itô map; Malliavin calculus; flow of diffeomorphisms.

1 Introduction

The theory of rough paths is now a standard tool to deal with stochastic differential equations (SDE) driven by continuous processes other than the Brownian motion such as the fractional Brownian motion. Even for standard SDE, it has been proved to be a convenient tool for dealing with large deviations or for numerical purposes. We refer the reader to [16, 22, 24, 26, 39, 40, 44, 45] for a presentation of this theory with several points of view. Here, we mainly rely on the notion of *controlled rough path* of M. Gubinelli [22, 26].

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For a Banach space U, a time horizon $T > 0$, a regularity indice $p \in [2, 3)$ and a control ω (See Section 4 for a definition), we denote by $\mathrm{C}_p(\mathrm{U})$ the space of paths from $[0, T]$ to U of finite p-variation with respect to the control ω. By this, we mean a path $x : [0, T] \to \mathrm{U}$ such that with $|x_t - x_s| \leq C\omega(s, t)^{1/p}$ for some constant C.

A *rough path* $\mathbf{x}$ is an extension in the non-commutative tensor space $\mathrm{T}_2(\mathrm{U}) := 1 \oplus \mathrm{U} \oplus (\mathrm{U} \otimes \mathrm{U})$ of a path x in $\mathrm{C}_p(\mathrm{U})$. This extension $\mathbf{x}$ is defined through algebraic and analytic properties. It is decomposed as $\mathbf{x} := 1+\mathbf{x}^1+\mathbf{x}^2$ with $\mathbf{x}^1 := x$ in U and $\mathbf{x}^2 \in \mathrm{U} \otimes \mathrm{U}$. The increments of $\mathbf{x}$ are defined by $\mathbf{x}_{s,t} := \mathbf{x}_s^{-1} \otimes \mathbf{x}_t$. It satisfies the multiplicative property $\mathbf{x}_{r,t} = \mathbf{x}_{r,s} \otimes \mathbf{x}_{s,t}$ for any $0 \leq r \leq s \leq t \leq T$. The space is equipped with the topology induced by the p-variation distance with respect to ω. The space of rough paths of finite p-variations with respect to the control ω is denoted by $\mathrm{R}_p(\mathrm{U})$. There exists a natural projection from $\mathrm{R}_p(\mathrm{U})$ onto $\mathrm{C}_p(\mathrm{U})$. Conversely, a path may be lifted from $\mathrm{C}_p(\mathrm{U})$ to $\mathrm{R}_p(\mathrm{U})$ [49], yet this cannot be done canonically.

Given a rough path $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$ and a vector field $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ for a Banach space V, a controlled differential equation

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s \tag{1}$$

is well defined provided that f is regular enough. This equation is called a *rough differential equation* (RDE). For a smooth path x, a rough path $\mathbf{x}$ could be naturally constructed using the iterated integrals of x. In this case, the solution to (1) corresponds to the solution to the ordinary differential equation $y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$. The theory of rough paths provides us with natural extension of controlled differential equations.

When (1) has a unique solution $y \in \mathrm{C}_p(\mathrm{V})$ for any $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$ given that the vector field f belongs to a proper subspace Fi, the map $\mathfrak{I} : (a, \mathbf{x}, f) \mapsto y$ from $\mathrm{U} \times \mathrm{R}_p(\mathrm{U}) \times \mathrm{Fi}$ to $\mathrm{C}_p(\mathrm{V})$ is called the *Itô map*. The Itô map is actually locally Lipschitz continuous on $\mathrm{U} \times \mathrm{R}_p(\mathrm{U}) \times \mathrm{Fi}$ when Fi is equipped with the proper topology [24, 36, 37].

Together with the Itô map $\mathfrak{I}$, we could consider for each $t \in [0, T]$ $\mathfrak{f}_t(a, \mathbf{x}, f) = \mathfrak{e}_t \circ \mathfrak{I}(a, \mathbf{x}, f)$ from $\mathrm{U} \times \mathrm{R}_p(\mathrm{U}) \times \mathrm{Fi}$ to V, where $\mathfrak{e}_t$ is the *evaluation map* $\mathfrak{e}_t(x) = x(t)$. The family $\{\mathfrak{f}_t(a, \mathbf{x}, f)\}_{t \in [0,T]}$ is the *flow* associated to the RDE (1). For ordinary differential equations, the flow defines a family of homeomorphisms or diffeomorphisms.

The differentiability properties of the Itô map or the flow are very important in view of applications. For SDE, Malliavin calculus opens the door to existence of a density and its regularity [50, 53], large deviation results [32], Monte Carlo methods [25, 50], ...

In this article, we then consider the differentiability properties, understood as Fréchet differentiability, of the Itô map under minimal regularity conditions on the vector field. Therefore, we extend the current results by proving Hölder continuity of the Itô map. This generalizes to $2 \leq p < 3$ the results of T. Lyons and X. Li [43].

Several strategies have been developed to consider the regularity of the Itô map and flows, which we review now. For this, we need to introduce some notations.

$\star$ The space of *geometric rough paths* $\mathrm{G}_p(\mathrm{U})$ is roughly described as the limit of the natural lift of smooth rough paths through their iterated integrals.

$\star$ For a path $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$ and a Banach space V, $\mathrm{P}_p(\mathbf{x}, \mathrm{V})$ is the space of *controlled rough path* (CRP). It contains paths $z : [0, T] \to \mathrm{V}$ whose increments are like those of $\mathbf{x}$, that is $z_{s,t} = z^\dagger_s x_{s,t} + z^\sharp_{s,t}$ for proper $(z^\dagger, z^\sharp)$ (for a formal definition, see Section 5). A CRP z is best identified with the pair $(z, z^\dagger)$ (See Remark 2 in Section 5).

$\star$ For a path $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$ and $h \in \mathrm{C}_q(\mathrm{U})$ with $1/p+1/q > 1$ (which means that $1 \leq q < 2$), there exists a natural way to construct a rough path $\mathbf{x}(h) \in \mathrm{R}_p(\Omega)$ such that $\pi(\mathbf{x}(h)) = \pi(\mathbf{x}) + h$, where π is the natural projection from $\mathrm{T}_2(\mathrm{U})$ onto U, and $\mathbf{x}(0) = \mathbf{x}$. The map $h \mapsto \mathbf{x}(h)$ is $\mathscr{C}^\infty$-Fréchet from $\mathrm{C}_q(\mathrm{U})$ to $\mathrm{R}_p(\mathrm{U})$.

$\star$ A vector field $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ is said to be γ-*Lipschitz* ($\gamma > 0$) if it is differentiable up to order $\lfloor\gamma\rfloor$ with a derivative of order $\lfloor\gamma\rfloor$ which is $(\gamma - \lfloor\gamma\rfloor)$-Hölder continuous (See [24], p. 213).

The flow and differentiability properties have already been dealt with in the following articles:

$\star$ In [47], T. Lyons and Z. Qian have studied the flow property for solutions to $y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s + \int_0^t g(y_s)\,\mathrm{d}h_s$ for a "regular" path h subject to a perturbation for $\mathrm{V} = \mathbb{R}^d$.

$\star$ In [48], T. Lyons and Z. Qian showed that the Itô map provides a flow of diffeomorphisms when the driving rough path is geometric for $\mathrm{V} = \mathbb{R}^d$.

$\star$ T. Lyons and Z. Qian [46] and more recently Z. Qian and J. Tudor [55] have studied the perturbation of the Itô map when the rough path is perturbed by a regular path h and the structure of the tangent spaces for finite dimensional Banach space. As these constructions hold in tensor spaces, quadratic terms are involved. They lead to rather intricate expressions.

$\star$ The properties of flow also arise directly from the constructions from almost flows, in the approach from I. Bailleul [3] in possibly infinite dimensional Banach spaces. Firstly, only Lipschitz flow were considered, but recently in [5], it was extended to Hölder continuous flows.

$\star$ In the case $1 \leq p < 2$, T. Lyons and Z. Li proved in [43] (see also [38]) that

$$x \in \mathrm{C}_p(\mathrm{U}) \mapsto \mathfrak{I}(a, x, f) \in \mathrm{C}_p(\mathrm{V}) \text{ is locally } \mathscr{C}^k\text{-Fréchet differentiable}$$

provided that U and V are finite dimensional Banach spaces and f is $\mathscr{C}^{k+\alpha+\epsilon}$, $k \geq 1$, $\alpha \in (p-1, 1-\epsilon)$, $\epsilon \in (0,1)$.

$\star$ In the book of P. Friz and N. Victoir [24], it is proved that

$$(b, h) \in \mathrm{U} \times \mathrm{C}_q(\mathrm{U}) \mapsto \mathfrak{I}(a + b, \mathbf{x}(h), f) \in \mathrm{C}_p(\mathrm{U})$$
$$\text{is locally } \mathscr{C}^k\text{-Fréchet at } (a, \mathbf{x}) \in \mathrm{U} \times \mathrm{G}_p(\mathrm{U})$$

provided that $\mathbf{x} \in \mathrm{G}_p(\mathrm{U})$, U is finite-dimensional and f is of class $\mathscr{C}^{k-1+\gamma}(\mathrm{U}, \mathrm{V})$ with $\gamma > p$ and $k \geq 1$ (See [24, Theorem 11.6, p. 287]). It is also proved that $\mathfrak{f} : [0, T] \times \mathrm{U} \to \mathrm{V}$ is a flow of $\mathscr{C}^k$-diffeomorphisms, and that $\mathfrak{f}$ and its derivatives are uniformly continuous with respect to $\mathbf{x} \in \mathrm{G}_p(\mathrm{U})$ (See [24, Section 11.2, p. 289]). Transposed in the context of CRP in [22], the flow $a \mapsto \mathfrak{f}_t(a, \mathbf{x}, f)$ is locally a diffeomorphism of class $\mathscr{C}^{k+1}$ for a vector field f is $\mathscr{C}^{k+3}$-Fréchet.

$\star$ In a series of articles [32–35] (See also [20]), Y. Inahama and H. Kawabi have studied various aspects of stochastic Taylor developments in ϵ of solutions to

$$y_t^\epsilon = a + \int_0^t f(y_s^\epsilon)\, \mathrm{d}\mathfrak{d}_\epsilon \mathbf{x}_s + \int_0^t g(y_s^\epsilon)\, \mathrm{d}h_s \qquad (2)$$

around the solutions to $z_t = a + \int_0^t g(z_s)\, \mathrm{d}h_s$, when $\mathbf{x} \in \mathrm{R}_p(\mathrm{W})$, $h \in \mathrm{C}_q(\mathrm{U})$ with $1/p + 1/q > 1$, $f : \mathrm{V} \to L(\mathrm{W}, \mathrm{V})$, $g : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ and $\mathfrak{d}_\epsilon : \mathrm{R}_p(\mathrm{W}) \to \mathrm{R}_p(\mathrm{W})$ is the dilatation operator defined by $\mathfrak{d}_\epsilon \mathbf{x} := 1 + \epsilon \mathbf{x}^1 + \epsilon^2 \mathbf{x}^2$. Up to the natural injection of $\mathbf{x}$ to $\mathrm{R}_p(\mathrm{W} \oplus \mathrm{U})$, h to $\mathrm{C}_p(\mathrm{W} \oplus \mathrm{U})$, $f : \mathrm{V} \to L(\mathrm{W} \oplus \mathrm{U}, \mathrm{V})$ and $g : \mathrm{V} \to L(\mathrm{W} \oplus \mathrm{U}, \mathrm{V})$, (2) is recast as

$$y^\epsilon = \mathfrak{I}(a, \mathfrak{d}_\epsilon \mathbf{x}(h), f + g).$$

$\star$ Using a Banach space version of the Implicit Functions Theorem, I. Bailleul recently proved in [4] that

$$(z, f) \in \mathrm{P}_p(\mathbf{b}, \mathrm{U}) \times \mathscr{C}^k(\mathrm{U}, \mathrm{V}) \mapsto \mathfrak{I}(a, \mathfrak{P}_{\mathbf{b}}(z), f) \in \mathrm{P}_p(\mathbf{b}, \mathrm{V})$$
$$\text{is } \mathscr{C}^{\lfloor k \rfloor - 2}\text{-Fréchet differentiable}$$

provided that $\mathbf{b} \in \mathrm{G}_p(\mathrm{B})$ and $k \geq 3$, where $\mathfrak{P}_{\mathbf{b}}(z)$ lifts $z \in \mathrm{P}_p(\mathbf{b}, \mathrm{U})$ to a geometric rough path.

$\star$ SDE driven by fractional Brownian motion and Gaussian processes attracted a lot of attention [6, 7, 9–13, 15, 18, 19, 28–30, 42, 51, 54]. Since integrability is the key to derive some integration by parts formula in Malliavin type calculus, several articles deal with moments estimates for solutions to linear equations driven by Gaussian rough paths [8, 14, 23, 27, 31, 56].

In this article,

$\star$ We establish that the Itô map $\mathfrak{I}$ is locally of class $\mathscr{C}^{\gamma-\epsilon}$ for all the types of perturbations of the driving rough path $\mathbf{x}$ seen above. We then generalized the results in [4], [22, Sect. 8.4] or [24] by providing the Hölder regularity, and not only differentiability. At the exception of the work of T. Lyons and X.D. Li [43] for the Young case ($1 \leq p < 2$), at the best of our knowledge, none of the works cited above deal with the Hölder regularity of the derivatives of the Itô map. The cited results are proved under stronger regularity conditions than ours on the vector field. In [4], the starting point is kept fixed. While in [22, 24], only the regularity with respect to the starting point and perturbation of the form $\mathbf{x}(h)$ of the driver are considered, as in [43]. Unlike [24], the chain rule may be applied as our solutions are constructed as CRP.

$\star$ By adding more flexibility in the notion of CRP, we define a bilinear continuous integral with CRP both as integrand and integrators. This simple trick allows one to focus on the effect of a non-linear function applied to a CRP and weaken the regularity assumptions imposed on the vector fields in [4, 22].

$\star$ Following the approach of [1], we provide a version of the so-called *Omega lemma* for paths of finite p-variations with $1 \leq p < 2$ and for CRP (for $2 \leq p < 3$). For a function f of given regularity, this lemma states the regularity of the map $\mathfrak{O}f : y \mapsto \{f(y_t)\}_{t\in[0,T]}$ when y is a path of finite p-variation or a CRP on $[0, T]$. When dealing with continuous paths with the sup-norms, where $\mathfrak{O}f$ has the regularity of f (for functions of class $\mathscr{C}^k$, but also Hölder or Sobolev). When dealing with paths of finite p-variation, the situation is more cumbersome. This explains the losses in the regularity observed in [43].

$\star$ We provide a "genuine rough path" approach which removes the restriction implied by smooth rough paths, the restriction to geometric rough paths (which could be dealt otherwise with $(p, p/2)$-rough paths [41]) as well as any restriction on the dimensions of the Banach spaces U and V. Although we use a CRP, the Duhamel formula shown in [17] could serve to prove a similar result for (partial) rough paths and not only CRP.

$\star$ We exemplify the difference between the "rough situation" and the smooth one. For ordinary differential equation, the spirit of the Omega lemma together with the Implicit Functions Theorem is that the regularity of the vector field is transported into the regularity of the Itô map. Once the Omega lemma is stated for our spaces of paths, its implication on the regularity of the Itô map is immediate.

2 Notations

Througout all the article, we denote by U, V and W Banach spaces.

For two such Banach spaces U and V, we denote by $L(\mathrm{U},\mathrm{V})$ the set of linear *continuous* maps from U to V. If $A \in L(\mathrm{U},\mathrm{V})$ is invertible, then its inverse A^{-1} is itself continuous and thus belongs to $L(\mathrm{V},\mathrm{U})$.

For a functional $\mathfrak{F}$ from U to V, we denote by $\mathrm{D}\mathfrak{F}$ its Fréchet derivative, which is a map from V to $L(\mathrm{V},\mathrm{U})$, and by $\mathrm{D}_y\mathfrak{F}$ its Fréchet derivative in the direction of a variable y in U.

For a function f and some $\alpha \in (0,1]$, we denote by $H_\alpha(f)$ its *α-Hölder semi-norm* $H_\alpha(f) := \sup_{x\neq y}|f(x)-f(y)|/|x-y|^\alpha$.

For any $\alpha > 0$, we denote by $\mathscr{C}^\alpha(U,\mathrm{V})$ the set of continuous, bounded functions from U to V with, bounded continuous (Fréchet) derivatives up to order $k := \lfloor \alpha \rfloor$, and its derivative of order k is $(\alpha-k)$-Hölder continuous. A function in $\mathscr{C}^\alpha(U,\mathrm{V})$ is simply said to be of class $\mathscr{C}^\alpha$ (in this article, we restrict ourselves to non-integer values of α).

With the convention that $\mathrm{D}^0 f := f$, we write for f in $\mathscr{C}^\alpha(U,\mathrm{V})$,

$$\|f\|_\alpha := \max_{j=0,\dots,k}\left\{\|\mathrm{D}^j f\|_\infty, H_{\alpha-k}(\mathrm{D}^k f)\right\}.$$

Our result are stated for bounded functions f. For proving existence and uniqueness, this boundedness condition may be relaxed by keeping only the boundedness of the derivatives of f (See *e.g.*, [24,36,37], ...). Once existence is proved under this linear growth condition, there is no problem in assuming that f itself is bounded since we only use estimates locally. This justifies our choice for the sake of simplicity.

3 The Implicit Functions Theorem

Let us consider two Banach spaces P and Λ as well as a functional $\mathfrak{F}$ from $\mathrm{P}\times\Lambda$ to P. Here, P plays the role of the spaces of paths, while Λ is the space of parameters of the equation.

We consider first solutions y to the fixed point problem

$$y = \mathfrak{F}(y,\lambda) + b,\ \ (\lambda, b) \in \Lambda \times \mathrm{P}. \tag{3}$$

This is an abstract way to consider equations of type $y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s + b_t$, whose parameters are $\lambda = (a, f, \mathbf{x}) \in \Lambda$ and $b \in \mathrm{P}$.

To ensure uniqueness of the solutions to (3), we slightly change the problem. We assume that P contains a Banach sub-space $\mathrm{P}^\star$, typically, the paths that start from 0. As $\mathrm{P}^\star$ is stable under addition, we consider the quotient space $\mathrm{P}^{\sim} := \mathrm{P}/\mathrm{P}^{\star}$ defined for the equivalence relation $x \sim y$ when $x - y \in \mathrm{P}^{\star}$. This quotient is nothing more than a way to encode the starting point.

We now consider instead of (3) the problem

$$y^* = \mathfrak{F}(y^* + z, \lambda) - z + b,\ b \in \mathrm{P}^{\star},\ z \in \mathrm{P}^{\sim},\ \lambda \in \Lambda. \tag{4}$$

There is clearly no problem in restricting b to $\mathrm{P}^{\star}$, since otherwise one has to change λ and z accordingly. Solving (4) implies that (3) is solved for $y = y^{\star} + z$.

For an integer $k \geq 0$ and $0 < \alpha \leq 1$, if $\mathfrak{G}(y^*, z, \lambda) := \mathfrak{F}(y^* + z, \lambda)$ and $\mathfrak{F}$ is of class $\mathscr{C}^{k+\alpha}$ with respect to (y, λ), then $\mathfrak{G}$ is of class $\mathscr{C}^{k+\alpha}$ with respect to (y^*, z, λ)

The reason for considering (4) instead of (3) is that for the cases we consider, $\mathfrak{G}$ will be strictly contractive in y^*, ensuring the existence of the unique solution to (4).

We use the following version of the Implicit Functions Theorem (See *e.g.* [1, § 2.5.7, p. 121] for a $\mathscr{C}^k$-version of the Implicit Functions Theorem[1].

Theorem 1 (Implicit Functions Theorem)**.** *Let us assume that*

i) The map $\mathfrak{G}(y^*, z, \lambda) := \mathfrak{F}(y^* + z, \lambda)$ *is of class* $\mathscr{C}^{k+\alpha}$ *from* $\mathrm{X} := \mathrm{P}^{\star} \times \mathrm{P}^{\sim} \times \Lambda$ *to* $\mathrm{P}^{\star}$ *for* $k \geq 1$, $0 < \alpha \leq 1$ *with respect to* $(y^*, z, \lambda) \in \mathrm{X}$.

ii) For some $(\widehat{y}^*, \widehat{z}, \widehat{\lambda}) \in \mathrm{X}$ *and any* $b^* \in \mathrm{P}^{\star}$*, the exists a unique solution* h^* *in* $\mathrm{P}^{\star}$ *to*

$$h^* = \mathrm{D}_{y^*}\mathfrak{G}(\widehat{y}^*, \widehat{z}, \widehat{\lambda})(h^*) + b^*$$

with $\|h^*\|_{\mathrm{P}} \leq C\|b^*\|_{\mathrm{P}}$ *for some constant* $C \geq 0$*. This means that* $\mathsf{Id} - \mathrm{D}_{\widehat{y}}\mathfrak{G}(\cdot, \widehat{z}, \widehat{\lambda})$ *is invertible from* $\mathrm{P}^{\star}$ *to* $\mathrm{P}^{\star}$ *with a bounded inverse.*

Then there exists a neighborhood U *of* $(\widehat{z}, \widehat{\lambda}) \in \mathrm{P}^{\sim} \times \Lambda$*, a neighborhood* V *of* $\mathfrak{G}(\widehat{y}^*, \widehat{z}, \widehat{\lambda})$*, as well as a unique map* $\mathfrak{H}$ *from* $V \times U$ *to* $\mathrm{P}^{\star}$ *which solves*

$$\mathfrak{H}(b, z, \lambda) = \mathfrak{G}(\mathfrak{H}(b, z, \lambda), z, \lambda) + b,\ \forall (b, z, \lambda) \in V \times U.$$

In other words, $\mathfrak{I}(b, z, \lambda) := z + \mathfrak{H}(b, z, \lambda)$ *is locally the solution in* $z + \mathrm{P}^{\star}$ *to* $\mathfrak{I}(b, z, \lambda) = -z + \mathfrak{F}(\mathfrak{I}(b, z, \lambda), \lambda) + b$.

[1]To extend it to $\mathscr{C}^{k+\alpha}$-Hölder continuous functions, we have just to note that the derivative of $(b, z, \lambda) \mapsto (\mathfrak{H}(b, z, \lambda), z, \lambda)$ below, which is given by the inverse function theorem, is the composition of $z \mapsto z^{-1}$, which is $\mathscr{C}^{\infty}$, $\mathfrak{G}$, which is $\mathscr{C}^{k-1+\alpha}$ and $\mathfrak{H}$ which is $\mathscr{C}^k$, see [52]).

Remark 1. Actually, we do not use this theorem in this form. We show that $\mathrm{D}_y\mathfrak{G}(\cdot, z, \lambda)$ is contractive when restricted to a bounded, closed, convex set C of $\mathrm{P}^\star \times \mathrm{P}^\sim \times \Lambda$, and only on a time interval τ which is small enough, in function of the radius of C. The controls we get allow us to solve iteratively the equations on abutting time intervals τ_i and to "stack them up" to get the result on any finite time interval (and even globally for suitable vector fields). As for this, we have only to re-use with slight adaptations what is already largely been done, we do not treat these issues.

4 The Omega lemma for paths of finite p-variation

We consider a time horizon $T > 0$. A *control* ω is a non-negative function defined on sub-intervals $[s,t] \subset [0,T]$ which is super-additive and continuous close to the diagonal $\{(t,t) \mid t \in [0,T]\}$. This means that is $\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t}$ for $0 \leq r \leq s \leq t \leq T$.

For a path x from $[0,T]$ to V, we set $x_{s,t} := x_{[s,t]} := x_t - x_s$. For some $p \geq 1$, a path x of finite p-variation controlled by ω satisfies

$$\|x\|_p := \sup_{\substack{[s,t]\subset[0,T]\\ s\neq t}} \frac{|x_t - x_s|}{\omega_{s,t}^{1/p}}.$$

We denote by $\mathrm{C}_p(\mathrm{V})$ the space of such paths, which is a Banach space with the norm

$$\|x\|_{\bullet p} := |x_0| + \|x\|_p.$$

The space $\mathrm{C}_p(\mathrm{V})$ is continuously embedded in the space of continuous functions $\mathrm{C}(\mathrm{V})$ with the sup-norm $\|\cdot\|_\infty$, with

$$\|x\|_\infty \leq |x_0| + \|x\|_p \omega_{0,T}^{1/p}. \tag{5}$$

For any $q \geq p$, $\mathrm{C}_p(\mathrm{V})$ is also continuously embedded in $\mathrm{C}_q(\mathrm{V})$.

We call a *universal constant* a constant that depends only on $\omega_{0,T}$ and the parameters p, q, κ, γ, ... that will appear later.

Proposition 1 (L.C. Young [57]). *Let $p, q \geq 1$ such that $1/p + 1/q > 1$. There exists a unique continuous, bilinear map*

$$\begin{aligned} \mathrm{C}_p(\mathrm{U}) \times \mathrm{C}_q(L(\mathrm{V},\mathrm{U})) &\to \mathrm{C}_p(\mathrm{V}) \\ (x,y) &\mapsto \int_0^{\cdot} y \,\mathrm{d}x \end{aligned}$$

which satisfies $\int_0^0 y_r \,\mathrm{d}x_r = 0$ *for any* (x, y) *and any* $[s, t] \subset [0, T]$,

$$\left| \int_s^t y_r \,\mathrm{d}x_r - y_s x_{s,t} \right| \leq K \|y\|_q \|x\|_p \omega_{s,t}^{\frac{1}{p}+\frac{1}{q}} \tag{6}$$

for some universal constant K.

For some $\kappa \in [0, 1]$, we set $\overline{\kappa} := 1 - \kappa$.

Lemma 1. *Let* $g \in \mathscr{C}^\gamma(\mathrm{V}, \mathrm{W})$. *Then for any* $\kappa \in [0, 1]$ *and* $\gamma \in [0, 1]$,

$$\begin{aligned} &|g(z) - g(y) - g(z') + g(y')| \\ &\qquad \leq H_\gamma(g)(|y' - y|^{\kappa\gamma} + |z' - z|^{\kappa\gamma})(|z' - y'|^{\gamma\overline{\kappa}} + |z - y|^{\gamma\overline{\kappa}}) \end{aligned} \tag{7}$$

for all $y, z, y', z' \in \mathrm{V}$.

Proof. First,

$$|g(z) - g(y) - g(z') + g(y')| \leq H_\gamma(g)(|y' - y|^\gamma + |z' - z|^\gamma).$$

By inverting the roles of z' and y in the above equation, we get a similar inequality. Choosing $\kappa \in [0, 1]$ and raising the first inequality to power κ and the second one to power $\overline{\kappa}$ leads to the result. □

We now fix $p \geq 1$, $\kappa \in (0, 1)$ and $\gamma \in (0, 1]$. We set $q := p/\kappa\gamma$. We define

$$\mathrm{C}_p^\star(\mathrm{V}) := \{y \in \mathrm{C}_p(\mathrm{V}) \mid y_0 = 0\} \text{ and } \mathrm{C}_q^\star(\mathrm{V}) := \{y \in \mathrm{C}_q(\mathrm{V}) \mid y_0 = 0\}.$$

An immediate consequence of this lemma is that for $\kappa \in (0, 1)$, $\gamma \in (0, 1]$, the map $\mathfrak{O}g : y \mapsto \{g(y_t)\}_{t\in[0,T]}$ is $\gamma\overline{\kappa}$-Hölder continuous from $\mathrm{C}_p^\star(\mathrm{V})$ to $\mathrm{C}_{p/\kappa\gamma}^\star$ with Hölder constant $2^\gamma H_\gamma(g)\omega_{0,T}^{\gamma\overline{\kappa}/p}$ when g is γ-Hölder continuous.

We now consider the case of higher differentiability of g.

We now give an alternative proof of the one in [43, Theorems 2.13 and 2.15], which mostly differs in the use of the converse of Taylor's theorem. For CRP in Section 5, the proof of Proposition 5 will be modelled on this one.

We still use the terminology of [1] regarding the Omega lemma. The Omega operator $\mathfrak{O}$ transforms a function f between two Banach spaces U and V to a function mapping continuous paths from $[0, T]$ to U to continuous paths from $[0, T]$ to V. The idea is then to study the regularity of $\mathfrak{O}f$ in function of the regularity of f and the one of the paths that are carried by $\mathfrak{O}f$. The difference with the results in [1] is that we use the p-variation norm instead of the sup-norm. This leads to a slight loss of regularity when transforming f to $\mathfrak{O}f$.

Proposition 2 (The Omega lemma for paths of finite p-variation)**.** *For $p \geq 1$, $k \geq 1$, $\gamma \in (0,1]$, $\kappa \in (0,1)$ and f of class $\mathscr{C}^{k+\gamma}$ from V to $\mathrm{W} := L(\mathrm{U},\mathrm{V})$, $\mathfrak{O}f(y) := (f(y_t))_{t\in[0,T]}$ is of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ from any ball of radius $\rho > 0$ of $\mathrm{C}_p(\mathrm{V})$ to $\mathrm{C}_q(\mathrm{W})$ with $q := p/\kappa\gamma$. Besides, $\mathrm{D}_y\mathfrak{O}f \cdot h = (\mathrm{D}f(y_t)\cdot h_t)_{t\in[0,T]} \in \mathrm{C}_q(\mathrm{W})$ for any $y, h \in \mathrm{C}_p(\mathrm{V})$. Finally, $\mathrm{D}\mathfrak{O}(y)\cdot h \in \mathrm{C}^\star_q(\mathrm{W})$ when $h \in \mathrm{C}^\star_p(\mathrm{V})$.*

Proof. I) Assume that $f \in \mathscr{C}^\gamma$. Thanks to the embedding from $\mathrm{C}_{p/\gamma}(L(\mathrm{U},\mathrm{V}))$ to $\mathrm{C}_q(L(\mathrm{U},\mathrm{V}))$, $\mathfrak{O}f$ maps $\mathrm{C}^\star_p(\mathrm{V})$ to $\mathrm{C}_q(L(\mathrm{U},\mathrm{V}))$ with $\|\mathfrak{O}f(y)\|_q \leq CH_\gamma(f)\|y\|_p$ for all $y \in \mathrm{C}^\star_p(\mathrm{V})$.

With Lemma 1, we easily obtain that

$$\|\mathfrak{O}f(y) - \mathfrak{O}f(z)\|_{\bullet q} \leq H_\gamma(f)(1+\omega_{0,T}^{\overline{\kappa}\gamma/p})\|y-z\|_{\bullet p}^{\overline{\kappa}\gamma}(\|y\|_p^{\kappa\gamma} + \|z\|_p^{\kappa\gamma}).$$

Then $\mathfrak{O}f$ is locally $\overline{\kappa}\gamma$-Hölder continuous from $\mathrm{C}_p(\mathrm{V})$ to $\mathrm{C}_q(\mathrm{W})$.

II) For some Banach space W', if $y \in \mathrm{C}_q(L(\mathrm{V}\otimes\mathrm{W}',\mathrm{V}))$ and $z \in \mathrm{C}_p(\mathrm{V})$ (resp. $y \in \mathrm{C}_q(\mathrm{V})$, $z \in \mathrm{C}_p(\mathrm{U})$), it is straightforward to show with (5) that $yz \in \mathrm{C}_q(L(\mathrm{W}',\mathrm{V}))$ (resp. $y \otimes z \in \mathrm{C}_q(\mathrm{V}\otimes\mathrm{U})$) with

$$\|y\cdot z\|_q \leq (1+2\omega_{0,T}^{1/q})\|y\|_{\bullet q}\|z\|_{\bullet q} \text{ and } \|y\cdot z\|_{\bullet q} \leq |y_0|\cdot|z_0| + \|y\cdot z\|_q,$$

where $y\cdot z = (y_t z_t)_{t\in[0,t]}$ (resp. $y \cdot z = (y_t \otimes z_t)_{t\in[0,T]}$) Besides, if $z_0 = 0$, then $(yz)_0 = 0$ (resp. $(y\otimes z)_0 = 0$).

III) Let us assume now that $f \in \mathscr{C}^{k+\gamma}$ for some $k \geq 1$. With the Taylor development of f up to order k,

$$f(y+z) = f(y) + \sum_{i=1}^k \frac{1}{i!}\mathrm{D}^i f(y) z^{\otimes k} + R(y,z)$$

with

$$R(y,z) = \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}(\mathrm{D}^k f(y+sz) - \mathrm{D}^k f(y))z^{\otimes k}\,\mathrm{d}s.$$

Since $\mathrm{D}^i f$ is of class $\mathscr{C}^{k-i+\gamma}$ from V to $L(\mathrm{V}^{\otimes i}, L(\mathrm{U},\mathrm{V}))$ identified with $L(\mathrm{V}^{\otimes i}\otimes \mathrm{U},\mathrm{V})$, then $y \in \mathrm{C}_p(\mathrm{V}) \mapsto (\mathrm{D}^i f(y_t))_{t\in[0,T]}$ takes its values in $\mathrm{C}_q(L(\mathrm{V}^{\otimes i}\otimes\mathrm{U},\mathrm{V}))$.

For $y, z^{(1)}, \dots, z^{(i)} \in \mathrm{C}_p(\mathrm{V})$, write

$$\phi_i(y)\cdot z^{(1)} \otimes \dots \otimes z^{(i)} := (\mathrm{D}^i f(y_t)\cdot z^{(1)} \otimes \dots \otimes z^{(i)})_{t\in[0,T]}.$$

Since $\mathrm{C}_p(\mathrm{V})$ is continuously embedded in $\mathrm{C}_q(\mathrm{V})$, II) implies that $\phi_i(y)$ is multi-linear and continuous from $\mathrm{C}_p(\mathrm{V})^{\otimes i}$ to $\mathrm{C}_q(\mathrm{W})$.

Similarly, for some constant C that depends only on ρ (the radius of the ball such that $\|y\|_{\bullet p} \leq \rho$), p, q and $\omega_{0,T}$,

$$\frac{\|R(y,z)\|_{\bullet q}}{\|z\|_{\bullet p}^k} \leq CH_\gamma(\mathrm{D}^k f)\|z\|_{\bullet p}^{\overline{\kappa}\gamma}.$$

The converse of the Taylor's theorem [2] implies that $\mathfrak{O}f$ is locally of class $\mathscr{C}^k$ from $\mathrm{C}_p(\mathrm{V})$ to $\mathrm{C}_q(\mathrm{W})$. In addition, it is easily shown from I) that since $\mathrm{D}^k f$ is locally of class $\mathscr{C}^\gamma$, $\mathfrak{O}f$ is locally of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ from $\mathrm{C}_p(\mathrm{V})$ to $\mathrm{C}_q(\mathrm{W})$. $\square$

Given $0 < \kappa < 1$ with $1 + \kappa\gamma > p$, we then define

$$\mathfrak{F}(y, x, f) := \int \mathfrak{O}f(y)\,\mathrm{d}x \text{ for } (y, x, f) \in \mathrm{C}_p(\mathrm{V}) \times \mathrm{C}_p(\mathrm{U}) \times \mathscr{C}^{k+\gamma}(\mathrm{V}, \mathrm{W}),$$

as the integral is well defined as a Young integral using our constraint on γ, κ and p. The map $\mathfrak{F}$ is linear and continuous with respect to (z, f). It is of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ with respect to y. When $1+\gamma > p$, it is well known that $y = \mathfrak{F}(y, z, f)$ has a unique solution (see *e.g.*, [22, 24, 44]). Besides, it is evident that for any $a \in \mathrm{V}$,

$$y_t = b_t + \int_0^t f(y_s)\,\mathrm{d}x_s,\ t \in [0, T]$$
$$\text{if and only if } y^\star_t = b^*_t + \int_0^t f(a + y^\star_s)\,\mathrm{d}x_s,\ t \in [0, T]$$

when $y = a + y^\star$, $b = a + b^\star$, $b^\star, y^\star \in \mathrm{C}^\star_p(\mathrm{V})$, so that $\mathrm{C}^\sim_p(\mathrm{V}) = \mathrm{C}_p(\mathrm{V})/\mathrm{C}^\star_p(\mathrm{V})$ is identified V and $a = y_0$.

The Fréchet derivatives of $\mathfrak{F}$ is the direction of the variable y is

$$\mathrm{D}_y\mathfrak{F}(y, z, f) \cdot h = \int (\mathfrak{O}\mathrm{D}f(y) \cdot h)\,\mathrm{d}z$$

which is also well defined as a Young integral.

From II), if $h \in \mathrm{C}^\star_p(\mathrm{V})$, then $\mathrm{D}f(y) \cdot h$ takes its values in $\mathrm{C}^\star_p(\mathrm{V})$. With (6), (5) and I), for T is small enough (depending only on $\|f\|_{1+\gamma}$ and $\|x\|_p$), $h \mapsto \int \mathrm{D}f(y) \cdot h\,\mathrm{d}x$ is strictly contractive on $\mathrm{C}^\star_p(\mathrm{V})$. This proves that $\mathfrak{G}(y^\star, a, x, f) := \mathfrak{F}(a + y^\star, x, f)$ satisfies the conditions of application of the Implicit Function Theorem 1 (See Remark 1). This is illustrated by Figure 1.

We then recover and extend the result in [43].

Theorem 2. *Fix $p \in [1, 2)$, $x \in \mathrm{C}_p(\mathrm{U})$, $a \in \mathrm{V}$ and $f \in \mathscr{C}^{k+\gamma}(\mathrm{V}, L(\mathrm{U}, \mathrm{V}))$, $\kappa \in (0, 1)$, $\gamma \in (0, 1]$, $k \geq 1$, provided that $1 + \kappa\gamma > p$, there exists a unique solution $\mathfrak{I}(a, f, x, b)$ in $\mathrm{C}_p(\mathrm{V})$ to*

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s + b_{0,t},\ t \in [0, T]. \tag{8}$$

Besides, $\mathfrak{I}$ is locally of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ from $\mathrm{V} \times \mathscr{C}^{k+\gamma}(\mathrm{V}, L(\mathrm{U}, \mathrm{V})) \times \mathrm{C}_p(\mathrm{U}) \times \mathrm{C}_p(\mathrm{V})$ to $\mathrm{C}_p(\mathrm{V})$.

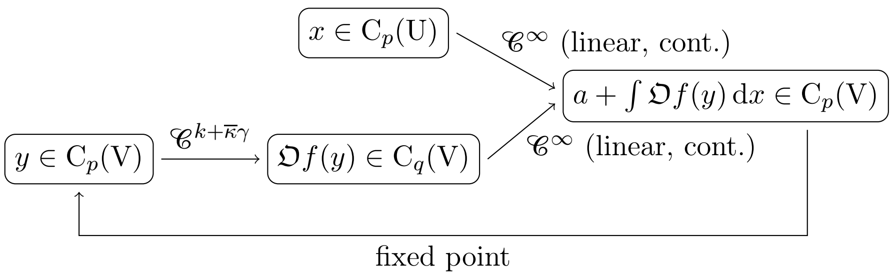

Figure 1: Schematic representation of the use of the Omega lemma.

For some $t \in [0, T]$, let $\mathfrak{e}_t : \mathrm{C}_p(\mathrm{V}) \to \mathrm{V}$ be the evaluation map $\mathfrak{e}_t(y) = y_t$. Thanks to (5), $\mathfrak{e}_t$ is continuous. We then define $\mathfrak{f}_t(a) := \mathfrak{e}_t \circ \mathfrak{I}(a, f, 0)$ for some vector field f.

Corollary 1. *Under the conditions of Theorem 2, $\mathfrak{f}_t : \mathrm{V} \to \mathrm{V}$ is locally a diffeomorphism of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ for any $t \in [0, T]$.*

Proof. Let us consider the solution to $y_{t,r}(a) = a + \int_r^t f(y_{s,r}(a))\,\mathrm{d}x_s$ for $0 \leq r \leq t$. Using the chain rule by combining the Omega lemma (Proposition 2) with the bilinearity of the Young integral (Proposition 1),

$$\begin{aligned} \mathrm{D}_a y_{t,r}(a) &= \mathsf{Id} + \int_r^t \mathrm{D}f(y_{s,r}(a))\mathrm{D}_a y_{s,r}(a)\,\mathrm{d}x_s \\ &= \mathsf{Id} + \int_r^t \mathrm{D}f(y_{s,r}(a))\mathfrak{e}_s \circ \mathrm{D}_a \mathfrak{I}(a, f, 0)\,\mathrm{d}x_s. \end{aligned} \tag{9}$$

Owing to the controls given above — namely (6), the boundedness of $\mathrm{D}f$ and Theorem 2 — the right-hand side of (9) is bounded when a belongs to a set $\{|a| \leq R\}$. Besides, due to (5), it is easily seen that for $t - r$ small enough (in function of f, γ, κ, $\omega_{r,t}$, p and R such that $|a| \leq R$), one may choose a constant $0 < \ell < 1$ such that

$$|\mathrm{D}_a y_{t,r}(a) - \mathsf{Id}| \leq \ell,$$

where $|\cdot|$ is the operator norm of $L(\mathrm{V}, \mathrm{V})$. It follows that $\mathrm{D}_a y_{t,r}$ is invertible at the point a. The Inverse Mapping Theorem (see *e.g.*, [1, Theorem 2.5.2, p. 116]) and Remark 1 assert that $y_{t,r}$ is then locally a $\mathscr{C}^{k+\overline{\kappa}\gamma}$-diffeomorphism around any point $a \in \mathrm{V}$.

Using the additive property of the integral and the uniqueness of the solution to (8), $y_{t,r}(a) = y_{t,s}(y_{s,r}(a))$. Hence, the regularity of $a \mapsto y_{t,r}(a)$ for a arbitrary times r, t is treated using the stability of $\mathscr{C}^{k+\overline{\kappa}\gamma}$ under composition and the flow property.

To conclude, it remains to remark that $\mathfrak{f}_t(a) = y_{t,0}(a)$. $\square$

5 The Omega lemma for controlled rough paths

Let us fix p, q, r with $2 \leq p < 3$, $q > 0$ and $0 < r < p$. We consider a rough path $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$ (see the Introduction for a definition).

A *Controlled Rough Path* (CRP) is a path $y : [0, T] \to \mathrm{V}$ which admits the following decomposition for any s, t:

$$y_{s,t} = y_s^\dagger \mathbf{x}_{s,t}^1 + y_{s,t}^\sharp \text{ with } y^\dagger \in \mathrm{C}_q(L(\mathrm{U}, \mathrm{V})) \text{ and } \|y^\sharp\|_r < +\infty. \tag{10}$$

Notation 1. A CRP y is identified as the pair of paths $(y, y^\dagger)$, as $y^\sharp$ may be computed from y and $y^\dagger$.

We write

$$\|y\|_{\mathbf{x}} := \|y^\dagger\|_q + \|y^\sharp\|_r \text{ and } \|y\|_{\bullet\mathbf{x}} := |y_0| + |y_0^\dagger| + \|y\|_{\mathbf{x}}.$$

The space of CRP is denoted by $\mathrm{P}_{p,q,r}(\mathbf{x}, \mathrm{V})$. This is a Banach space with the norm $\|\cdot\|_{\bullet\mathbf{x}}$.

Useful inequalities are

$$\|y^\dagger\|_\infty \leq (1 + \omega_{0,T}^{1/q}) \|y\|_{\bullet\mathbf{x}}, \tag{11}$$

$$\begin{aligned} \|y\|_p &\leq \|y^\dagger\|_\infty \|\mathbf{x}\|_p + \|y^\sharp\|_r \omega_{0,T}^{1/r - 1/p} \\ &\leq \|y\|_{\bullet\mathbf{x}} (1 + \|\mathbf{x}\|_p)(1 + \omega_{0,T}^{1/q} + \omega_{0,T}^{1/r - 1/q}) \text{ when } r \leq p \end{aligned} \tag{12}$$

$$\text{and } \|y\|_\infty \leq (1 + \omega_{0,T}^{1/q} + \omega_{0,T}^{1/r - 1/p})(1 + \omega_{0,T}^{1/p}) \|y\|_{\bullet\mathbf{x}} (1 + \|\mathbf{x}\|_p). \tag{13}$$

Remark 2. This definition, which involves three indices, is more general than the one in [22], in which $(p, q, r) = (p, p, p/2)$, and the seminal article [26], in which $(p, q, r) = (p, q, (p^{-1} + q^{-1})^{-1})$. The reason of this flexibility will appear with the Omega lemma.

For any $s \in [0, T]$, we extend $y_s^\dagger$ as an operator in $L(\mathrm{U} \otimes \mathrm{U}, \mathrm{U} \otimes \mathrm{V})$ by $y_s^\dagger(a \otimes b) = a \otimes y_s^\dagger b$ for all $a, b \in \mathrm{U}$.

Proposition 3. *Assume that*

$$\theta := \min\left\{\frac{2}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}\right\} > 1.$$

There exists a continuous linear map $y \mapsto y^\flat$ on $\mathrm{P}_{p,q,r}(\mathbf{x}, \mathrm{V})$ which transforms y to $\{y_{s,t}^\flat\}_{[s,t] \subset [0,T]}$ with values in $\mathrm{U} \otimes \mathrm{V}$ such that

$$y_{r,s}^\flat + y_{s,t}^\flat + \mathbf{x}_{r,s}^1 \otimes y_{s,t} = y_{r,t}^\flat, \text{ for all } 0 \leq r \leq s \leq t \leq T. \tag{14}$$

Moreover, for some universal constants K and K',

$$|y^\flat_{s,t} - y^\dagger_s \mathbf{x}^2_{s,t}| \leq K \|y\|_{\mathbf{x}} (\|\mathbf{x}\|_p \vee \|\mathbf{x}\|_p^2) \omega^\theta_{s,t} \tag{15}$$

and

$$\begin{aligned} \|y^\flat\|_{p/2} &\leq \big(|y^\dagger_0| + (\omega_{0,T}^{1/q} + K\omega_{0,T}^{\theta-2/p}\|y\|_{\mathbf{x}})\big)(\|\mathbf{x}\|_p \vee \|\mathbf{x}\|_p^2) \\ &\leq K'\|y\|_{\bullet\mathbf{x}}(\|\mathbf{x}\|_p \vee \|\mathbf{x}\|_p^2). \end{aligned} \tag{16}$$

Proof. Let us introduce on the linear space $\mathrm{W} := \mathrm{U} \oplus \mathrm{V} \oplus (\mathrm{U} \otimes \mathrm{V})$ the (non-commutative) operation

$$(a, b, c) \boxtimes (a', b', c') = (a + a', b + b', c + c' + a \otimes b')$$

and the norm $|(a, b, c)| := \max\{|a|, |b|, |c|\}$.

For $Y, X, Z \in \mathrm{W}$, it is clear that $|\cdot|$ is Lipschitz continuous and

$$|Z \boxtimes Y - Z \boxtimes X| \leq (1+|Z|)|Y - X| \text{ and } |Y \boxtimes Z - X \boxtimes Y| \leq (1+|Z|)|Y - X|.$$

With $\boxtimes$, W is a monoid for which the hypotheses of the Multiplicative Sewing Lemma are fulfilled [21].

We then define for $y \in \mathrm{P}_{p,q,r}(\mathbf{x}, L(\mathrm{U}, \mathrm{V}))$ the family of operators

$$\phi_{s,t}(y) := (\mathbf{x}^1_{s,t}, y_{s,t}, y^\dagger_s \mathbf{x}^2_{s,t}),\ [s, t] \subset [0, T].$$

Thus,

$$\phi_{r,s}(y) \boxtimes \phi_{s,t}(y) - \phi_{r,t}(y) = (0, 0, y^\dagger_{r,s}\mathbf{x}^2_{s,t} + \mathbf{x}^1_{r,s} y^\sharp_{s,t} - \mathbf{x}^1_{r,s} \otimes y^\dagger_{r,s}\mathbf{x}^1_{s,t}). \tag{17}$$

The Multiplicative Sewing Lemma [21] on $(\phi_{s,t}(y))_{[s,t]\subset[0,T]}$ yields the existence of a family $\{Y_{s,t}\}_{[s,t]\subset[0,T]}$ taking its values in W with $|Y_{s,t} - \phi_{s,t}(y)| \leq C\omega^\theta_{s,t}$ for any $[s, t] \subset [0, T]$. We define $y^\flat$ as the part in $\mathrm{U} \otimes \mathrm{V}$ in the decomposition of Y as $Y_{s,t} = (\mathbf{x}^1_{s,t}, y_{s,t}, y^\flat_{s,t})$. Since Y satisfies $Y_{r,s} \boxtimes Y_{s,t} = Y_{r,t}$, $y^\flat$ satisfies (14). From (17), we easily obtain (15) and (16).

For $y, z \in \mathrm{P}_{p,q,r}(\mathbf{x}, \mathrm{V})$,

$$\phi_{r,s}(y + z) \boxtimes \phi_{s,t}(y + z) = \phi_{r,s}(y) \boxtimes \phi_{s,t}(y) + \phi_{r,s}(z) \boxtimes \phi_{s,t}(z).$$

From this additivity property, the construction of the Multiplicative Sewing Lemma and (16), $y \mapsto y^\flat$ is linear and continuous. □

When y takes its values in $L(\mathrm{V}, \mathrm{W})$, $y^\dagger$ takes its values in $L(\mathrm{U}, L(\mathrm{V}, \mathrm{W})) \simeq L(\mathrm{U} \otimes \mathrm{V}, \mathrm{W})$.

Proposition 4. *Fix (p,q,r) and (p,q',r') with $2 \le r < p < 3$. Assume that*

$$\widehat{\theta} := \min\left\{\frac{2}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}\right\} > 1 \text{ and } \theta' := \min\left\{\frac{2}{p} + \frac{1}{q'}, \frac{1}{p} + \frac{1}{r'}\right\} > 1.$$

There exists a bilinear continuous mapping

$$\begin{aligned} \mathrm{P}_{p,q,r}(\mathbf{x}, L(\mathrm{W},\mathrm{V})) \times \mathrm{P}_{p,q',r'}(\mathbf{x},\mathrm{W}) &\mapsto \mathrm{P}_{p,p\vee q',p/2}(\mathbf{x},\mathrm{V}) \\ (y,z) &\to \int y \,\mathrm{d}z \end{aligned}$$

such that $\left(\int y \,\mathrm{d}z\right)^\dagger_s = y_s z^\dagger_s$,

$$\left|\int_s^t y_r \,\mathrm{d}z_r - y_s z_{s,t} - y^\dagger_s z^\flat_{s,t}\right| \le K\|y\|_{\mathbf{x}}\|z\|_{\bullet\mathbf{x}}(1 + \|\mathbf{x}\|_p \vee \|\mathbf{x}\|_p^2)\omega^{\widehat{\theta}}_{s,t} \tag{18}$$

and

$$\left\|\int y \,\mathrm{d}z\right\|_{\mathbf{x}} \le K'\|y\|_{\bullet\mathbf{x}}\|z\|_{\bullet\mathbf{x}}(1 + \|\mathbf{x}\|_p \vee \|\mathbf{x}\|_p^2) \tag{19}$$

for some universal constants K and K'.

Proof. We set $Y_{s,t} := y_s z_{s,t} + y^\dagger_s z^\flat_{s,t}$. Thus, for any $r \le s \le t$,

$$Y_{r,s} + Y_{s,t} - Y_{r,t} = y^\sharp_{r,s} z_{s,t} + y^\dagger_{r,s} z^\flat_{s,t}.$$

The existence of the integrals follows from the Additive Sewing Lemma [21]. Therefore, the inequalities (18) and (19) are straightforward. □

From now, let us fix $2 \le p < 3$, $q,r \ge 1$ as well as $0 < \gamma \le 1$ and $0 < \kappa < 1$. For two Banach spaces V and W and a rough path $\mathbf{x} \in \mathrm{R}_p(\mathrm{U})$, we set

$$\begin{gathered} \mathrm{PV} := \mathrm{P}_{p,q,r}(\mathbf{x},\mathrm{V}),\ \mathrm{P}^\star\mathrm{V} := \{y \in \mathrm{PV} \mid (y_0, y^\dagger_0) = (0,0)\}, \\ \mathrm{QW} := \mathrm{P}_{p,\frac{q\vee p}{\kappa\gamma},r\vee\frac{p}{1+\kappa\gamma}}(\mathbf{x},\mathrm{W}) \text{ and } \mathrm{Q}^\star\mathrm{W} := \{y \in \mathrm{QW} \mid (y_0, y^\dagger_0) = (0,0)\}. \end{gathered}$$

The spaces $\mathrm{P}^\star\mathrm{V}$ and $\mathrm{Q}^\star\mathrm{W}$ are Banach sub-spaces of PV and QW.

Proposition 5 (The Omega lemma for CRP)**.** *Assume that $f \in \mathscr{C}^{k+1+\gamma}(\mathrm{V},\mathrm{W})$. Then $\mathfrak{O}f := (f(y_t))_{t\in[0,T]}$ is locally of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ from* PV *to* QW *with $\mathfrak{O}f(y)^\dagger = \mathrm{D}f(y)y^\dagger$. Besides,*

$$\mathrm{D}\mathfrak{O}f(y)\cdot z = (\mathrm{D}f(y)_t \cdot z_t)_{t\in[0,T]} \in \mathrm{QW},\ \forall y,z \in \mathrm{PV}.$$

In addition, if $z \in \mathrm{P}^\star\mathrm{V}$, then $\mathrm{D}\mathfrak{O}f(y)z \in \mathrm{Q}^\star\mathrm{W}$ for any $y \in \mathrm{PV}$.

Proof. I) Let us prove first that $\mathfrak{O}f$ maps PV to $\mathrm{P}_{p,\frac{p}{\gamma},r\vee\frac{p}{1+\gamma}}(\mathbf{x},\mathrm{W})$. As the latter space is continuously embedded in QW, this proves that $\mathfrak{O}f$ maps PV to QW.

Set for $0 \leq s \leq t \leq T$,

$$Y_t := f \circ y_t,\ Y_t^\dagger := \mathrm{D}f(y_t)y_t^\dagger$$
$$\text{and } Y_{s,t}^\sharp := \mathrm{D}f(y_s)y_{s,t}^\sharp + \int_0^1 (\mathrm{D}f(y_s+\theta y_{s,t}) - \mathrm{D}f(y_s))y_{s,t}\,\mathrm{d}\theta.$$

For Y, the decomposition (10) is $Y_{s,t} = Y_s^\dagger x_{s,t}^1 + Y_{s,t}^\sharp$. Besides,

$$|Y_{s,t}^\sharp| \leq \|\mathrm{D}f\|_\infty \|y\|_{\mathbf{x}} \omega_{s,t}^{1/r} + H_\gamma(\mathrm{D}f)\|y\|_p^{1+\gamma}\omega_{s,t}^{(1+\gamma)/p},$$
$$|Y_{s,t}^\dagger| \leq H_\gamma(\mathrm{D}f)|y_{s,t}|^\gamma \cdot \|y^\dagger\|_\infty + \|\mathrm{D}f\|_\infty \cdot |y_{s,t}^\dagger|.$$

We deduce that $Y \in \mathrm{P}_{p,q\vee\frac{p}{\gamma},r\vee\frac{p}{1+\gamma}}(\mathbf{x},\mathrm{W})$ with

$$\|Y\|_{\mathbf{x}} \leq C\|f\|_{1+\gamma} \max\{\|y\|_{\mathbf{x}}, \|y\|_{\bullet\mathbf{x}}^{1+\gamma}\},$$

for some universal constant C.
II) Setting $f(y,z) = yz$ (resp. $f(y,z) = y \otimes z$) for $y \in \mathrm{P}L(\mathrm{W},\mathrm{V})$ (resp. $y \in \mathrm{PV}$) and $z \in \mathrm{PW}$ shows that $yz \in \mathrm{PV}$ (resp. $y \otimes z \in \mathrm{PW} \otimes \mathrm{V}$). Moreover, $(yz)_t^\dagger = y_t^\dagger z_t + y_t z_t^\dagger$ (resp. $(y\otimes z)_t^\dagger = y_t^\dagger \otimes z_t + y_t \otimes z_t^\dagger$) for any $t \in [0,T]$.

In particular, $z_0 = 0$ and $z_0^\dagger = 0$ implies that $(yz) \in \mathrm{P}^\star\mathrm{V}$ (resp. $y \otimes z \in \mathrm{P}^\star\mathrm{W}\otimes\mathrm{V}$) when $z \in \mathrm{P}^\star\mathrm{W}$.

In addition, it is easily obtained that for the product $y \cdot z = yz$ or $y \cdot z = y \otimes z$,

$$\|yz\|_{\bullet\mathbf{x}} \leq C\|y\|_{\bullet\mathbf{x}}\|z\|_{\bullet\mathbf{x}}\|\mathbf{x}\|_p$$

for some universal constant C.
III) We consider that $f \in \mathscr{C}^{1+\gamma}(\mathrm{V},\mathrm{W})$. Let $y,z \in \mathrm{PV}$ and set $Y_t := f(y_t)$, $Z_t := f(z_t)$. According to the definition of $Z^\dagger$ and $Y^\dagger$,

$$\begin{aligned} Z_{s,t}^\dagger - Y_{s,t}^\dagger \\ &= (\mathrm{D}f(z_t) - \mathrm{D}f(z_s) - \mathrm{D}f(y_t) + \mathrm{D}f(y_s))z_t^\dagger + (\mathrm{D}f(z_s) - \mathrm{D}f(y_s))z_{s,t}^\dagger \\ &\quad + \mathrm{D}f(y_s)(z_{s,t}^\dagger - y_{s,t}^\dagger) + (\mathrm{D}f(y_t) - \mathrm{D}f(y_s))(z_t^\dagger - y_t^\dagger). \end{aligned}$$

Applying (7) in Lemma 1, for $0 < \kappa < 1$,

$$\begin{aligned} |Z_{s,t}^\dagger - Y_{s,t}^\dagger| &\leq H_\gamma(\mathrm{D}f)\|z-y\|_\infty^{\overline{\kappa}\gamma}(\|z\|_p^{\kappa\gamma} + \|y\|_p^{\kappa\gamma})\|z^\dagger\|_\infty \omega_{s,t}^{\kappa\gamma/p} \\ &\quad + H_\gamma(\mathrm{D}f)\|z-y\|_\infty^\gamma \|z^\dagger\|_q \omega_{s,t}^{\gamma/q} + \|\mathrm{D}f\|_\infty \|z^\dagger - y^\dagger\|_q^\gamma \omega_{s,t}^{\gamma/q} \\ &\quad + H_\gamma(\mathrm{D}f)\|y\|_p^\gamma \|y^\dagger - z^\dagger\|_\infty \omega_{s,t}^{\gamma/p}. \end{aligned}$$

From this, and (11)-(13),

$$\begin{aligned}\frac{|Z^{\dagger}_{s,t}-Y^{\dagger}_{s,t}|}{\omega_{s,t}^{\frac{\kappa\gamma}{p\vee q}}} \leq K_1 H_\gamma(\mathrm{D}f)\|z-y\|_{\bullet\mathbf{x}}^{\overline{\kappa}\gamma}(\|z\|_{\mathbf{x}}^{\kappa\gamma}+\|y\|_{\mathbf{x}}^{\kappa\gamma})\|z\|_{\bullet\mathbf{x}}(1+\|\mathbf{x}\|_p)^{1+\overline{\kappa}\gamma}\\ + K_2 H_\gamma(\mathrm{D}f)\|z-y\|_{\bullet\mathbf{x}}^{\gamma}\|z\|_{\mathbf{x}}(1+\|\mathbf{x}\|_p)^{1+\gamma}\\ + K_3\|\mathrm{D}f\|_\infty\|z-y\|_{\mathbf{x}}^{\gamma} + K_4 H_\gamma(\mathrm{D}f)(1+\|\mathbf{x}\|_p)^{\gamma}\|y\|_{\bullet\mathbf{x}}^{\gamma}\|y-z\|_{\bullet\mathbf{x}} \quad (20)\end{aligned}$$

for some universal constants K_1, K_2, K_3 and K_4. Besides,

$$\begin{aligned}Z^{\sharp}_{s,t}-Y^{\sharp}_{s,t} = \mathrm{D}f(z_s)z^{\sharp}_{s,t}-\mathrm{D}f(y_s)y^{\sharp}_{s,t}+\int_0^1 \big(\mathrm{D}f(y_s+\tau y_{s,t})-\mathrm{D}f(y_s)\big)(z_{s,t}-y_{s,t})\,\mathrm{d}\tau\\ +\int_0^t \big(\mathrm{D}f(z_s+\tau z_{s,t})-\mathrm{D}f(z_s)-\mathrm{D}f(y_s+\tau y_{s,t})+\mathrm{D}f(y_s)\big)y_{s,t}\,\mathrm{d}\tau.\end{aligned}$$

With Lemma 1 and (11),

$$\begin{aligned}|Z^{\sharp}_{s,t}-Y^{\sharp}_{s,t}| \leq \|z\|_{\mathbf{x}} H_\gamma(\mathrm{D}f)\|z-y\|_\infty^{\gamma}\omega_{s,t}^{1/r}\\ +\|\mathrm{D}f\|_\infty\|z-y\|_{\mathbf{x}}\omega_{s,t}^{1/r}+H_\gamma(\mathrm{D}f)\|y\|_p\|y-z\|_p\omega_{s,t}^{(1+\gamma)/p}\\ +4^{\overline{\kappa}\gamma}H_\gamma(\mathrm{D}f)\|y\|_p\|z-y\|_\infty^{\overline{\kappa}\gamma}(\|z\|_p^{\kappa\gamma}+\|y\|_p^{\kappa\gamma})\omega_{s,t}^{(1+\kappa\gamma)/p}. \quad (21)\end{aligned}$$

With (11) and (13), we deduce that

$$|Z^{\sharp}_{s,t}-Y^{\sharp}_{s,t}| \leq C\omega_{s,t}^{\frac{1+\kappa\gamma}{p}\wedge\frac{1}{r}},$$

for some constant C that depends on $\omega_{0,T}$, $\|\mathbf{x}\|_p$, $\|y\|_{\bullet\mathbf{x}}$, $\|z\|_{\bullet\mathbf{x}}$ and the parameters γ, κ, p, q and r.

With (13) applied to $\|y-z\|_\infty$, (21) and (20) could be summarized as

$$\|Z-Y\|_{\mathbf{x}} \leq K\|f\|_{1+\gamma}\|z-y\|_{\bullet\mathbf{x}}^{\overline{\kappa}\gamma},$$

where $\|Z-Y\|_{\mathbf{x}}$ refers to the norm in QW and K is a constant which depends on $\|x\|_p$, $\|z\|_{\bullet\mathbf{x}}$, $\|y\|_{\bullet\mathbf{x}}$, κ, γ, (p,q,r) and $\omega_{0,T}$.

Moreover, $|f(y_0)-f(z_0)| \leq \|\mathrm{D}f\|_\infty|y_0-z_0|$ and

$$|f(y)^{\dagger}_0-f(z)^{\dagger}_0| \leq H_\gamma(\mathrm{D}f)|y_0-z_0|+\|\mathrm{D}f\|_\infty|y^{\dagger}_0-z^{\dagger}_0|.$$

Up to changing K, we get a similar inequality as $\|Z-Y\|_{\mathbf{x}}$ is replaced by $\|Z-Y\|_{\bullet\mathbf{x}}$.

We have then proved that $\mathfrak{D}f$ is locally of class $\mathscr{C}^{\overline{\kappa}\gamma}$ from PV to QW.

IV) For dealing with the general case $f\in\mathscr{C}^{k+1+\gamma}(\mathrm{V},\mathrm{W})$, we apply the converse of the Taylor theorem as in the proof of Proposition 2, using II), the proof being in all points similar. □

When considering a fixed point for $y \mapsto \mathfrak{F}(y, z, f) := \int f(y)\,\mathrm{d}z$, $\mathfrak{F}$ should map PV to PV. Owing to Propositions 4 and 5, a suitable choice is

$$q = q' = p,\ r = r' = p/2.$$

From now, we use these values $(q', r') = (p, p/2)$.

We now state an existence and uniqueness result for solutions of RDE. Its proof may be found, up to a straightforward modification for dealing with $b \neq 0$, in [22, 26].

Proposition 6. *For any $f \in \mathscr{C}^{k+1+\gamma}(\mathrm{V}, L(\mathrm{W}, \mathrm{V}))$ with $k \geq 0$, any $z \in \mathrm{PW}$ and any $a \in \mathrm{V}$, $b \in L(\mathrm{W}, \mathrm{V})$, then there exists a unique CRP $y \in \mathrm{PV}$ which solves*

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}z_s + bz_{0,t} \text{ with } y^\dagger_t = f(y_t)z^\dagger_t + bz^\dagger_t \tag{22}$$

for any $t \in [0, T]$. Besides, when $|a| + |b| \leq R$, then y belongs to a closed ball of PV whose radius depends only on $\|f\|_{k+1+\gamma}$, $\omega_{0,T}$, p, γ, $\|z\|_{\mathbf{x}}$ and R.

Let us set

$$\mathrm{X}_{k,\gamma} := \mathrm{V} \times L(\mathrm{V}, \mathrm{W}) \times \mathscr{C}^{k+1+\gamma}(\mathrm{V}, L(\mathrm{W}, \mathrm{V})) \times \mathrm{PW}.$$

We then define the *Itô map* $\mathfrak{I}$ as the map sending $(a, b, f, z) \in \mathrm{X}_{k,\gamma}$ to $(y, y^\dagger)$ given by (22). Besides, (22) is equivalent in finding $y^* \in \mathrm{P}^\star\mathrm{V}$ which solves

$$y^*_t = \int_0^t f(y^*_s + a + bz_{0,s})\,\mathrm{d}z_s$$

and then to set $y_t := y^*_t + a + bz_{0,t}$.

The proof of the regularity result is now in all points similar to the one for paths of finite p-variation so that we skip it.

Theorem 3. *Under the above conditions, the Itô map $\mathfrak{I} : \mathrm{X}_{k,\gamma} \to \mathrm{PV}$ is locally of class $\mathscr{C}^{k+\overline{\kappa}\gamma}$ for any $\kappa \in (0, 1)$ with $1 + \kappa\gamma > p$.*

For $t \in [0, T]$, let $\mathfrak{e}_t$ be the evaluation map $\mathfrak{e}_t(y, y^\dagger) = y_t$ (this choice forces the value $y^\dagger_t = f(y_t)$).

The proof of the next result is in all points similar to the one of Corollary 1.

Corollary 2. *Under the hypotheses of Theorem 3, for any $t > 0$, $\mathfrak{f}_t(a) := \mathfrak{e}_t \circ \mathfrak{I}(a, 0, f, 0)$ is locally a $\mathscr{C}^{k+\overline{\kappa}\gamma}$-diffeomorphism from V to V.*

Since $\mathrm{P}_{p,p,p/2}(\mathbf{x}, \mathrm{V})$ is continuously embedded in $\mathrm{C}_p(\mathrm{V})$, we could consider this approach for studying the regularity of

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}z_s + \int_0^t g(y_s)\,\mathrm{d}h_s + b_t,\ t \geq 0, \tag{23}$$

where for a rough path $\mathbf{x}$, $z \in \mathrm{P}_{p,p,p/2}(\mathbf{x}, \mathrm{W})$, $h \in \mathrm{C}_q(\mathrm{W}')$, f and g are maps respectively of class $\mathscr{C}^{k+1+\gamma}$ from V to $L(\mathrm{W}, \mathrm{V})$ and of class $\mathscr{C}^{k+\delta}$ from V to $L(\mathrm{W}', \mathrm{V})$, provided that

$$\frac{1}{p} + \frac{1}{q} > 1,\ 1 + \gamma > p \text{ and } 1 + \delta > q \text{ for } 0 < \gamma, \delta \leq 1.$$

The map $\mathfrak{I}(f, g, b, y_0, y_0^\dagger)$ giving the solution to (23) is then of class $\mathscr{C}^{k+(1-\kappa)\min\{\delta,\gamma\}}$.

Using for z the decomposition $z^\dagger = 1$ and $z^\sharp = 0$, and replacing the vector field f by ϵf, it is easily seen that we may consider the problem

$$y_t^\epsilon = a + \int_0^t f(y_s^\epsilon)\,\mathrm{d}\mathfrak{d}_\epsilon \mathbf{x}_s + \int_0^t g(y_s^\epsilon)\,\mathrm{d}h_s \text{ for } a = (y_0, y_0^\dagger) \text{ given.}$$

Asymptotic expansions in ϵ can then be performed as in [5, 32–35].

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