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Abstract

We prove that the dimension of trivariate tensor-product spline space of tri-degree (m, m, m) with maximal order of smoothness over a three-dimensional domain coincides with the number of tensor-product B-spline basis functions acting effectively on the domain considered. A domain is required to belong to a certain class. This enables us to show that, for a certain assumption about the configuration of a hierarchical mesh, hierarchical B-splines span the spline space.

This paper presents an extension to three-dimensional hierarchical meshes of results proposed recently by Giannelli and Jüttler for two-dimensional hierarchical meshes.

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1. Introduction

The spline representations that admit local refinement appeared originally within the framework of geometric design. A new interest in this issue has emerged recently in connection with isogeometric analysis [1]. Let us recall some recent advances in spline representation techniques admitting local refinement.

A well-known T-splines representation defined over two-dimensional T-meshes has been introduced by Sederberg et al. [2]. A practicable mesh refinement algorithm based on the concept of T-splines is described in [3]. The issue of the linear independence of T-splines [4], which is relevant to isogeometric analysis, has been resolved lately by considering analysis-suitable T-splines [5], which are a restricted subset of T-splines. The refinement algorithm for analysis-suitable T-splines is given in [6]. Being an equivalent definition of analysis-suitable T-splines, dual-compatible T-splines [7] are natural candidates for a three-dimensional generalization of analysis-suitable T-splines. However, the refinement algorithm for three-dimensional dual-compatible meshes is still an open problem.

Deng et al. [8] introduced splines over T-meshes. In the case of reduced regularity, with splines of bi-degree $(3, 3)$ and order of smoothness $(1, 1)$, the refinement technique for polynomial splines over hierarchical T-meshes (called PHT-splines) [9] has been given in terms of Bézier ordinates. PHT-splines have been generalized in the form of B-splines for an arbitrary T-mesh [10]. Wu et al. [11] have recently introduced a consistent hierarchical T-mesh. The construction of spline basis functions has been achieved for a particular type of consistent hierarchical T-meshes [12]. For a three-dimensional hierarchical T-mesh the dimension formula for a spline space of reduced regularity has been derived [13].

Dokken et al. [14] have lately proposed the concept of locally refined splines (LR-splines). LR B-splines are tensor-product B-splines with minimal supports. The refining process is based on so-called hand-in-hand LR-refinement and starts with a tensor-product mesh, which guarantees that the collection of LR B-splines spans the spline space. So far, the refinement algorithm for LR-splines has mostly been developed for the two-dimensional case.

Multilevel B-splines for surface modeling were originally introduced by Forsey and Bartels [15]. Kraft [16] suggested a selection mechanism for hierarchical B-splines that ensures their linear independence as well as local refinement control. In addition, a quasi-interpolant that achieves the optimal local approximation order has been introduced [16]. Vuong et al. [17] looked more clearly at hierarchical B-splines to consider subdomains with partly overlapping boundaries and their applications in isogeometric analysis. Recently, truncated hierarchical B-splines, which are modified to satisfy the partition-of-unity property, have been introduced by Giannelli et al. [18].

In the three-dimensional case, hierarchical B-splines remain the basic approach, allowing a feasible refinement algorithm that guarantees locality of the refinement and linear independence of the blending functions. This paper is inspired by the theoretical results [19] obtained recently by Giannelli and Jüttler for bivariate hierarchical B-splines. For a domain that is a set of cells from an infinite two-dimensional tensor-product grid, it has been proved [19] that the dimension of bivariate tensor-product spline spaces of bi-degree (m, m) with maximal order of smoothness on the domain is equal to the number of tensor-product B-spline basis functions, defined by single knots in both directions, acting effectively on the domain. A reasonable assumption about the configuration of the domain is required. Based on these observations, Giannelli and Jüttler have proved that hierarchical B-splines, produced by Kraft's selection mechanism, span the spline space defined over a hierarchical mesh generated by a decreasingly nested sequence of domains: $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1}$ associated with an increasingly nested sequence of tensor-product spline spaces $V^0 \subset \cdots \subset V^{N-1}$. Again, a reasonable assumption is required about the configuration of the domains $\Omega^0, \ldots, \Omega^{N-1}$ associated with $V^0, \ldots, V^{N-1}$, respectively.

In this paper, we will prove the analogous results for trivariate hierarchical B-splines on the basis of standard homological algebraic techniques and Mourrain's paper [20]. For a domain that is a set of cells from an infinite three-dimensional tensor-product grid, we will obtain the dimension of the trivariate tensor-product spline space of tri-degree (m, m, m) with maximal order of smoothness on the domain, under the assumption that the domain is homeomorphic to a three-dimensional ball and two-dimensional slices of this domain are simply connected. In addition, we will obtain the dimension of the spline space under the condition that the domain belongs to a certain class (which will be defined in Subsection 4.1). In this case, we will not impose topological restrictions on the domain itself, apart from restrictions imposed on its configuration with respect to the associated grid. Moreover, we will prove that the dimension of the spline space is equal to the number of tensor-product B-splines, defined by single knots in three directions, acting effectively on a domain of this class.

These results will enable us to prove that hierarchical B-splines, produced by Kraft's selection mechanism [16], span the spline space of tri-degree (m, m, m) with maximal order of smoothness over a three-dimensional hierarchical mesh generated by a decreasingly nested sequence of domains associated with an increasingly nested sequence of tensor-product spline spaces. As in the two-dimensional case, a reasonable assumption about the configuration of the domains will be required. By following our approach, we will also confirm the results obtained by Giannelli and Jüttler [19]. We present our approach gradually, starting from the simplest one-dimensional case.

The rest of this paper is organized as follows: In Sections 2–4 we consider the one-dimensional, two-dimensional, and three-dimensional cases, respectively. Subsections 3.1 and 4.1 introduce classes of two-dimensional and three-dimensional domains, respectively, for which we obtain the numbers of tensor-product B-splines acting effectively. In Subsections 3.2 and 4.2 we derive the dimension of the tensor-product spline space under certain topological assumptions about two-dimensional and three-dimensional domains, respectively. In addition, we obtain the dimension and a basis of the spline space on a domain of the classes introduced in Subsections 3.1 and 4.1. Based on Sections 2–4, we provide in Section 5 a unified proof that B-spline functions, produced by Kraft's selection mechanism [16], span the spline space defined over a hierarchical mesh. Since the final part of the proof is the same as that in [19], we adopt the notation used by Giannelli and Jüttler. We conclude the paper in Section 6. In order to simplify the notation, we will avoid using extra indices related to univariate, bivariate, and trivariate splines, unless otherwise stated. Throughout the paper we will use $\operatorname{int}\Omega$ to denote the interior of a domain Ω, and $\operatorname{supp} b$ to denote the support of a function b.

2. Univariate splines

Let T' be an infinite one-dimensional grid. Without loss of generality, we will suppose that the distances between adjacent grid nodes of T' are equal to 1. A cell of T' is a closed segment of length 1 between adjacent grid nodes.

Let Ω be a closed domain formed by a finite number of cells of T' (see Fig. 1). Then, Ω consists of a number of segments of finite length. A vertex of a domain Ω is a grid node of T' that belongs to Ω. We say that a vertex of Ω is an inner vertex if it belongs to $\operatorname{int}\Omega$. We define the distance between two neighboring segments of Ω as the number of cells between them. For a given integer $k \geqslant 0$, the class $\mathcal{A}_k^1$ of one-dimensional domains is defined as follows:

Definition 1. *Let k be a nonnegative integer. We say that a domain Ω admits an offset at a distance of $\frac{k}{2}$ if the distance between any two neighboring segments exceeds k. We denote by $\mathcal{A}_k^1$ the class of one-dimensional domains that admit an offset at a distance of $\frac{k}{2}$.*

Figure 1: The grid nodes of T' are denoted by black bars. The cells of a domain Ω are denoted by thick solid lines. We note that Ω admits an offset at a distance of 1 but does not admit an offset at a distance of $\frac{3}{2}$.

For a given integer $m \geqslant 1$, let $\widehat{\mathcal{B}}$ be the set of segments formed by $m+1$ consecutive cells of T', so $\widehat{\mathcal{B}}$ is the set of all possible minimal supports for B-splines of degree m defined over T' and with knot multiplicities equal to 1. We denote by $\mathcal{B}$ the collection of B-splines $b(x)$ whose supports become the elements of $\widehat{\mathcal{B}}$. Let $\mathcal{N}$ be the number of elements of $\widehat{\mathcal{B}}$ that have at least one cell in common with a domain Ω. We denote by f_1 and f_0^0 the number of cells forming a domain Ω and the number of inner vertices of Ω, respectively. It can be seen that $f_1 - f_0^0$ coincides with the number of connected components of Ω. The following proposition is a simple observation, so we omit its proof:

Proposition 1. *For a given integer $m \geqslant 1$, let Ω be a one-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}_{m-1}^1$. Then, the following identity holds:*

$$\mathcal{N} = (m+1)f_1 - mf_0^0. \tag{1}$$

Let R_m be the vector space of univariate polynomials of degree m. Let $\mathcal{T}$ be a mesh, which is a portion of T' over a domain Ω. We denote by $\mathcal{S}_m(\mathcal{T})$ the vector space of C^{m-1} smooth functions defined on Ω that are polynomials in R_m on each cell of a domain Ω. For the sake of brevity, let us omit the proof of the following proposition:

Proposition 2. *For a given domain Ω, the dimension of the corresponding spline space is*

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)f_1 - mf_0^0.$$

As a result of Propositions 1 and 2 we obtain the following:

Theorem 1. *Suppose that $\Omega \in \mathcal{A}^1_{m-1}$. Then, the basis of a space $\mathcal{S}_m(\mathcal{T})$ can be obtained as follows:*

$$\{b|_\Omega : b(x) \in \mathcal{B} \ \wedge \ \operatorname{supp} b(x) \cap \operatorname{int}\Omega \neq \varnothing\}. \tag{2}$$

Corollary 1. *For a given integer $m \geqslant 1$, let Ω be a one-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}^1_{m-1}$. Let $f \in \mathcal{S}_m(\mathcal{T})$ be a spline function defined over the corresponding mesh $\mathcal{T}$. Then, there exists a spline function $\widetilde{f}$ of degree m defined globally over T' such that $\widetilde{f}|_\Omega = f$.*

Proof. The splines from $\mathcal{B}$ are defined globally over T'. Thus, by Theorem 1, the corollary is proved. $\square$

3. Bivariate splines

Let T' be a two-dimensional infinite grid. Without loss of generality, we will suppose that the distances between adjacent grid nodes of T' are equal to 1. A cell of T' is a closed square with sides of length 1 aligned with the grid lines of T'.

Let Ω be a closed domain formed by a finite number of cells of T' (see Fig. 2). A vertex of a domain Ω is a grid node of T' that belongs to Ω. We say that a vertex is a boundary vertex if it belongs to $\partial\Omega$, and we say that a vertex is an inner vertex if it belongs to $\operatorname{int}\Omega$. An edge of a domain Ω is a closed segment between two adjacent grid nodes of T', which is a subset of Ω. We say that an edge is a boundary edge if it is a subset of $\partial\Omega$, and we say that an edge is an inner edge if it is not a boundary edge.

Throughout this section we will suppose that Ω is a two-dimensional topological manifold with a boundary. A violation of this restriction can occur only in the neighborhood of a boundary vertex. The admissible and inadmissible configurations for a neighborhood of a boundary vertex are shown in Fig. 3. Additionally, we remark that Ω might have several connected components.

For a given integer $m \geqslant 1$, let $\widehat{\mathcal{B}}$ be the set of $(m+1) \times (m+1)$ squares formed by $(m+1)^2$ cells of T', so $\widehat{\mathcal{B}}$ is the set of all possible minimal supports for tensor-product B-splines of bi-degree (m, m) defined over T' with knot multiplicities equal to 1. We denote by $\mathcal{B}$ the collection of B-splines $b(x, y)$ whose supports become the elements of $\widehat{\mathcal{B}}$. Let $\mathcal{N}$ be the number of elements of $\widehat{\mathcal{B}}$ that have at least one cell in common with a domain Ω. In Subsection 3.1 we will prove Corollary 2, in which $\mathcal{N}$ will be obtained for a domain $\Omega \in \mathcal{A}^2_{m-1}$ (see Definitions 3 and 4 in Subsection 3.1 for the specification of class $\mathcal{A}^2_k$, for an integer $k \geqslant 0$).

Let R_m be the vector space of polynomials of bi-degree (m, m) with respect to two variables x and y. Let $\mathcal{T}$ be a T-mesh [8, 20], which is a portion of T' over a domain Ω. We denote by $\mathcal{S}_m(\mathcal{T})$ the vector space of C^{m-1} smooth functions defined on Ω that are polynomials in R_m on each cell of a domain Ω. In Subsection 3.2 we will prove Corollary 4, in which $\dim \mathcal{S}_m(\mathcal{T})$ will be obtained for a domain $\Omega \in \mathcal{A}^2_{m-1}$. Based on the observation that $\dim \mathcal{S}_m(\mathcal{T}) = \mathcal{N}$ for $\Omega \in \mathcal{A}^2_{m-1}$, we will prove Theorem 3, in which we will obtain a basis of the space $\mathcal{S}_m(\mathcal{T})$.

3.1. The dilatation of a two-dimensional domain

In this subsection we will introduce the classes of two-dimensional domains $\mathcal{A}^2_k$, for any integer $k \geqslant 0$. For a domain $\Omega \in \mathcal{A}^2_{m-1}, m \geqslant 1$, we will obtain the formula for the number $\mathcal{N}$ (that is the number of tensor-product B-splines of bi-degree (m, m) acting effectively on Ω) in terms of the numbers of cells, inner edges and inner vertices of Ω.

Let T'' be the grid that is obtained by shifting T' by the vector $\{\frac{1}{2}, \frac{1}{2}\}$ (see Fig. 2). We first define dilatation domains Ω^e_k of Ω in a recursive manner for $0 \leqslant k \in \mathbb{Z}$.

Definition 2. *If $k = 0$, the dilatation domain $\Omega^e_0 := \Omega$. If k is odd, the dilatation domain Ω^e_k is the union of the cells of T'' with vertices of Ω^e_{k-1} as their centroids. If k is even, the dilatation domain Ω^e_k is the union of the*

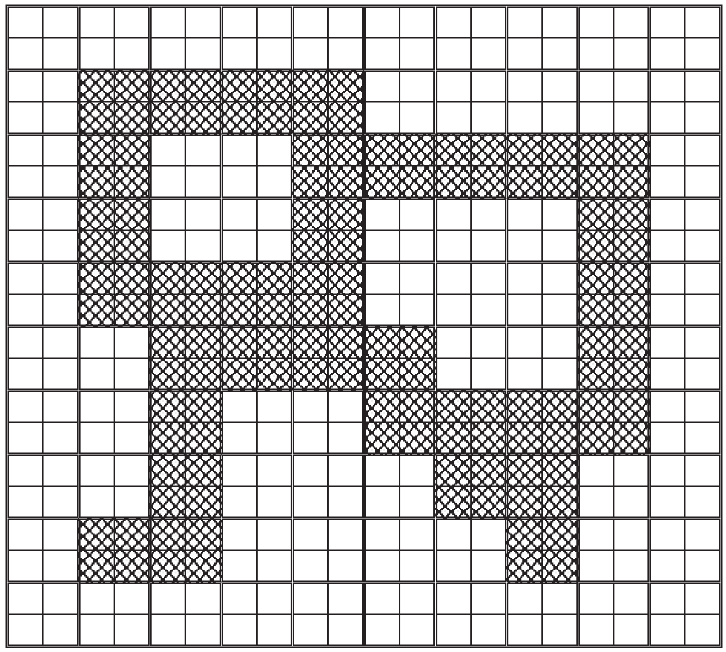

Figure 2: The grid T' is aligned with thick solid lines. The cells of a domain Ω are diagonally hatched. The grid T'', which is shifted by the vector $\{\frac{1}{2}, \frac{1}{2}\}$ from T', is aligned with thin solid lines.

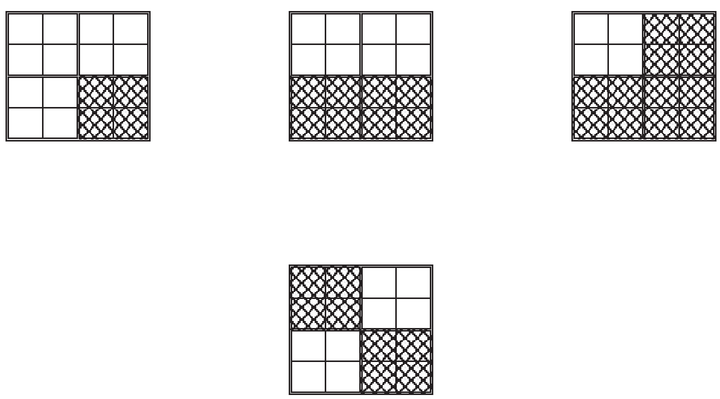

Figure 3: The boundary vertices are at the centroids of 2×2 squares that are formed by four cells of T'. The diagonally hatched cells belong to Ω. Three admissible configurations are shown at the top. The inadmissible configuration is shown at the bottom.

cells of T' with vertices of Ω^e_{k-1} as their centroids. By the centroid of a cell we mean the vertex corresponding to the intersection of diagonals of this cell.

We observe that $\Omega^e_{k-1} \subset \Omega^e_k, k \geqslant 1$. An example of a domain Ω and its dilatation Ω^e_1 are shown in Fig. 4.

Definition 3. *We say that a domain Ω admits an offset at a distance of* 0 *if Ω is a two-dimensional topological manifold with boundary. We say that a domain Ω admits an offset at a distance of $\frac{1}{2}$ if the following requirements are satisfied:*

1. *The dilatation domain Ω^e_1 is a two-dimensional topological manifold with boundary.*
2. *All the inner vertices of Ω^e_1 are exactly the centroids of cells of the domain Ω.*
3. *All the inner edges of Ω^e_1 are intersected by edges of the domain Ω.*

We say that a domain Ω admits an offset at a distance of $\frac{k}{2}$ for $k \geqslant 2$ if Ω admits an offset at a distance of $\frac{k-1}{2}$ and Ω^e_{k-1} admits an offset at a distance of $\frac{1}{2}$. We denote by $\mathcal{A}^2_k$ the class of two-dimensional domains that admit an offset at a distance of $\frac{k}{2}$.

Giannelli and Jütller [19] stated the original definition of the class $\mathcal{A}^2_1$ in terms of the admissible types of intersections between Ω and cells in the complement $\mathbb{R}^2 \setminus \Omega$:

Definition 4 ([19], Definition 15)**.** *The offset region R from Ω is defined as the set of cells of T' that are not in Ω but have at least one point in common with Ω. We say that a domain Ω has an offset at a distance of $\frac{1}{2}$ if every cell in the offset region R intersects with the domain Ω in any of the following three ways:*

1. *a cell from R shares only one vertex with Ω (see Fig. 5, left);*
2. *a cell from R shares only one edge with Ω (see Fig. 5, center);*
3. *a cell from R shares two adjacent edges with Ω (see Fig. 5, right).*

These three types of admissible intersections are shown in Fig. 5.

The proof of the equivalence of Definition 3 and Definition 4 for the class $\mathcal{A}^2_1$ is straightforward. Indeed, let Ω have an offset at a distance of $\frac{1}{2}$ according to Definition 4. Then, Items 1 and 2 of Definition 3 follow from the observation that the set of boundary vertices of Ω_1^e coincides with the set of centroids of the cells of the offset region R, and all the neighborhoods of boundary vertices of Ω_1^e have admissible configurations (see Fig. 3). It can be seen that Item 3 of Definition 3 is equivalent to the following: if two vertices of an edge belong to the intersection between Ω and a cell from R, then the whole edge belongs to this intersection. Therefore, Item 3 of Definition 3 is satisfied because of the types of intersections between Ω and cells of R (see Fig. 5).

Conversely, let Ω admit an offset at a distance of $\frac{1}{2}$ according to Definition 3. Then, it can be verified that the types of intersections between Ω and a cell in R shown in Fig. 5 are the only ones to satisfy all the items of Definition 3.

An example of a domain Ω that belongs to $\mathcal{A}_1^2$ but does not belong to $\mathcal{A}_2^2$ is shown in Fig. 4.

Remark 1. *The main objective of using Definition 3 for the class $\mathcal{A}_k^2$ rather than the original one [19] is being able to define the classes $\mathcal{A}_k^2$ more formally and accurately for all nonnegative integers k and provide a simple proof of the formula for the number of vertices in the dilatation domain Ω_k^e.*

Remark 2. *We note that Definition 1 for the class $\mathcal{A}_1^1$ can be given in the same way as Definition 4: a one-dimensional domain Ω has an offset at a distance of $\frac{1}{2}$ if every cell in the offset region intersects with the domain Ω such that a cell from the offset region shares only one vertex with Ω. The offset region consists of the cells of T' that are not in Ω but have at least one common vertex with this domain. The class $\mathcal{A}_k^1$ can be analogously defined by induction for any integer $k \geqslant 0$.*

Proposition 3. *Let f_2, f_1, and f_0 be the numbers of cells, edges and vertices of Ω. Then, the following identities hold:*

$$f_1 = 4f_2 - f_1^0, \tag{3}$$

$$f_0 = 4f_2 - 2f_1^0 + f_0^0, \tag{4}$$

where f_1^0 and f_0^0 are the numbers of inner edges and inner vertices of Ω, respectively.

Proof. It is easy to see that the number of boundary edges equals $4f_2 - 2f_1^0$. Thus, $4f_2 - 2f_1^0 = f_1 - f_1^0$, which implies (3). As long as Ω is a two-dimensional

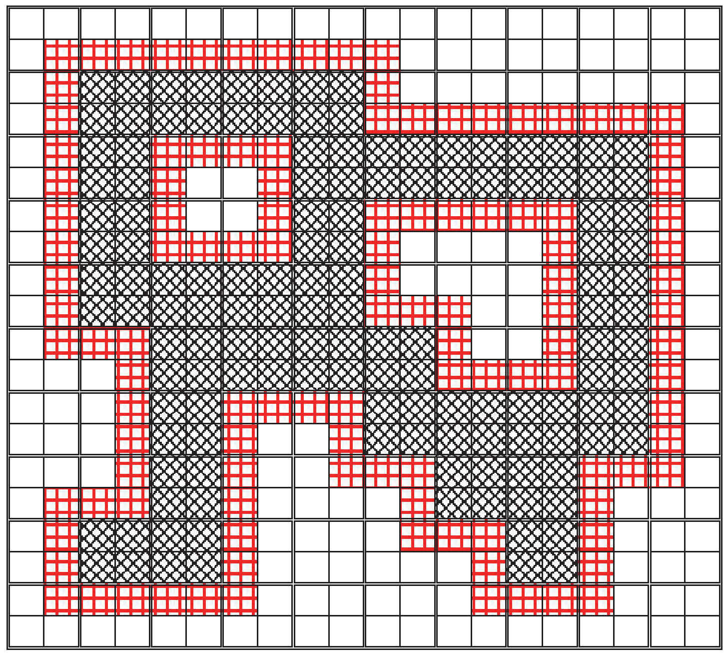

Figure 4: The cells of a domain Ω are diagonally hatched. The domain Ω_1^e is the whole shaded area and might be formed by cells of T''.

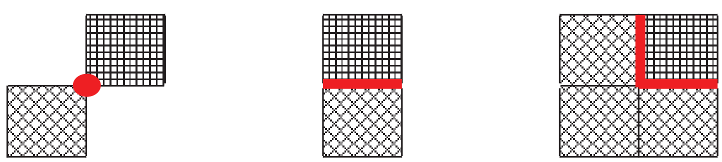

Figure 5: Three admissible types of intersections: one shared vertex (left), one shared edge (center), and two shared adjacent edges (right). The cells of a domain Ω are diagonally hatched. The cells from an offset region R are gridded.

topological manifold with boundary, the boundary $\partial\Omega$ falls into piecewise linear curves that are connected, closed, and free of self-intersections. For each of these curves the number of edges is equal to the number of vertices. Thus, $f_1 - f_1^0 = f_0 - f_0^0$, which implies (4). □

Theorem 2. *Let $\Omega \in A_k^2$ for a nonnegative integer k. Let $f_{2,k}$, $f_{1,k}$, and $f_{0,k}$ be the numbers of cells, edges, and vertices of the dilatation domain Ω_k^e. Then, the following identities hold:*

$$f_{2,k} = (k+1)^2 f_2 - k(k+1) f_1^0 + k^2 f_0^0, \tag{5}$$

$$f_{1,k} = 2(k+2)(k+1) f_2 - (1+4k+2k^2) f_1^0 + 2k(k+1) f_0^0, \tag{6}$$

$$f_{0,k} = (k+2)^2 f_2 - (k+2)(k+1) f_1^0 + (k+1)^2 f_0^0, \tag{7}$$

where f_2, f_1^0, and f_0^0 are the numbers of cells, inner edges, and inner vertices of Ω, respectively.

Proof. We will prove the theorem by induction on k. If $k = 0$, then (5) is straightforward, while (6) and (7) are the direct consequences of (3) and (4), respectively. Suppose that the theorem is proved for an integer $k-1$, $k \geqslant 1$. By Definition 3, we have

$$f_{2,k} = f_{0,k-1} = (k+1)^2 f_2 - (k+1)k f_1^0 + k^2 f_0^0, \tag{8}$$

$$f_{1,k}^0 = f_{1,k-1} = 2(k+1)k f_2 - (1+4(k-1)+2(k-1)^2) f_1^0 + 2k(k-1) f_0^0, \tag{9}$$

$$f_{0,k}^0 = f_{2,k-1} = k^2 f_2 - k(k-1) f_1^0 + (k-1)^2 f_0^0, \tag{10}$$

where $f_{1,k}^0$ and $f_{0,k}^0$ are the numbers of inner edges and inner vertices of the dilatation domain Ω_k^e, respectively. Since $\Omega \in \mathcal{A}_k^2$, the dilatation domain Ω_k^e is a topological manifold with boundary, and from Proposition 3 we obtain

$$f_{1,k} = 4 f_{2,k} - f_{1,k}^0, \tag{11}$$

$$f_{0,k} = 4 f_{2,k} - 2 f_{1,k}^0 + f_{0,k}^0. \tag{12}$$

Substituting (8), (9), and (10) into (11) and (12), we prove (6) and (7). The identity (5) was proved already by (8). □

Corollary 2. *Let Ω be a two-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}_{m-1}^2$. Then, the following identity holds:*

$$\mathcal{N} = (m+1)^2 f_2 - m(m+1) f_1^0 + m^2 f_0^0. \tag{13}$$

Proof. Each $(m+1)\times(m+1)$ square from $\widehat{\mathcal{B}}$ is associated with its centroid. If m is odd, then this centroid is a grid node of T', and if m is even, then this centroid is a grid node of T''. It is clear that an element of $\widehat{\mathcal{B}}$ has at least one cell in common with Ω if and only if its centroid is a vertex of the dilatation domain Ω^e_{m-1}. Thus, $\mathcal{N} = f_{0,m-1}$, and from (7) we obtain (13). $\square$

3.2. Dimension and basis of a spline space over a two-dimensional domain

Following Mourrain [20], in this subsection we will use standard homological algebraic techniques (e.g., see [21]) to find the dimension of a spline space $\mathcal{S}_m(\mathcal{T})$, for $\mathcal{T}$ defined over a simply connected domain Ω. Then, supposing that $\Omega \in \mathcal{A}^2_{m-1}$, we will prove Corollary 4, where $\dim \mathcal{S}_m(\mathcal{T})$ will be derived. We will finish this subsection by proving Theorem 3 in which a basis of $\mathcal{S}_m(\mathcal{T})$ for $\Omega \in \mathcal{A}^2_{m-1}$ will be obtained.

Let us mention briefly the notation and definitions used in [20]. For more details and examples, refer to [20]. For a given two-dimensional domain Ω and the corresponding T-mesh $\mathcal{T}$, let $\mathcal{T}_2$, $\mathcal{T}^0_1$, and $\mathcal{T}^0_0$ be the sets of cells, inner edges, and inner vertices of a domain Ω. For a given horizontal or vertical inner edge $\tau \in \mathcal{T}^0_1$, let $\mathcal{J}_m(\tau) = I[\tau] \cap R_m$, where $I[\tau]$ is the ideal generated by the polynomial $(y-y_\tau)^m$ or $(x-x_\tau)^m$, where $y = y_\tau$ or $x = x_\tau$ is the equation of the horizontal or vertical line containing τ. For a given inner vertex $\gamma \in \mathcal{T}^0_0$, let $\mathcal{J}_m(\gamma) = I[\gamma] \cap R_m$, where $I[\gamma]$ is the ideal generated by the polynomials $(x-x_\gamma)^m$ and $(y-y_\gamma)^m$, where x_γ and y_γ are the coordinates of γ. We use the following short exact sequence of chain complexes [20]:

$$
\begin{array}{lccccccc}
 & & & \mathbf{0} & & \mathbf{0} & & \\
 & & & \downarrow & & \downarrow & & \\
\mathfrak{I}_m(\mathcal{T}^0): & \mathbf{0} & \xrightarrow{\widehat{\partial}_2} & \bigoplus\limits_{\tau\in\mathcal{T}^0_1} [\tau]\,\mathcal{J}_m(\tau) & \xrightarrow{\widehat{\partial}_1} & \bigoplus\limits_{\gamma\in\mathcal{T}^0_0} [\gamma]\,\mathcal{J}_m(\gamma) & \xrightarrow{\widehat{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \\
R_m(\mathcal{T}^0): & \bigoplus\limits_{\sigma\in\mathcal{T}_2} [\sigma]\,R_m & \xrightarrow{\partial_2} & \bigoplus\limits_{\tau\in\mathcal{T}^0_1} [\tau]\,R_m & \xrightarrow{\partial_1} & \bigoplus\limits_{\gamma\in\mathcal{T}^0_0} [\gamma]\,R_m & \xrightarrow{\partial_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{S}_m(\mathcal{T}^0): & \bigoplus\limits_{\sigma\in\mathcal{T}_2} [\sigma]\,R_m & \xrightarrow{\overline{\partial}_2} & \bigoplus\limits_{\tau\in\mathcal{T}^0_1} [\tau]\,R_m/\mathcal{J}_m(\tau) & \xrightarrow{\overline{\partial}_1} & \bigoplus\limits_{\gamma\in\mathcal{T}^0_0} [\gamma]\,R_m/\mathcal{J}_m(\gamma) & \xrightarrow{\overline{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & &
\end{array}
$$

The vector spaces of these complexes are the modules generated by formal elements $[\sigma], [\tau]$, and $[\gamma]$ associated with the oriented cells, inner edges, and inner vertices of the T-mesh $\mathcal{T}$. The differentials of the chain complex $R_m\left(\mathcal{T}^0\right)$ are similar to some for cellular homologies (e.g., see [21]) and defined as follows:

- for each interior vertex $\gamma \in \mathcal{T}_0^0$, $\partial_0([\gamma]) = 0$;
- for each interior edge $\tau \in \mathcal{T}_1^0$ from γ_1 to $\gamma_2 \in \mathcal{T}_0$, $\partial_1([\tau]) = [\gamma_2] - [\gamma_1]$, where $[\gamma] = 0$ if $\gamma \notin \mathcal{T}_0^0$;
- for each cell $\sigma \in \mathcal{T}_2$ with its counter-clockwise boundary formed by four edges $a_1a_2, \ldots, a_4a_1$, $\partial_2([\sigma]) = [a_1a_2] \oplus [a_2a_3] \oplus [a_3a_4] \oplus [a_4a_1]$.

The only nontrivial differential $\widehat{\partial}_1$ of the complex $\mathfrak{J}_m\left(\mathcal{T}^0\right)$ is obtained from ∂_1 by restriction to the subspace $\bigoplus\limits_{\tau\in\mathcal{T}_1^0} [\tau]\, \mathcal{J}_m\left(\tau\right) \subset \bigoplus\limits_{\tau\in\mathcal{T}_1^0} [\tau]\, R_m$. The differentials $\overline{\partial}_i, i = 0, 1, 2$ of the complex $\mathfrak{G}_m\left(\mathcal{T}^0\right)$ are obtained from $\partial_i, i = 0, 1, 2$ by taking the quotient with the corresponding vector spaces of the complex $\mathfrak{J}_m\left(\mathcal{T}^0\right)$. The vertical maps of the diagram above are respectively the inclusion maps and the quotient maps. A reader familiar with algebraic topology (e.g., see [21]) may easily recognize that $R_m\left(\mathcal{T}^0\right)$ is the chain complex $R_m\left(\mathcal{T}\right)$ (defined by the T-mesh $\mathcal{T}$) relative to $R_m\left(\partial\mathcal{T}\right)$ (defined by the T-mesh boundary $\partial\mathcal{T}$).

The following lemma is the key that enables us to apply the homology technique to the theory of splines:

Lemma 1 ([22] and [20], Lemma 1.6)**.** *Let $\tau \in \mathcal{T}_1^0$ and let $p_1, p_2 \in R_m$ be two polynomials defined over cells of a T-mesh $\mathcal{T}$ sharing the edge τ. Their derivatives coincide on τ up to order $m-1$ if and only if $p_1 - p_2 \in \mathcal{J}_m(\tau)$.*

The following proposition establishes the one-to-one correspondence between the spline space $\mathcal{S}_m(\mathcal{T})$ of a T-mesh $\mathcal{T}$ and the homology module $H_2(\mathfrak{G}_m(\mathcal{T}^0))$:

Proposition 4 ([20], Proposition 2.9)**.**

$$H_2(\mathfrak{G}_m(\mathcal{T}^0)) = \ker\overline{\partial}_2 = S_m(\mathcal{T}). \tag{14}$$

Proof. The first equality follows from the definition of a homology module (e.g., see [21]). An element $\sum\limits_{\sigma\in\mathcal{T}_2} p_\sigma\,[\sigma]$ is in the kernel of $\overline{\partial}_2$ if $p_\sigma \equiv$

$p_{\sigma'} \bmod \mathcal{J}_m(\tau)$ for any σ and σ' sharing an edge $\tau \in \mathcal{T}_1^0$. It follows from Lemma 1 that the piecewise polynomial function (which is p_σ on a cell σ) is a function in class C^{m-1} and thus is an element of $\mathcal{S}_m(\mathcal{T})$. □

In the following lemma we will obtain the general formula for the dimension of the spline space $\mathcal{S}_m(\mathcal{T})$ in terms of the numbers of cells, inner edges, and inner vertices as well as the dimensions of the corresponding homology modules $H_1(\mathfrak{G}_m(\mathcal{T}^0))$ and $H_0(\mathfrak{G}_m(\mathcal{T}^0))$.

Lemma 2 ([20], proof of Theorem 3.1)**.**

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^2 f_2 - m(m+1) f_1^0 + m^2 f_0^0 + \\ \dim(H_1(\mathfrak{G}_m(\mathcal{T}^0))) - \dim(H_0(\mathfrak{G}_m(\mathcal{T}^0))).$$

Proof. It follows from the definition of a chain complex (e.g., see [21]) that

$$\dim(H_2(\mathfrak{G}_m(\mathcal{T}^0))) - \dim(H_1(\mathfrak{G}_m(\mathcal{T}^0))) + \dim(H_0(\mathfrak{G}_m(\mathcal{T}^0))) = \\ \dim(\oplus_{\sigma\in\mathcal{T}_2}[\sigma]R_m) - \dim(\oplus_{\tau\in\mathcal{T}_1^0}[\tau]R_m/\mathfrak{J}_m(\tau)) + \dim(\oplus_{\gamma\in\mathcal{T}_0^0}[\gamma]R_m/\mathfrak{J}_m(\gamma)).$$

The dimensions of the vector spaces R_m, $R_m/\mathfrak{J}_m(\tau)$ and $R_m/\mathfrak{J}_m(\gamma)$ are $(m+1)^2$, $m(m+1)$, and m^2, respectively (see [20], Lemma 1.5). By Proposition 4, the lemma is proved. □

To be able to prove that homology modules $H_1(\mathfrak{G}_m(\mathcal{T}^0))$ and $H_0(\mathfrak{G}_m(\mathcal{T}^0))$ are zero for a certain type of T-mesh $\mathcal{T}$, let us recall one of the basic algebraic consequences of the short exact sequence of chain complexes $\mathfrak{J}_m(\mathcal{T}^0)$, $R_m(\mathcal{T}^0)$, and $\mathfrak{G}_m(\mathcal{T}^0)$.

Proposition 5 ([20], proof of Proposition 2.7)**.** *There exists a long exact sequence*

$$\begin{aligned} 0 \longrightarrow H_2(R_m(\mathcal{T}^0)) \longrightarrow H_2(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow H_1(\mathfrak{J}_m(\mathcal{T}^0)) \longrightarrow \\ H_1(R_m(\mathcal{T}^0)) \longrightarrow H_1(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow H_0(\mathfrak{J}_m(\mathcal{T}^0) \longrightarrow \\ H_0(R_m(\mathcal{T}^0)) \longrightarrow H_0(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow 0. \end{aligned} \tag{15}$$

Proof. The proof of the long exact sequence can be found in [21], for example. □

Lemma 3 ([20], Proposition D.1 and Corollary 3.2)**.**

$$H_0(R_m(\mathcal{T}^0)) = H_0(\mathfrak{G}_m(\mathcal{T}^0)) = H_0(\mathfrak{J}_m(\mathcal{T}^0)) = 0. \tag{16}$$

Proof. Let $\gamma \in \mathcal{T}_0^0$ be an inner vertex defined by its coordinates (x_γ, y_γ). There is a sequence of horizontal edges $\tau_0^h = \gamma_0^h\gamma_1^h, \tau_1^h = \gamma_1^h\gamma_2^h, \dots, \tau_l^h = \gamma_l^h\gamma$ such that $\gamma_0^h \in \partial\mathcal{T}$. Then,

$$\partial_1([\tau_0^h] + \dots + [\tau_l^h]) = [\gamma_1^h] + [\gamma_2^h] - [\gamma_1^h] + \dots + [\gamma] - [\gamma_l^h] = [\gamma]. \tag{17}$$

Thus, for a given polynomial $p \in R_m$, we have $p[\gamma] \in \operatorname{im}\partial_1$, and therefore the homology module $H_0(R_m(\mathcal{T}^0))$ is zero. In fact, $H_0(R_m(\mathcal{T}^0)) = 0$ for any T-mesh $\mathcal{T}$ (see [20], Proposition D.1). It follows from the long exact sequence (15) that the homology module $H_0(\mathfrak{G}_m(\mathcal{T}^0))$ must be zero.

There is a sequence of vertical edges $\tau_0^v = \gamma_0^v\gamma_1^v, \tau_1^v = \gamma_1^v\gamma_2^v, \dots, \tau_k^v = \gamma_k^v\gamma$ such that $\gamma_0^v \in \partial\mathcal{T}$. For a given polynomial $p = p_1(x - x_\gamma)^m + p_2(y - y_\gamma)^m \in \mathcal{J}_m(\gamma)$, the following can be obtained:

$$\begin{aligned}\partial_1((p_1(x - x_\gamma)^m[\tau_0^v] + \dots + p_1(x - x_\gamma)^m[\tau_k^v]) +\\ (p_2(y - y_\gamma)^m[\tau_0^h] + \dots + p_2(y - y_\gamma)^m[\tau_l^h])) =\\ (p_1(x - x_\gamma)^m + p_2(y - y_\gamma)^m)[\gamma] = p[\gamma].\end{aligned} \tag{18}$$

Thus, the homology module obeys $H_0(\mathfrak{J}_m(\mathcal{T}^0)) = 0$. $\square$

In the following lemma we will prove that $H_1(\mathfrak{G}_m(\mathcal{T}^0))$ is zero for a T-mesh $\mathcal{T}$ defined over a simply connected domain Ω.

Lemma 4 ([20], Proposition D.2)**.** *Suppose that a domain Ω corresponding to the T-mesh $\mathcal{T}$ is simply connected. Then,*

$$H_1(R_m(\mathcal{T}^0)) = H_1(\mathfrak{G}_m(\mathcal{T}^0)) = 0. \tag{19}$$

Proof. Since Ω is simply connected, the factor space $\Omega/\partial\Omega$ is homeomorphic to the two-dimensional sphere S^2 and the relative homology group [21] obeys $H_1(\Omega, \partial\Omega) \cong H_1(\Omega/\partial\Omega) \cong H_1(S^2) = 0$, which implies that the homology module obeys $H_1(R_m(\mathcal{T}^0)) = 0$. It follows from the long exact sequence (15) and Lemma 3 that the homology module $H_1(\mathfrak{G}_m(\mathcal{T}^0))$ is zero, and so the lemma is proved.$\square$

Corollary 3. *Suppose that a domain Ω corresponding to the T-mesh $\mathcal{T}$ is simply connected. Then,*

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^2 f_2 - m(m+1) f_1^0 + m^2 f_0^0. \tag{20}$$

where f_2, f_1^0, and f_0^0 are the numbers of cells, inner edges, and inner vertices of a domain Ω, respectively.

Proof. This corollary is a direct consequence of Lemmas 3 and 4 (i.e., the facts that $H_0(\mathfrak{G}_m(\mathcal{T}^0)) = 0$ and $H_1(\mathfrak{G}_m(\mathcal{T}^0)) = 0$) and Lemma 2. $\square$

Remark 3. *We note that Corollary 3 can be also obtained as a direct consequence of Theorem 3.3 and Corollary 3.2 [20].*

Remark 4. *The main reason for us to recall some elements of homological algebra used in [20] is to prepare for proving the dimension formula for trivariate splines similar to those given in Corollary 3.*

In the following lemma we will obtain the dimension of a spline space $\dim \mathcal{S}_m(\mathcal{T})$ if the corresponding domain Ω is split into two domains Ω_1 and Ω_2 (see Fig. 6). Let U' be a grid line of T'. We say that U' splits a domain Ω into two nonempty domains if $\Omega = \Omega_1 \bigcup \Omega_2$, where Ω_1 and Ω_2 are contained in different half-spaces divided by U'. We denote by U the corresponding one-dimensional domain $U = \Omega_1 \cap \Omega_2$ formed by one-dimensional cells of U'.

Lemma 5. *Let a grid line U' split a domain Ω into two domains: $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = U$. Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$ be the T-meshes corresponding to Ω_1, Ω_2, and Ω, respectively. For a given integer $m \geqslant 1$, suppose that the dimensions of the spaces $\mathcal{S}_m(\mathcal{T}_1)$ and $\mathcal{S}_m(\mathcal{T}_2)$ can be obtained from* (20)*. In addition, suppose that $U \in \mathcal{A}^1_{m-1}$ with respect to the infinite one-dimensional grid U'. Then, the dimension of the spline space $\mathcal{S}_m(\mathcal{T})$ is given by* (20) *as well.*

Proof. Without loss of generality, we suppose that U' is a vertical line $x = 0$. Let $\mathcal{U}$ be the one–dimensional mesh, which is the portion of the grid U' over a domain U. We denote by $\mathcal{S}_m(\mathcal{U})$ the space of C^{m-1} smooth functions defined on U that are univariate polynomials of degree m on each cell of U. Let $\mathcal{S}_m(\mathcal{U})^m$ be the direct sum of m copies of the space $\mathcal{S}_m(\mathcal{U})$. We can define the linear operator

$$\mathcal{G} : \mathcal{S}_m(\mathcal{T}_1) \oplus \mathcal{S}_m(\mathcal{T}_2) \to \mathcal{S}_m(\mathcal{U})^m$$

as follows: for given splines $\phi_1 \in \mathcal{S}_m(\mathcal{T}_1)$ and $\phi_2 \in \mathcal{S}_m(\mathcal{T}_2)$ the corresponding spline-vector $\mathcal{G}(\langle \phi_1, \phi_2 \rangle) \in \mathcal{S}_m(\mathcal{U})^m$ equals

$$\Big\langle (\phi_1 - \phi_2)|_{x=0}, \frac{\partial(\phi_1 - \phi_2)}{\partial x}|_{x=0}, \ldots, \frac{\partial^{m-1}(\phi_1 - \phi_2)}{\partial x^{m-1}}|_{x=0} \Big\rangle,$$

so $\ker \mathcal{G} = \mathcal{S}_m(\mathcal{T})$. Thus, we obtain

$$\begin{gathered} \dim \mathcal{S}_m(\mathcal{T}) = \dim \mathcal{S}_m(\mathcal{T}_1) + \dim \mathcal{S}_m(\mathcal{T}_2) - \dim \operatorname{im} \mathcal{G} = \\ (m+1)^2 f_2 - m(m+1)(f_1^0 - h_1) + m^2(f_0^0 - h_0^0) - \dim \operatorname{im} \mathcal{G} \end{gathered} \tag{21}$$

where f_2, f_1^0, and f_0^0 are the numbers of cells, inner edges, and inner vertices of Ω, while h_1 and h_1^0 are the numbers of cells and inner vertices of the one-dimensional domain U. We remark that cells and inner vertices of U are inner edges and inner vertices of Ω, respectively, but are not inner edges and inner vertices of Ω_1 and Ω_2. Therefore, $f_1^0 - h_1$ and $f_0^0 - h_0^0$ are the numbers of inner edges and inner vertices contained in either Ω_1 or Ω_2.

In order to prove that $\mathcal{G}$ is an epimorphism, let us take an element of $\mathcal{S}_m(\mathcal{U})^m$: $\psi = \langle \psi_1(y), \ldots, \psi_m(y) \rangle$. It follows from Corollary 1 that there exist splines $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_m$ defined globally over the infinite one-dimensional grid U' such that $\widetilde{\psi}_i|_U = \psi_i, i = 1 \ldots m$. We define a bivariate spline $\phi(x, y)$ globally over T' as follows:

$$\phi(x, y) := \sum_{i=1}^{m} \widetilde{\psi}_i(y) \frac{x^{i-1}}{(i-1)!} + x^m.$$

Let $\phi_1 := \phi|_{\Omega_1}$ and $\phi_2 \equiv 0$ on Ω_2. Then, $\mathcal{G}(\langle \phi_1, \phi_2 \rangle) = \psi$. Thus, by virtue of Proposition 2, we obtain $\dim \operatorname{im} \mathcal{G} = m \dim \mathcal{S}_m(\mathcal{U}) = m(m+1)h_1 - m^2 h_0^0$, and so the lemma is proved. $\square$

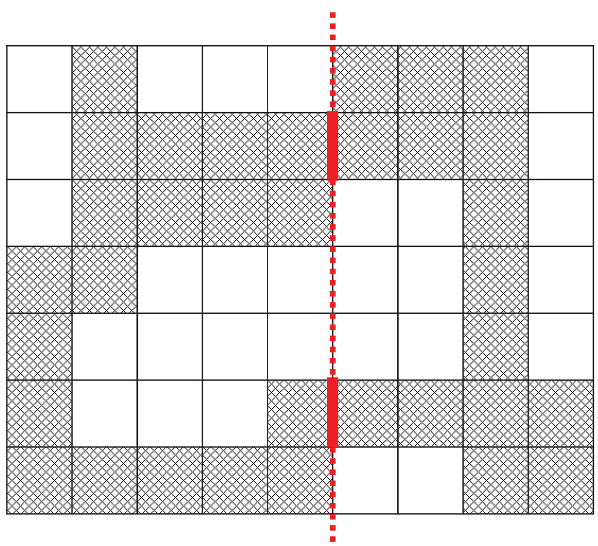

Figure 6: The cells of a domain Ω are diagonally hatched. The grid line U' is denoted by a dotted red line. The domains Ω_1 and Ω_2 are to the left and right of U', respectively. The one-dimensional domain $U = \Omega_1 \bigcap \Omega_2$ is denoted by solid red lines.

Corollary 4. *Let $\Omega \in \mathcal{A}^2_{m-1}$ be a two-dimensional domain and $\mathcal{T}$ be the corresponding T-mesh. Then, the dimension of a space $\mathcal{S}_m(\mathcal{T})$ is*

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^2 f_2 - m(m+1) f_1^0 + m^2 f_0^0, \tag{22}$$

where f_2, f_1^0, and f_0^0 are the numbers of cells, inner edges, and inner vertices of a domain Ω, respectively.

Proof. Suppose that a domain Ω is split into two domains Ω_1 and Ω_2 by a vertical grid line U' of T' (see Fig. 6).

If Ω is a two-dimensional topological manifold with boundary, then $U = \Omega_1 \bigcap \Omega_2$ is a one-dimensional topological manifold with boundary. That is, if $\Omega \in \mathcal{A}^2_0$, then $U \in \mathcal{A}^1_0$. From Definition 4 and Remark 2, it can be seen that if $\Omega \in \mathcal{A}^2_1$, then $U \in \mathcal{A}^1_1$. Moreover, it can be proven by induction on m that if $\Omega \in \mathcal{A}^2_{m-1}$ for a given integer $m \geqslant 1$, then $U \in \mathcal{A}^1_{m-1}$.

Using a sufficient number of vertical grid lines, a domain Ω can be split into pieces that are simply connected. By Lemma 5 and Corollary 3, the corollary is proved. □

Theorem 3. *Suppose that $\Omega \in \mathcal{A}^2_{m-1}$. Then, the basis of a space $\mathcal{S}_m(\mathcal{T})$ can be obtained as follows:*

$$\{b|_\Omega : b(x,y) \in \mathcal{B} \ \wedge \ \operatorname{supp} b(x,y) \cap \operatorname{int} \Omega \neq \varnothing\}. \tag{23}$$

Proof. From Corollaries 2 and 4 we obtain $\mathcal{N} = \dim \mathcal{S}_m(\mathcal{T})$. Finally, the fact that tensor-product B-splines from $\mathcal{B}$ are locally linearly independent proves the theorem. □

Corollary 5. *For a given integer $m \geqslant 1$, let Ω be a two-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}^2_{m-1}$. Let $f \in \mathcal{S}_m(\mathcal{T})$ be a spline function defined over the corresponding T-mesh $\mathcal{T}$. Then, there exists a spline function $\widetilde{f}$ of bi-degree (m,m) defined globally over T' such that $\widetilde{f}|_\Omega = f$.*

Proof. The splines from $\mathcal{B}$ are defined globally over T'. Thus, by Theorem 3, the corollary is proved. □

Remark 5. *The result of Theorem 3 was originally proved by Giannelli and Jüttler [19]. However, their approach to calculating the dimension of a spline space and the number $\mathcal{N}$ is different from ours. The advantage of the technique used in this paper is the ability to generalize Theorem 3 easily to the three-dimensional case.*

4. Trivariate splines

Let T' be a three-dimensional infinite grid. Without loss of generality, we will suppose that the distances between adjacent grid nodes T' are equal to 1. A cell of T' is a closed cube with sides of length 1 aligned with the grid lines of T'.

Let Ω be a closed domain formed by a finite number of cells of T'. A vertex of a domain Ω is a grid node of T' that belongs to Ω. We say that a vertex is a boundary vertex if it belongs to $\partial\Omega$, and we say that a vertex is an inner vertex if it belongs to $\operatorname{int}\Omega$. An edge of a domain Ω is a closed segment between two adjacent grid nodes of T', which is a subset of Ω. We say that an edge is a boundary edge if it is a subset of $\partial\Omega$, and we say that an edge is an inner edge if it is not a boundary edge. A facet of a domain Ω is a closed square with sides of length 1 aligned with the grid lines of T', which is a subset of Ω. We say that a facet is a boundary facet if it is a subset of $\partial\Omega$, and we say that a facet is an inner facet if it is not a boundary facet.

Throughout this section we will suppose that Ω is a three-dimensional topological manifold with boundary. To avoid excessive drawing, we omit the graphical description of the admissible configurations for neighborhoods of boundary vertices which ensure that Ω is a topological manifold with boundary. Additionally, we remark that Ω might have several connected components.

For a given integer $m \geqslant 1$, let $\widehat{\mathcal{B}}$ be the set of $(m+1)\times(m+1)\times(m+1)$ cubes formed by $(m+1)^3$ cells of T', so $\widehat{\mathcal{B}}$ is the set of all possible minimal supports for tensor-product B-splines of tri-degree (m,m,m) defined over T' and with knot multiplicities equal to 1. We denote by $\mathcal{B}$ the collection of B-splines $b(x,y,z)$ whose supports become the elements of $\widehat{\mathcal{B}}$. Let $\mathcal{N}$ be the number of elements of $\widehat{\mathcal{B}}$ that have at least one cell in common with a domain Ω. In Subsection 4.1 we will prove Corollary 6, in which $\mathcal{N}$ will be obtained for a domain $\Omega \in \mathcal{A}^3_{m-1}$ (see Definitions 6 and 7 in Subsection 4.1 for the specification of the class $\mathcal{A}^3_k$, for an integer $k \geqslant 0$).

Let R_m be the vector space of polynomials of tri-degree (m,m,m) with respect to three variables x, y, and z. Let $\mathcal{T}$ be the three-dimensional mesh, which is the portion of T' over a domain Ω. We denote by $\mathcal{S}_m(\mathcal{T})$ the vector space of C^{m-1} smooth functions defined on Ω that are polynomials in R_m on each cell of a domain Ω. In Subsection 4.2 we will prove Corollary 8 in which $\dim\mathcal{S}_m(\mathcal{T})$ will be obtained for a domain $\Omega \in \mathcal{A}^3_m$. Based on the observation that $\dim\mathcal{S}_m(\mathcal{T}) = \mathcal{N}$ for $\Omega \in \mathcal{A}^3_m$, we will prove Theorem 5, in which we will obtain a basis of the space $\mathcal{S}_m(\mathcal{T})$.

4.1. The dilatation of a three-dimensional domain

In this subsection we will introduce the classes of three-dimensional domains $\mathcal{A}_k^3$, for any integer $k \geqslant 0$. For a domain $\Omega \in \mathcal{A}_{m-1}^3, m \geqslant 1$, we will obtain the formula for the number $\mathcal{N}$ (that is the number of tensor-product B-splines of tri-degree (m, m, m) acting effectively on this domain Ω) in terms of the numbers of cells, inner facets, inner edges and inner vertices of Ω.

Let T'' be the infinite grid that is obtained by shifting T' by the vector $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. We first define dilatation domains Ω_k^e of Ω in a recursive manner for $0 \leqslant k \in \mathbb{Z}$.

Definition 5. *If $k = 0$, the dilatation domain $\Omega_0^e := \Omega$. If k is odd, the dilatation domain Ω_k^e is the union of the cells of T'' having vertices of Ω_{k-1}^e as their centroids. If k is even, the dilatation domain Ω_k^e is the union of the cells of T' having vertices of Ω_{k-1}^e as their centroids. By the centroid of a cell we mean the vertex corresponding to the intersection of diagonals of this cell.*

In order to save space, we omit any example of the three-dimensional domain Ω or its dilatation Ω_1^e.

Definition 6. *We say that a domain Ω admits an offset at a distance of 0 if Ω is a three-dimensional topological manifold with boundary. We say that a domain Ω admits an offset at a distance of $\frac{1}{2}$ if the following requirements are satisfied:*

1. *The dilatation domain Ω_1^e is a three-dimensional topological manifold with boundary.*
2. *All the inner vertices of Ω_1^e are exactly the centroids of cells of a domain Ω.*
3. *All the inner edges of Ω_1^e are intersected by facets of a domain Ω.*
4. *All the inner facets of Ω_1^e are intersected by edges of a domain Ω.*

We say that a domain Ω admits an offset at a distance of $\frac{k}{2}$ (for an integer $k \geqslant 2$) if Ω admits an offset at a distance of $\frac{k-1}{2}$ and Ω_{k-1}^e admits an offset at a distance of $\frac{1}{2}$. We denote by $\mathcal{A}_k^3$ the class of three-dimensional domains that admit an offset at a distance of $\frac{k}{2}$.

Reminiscent of Definition 4, the class $\mathcal{A}_1^3$ can be redefined in terms of the admissible types of intersections between Ω and cells in the complement $\mathbb{R}^3 \setminus \Omega$:

Definition 7. *The offset region R from Ω is defined as the set of cells of T' that are not in Ω but have at least one point in common with Ω.*

We say that a domain Ω admits an offset at a distance of $\frac{1}{2}$ if every cell in the offset region R intersects with the domain Ω in any of the following nine ways:

1. *a cell from R shares only one vertex with Ω (see Fig. 7, left);*
2. *a cell from R shares only one edge with Ω (see Fig. 7, center);*
3. *a cell from R shares two adjacent edges with Ω (see Fig. 7, right);*
4. *a cell from R shares with Ω three edges intersecting in one vertex (see Fig. 8, left);*
5. *a cell from R shares with Ω three edges of different directions linked together in consecutive order (see Fig. 8, center);*
6. *a cell from R shares only one facet with Ω (see Fig. 8, right);*
7. *a cell from R shares with Ω one facet and one edge that is orthogonal to this facet (see Fig. 9, left);*
8. *a cell from R shares with Ω two facets intersecting in one edge (see Fig. 9, center);*
9. *a cell from R shares with Ω three facets intersecting in one vertex (see Fig. 9, right)*

All nine types of admissible intersections are shown in Figs. 7–9.

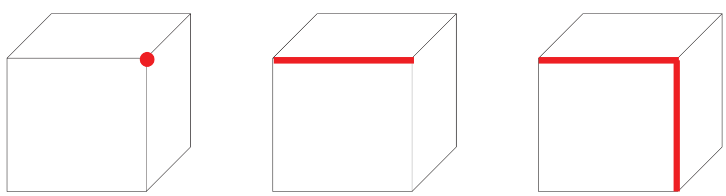

Figure 7: Three of nine admissible types of intersections: one shared vertex (left), one shared edge (center), and two shared adjacent edges (right). The shared vertex (left) is drawn with a bold red dot. The shared edges (center, right) are drawn in bold red lines.

The proof of the equivalence of Definition 6 and Definition 7 for the class $\mathcal{A}_1^3$ is straightforward. Indeed, let Ω have an offset at a distance of $\frac{1}{2}$

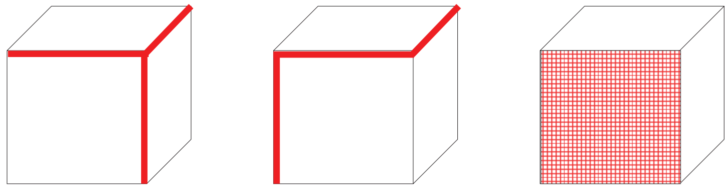

Figure 8: Three of nine admissible types of intersections: three edges intersecting in one vertex (left), three shared edges of different directions linked together in consecutive order (center), and one shared facet (right). The shared edges (left, center) are drawn in bold red lines. The shared facet (right) is shaded by a red grid.

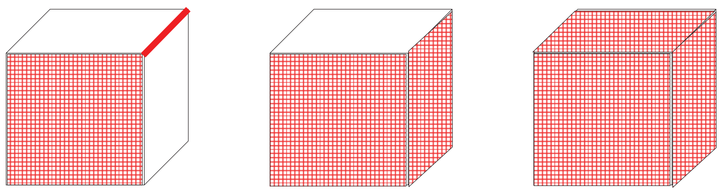

Figure 9: Three of nine admissible types of intersections: one shared facet and one shared edge that is orthogonal to this facet (left), two shared facets intersecting in one edge (center), and three shared facets intersecting in one vertex (right). The shared edge (left) is drawn with a bold red line. The shared facets are shaded by a red grid.

according to Definition 7. Then, Items 1 and 2 of Definition 6 follow from the observation that the set of boundary vertices of Ω_1^e coincides with the set of centroids of cells of the offset region R, and all the neighborhoods of the boundary vertices of Ω_1^e have admissible configurations, so Ω_1^e is a three-dimensional topological manifold with boundary. It can be seen that Item 3 of Definition 6 is equivalent to the following: if four vertices of a facet belong to the intersection between Ω and a cell from R, then the whole facet belongs to this intersection. In addition, Item 4 of Definition 6 is equivalent to the following: if two vertices of an edge belong to the intersection between Ω and a cell from R, then the whole edge belongs to this intersection. Therefore, Items 3 and 4 of Definition 6 are satisfied, because of the types of intersections between Ω and cells of R (see Figs. 7, 8, and 9).

Conversely, let Ω admit an offset at a distance of $\frac{1}{2}$ according to Definition 6. Then, it can be verified that the types of intersections between Ω and a cell from R shown in Figs. 7, 8, and 9 are the only ones to satisfy all the items of Definition 6.

Proposition 6. *Let f_3, f_2, f_1, and f_0 be the numbers of cells, facets, edges,* *and vertices of* Ω*. Then, the following identities hold:*

$$f_2 = 6f_3 - f_2^0, \tag{24}$$

$$f_1 = 12f_3 - 4f_2^0 + f_1^0, \tag{25}$$

$$f_0 = 8f_3 - 4f_2^0 + 2f_1^0 - f_0^0, \tag{26}$$

where f_2^0*,* f_1^0*, and* f_0^0 *are the numbers of inner facets, inner edges, and inner vertices of* Ω*, respectively.*

Proof. It is easy to see that the number of boundary facets equals $6f_3 - 2f_2^0$. Thus, $6f_3 - 2f_2^0 = f_2 - f_2^0$, which implies (24). As long as Ω is a topological manifold with boundary, the boundary $\partial\Omega$ falls into four-vertex polygonal surfaces that are connected, closed, and free of self-intersections. For each of these surfaces, the number of facets is equal to half the number of edges: $f_1 - f_1^0 = 2\,(f_2 - f_2^0)$. Thus, by using (24) we obtain (25). To prove (26), we use the fact that the Euler characteristic of an odd-dimensional compact manifold with boundary M is defined by the Euler characteristic of its boundary ∂M: $2\chi(M) = \chi(\partial M)$[1]. Thus, we have the following identity:

$$2(-f_3 + f_2 - f_1 + f_0) = (f_2 - f_2^0) - (f_1 - f_1^0) + (f_0 - f_0^0). \tag{27}$$

By substituting (24) and (25) into (27) we obtain (26). □

Theorem 4. *Let* $\Omega \in A_k^3$ *for a nonnegative integer* k*. Let* $f_{3,k}$*,* $f_{2,k}$*,* $f_{1,k}$*, and* $f_{0,k}$ *be the numbers of cells, facets, edges, and vertices of the dilatation domain* Ω_k^e*. Then, the following identities hold:*

$$f_{3,k} = (k+1)^3 f_3 - k(k+1)^2 f_2^0 + k^2(k+1) f_1^0 - k^3 f_0^0, \tag{28}$$

$$\begin{aligned} f_{2,k} = 3(k+1)^2(k+2) f_3 - (3k^3 + 9k^2 + 7k + 1) f_2^0 + \\ k(3k^2 + 6k + 2) f_1^0 - 3k^2(k+1) f_0^0, \end{aligned} \tag{29}$$

$$\begin{aligned} f_{1,k} = 3(k+1)(k+2)^2 f_3 - (3k^3 + 12k^2 + 14k + 4) f_2^0 + \\ (3k^3 + 9k^2 + 7k + 1) f_1^0 - 3k(k+1)^2 f_0^0, \end{aligned} \tag{30}$$

[1]For our particular case this observation can be obtained as follows. Suppose that Ω is connected and the boundary surface $\partial\Omega$ has genus g. Then, $\chi(\partial\Omega) = 2 - 2g$. On the other hand, Ω is homotopically equivalent to the wedge sum of g circles, which implies that $\chi(\Omega) = 1 - g$.

$$f_{0,k} = (k+2)^3 f_3 - (k+1)(k+2)^2 f_2^0 + (k+1)^2(k+2) f_1^0 - (k+1)^3 f_0^0, \quad (31)$$

where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges and inner vertices of Ω, respectively.

Proof. We will prove the theorem by induction on k. If $k = 0$, then (28) is straightforward, while (29), (30), and (31) are direct consequences of (24), (25), and (26), respectively. Suppose that the theorem is proved for an integer $k-1$, $k \geqslant 1$. By Definition 6, we have

$$f_{3,k} = f_{0,k-1} = (k+1)^3 f_3 - k(k+1)^2 f_2^0 + k^2(k+1) f_1^0 - k^3 f_0^0, \quad (32)$$

$$f_{2,k}^0 = f_{1,k-1} = 3k(k+1)^2 f_3 + (-3k^3 - 3k^2 + k + 1) f_2^0 + k(3k^2 - 2) f_1^0 - 3k^2(k-1) f_0^0, \quad (33)$$

$$f_{1,k}^0 = f_{2,k-1} = 3k^2(k+1) f_3 + k(2 - 3k^2) f_2^0 + (3k^3 - 3k^2 - k + 1) f_1^0 - 3k(k-1)^2 f_0^0, \quad (34)$$

$$f_{0,k}^0 = f_{3,k-1} = k^3 f_3 - (k-1)k^2 f_2^0 + (k-1)^2 k f_1^0 - (k-1)^3 f_0^0, \quad (35)$$

where $f_{2,k}^0$, $f_{1,k}^0$, and $f_{0,k}^0$ are the numbers of inner facets, inner edges, and inner vertices of the dilatation domain Ω_k^e, respectively. Since $\Omega \in \mathcal{A}_k^3$, the dilatation domain Ω_k^e is a three-dimensional topological manifold with boundary, and from Proposition 6 we obtain

$$f_{2,k} = 6 f_{3,k} - f_{2,k}^0 \quad (36)$$

$$f_{1,k} = 12 f_{3,k} - 4 f_{2,k}^0 + f_{1,k}^0, \quad (37)$$

$$f_{0,k} = 8 f_{3,k} - 4 f_{2,k}^0 + 2 f_{1,k}^0 - f_{0,k}^0. \quad (38)$$

Substituting (32), (33), (34), and (35) into (36), (37), and (38), we prove (29), (30), and (31). The identity (28) is proved already by (32). □

Corollary 6. *Let Ω be a three-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}_{m-1}^3$. Then, the following identity holds:*

$$\mathcal{N} = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0. \quad (39)$$

Proof. Each $(m+1) \times (m+1) \times (m+1)$ cube from $\widehat{\mathcal{B}}$ is associated with its centroid. If m is odd, then this centroid is a grid node of T', and if m is even, then this centroid is a grid node of T''. It is clear that an element of $\widehat{\mathcal{B}}$ has at least one cell in common with Ω if and only if its centroid is a vertex of the dilatation domain Ω_{m-1}^e. Thus, $\mathcal{N} = f_{0,m-1}$ and from (31) we obtain (39). □

4.2. Dimension and basis of a spline space over a three-dimensional domain

The approach used in this subsection is analogous to the one used in Subsection 3.2. Applying the homology technique, we will obtain the dimension of a spline space $\mathcal{S}_m(\mathcal{T})$ for $\mathcal{T}$ defined over a domain Ω such that $\partial\Omega$ is a two-dimensional sphere (topologically) and for each grid plane splitting Ω the corresponding two-dimensional slice is simply connected. Then, assuming that $\Omega \in \mathcal{A}^3_{m-1}$, we will prove Corollary 8, whereby $\dim \mathcal{S}_m(\mathcal{T})$ will be derived. We will finish this subsection by proving Theorem 5 in which a basis of $\mathcal{S}_m(\mathcal{T})$ for $\Omega \in \mathcal{A}^3_{m-1}$ will be obtained.

Let us specify the notation used in this subsection. For a given three-dimensional domain Ω and the corresponding three-dimensional mesh $\mathcal{T}$, let $\mathcal{T}^3$, $\mathcal{T}^0_2$, $\mathcal{T}^0_1$, and $\mathcal{T}^0_0$ be the sets of cells, inner facets, inner edges, and inner vertices of a domain Ω. For a given inner facet $\sigma \in \mathcal{T}^0_2$ that is parallel to the yz-plane, xz-plane, or xy-plane, let $\mathcal{J}_m(\sigma) = I[\sigma] \cap R_m$, where $I[\sigma]$ is the ideal generated by the polynomial $(x - x_\sigma)^m$, $(y - y_\sigma)^m$, or $(z - z_\sigma)^m$, where $x = x_\sigma$, $y = y_\sigma$, or $z = z_\sigma$ is the equation of the plane containing σ.

For a given inner edge $\tau \in \mathcal{T}^0_1$ that is parallel to the x-axis, y-axis, or z-axis, let $\mathcal{J}_m(\tau) = I[\tau] \cap R_m$, where $I[\tau]$ is the ideal generated by the polynomials $(y - y_\tau)^m$ and $(z - z_\tau)^m$, $(x - x_\tau)^m$ and $(z - z_\tau)^m$, or $(x - x_\tau)^m$ and $(y - y_\tau)^m$, where $y = y_\tau$ and $z = z_\tau$, $x = x_\tau$ and $z = z_\tau$, or $x = x_\tau$ and $y = y_\tau$ are the equations of the lines containing τ. For a given inner vertex $\gamma \in \mathcal{T}^0_0$, let $\mathcal{J}_m(\gamma) = I[\gamma] \cap R_m$, where $I[\gamma]$ is the ideal generated by the polynomials $(x - x_\gamma)^m$, $(y - y_\gamma)^m$, and $(z - z_\gamma)^m$, where x_γ, y_γ, and z_γ are the coordinates of γ.

As in Subsection 3.2, we construct the following short exact sequence of chain complexes $\mathfrak{J}_m(\mathcal{T}^0)$, $R_m(\mathcal{T}^0)$, and $\mathfrak{G}_m(\mathcal{T}^0)$:

$$\begin{array}{ccccccccccc}
 & & & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{J}_m(\mathcal{T}^o): & \mathbf{0} & \xrightarrow{\widehat{\partial}_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{J}_m(\sigma) & \xrightarrow{\widehat{\partial}_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{J}_m(\tau) & \xrightarrow{\widehat{\partial}_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{J}_m(\gamma) & \xrightarrow{\widehat{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{R}_m(\mathcal{T}^o): & \bigoplus_{\mu\in\mathcal{T}_3}[\mu]\,\mathcal{R}_m & \xrightarrow{\partial_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{R}_m & \xrightarrow{\partial_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{R}_m & \xrightarrow{\partial_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{R}_m & \xrightarrow{\partial_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{G}_m(\mathcal{T}^o): & \bigoplus_{\mu\in\mathcal{T}_3}[\mu]\,\mathcal{R}_m & \xrightarrow{\overline{\partial}_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{R}_m/\mathcal{J}_m(\sigma) & \xrightarrow{\overline{\partial}_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{R}_m/\mathcal{J}_m(\tau) & \xrightarrow{\overline{\partial}_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{R}_m/\mathcal{J}_m(\gamma) & \xrightarrow{\overline{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & &
\end{array}$$

The proofs of Lemmas 6 and 7 and Propositions 7 and 8 are similar to those for Lemmas 1 and 2, and Propositions 4 and 5, respectively. Therefore, we omit the proofs for the sake of brevity.

Lemma 6. *Let $\sigma \in \mathcal{T}_2^0$ and let $p_1, p_2 \in R_m$ be two polynomials defined over cells of a three-dimensional mesh $\mathcal{T}$ sharing the facet σ. Their derivatives coincide on σ up to order $m-1$ if and only if $p_1 - p_2 \in \mathcal{J}_m(\sigma)$.*

Proposition 7.

$$H_3(\mathfrak{G}_m(\mathcal{T}^0)) = \ker\overline{\partial}_3 = S_m(\mathcal{T}). \tag{40}$$

Lemma 7.

$$\begin{aligned}\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0 + \\ \dim(H_2(\mathfrak{G}_m(\mathcal{T}^0))) - \dim(H_1(\mathfrak{G}_m(\mathcal{T}^0))) + \dim(H_0(\mathfrak{G}_m(\mathcal{T}^0))).\end{aligned}$$

Proposition 8. *There exists a long exact sequence*

$$\begin{aligned} 0 \longrightarrow H_3(R_m(\mathcal{T}^0)) \longrightarrow H_3(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow H_2(\mathfrak{J}_m(\mathcal{T}^0)) \longrightarrow \\ H_2(R_m(\mathcal{T}^0)) \longrightarrow H_2(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow H_1(\mathfrak{J}_m(\mathcal{T}^0)) \longrightarrow \\ H_1(R_m(\mathcal{T}^0)) \longrightarrow H_1(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow H_0(\mathfrak{J}_m(\mathcal{T}^0)) \longrightarrow \\ H_0(R_m(\mathcal{T}^0)) \longrightarrow H_0(\mathfrak{G}_m(\mathcal{T}^0)) \longrightarrow 0 \end{aligned} \tag{41}$$

The proof of Lemma 8 below is straightforward and similar to that of Lemma 3. However, let us give a short proof of the identity $H_0(\mathfrak{J}_m(\mathcal{T}^0)) = 0$ to show a particularity of the three-dimensional case.

Lemma 8.

$$H_0(R_m(\mathcal{T}^0)) = H_0(\mathfrak{G}_m(\mathcal{T}^0)) = H_0(\mathfrak{J}_m(\mathcal{T}^0)) = 0. \tag{42}$$

Proof. It is easy to see that $H_0(R_m(\mathcal{T}^0)) = 0$ (see also the proof in Lemma 3). It follows from the long exact sequence (41) that the homology module $H_0(\mathfrak{G}_m(\mathcal{T}^0))$ must be zero.

For a given inner vertex $\gamma \in \mathcal{T}_0^0$, there are three sequences of edges (parallel to the x-axis, y-axis, and z-axis, respectively):

$$\begin{aligned} \tau_0^x = \gamma_0^x\gamma_1^x, \ldots, \tau_{k_1}^x = \gamma_{k_1}^x\gamma, \\ \tau_0^y = \gamma_0^y\gamma_1^y, \ldots, \tau_{k_2}^y = \gamma_{k_2}^y\gamma, \\ \tau_0^z = \gamma_0^z\gamma_1^z, \ldots, \tau_{k_3}^z = \gamma_{k_3}^z\gamma, \end{aligned} \tag{43}$$

such that $\{\gamma_0^x, \gamma_0^y, \gamma_0^z\} \subset \partial\mathcal{T}$. For a given polynomial $p = p_1(x-x_\gamma)^m + p_2(y-y_\gamma)^m + p_3(z-z_\gamma)^m \in \mathcal{J}_m(\gamma)$, the following can be obtained:

$$
\begin{aligned}
\partial_1\Big(\frac{p_2(y-y_\gamma)^m + p_3(z-z_\gamma)^m}{2}([\tau_0^x] + \cdots + [\tau_{k_1}^x]) +& \\
\frac{p_1(x-x_\gamma)^m + p_3(z-z_\gamma)^m}{2}([\tau_0^y] + \cdots + [\tau_{k_2}^y]) +& \\
\frac{p_1(x-x_\gamma)^m + p_2(y-y_\gamma)^m}{2}([\tau_0^z] + \cdots + [\tau_{k_3}^z])\Big) = p[\gamma]&.
\end{aligned}
\tag{44}
$$

Thus, the homology module obeys $H_0(\mathfrak{J}_m(\mathcal{T}^0)) = 0$. □

Proposition 9. *For a given domain Ω, suppose that $\partial\Omega$ is a two-dimensional sphere (topologically). Then,*

$$
H_1(R_m(\mathcal{T}^0)) = H_2(R_m(\mathcal{T}^0)) = 0. \tag{45}
$$

Proof. The factor space $\Omega/\partial\Omega$ is homeomorphic to the three-dimensional sphere S^3 and the relative homology group: $H_2(\Omega, \partial\Omega) \cong H_2(\Omega/\partial\Omega) \cong H_2(S^3) = 0$ and $H_1(\Omega, \partial\Omega) \cong H_1(\Omega/\partial\Omega) \cong H_1(S^3) = 0$, which implies that the homology modules obey $H_2(R_m(\mathcal{T}^0)) = H_1(R_m(\mathcal{T}^0)) = 0$, so the proposition is proved. □

Let S' be a two-dimensional grid plane of T'. We say that S' splits a domain Ω into two nonempty domains if $\Omega = \Omega_1 \bigcup \Omega_2$, where Ω_1 and Ω_2 are contained in different half-spaces divided by S' (see Fig. 10 for a domain split by two planes orthogonal to the z-axis). We denote by S the corresponding two-dimensional domain $S = \Omega_1 \cap \Omega_2$ formed by the two-dimensional cells of S'.

Lemma 9. *Suppose that for any plane S' splitting Ω the corresponding two-dimensional domain S is simply connected. Then,*

$$
H_1(\mathfrak{J}_m(\mathcal{T}^0)) = \ker\widehat{\partial}_1/\operatorname{im}\widehat{\partial}_2 = 0. \tag{46}
$$

We recall that the differentials $\widehat{\partial}_1$ and $\widehat{\partial}_2$ of the chain complex $\mathfrak{J}_m(\mathcal{T}^0)$ are the restrictions of ∂_1 and ∂_2 to the modules $\bigoplus_{\tau\in\mathcal{T}_1^0} [\tau]\,\mathcal{J}_m(\tau)$ and $\bigoplus_{\sigma\in\mathcal{T}_2^0} [\sigma]\,\mathcal{J}_m(\sigma)$, respectively.

Proof. For a given $p \in \bigoplus_{\tau\in\mathcal{T}_1^0} [\tau]\,\mathcal{J}_m(\tau)$ we denote by p_τ the polynomial from $\mathcal{J}_m(\tau)$ corresponding to an inner edge τ, so $p = \sum_{\tau\in\mathcal{T}_1^0} p_\tau[\tau]$.

Suppose that $\partial_1 p_0 = 0$ for some $p_0 \in \bigoplus_{\tau \in \mathcal{T}_1^0} [\tau]\, \mathcal{J}_m(\tau)$. We will prove that for p_0 there exists a homologically equivalent element $p_2 \in \bigoplus_{\tau \in \mathcal{T}_1^0} [\tau]\, \mathcal{J}_m(\tau)$ (i.e. $p_2 - p_0 = \partial_2 q$ for some $q \in \bigoplus_{\sigma \in \mathcal{T}_2^0} [\sigma]\, \mathcal{J}_m(\sigma)$) such that for any inner edge τ the degree of the polynomial $p_{2\tau}$ with respect to the variable z is less than m.

In order to obtain p_2, let us first obtain $p_1 \in \bigoplus_{\tau \in \mathcal{T}_1^0} [\tau]\, \mathcal{J}_m(\tau)$ that is homologically equivalent to p_0 and is such that $p_{1\tau} = 0$ for any inner edge τ parallel to the z-axis. The procedure to obtain p_1 is straightforward. However, for clarity, let us refer to Fig. 10. Let $\tau = [C_1 O]$ be an inner edge of a domain Ω, and let the corresponding polynomial be $p_{0\tau} = r(x - x_0)^m + s(y - y_0)^m$, where x_0 and y_0 are the x-coordinate and y-coordinate of the vertex O, respectively, and $r = r(y, z)$ and $s = s(x, z)$ are some polynomials of bi-degree (m, m). Let σ_1, σ_2, and σ_3 be the oriented facets specified by the following sequences of vertices: $A_3 A_2 A_1$, $C_1 O A_2$, and $C_1 O D_1$. Then, for $p = p_0 - \partial_2\big(s(y - y_0)^m\, ([\sigma_1] + [\sigma_2]) + r(x - x_0)^m [\sigma_3]\big)$, the corresponding polynomial obeys $p_\tau = 0$. Repeating this procedure for the other inner edges $[A_2 A_3]$, $[O C_2]$, and $[B_1 B_2]$, we obtain p_1.

Let S' be a grid plane of T' that is given by the equation $z = h$ and splits a domain Ω. Let $\mathcal{T}_S$ be the corresponding two-dimensional mesh defined over S. For any inner edge $\tau \in \mathcal{T}_{S,1}^0$ of $\mathcal{T}_S$, let us take an expansion $p_{1\tau} = r_1^\tau z^m + s_1^\tau$, where $r_1^\tau = r_1^\tau(x, y)$ is a polynomial of bi-degree (m, m) and $s_1^\tau = s_1^\tau(x, y, z)$ is a polynomial of bi-degree $(m, m, m-1)$. Thus, $\partial_1 \sum_{\tau \in \mathcal{T}_{S,1}^0} r_1^\tau [\tau] = 0$ because $\partial_1 p_1 = 0$ and $p_{1\tau} = 0$ for any inner edge τ parallel to the z-axis. Since S is simply connected, Lemma 4 implies that there exist polynomials $t^\sigma = t^\sigma(x, y)$ of bi-degree (m, m), for $\sigma \in \mathcal{T}_{S,2}^0$, such that $\partial_2 \left(\sum_{\sigma \in \mathcal{T}_{S,2}^0} t^\sigma [\sigma] \right) = \sum_{\tau \in \mathcal{T}_{S,1}^0} r_1^\tau [\tau]$. Then, for $p = p_1 - \partial_2 \left(\sum_{\sigma \in \mathcal{T}_{S,2}^0} t^\sigma (z - h)^m [\sigma] \right)$, the polynomial p_τ corresponding to an inner edge $\tau \in \mathcal{T}_{S,1}^0$ has a degree of less than m with respect to the variable z. Repeating this procedure for all planes (orthogonal to the z-axis) that split Ω, we obtain p_2. For example, in Fig. 10 there exist two planes (orthogonal to the z-axis) that split a domain Ω.

Repeating the procedure above for the variables x and y, we obtain a

$p_3 \in \bigoplus_{\tau \in \mathcal{T}_1^0} [\tau] \, \mathcal{J}_m(\tau)$ that is homologically equivalent to p_0 and is such that for any inner edge τ the degree of the polynomial $p_{3\tau}$ with respect to each variable (x, y, or z) is less than m. This observation implies that $p_{3\tau} = 0$ for any inner edge τ. Thus, p_3 equals zero and is homologically equivalent to p_0, so the lemma is proved. $\square$

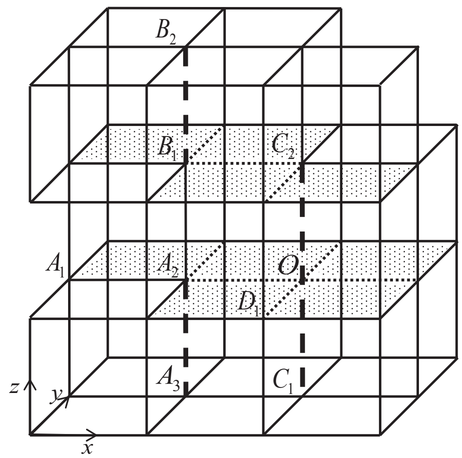

Figure 10: Thick dotted lines denote the inner edges parallel to the z-axis. Thin dotted lines denote the other inner edges. Dotted areas denote the two-dimensional domains corresponding to two grid planes that are orthogonal to the z-axis and split a domain.

Corollary 7. *For a given domain Ω, suppose that $\partial\Omega$ is a two-dimensional sphere (topologically). In addition, suppose that for any plane S' splitting Ω the corresponding two-dimensional domain S is simply connected. Then,*

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0, \quad (47)$$

where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges, and inner vertices of a domain Ω, respectively.

Proof. By virtue of the long exact sequence (41), it follows from Lemmas 8 and 9 and Proposition 9 that $H_2(\mathfrak{G}_m(\mathcal{T}^0)) = 0$, $H_1(\mathfrak{G}_m(\mathcal{T}^0)) = 0$, and $H_0(\mathfrak{G}_m(\mathcal{T}^0)) = 0$. Thus, Lemma 7 proves the corollary. $\square$

Reminiscent of Lemma 5, in the following lemma we will obtain the dimension of a spline space $\dim \mathcal{S}_m(\mathcal{T})$ when the corresponding domain Ω is split into two domains Ω_1 and Ω_2.

Lemma 10. *Let a grid plane S' split a domain Ω into two domains $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = S$. Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$ be the three–dimensional meshes corresponding to Ω_1, Ω_2, and Ω, respectively. For a given integer $m \geqslant 1$, suppose that the dimensions of the spaces $\mathcal{S}_m(\mathcal{T}_1)$ and $\mathcal{S}_m(\mathcal{T}_2)$ can be obtained from (47). In addition, suppose that $S \in \mathcal{A}^2_{m-1}$ with respect to the infinite two-dimensional grid S'. Then, the dimension of the spline space $\mathcal{S}_m(\mathcal{T})$ is given by (47) as well.*

Proof. Without loss of generality, we suppose that S' is the plane $z = 0$. Let $\mathcal{S}$ be the two–dimensional mesh, which is the portion of the grid S' over a domain S. We denote by $\mathcal{S}_m(\mathcal{S})$ the space of C^{m-1} smooth functions defined on S that are bivariate polynomials of degree (m, m) on each cell of S. Let $\mathcal{S}_m(\mathcal{S})^m$ be the direct sum of m copies of the space $\mathcal{S}_m(\mathcal{S})$. We define the linear operator

$$\mathcal{G} : \mathcal{S}_m(\mathcal{T}_1) \oplus \mathcal{S}_m(\mathcal{T}_2) \to \mathcal{S}_m(\mathcal{S})^m$$

as follows: for given splines $\phi_1 \in \mathcal{S}_m(\mathcal{T}_1)$ and $\phi_2 \in \mathcal{S}_m(\mathcal{T}_2)$, the corresponding spline-vector $\mathcal{G}(\langle\phi_1, \phi_2\rangle) \in \mathcal{S}_m(\mathcal{S})^m$ equals

$$\Big\langle (\phi_1 - \phi_2)|_{z=0}, \frac{\partial(\phi_1 - \phi_2)}{\partial z}|_{z=0}, \dots, \frac{\partial^{m-1}(\phi_1 - \phi_2)}{\partial z^{m-1}}|_{z=0} \Big\rangle,$$

so $\ker \mathcal{G} = \mathcal{S}_m(\mathcal{T})$. Thus, we obtain

$$\begin{gathered}\dim \mathcal{S}_m(\mathcal{T}) = \dim \mathcal{S}_m(\mathcal{T}_1) + \dim \mathcal{S}_m(\mathcal{T}_2) - \dim \operatorname{im} \mathcal{G} = (m+1)^3 f_3 - \\ m(m+1)^2(f_2^0 - h_2) + m^2(m+1)(f_1^0 - h_1^0) - m^3(f_0^0 - h_0^0) - \dim \operatorname{im} \mathcal{G}\end{gathered} \tag{48}$$

where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges, and inner vertices of Ω, while h_2, h_1^0, and h_0^0 are the numbers of cells, inner edges, and inner vertices of S. We remark that cells, inner edges, and inner vertices of S are inner facets, inner edges, and inner vertices of Ω, respectively, but are not inner facets, inner edges, and inner vertices of Ω_1 and Ω_2. Therefore, $f_2^0 - h_2^0$, $f_1^0 - h_1^0$, and $f_0^0 - h_0^0$ are the numbers of inner facets, inner edges, and inner vertices contained in either Ω_1 or Ω_2.

In order to prove that $\mathcal{G}$ is an epimorphism, let us take an element of $\mathcal{S}_m(\mathcal{S})^m$: $\psi = \langle\psi_1(x, y), \dots, \psi_m(x, y)\rangle$. It follows from Corollary 5 that there exist splines $\widetilde{\psi}_1, \dots, \widetilde{\psi}_m$ defined globally over the infinite two-dimensional grid S' such that $\widetilde{\psi}_i|_S = \psi_i, i = 1 \dots m$. We define a trivariate spline $\phi(x, y, z)$ globally over T' as follows:

$$\phi(x, y, z) := \sum_{i=1}^{m} \widetilde{\psi}_i(x, y) \frac{z^{i-1}}{(i-1)!} + z^m.$$

Let $\phi_1 := \phi|_{\Omega_1}$ and $\phi_2 \equiv 0$ on Ω_2. Then, $\mathcal{G}(\langle\phi_1, \phi_2\rangle) = \psi$. Thus, by virtue of Corollary 4, we obtain

$$\dim \operatorname{im} \mathcal{G} = m \dim \mathcal{S}_m(\mathcal{S}) = m(m+1)^2 h_2 - m^2(m+1)h_1^0 + m^3 h_0^0, \tag{49}$$

so the lemma is proved. □

Corollary 8. *For a given $m \geqslant 1$, suppose that $\Omega \in \mathcal{A}^3_{m-1}$, and let $\mathcal{T}$ be the corresponding three-dimensional mesh. Then, the dimension of a spline space $\mathcal{S}_m(\mathcal{T})$ is*

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0, \tag{50}$$

where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges, and inner vertices of a domain Ω, respectively.

Proof. Suppose that a domain Ω is split into two domains Ω_1 and Ω_2 by a grid plane S' of T'. Without loss of generality, we suppose that S' is parallel to the xy-plane.

If Ω is a three-dimensional topological manifold with boundary, then $S = \Omega_1 \bigcap \Omega_2$ is a two-dimensional topological manifold with boundary. That is, if $\Omega \in \mathcal{A}^3_0$, then $S \in \mathcal{A}^2_0$. From Definitions 7 and 4, it can be seen that if $\Omega \in \mathcal{A}^3_1$, then $S \in \mathcal{A}^2_1$. Moreover, the following can be proven by induction on m: if $\Omega \in \mathcal{A}^3_{m-1}$ for a given integer $m \geqslant 1$, then $S \in \mathcal{A}^2_{m-1}$.

Using a sufficient number of grid planes parallel to the xy-plane, a domain Ω can be split into pieces that each either satisfy the condition of Corollary 7 or have only a single level of cells with respect to the z-direction. For such a domain the dimension formula follows from Corollary 4. By Lemma 10 and Corollary 7, the corollary is proved. □

Theorem 5. *Suppose that $\Omega \in \mathcal{A}^3_{m-1}$. Then, the basis of a space $\mathcal{S}_m(\mathcal{T})$ can be obtained as follows:*

$$\{b|_\Omega : b(x,y,z) \in \mathcal{B} \ \wedge \ \operatorname{supp} b(x,y,z) \cap \operatorname{int}\Omega \neq \varnothing\}. \tag{51}$$

Proof. From Corollaries 6 and 8 below we obtain $\mathcal{N} = \dim \mathcal{S}_m(\mathcal{T})$. Finally, the fact that tensor-product B-splines from $\mathcal{B}$ are locally linearly independent proves the theorem. □

Corollary 9. *For a given integer $m \geqslant 1$, let Ω be a three-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}^3_{m-1}$. Let $f \in \mathcal{S}_m(\mathcal{T})$ be a spline function defined over the corresponding three-dimensional mesh $\mathcal{T}$. Then, there exists a spline function $\widetilde{f}$ of tri-degree (m, m, m) defined globally over T' such that $\widetilde{f}|_\Omega = f$.*

Proof. The splines from $\mathcal{B}$ are defined globally over T'. Thus, by Theorem 5, the corollary is proved. $\square$

5. Hierarchical splines

In this section we will recall Kraft's selection mechanism (see [16] and [19]) for constructing hierarchical B-splines on a nested hierarchy of d-dimensional domains ($d = 1$, $d = 2$, and $d = 3$) in a unified way. We will prove that these hierarchical B-splines span the spline space defined over the corresponding d-dimensional hierarchical mesh. For the sake of simplicity, we will not distinguish between the one-dimensional, two-dimensional, and three-dimensional cases in the notation. Besides the notation for a hierarchical mesh $\mathcal{H}$ and for d-variate tensor-product B-splines b, we will follow the notation introduced by Giannelli and Jüttler [19].

Let $\{V^l\}_{l=0,\ldots,N-1}$ be a sequence of N tensor-product spline spaces such that $V^l \subset V^{l+1}$ for $l = 0, \ldots, N-2$. We recall that the knot multiplicities are equal to 1. Each space V^l has an associated infinite grid G^l for $l = 0, \ldots, N-1$. We denote by T^l a tensor-product B-spline basis that spans the spline space V^l, $l = 0, \ldots, N-1$.

Let us consider a nested sequence of domains $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing$ such that each domain Ω^l is formed by a finite number of cells of $G^l, l = 0, \ldots, N-1$. We require that for each $l = 1, \ldots, N-1$ the boundary $\partial\Omega^l$ is aligned with grid lines of G^{l-1}. A domain $R^l = \Omega^0 \backslash \Omega^{l+1}, l = 0, \ldots, N-1$ will be called a ring. We remark that R^l consists of cells of $G^l, l = 0, \ldots, N-1$.

Let $\mathcal{H}$ be the hierarchical mesh determined by a nested sequence of domains $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing$ associated with the sequence of grids $G^l, l = 0, \ldots, N-1$. For a given integer $m \geqslant 1$, let W be the space of splines of degree m, with the highest order of smoothness, defined over the hierarchical mesh $\mathcal{H}$. In our previous notation, $W = \mathcal{S}_m(\mathcal{H})$. Figure 11 shows a simple example of a hierarchical mesh $\mathcal{H}$ determined by the nested sequence of two-dimensional domains: $\Omega^0 \supset \Omega^1 \supset \Omega^2 = \varnothing$.

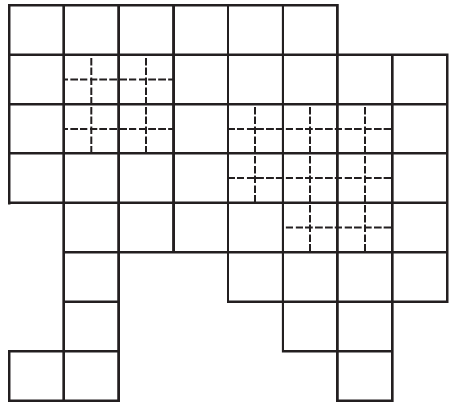

Figure 11: For the hierarchical mesh $\mathcal{H}$ shown, the domain Ω^0 is the union of all shown cells of G^0, and the domain Ω^1 is the union of the cells of G^0 that are subdivided by dashed lines. The underlying grid G^0 is aligned with the solid lines, and the dashed lines belong to the underlying grid $G^1 \supset G^0$.

Definition 8 ([16] and [19], Definition 1)**.** *The hierarchical basis $\mathcal{K}$ is defined as*

$$\mathcal{K} = \bigcup_{l=0}^{N-1} \mathcal{K}^l,$$

with $\mathcal{K}^l = \{b \in T^l : \operatorname{supp} b \bigcap \operatorname{int} R^{l-1} = \varnothing \ \wedge \ \operatorname{supp} b \bigcap \operatorname{int} R^l \neq \varnothing\}$. We define $R^{-1} = \varnothing$ to include the case $l = 0$.

In order to avoid repetition, we refer the reader to [19] for detailed explanations of the notation and concepts used in this section. Based on the analysis above (see Theorems 1, 3, and 5), we are now ready to prove the following theorem:

Theorem 6. *Let $m \geqslant 1$ be an integer. For a given nested sequence of domains $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing$ suppose that the domain $R^l = \Omega^0 \backslash \Omega^{l+1}$ (with respect to the grid G^l) admits an offset at a distance of $\frac{m-1}{2}$ for each $l = 0, \ldots, N-1$. Then, the set of B-splines from $\mathcal{K}$ restricted on Ω^0 is a basis of the spline space W defined over the corresponding hierarchical mesh $\mathcal{H}$. The theorem holds for $d = 1, 2, 3$, if that is the case.*

Proof. The linear independence of B-splines from $\mathcal{K}$ is a trivial observation due to the local linear independence of tensor-product B-splines.

Let us prove that B-splines from $\mathcal{K}$ span W. Let $f \in W$. Since R^0 admits an offset at a distance of $\frac{m-1}{2}$ with respect to the grid G^0, there exists $f^0 = \sum_{b \in \mathcal{K}^0} c_b^0 b \in V^0$, for proper real numbers $c_b^0, b \in \mathcal{K}^0$, such that $f|_{R^0} = f^0|_{R^0}$ (for the proof, see Theorems 1, 3, and 5 in the one-, two-, and three-dimensional cases, respectively). Since R^1 admits an offset at a distance of $\frac{m-1}{2}$ with respect to the grid G^1, there exists $f^1 = \sum_{b \in T^1} c_b^1 b \in V^1$ such that $f^1|_{R^1} = (f - f^0)\,|_{R^1}$. Since $f^1|_{R^0} = 0$, we obtain $c_b^1 = 0$ for any $b \notin \mathcal{K}^1$, and thus $f^1 = \sum_{b \in \mathcal{K}^1} c_b^1 b$. Repeating this procedure, we obtain $f = \sum_{l=0}^{N-1} f^l|_{\Omega_0}$ such that $f^l|_{R^l} = f|_{R^l} - \sum_{i=0}^{l-1} f^i|_{R^l}$ and $f^l = \sum_{b \in \mathcal{K}^l} c_b^l b$ for $l = 0, \ldots, N-1$. Theorem is proved.□

Remark 6. *We note that the final part of the proof of Theorem 6 presented above coincides with that of Theorem 20 in [19]. In the three-dimensional case, the key is Theorem 5.*

Remark 7. *Similar to Remark 21 of [19], a simple configuration satisfying the condition of Theorem 6 for three-dimensional hierarchical meshes is one in which the domains $\Omega^{l+1}, l = 0, \ldots, N-2$ are formed by aligned disjoint boxes composed of $m \times m \times m$ cells of G^l.*

Remark 8. *A trivariate hierarchical B-spline basis can also be modified to introduce a normalized weighted basis [17] or a truncated basis [18].*

6. Conclusions and future work

We studied the dimension of the d-variate tensor-product splines space ($1 \leqslant d \leqslant 3$) on a domain that consists of a finite number of cells of an infinite d-dimensional grid. Further, we analyzed the restrictions on the configuration of such domains and under which the dimension of the spline space is derived.

We also proved that under the assumption given in Theorem 6 about the configuration of the nested sequence of domains, the spline space over a d-dimensional hierarchical mesh (determined by this nested sequence of domains) is spanned by the hierarchical B-splines introduced by Kraft [16]. Using our unified approach, we confirmed the results obtained by Giannelli and Jüttler for bivariate splines [19], and we generalized those for trivariate splines.

In the future, we will develop an efficient refinement algorithm based on our theoretical results.

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