State Control Design for Linear Systems with Distributed Time Delays

Daniel Gontkovic, Robert Fonod

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Daniel Gontkovič and Róbert Fónod
Technical University of Košice, Faculty of Electrical Engineering and Informatics
Department of Cybernetics and Artificial Intelligence
Košice, Slovakia
e-mail: damiel.gontkovic@tuke.sk; robert.fonod@student.tuke.sk

***Abstract*—This paper is concerned with the problem of stabilization for continuous-time systems with distributed time delays. Using an extended form of the Lyapunov-Krasovskii functional the controller design conditions are derived with respect to application of structured matrix variables in linear matrix inequalities. The result giving a sufficient condition for stabilization of the system with distributed time delays is illustrated with a numerical example to note reduced conservatism in the system structure.**

***Keywords*— Linear matrix inequality, systems with distributed time delays, Lyapunov-Krasovskii functional, state control, asymptotic stability.**

I. Introduction

Control systems are used in many industrial applications, where time delays can take a deleterious effect on both the stability and the dynamic performance in open and closed-loop systems. Therefore the stability and control of the dynamical systems involving distributed time delays is a problem of large practical interest where intensive activity are done to develop control laws for systems stabilization.

During the last decades, considerable attention has been devoted to the problem of stability analysis and controller design for systems with time-delay. The existing stabilization results for time-delay systems can be delay independent or delay-dependent. The delay-dependent stabilization is concerned with the size of the delay and usually provides an upper bound of the delay such that the closed loop system is stable for any delay less than the given upper bound. On the other side, the delay-independent stabilization provides a controller, which can stabilize given system irrespective of the size of the delay.

The use of Lyapunov method for stability analysis of the time delay systems has been ever growing subject of interest, starting with the pioneering works of Krasovskii [9], [10]. Usually nowadays for the stability issue some modified Lyapunov-Krasovskii functionals are used (e.g. see [4], [5]) to obtain delay-independent stabilization and the results based on these functionals are applied to controller synthesis and observer design. This time-delay independent methodology and the bounded inequality techniques are sources of a conservatism that can cause higher norm of the state feedback gain. Progres review in this research field can be found e.g. in [18], [19], and the references therein.

Despite the significance, as the controllers are usually digitally implemented, systems with distributed time delays have not been paid due attention, and in contradiction to results given e.g. in [13], [15] there didn't exist much structures to solve this problem, formulated with respect to LMI ([1], [6], [8], [16]). However, standard schemes are not applicable to systems with distributed time delays and new design conditions have to be derived [7]. By introducing triple integral terms into Lyapunov-Krasovskii functional [17], the conservatism of conditions is further reduced but the design task still state singular. The presented LMI approach is based on the form of Lyapunov-Krasovskii functional used in [2] but the stability conditions as well as the controller design condition are reformulated with respect the application of structured matrix variables in LMI solution. Generally, since Lyapunov-Krasovskii functional is used only sufficient conditions for system stability are obtained. Used modification was motivated by [3], in here presented form enables to design systems with standard structures. It seems, another applications based on the bounded real lemma are immediate.

II. System Model

The systems under consideration are understood as multi-input and multi-output linear (MIMO) dynamic systems with distributed time delay. Without lose of generalization this class of systems can be represented in a state-space form by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_h \int\limits_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s + \boldsymbol{B}\boldsymbol{u}(s) \qquad (1)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{u}(t) \qquad (2)$$

with the initial condition

$$\boldsymbol{q}(\theta) = \varphi(\theta), \quad \forall\theta \in \langle -(h+\frac{h}{m}), 0\rangle \qquad (3)$$

where $h > 0$ represents the system distributed delay, $m > 0$ is a partitioning factor, $\boldsymbol{q}(t) \in I\!R^n$, $\boldsymbol{u}(t) \in I\!R^r$, and $\boldsymbol{y}(t) \in I\!R^p$ are vectors of the state, input and output variables, respectively, and matrices $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{A}_h \in I\!R^{n\times n}$, $\boldsymbol{B} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{p\times n}$, and $\boldsymbol{D} \in I\!R^{p\times r}$ are real matrices. Throughout the paper it is assumed that the couple $(\boldsymbol{A}, \boldsymbol{B})$ is controllable.

Using a linear memoryless state feedback controller

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{4}$$

where the matrix $\boldsymbol{K} \in I\!R^{r\times n}$ is the gain matrix, problem of the interest is to design $\boldsymbol{K}$ such that the closed-loop system

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{q}(t) + \boldsymbol{A}_h \int\limits_{t-h}^{t} \boldsymbol{q}(s)\mathrm{d}s \tag{5}$$

is asymptotically stable for given h.

III. Basic Preliminaries

Proposition 1: If $\boldsymbol{N}$ is a positive definite symmetric matrix, and $\boldsymbol{M}$ is a square matrix of the same dimension then

$$\boldsymbol{M}^{-T}\boldsymbol{N}\boldsymbol{M}^{-1} \geq \boldsymbol{M}^{-1} + \boldsymbol{M}^{-T} - \boldsymbol{N}^{-1} \tag{6}$$

Proof: Since $\boldsymbol{N}$ is positive definite then it yields

$$(\boldsymbol{M}^{-1}-\boldsymbol{N}^{-1})^T\boldsymbol{N}(\boldsymbol{M}^{-1}-\boldsymbol{N}^{-1}) \geq 0 \tag{7}$$

$$\boldsymbol{M}^{-T}\boldsymbol{N}\boldsymbol{M}^{-1} - \boldsymbol{M}^{-T} - \boldsymbol{M}^{-1} + \boldsymbol{N}^{-1} \geq 0 \tag{8}$$

respectively, and evidently (8) implies (6). This concludes the proof. ∎

Proposition 2: (Schur Complement) Let $\boldsymbol{S}$, $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{R} = \boldsymbol{R}^T$, $\det \boldsymbol{R} \neq 0$ are real matrices of appropriate dimensions, then the next inequalities are equivalent

$$\begin{gathered}\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} \mathbf{Q}-\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} > 0 \\ \Updownarrow \\ \mathbf{Q}-\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T > 0, \quad \mathbf{R} > 0 \end{gathered} \tag{9}$$

Proof: (see e.g. [11]) Let the linear matrix inequality takes form

$$\begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} > 0 \tag{10}$$

then using Gauss elimination principle it yields

$$\begin{aligned}&\begin{bmatrix} \mathbf{I} & -\mathbf{S}\mathbf{R}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{I} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{Q}-\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\end{aligned} \tag{11}$$

Since

$$\det \begin{bmatrix} \mathbf{I} & -\mathbf{S}\mathbf{R}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = 1 \tag{12}$$

it is evident that this transform doesn't change positivity of (10), and so (11) implies (9). This concludes the proof (compare e.g. [11]). ∎

Proposition 3: (Symmetric upper-bounds inequalities) Let $f(\boldsymbol{x}(p))$, $\boldsymbol{x}(p) \in I\!R^n$, $\boldsymbol{X} = \boldsymbol{X}^T > 0$, $\boldsymbol{X} \in I\!R^{n\times n}$ is a real positive definite and integrable vector function of the form

$$f(\boldsymbol{x}(p)) = \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) \tag{13}$$

such that there exist well defined integrations as following

$$\int_{-b}^{0}\int_{t+r}^{t} f(\boldsymbol{x}(p))\mathrm{d}p\,\mathrm{d}r > 0 \tag{14}$$

$$\int_{t-b}^{t} f(\boldsymbol{x}(p))\mathrm{d}p > 0 \tag{15}$$

with $b > 0$, $b \in I\!R$, $t \in \langle 0, \infty)$, then

$$\begin{aligned}&\int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r \geq \\ &\geq \tfrac{2}{b^2}\int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\mathrm{d}r\,\boldsymbol{X}\int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r\end{aligned} \tag{16}$$

$$\int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p \geq \frac{1}{b}\int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\boldsymbol{X}\int\limits_{t-b}^{t} \boldsymbol{x}(p)\mathrm{d}p \tag{17}$$

Proof: (see e.g. [12]) Since with (13) it can be written

$$\boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) - \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) = 0 \tag{18}$$

and according to Schur complement (9) it is true that

$$\begin{bmatrix} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p) & \boldsymbol{x}^T(p) \\ \boldsymbol{x}(p) & \boldsymbol{X}^{-1} \end{bmatrix} = 0 \tag{19}$$

then the double integration of (19) leads to

$$\begin{bmatrix} \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r & \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\mathrm{d}r \\ \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}(p)\mathrm{d}p\mathrm{d}r & \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{X}^{-1}\mathrm{d}p\,\mathrm{d}r \end{bmatrix} \geq 0 \tag{20}$$

Using the equalities

$$\int_{t+r}^{t} \boldsymbol{X}^{-1}\mathrm{d}p = -r\boldsymbol{X}^{-1}, \quad \int\limits_{-b}^{0} -r\boldsymbol{X}^{-1}\mathrm{d}r = \frac{r^2}{2}\boldsymbol{X}^{-1} \tag{21}$$

inequality (20) can be rewritten as

$$\begin{bmatrix} \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p\,\mathrm{d}r & \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}^T(p)\mathrm{d}p\,\mathrm{d}r \\ \int\limits_{-b}^{0}\int\limits_{t+r}^{t} \boldsymbol{x}(p)\mathrm{d}p\mathrm{d}r & \frac{r^2}{2}\boldsymbol{X}^{-1} \end{bmatrix} \geq 0 \tag{22}$$

It is evident, that (22) implies (16).

Analogously using (19) it yields

$$\begin{bmatrix} \int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p & \int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\mathrm{d}p \\ \int\limits_{t-b}^{t} \boldsymbol{x}(p)\mathrm{d}p & \int\limits_{t-b}^{t} \boldsymbol{X}^{-1}\mathrm{d}p \end{bmatrix} \geq 0 \tag{23}$$

and since

$$\int_{t-b}^{t} \boldsymbol{X}^{-1}\mathrm{d}p = b\boldsymbol{X}^{-1} \tag{24}$$

the following is obtained

$$\begin{bmatrix} \int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\boldsymbol{X}\boldsymbol{x}(p)\mathrm{d}p & \int\limits_{t-b}^{t} \boldsymbol{x}^T(p)\mathrm{d}p \\ \int\limits_{t-b}^{t} \boldsymbol{x}(p)\mathrm{d}p & b\boldsymbol{X}^{-1} \end{bmatrix} \geq 0 \tag{25}$$

which implies (17). This concludes the proof. ∎

IV. Stability of the Autonomous System

Theorem 1: The autonomous system of (1) is asymptotically stable if for given $h > 0$, $m > 0$ there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{n\times n}$, $\boldsymbol{W} \in \mathbb{R}^{mn\times mn}$ such that

$$\boldsymbol{P}{=}\boldsymbol{P}^T{>}0,\ \boldsymbol{U}{=}\boldsymbol{U}^T{>}0,\ \boldsymbol{V}{=}\boldsymbol{V}^T{>}0,\ \boldsymbol{W}{=}\boldsymbol{W}^T{>}0 \quad (26)$$

$$\begin{gathered}\boldsymbol{P}^\circ = \boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A + \\ +\boldsymbol{T}_U^T\boldsymbol{U}^\circ\boldsymbol{T}_U + \boldsymbol{T}_V^T\boldsymbol{V}^\circ\boldsymbol{T}_V + \boldsymbol{T}_W^T\boldsymbol{W}^\circ\boldsymbol{T}_W < 0\end{gathered} \quad (27)$$

where

$$\boldsymbol{T}_U = \begin{bmatrix}\sqrt{\frac{h}{m}}\boldsymbol{I}_n & \\ & \sqrt{\frac{m}{h}}\boldsymbol{I}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0} & [\,\boldsymbol{0}\cdots\boldsymbol{0}\,] & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_n & [\,\boldsymbol{0}\cdots\boldsymbol{0}\,] & \boldsymbol{0}\end{bmatrix} \quad (28)$$

$$\boldsymbol{T}_V{=}\begin{bmatrix}\frac{h}{\sqrt{2}m}\boldsymbol{I}_n & \\ & \frac{\sqrt{2}m}{h}\boldsymbol{I}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{A}_h & [\boldsymbol{A}_h\cdots\boldsymbol{A}_h] & \boldsymbol{0}\\ \frac{h}{m}\boldsymbol{I}_n & -\boldsymbol{I}_n & [\boldsymbol{0}\ \cdots\ \boldsymbol{0}] & \boldsymbol{0}\end{bmatrix} \quad (29)$$

$$\boldsymbol{T}_W = \begin{bmatrix}\boldsymbol{0}_w & [\,\boldsymbol{I}_{mn}\ \boldsymbol{0}_w\,]\\ \boldsymbol{0}_w & [\,\boldsymbol{0}_w\ \boldsymbol{I}_{mn}\,]\end{bmatrix} \quad (30)$$

$$\boldsymbol{U}^\circ{=}\begin{bmatrix}\boldsymbol{U} & \\ & -\boldsymbol{U}\end{bmatrix},\ \boldsymbol{V}^\circ{=}\begin{bmatrix}\boldsymbol{V} & \\ & -\boldsymbol{V}\end{bmatrix},\ \boldsymbol{W}^\circ{=}\begin{bmatrix}\boldsymbol{W} & \\ & -\boldsymbol{W}\end{bmatrix} \quad (31)$$

$$\boldsymbol{T}_A = [\ \boldsymbol{A}\quad \boldsymbol{A}_h\quad [\ \boldsymbol{A}_h\ \cdots\ \boldsymbol{A}_h\ ]\quad \boldsymbol{0}\ ] \quad (32)$$

$$\boldsymbol{T}_I = [\ \boldsymbol{I}_n\quad \boldsymbol{0}\quad [\ \boldsymbol{0}\ \cdots\ \boldsymbol{0}\ ]\quad \boldsymbol{0}\ ] \quad (33)$$

$\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$, $\boldsymbol{I}_{mn} \in \mathbb{R}^{mn\times mn}$ are identity matrices, $\boldsymbol{0} \in \mathbb{R}^{n\times n}$, $\boldsymbol{0}_w \in \mathbb{R}^{mn\times n}$ are zero matrices, respectively, and $\boldsymbol{U}^\circ, \boldsymbol{V}^\circ \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{W}^\circ \in \mathbb{R}^{2mn\times 2mn}$ are structured matrix variables.

Proof: Defining Lyapunov-Krasovskii functional candidate as follows

$$\begin{gathered}v(\boldsymbol{q}(t)) = \int\limits_{t-\frac{h}{m}}^{t} \boldsymbol{p}^T(s)\boldsymbol{W}\boldsymbol{p}(s)\mathrm{d}s + \\ +\boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}s\mathrm{d}\vartheta + \\ + \int\limits_{-\frac{h}{m}}^{0}\int\limits_{\vartheta}^{0}\int\limits_{t+\lambda}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\vartheta +\end{gathered} \quad (34)$$

with

$$\boldsymbol{p}^T(s) = [\ \boldsymbol{p}_1^T(s)\quad \boldsymbol{p}_2^T(s)\ ] \quad (35)$$

$$\boldsymbol{p}_1^T(s) = \int_{t-\frac{h}{m}}^{t}\boldsymbol{q}^T(s)\mathrm{d}s \quad (36)$$

$$\boldsymbol{p}_2^T(s) = \begin{bmatrix}\int\limits_{t-\frac{2h}{m}}^{t-\frac{h}{m}}\boldsymbol{q}^T(s)\mathrm{d}s & \cdots & \int\limits_{t-h}^{t-(m-1)\frac{h}{m}}\boldsymbol{q}^T(s)\mathrm{d}s\end{bmatrix} \quad (37)$$

then evaluating the derivative of $v(\boldsymbol{q}(t)$ along a solution of (1) it can be obtained

$$\begin{gathered}\dot{v}(\boldsymbol{q}(t)) = \dot{v}_1(\boldsymbol{q}(t)) + \dot{v}_2(\boldsymbol{q}(t)) + \\ +\dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \\ +\boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{p}(t) - \boldsymbol{p}^T(t-\tfrac{h}{m})\boldsymbol{W}\boldsymbol{p}(t-\tfrac{h}{m})\end{gathered} \quad (38)$$

where

$$\begin{gathered}\dot{v}_1(\boldsymbol{q}(t)) = \\ = \int\limits_{-\frac{h}{m}}^{0}\left\{\int\limits_{t}^{t}\boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}\vartheta - \int\limits_{t+\vartheta}^{t}\boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}\vartheta\right\}\mathrm{d}s = \\ = \int\limits_{-\frac{h}{m}}^{0}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t)\mathrm{d}s - \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}\vartheta\mathrm{d}s = \\ = \tfrac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{q}^T(s)\boldsymbol{U}\boldsymbol{q}(s)\mathrm{d}s\end{gathered} \quad (39)$$

$$\begin{gathered}\dot{v}_2(\boldsymbol{q}(t)) = \\ = \int\limits_{-\frac{h}{m}}^{0}\int\limits_{\vartheta}^{0}\left\{\int\limits_{t}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}\lambda - \int\limits_{t+\lambda}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}\lambda\right\}\mathrm{d}s\mathrm{d}\vartheta = \\ = \int\limits_{-\frac{h}{m}}^{0}-\vartheta\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t)\mathrm{d}\vartheta - \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta = \\ = \tfrac{1}{2}\left(\tfrac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\boldsymbol{V}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta\end{gathered} \quad (40)$$

and subsequently, using (16), (17), it yields

$$\begin{gathered}\dot{v}_1(\boldsymbol{q}(t)) \le \\ \le \tfrac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \tfrac{m}{h}\int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{q}^T(s)\mathrm{d}s\ \boldsymbol{U}\int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{q}(s)\mathrm{d}s = \\ = \tfrac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t) - \tfrac{m}{h}\boldsymbol{p}_1^T(t)\boldsymbol{U}\boldsymbol{p}_1(t)\end{gathered} \quad (41)$$

$$\begin{gathered}\dot{v}_2(\boldsymbol{q}(t)) \le \tfrac{1}{2}\left(\tfrac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \\ -2\left(\tfrac{m}{h}\right)^2\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\mathrm{d}s\mathrm{d}\vartheta\ \boldsymbol{V}\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{q}}(s)\mathrm{d}s\mathrm{d}\vartheta = \\ = \tfrac{1}{2}\left(\tfrac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \\ -2\left(\tfrac{m}{h}\right)^2\int\limits_{-\frac{h}{m}}^{0}\int\limits_{t+\vartheta}^{t}\dot{\boldsymbol{q}}^T(s)\mathrm{d}s\mathrm{d}\vartheta\ \boldsymbol{V}\left\{\begin{array}{c}\frac{h}{m}\boldsymbol{q}(t) - \\ -\int\limits_{t-\frac{h}{m}}^{t}\boldsymbol{q}(s)\mathrm{d}s\end{array}\right\} = \\ = \tfrac{1}{2}\left(\tfrac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t) - \\ -2\left(\tfrac{m}{h}\right)^2(\tfrac{h}{m}\boldsymbol{q}^T(t) - \boldsymbol{p}_1^T(t))\boldsymbol{V}(\tfrac{h}{m}\boldsymbol{q}(t) - \boldsymbol{p}_1(t))\end{gathered} \quad (42)$$

Thus, using the notation

$$\boldsymbol{q}^{\circ T}(t) = [\ \boldsymbol{q}^T(t)\quad \boldsymbol{p}_1^T(t)\quad \boldsymbol{p}_2^T(t)\quad \boldsymbol{p}_3^T(t)\ ] \quad (43)$$

$$\boldsymbol{p}_3^T(t) = \int_{t-h-\frac{h}{m}}^{t-h}\boldsymbol{q}^T(s)\mathrm{d}s \quad (44)$$

then with (32), (33) it can be written

$$\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{A}_h\int\limits_{t-h}^{t}\boldsymbol{q}(s)\mathrm{d}s = \boldsymbol{T}_A\boldsymbol{q}^\circ(t) \quad (45)$$

$$\boldsymbol{q}(t) = \boldsymbol{T}_I\boldsymbol{q}^\circ(t) \quad (46)$$

and

$$\begin{gathered}\dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) = \\ = \boldsymbol{q}^{\circ T}(t)(\boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I + \boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A)\boldsymbol{q}^\circ(t)\end{gathered} \quad (47)$$

In the same sense using (28)–(31) it can be obtained

$$\begin{gathered}\boldsymbol{p}^T(t)\boldsymbol{W}\boldsymbol{p}(t) - \boldsymbol{p}^T(t-\tfrac{h}{m})\boldsymbol{W}\boldsymbol{p}(t-\tfrac{h}{m}) = \\ = \boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_W^T\boldsymbol{W}^\circ\boldsymbol{T}_W\boldsymbol{q}^\circ(t)\end{gathered} \quad (48)$$

$$\frac{h}{m}\boldsymbol{q}^T(t)\boldsymbol{U}\boldsymbol{q}(t)-\frac{m}{h}\boldsymbol{p}_1^T(t)\boldsymbol{U}\boldsymbol{p}_1(t)= \\ =\boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_U^T\boldsymbol{U}^\circ\boldsymbol{T}_U\boldsymbol{q}^\circ(t) \tag{49}$$

$$\frac{1}{2}\left(\frac{h}{m}\right)^2\dot{\boldsymbol{q}}^T(t)\boldsymbol{V}\dot{\boldsymbol{q}}(t)- \\ -2\left(\frac{m}{h}\right)^2\left(\frac{h}{m}\boldsymbol{q}^T(t)-\boldsymbol{p}_1^T(t)\right)\boldsymbol{V}\left(\frac{h}{m}\boldsymbol{q}(t)-\boldsymbol{p}_1(t)\right)= \\ =\boldsymbol{q}^{\circ T}(t)\boldsymbol{T}_V^T\boldsymbol{V}^\circ\boldsymbol{T}_V\boldsymbol{q}^\circ(t) \tag{50}$$

Thus, with $\boldsymbol{P}^\circ$ given in (27) it yields

$$\dot{v}(\boldsymbol{q}(t))\leq\boldsymbol{q}^{\circ T}(t)\boldsymbol{P}^\circ\boldsymbol{q}^\circ(t)<0 \tag{51}$$

and it is obvious that $\boldsymbol{P}^\circ$ has to be negative definite. ■

V. Control Law Parameter Design

Theorem 2: The closed-loop system (1) controlled by the control law (4) is asymptotically stable if for given $h>0$, $m>0$ there exist symmetric positive definite matrices $\boldsymbol{Y},\boldsymbol{U}^\bullet,\boldsymbol{V}^\bullet\in I\!R^{n\times n}$, $\boldsymbol{W}^\bullet\in I\!R^{mn\times mn}$, and a matrix $\boldsymbol{Z}\in I\!R^{r\times n}$ such that

$$\boldsymbol{Y}=\boldsymbol{Y}^T>0 \\ \boldsymbol{U}^\bullet=\boldsymbol{U}^{\bullet T}>0,\ \boldsymbol{V}^\bullet=\boldsymbol{V}^{\bullet T}>0,\ \boldsymbol{W}^\bullet=\boldsymbol{W}^{\bullet T}>0 \tag{52}$$

$$\begin{bmatrix}\boldsymbol{P}^\bullet & *\\ \boldsymbol{T}_A\boldsymbol{Y}^\diamond & -2\left(\frac{m}{h}\right)^2\boldsymbol{V}^\bullet\end{bmatrix}<0 \tag{53}$$

where

$$\boldsymbol{P}^\bullet=\boldsymbol{Y}^{\diamond T}\boldsymbol{T}_A^{\bullet T}\boldsymbol{T}_I+\boldsymbol{T}_I^T\boldsymbol{T}_A^\bullet\boldsymbol{Y}^\diamond+ \\ +\boldsymbol{T}_U^T\boldsymbol{U}^\diamond\boldsymbol{T}_U+\boldsymbol{T}_{V2}^T\boldsymbol{V}_2^\bullet\boldsymbol{T}_{V2}+\boldsymbol{T}_W^T\boldsymbol{W}^\diamond\boldsymbol{T}_W \tag{54}$$

$$\boldsymbol{U}^\diamond=\begin{bmatrix}\boldsymbol{U}^\bullet & \\ & -\boldsymbol{U}^\bullet\end{bmatrix},\ \boldsymbol{W}^\diamond=\begin{bmatrix}\boldsymbol{W}^\bullet & \\ & -\boldsymbol{W}^\bullet\end{bmatrix},\ \boldsymbol{V}_2^\bullet=\boldsymbol{V}^\bullet-2\boldsymbol{Y} \tag{55}$$

$$\boldsymbol{T}_{V2}=\sqrt{2}\frac{m}{h}\left[\frac{h}{m}\boldsymbol{I}_n\ -\boldsymbol{I}_n\left[\boldsymbol{0}\ \cdots\ \boldsymbol{0}\right]\boldsymbol{0}\right] \tag{56}$$

$$\boldsymbol{T}_A^\bullet=\left[\,[\boldsymbol{A}\ -\boldsymbol{B}]\quad \boldsymbol{A}_h\quad \left[\,\boldsymbol{A}_h\ \cdots\ \boldsymbol{A}_h\,\right]\quad \boldsymbol{0}\,\right] \tag{57}$$

$$\boldsymbol{Y}^\diamond=\text{diag}\left[\begin{bmatrix}\boldsymbol{Y}\\ \boldsymbol{Z}\end{bmatrix}\boldsymbol{Y}\left[\boldsymbol{Y}\cdots\boldsymbol{Y}\right]\boldsymbol{Y}\right] \tag{58}$$

$\boldsymbol{Y}^\diamond\in I\!R^{(n(m+2)+r)\times n(m+2)}$, $\boldsymbol{W}^\diamond\in I\!R^{2rn\times 2rn}$, $\boldsymbol{U}^\circ\in I\!R^{2n\times 2n}$ are structured matrix variables, and $\boldsymbol{T}_U$, $\boldsymbol{T}_W$, and $\boldsymbol{T}_I$ are used as in (28), (30), (33), respectively.

Now, the control gain is given as

$$\boldsymbol{K}=\boldsymbol{Z}\boldsymbol{Y}^{-1} \tag{59}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: Using Schur complement property then (27) can be rewritten as

$$\begin{bmatrix}\boldsymbol{P}^\diamond & \boldsymbol{T}_{V01}^T\\ \boldsymbol{T}_{V01} & -\left(\frac{2m}{h}\right)^2\boldsymbol{V}^{-1}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}^\diamond & \boldsymbol{T}_A^T\\ \boldsymbol{T}_A & -\left(\frac{2m}{h}\right)^2\boldsymbol{V}^{-1}\end{bmatrix}<0 \tag{60}$$

where

$$\boldsymbol{P}^\diamond=\boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I+\boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A+ \\ +\boldsymbol{T}_U^T\boldsymbol{U}^\circ\boldsymbol{T}_U+\boldsymbol{T}_{V2}^T\boldsymbol{V}^\circ\boldsymbol{T}_{V2}+\boldsymbol{T}_W^T\boldsymbol{W}^\circ\boldsymbol{T}_W \tag{61}$$

Then defining the congruence transform matrix

$$\boldsymbol{T}_C=\text{diag}\left[\boldsymbol{T}_{C1}\quad \boldsymbol{I}_n\right]= \\ =\text{diag}\left[\boldsymbol{P}^{-1}\ \boldsymbol{P}^{-1}\left[\boldsymbol{P}^{-1}\cdots\boldsymbol{P}^{-1}\right]\boldsymbol{P}^{-1}\ \boldsymbol{I}_n\right] \tag{62}$$

and pre-multiplying right-hand side and left-hand side of (60) by (62) gives the next result

$$\begin{bmatrix}\boldsymbol{T}_{C1}\boldsymbol{P}^\diamond\boldsymbol{T}_{C1} & \boldsymbol{T}_{C1}\boldsymbol{T}_A^T\\ \boldsymbol{T}_A\boldsymbol{T}_{C1} & -\left(\frac{2m}{h}\right)^2\boldsymbol{V}^{-1}\end{bmatrix}<0 \tag{63}$$

Using notation $\boldsymbol{P}^{-1}=\boldsymbol{Y}$ then (63) implies

$$\boldsymbol{T}_A\boldsymbol{T}_{C1}=\boldsymbol{T}_A\boldsymbol{Y}^\diamond \tag{64}$$

$$\boldsymbol{Y}^\diamond=\boldsymbol{T}_{C1}^\diamond=\text{diag}\left[\boldsymbol{Y}\ \boldsymbol{Y}\left[\boldsymbol{Y}\cdots\boldsymbol{Y}\right]\boldsymbol{Y}\right] \tag{65}$$

$$\boldsymbol{T}_{C1}(\boldsymbol{T}_A^T\boldsymbol{P}\boldsymbol{T}_I+\boldsymbol{T}_I^T\boldsymbol{P}\boldsymbol{T}_A)\boldsymbol{T}_{C1}= \\ =\boldsymbol{Y}^\diamond\boldsymbol{T}_A^T\boldsymbol{T}_I+\boldsymbol{T}_I^T\boldsymbol{T}_A\boldsymbol{Y}^\diamond \tag{66}$$

$$\boldsymbol{T}_{C1}\boldsymbol{T}_U^T\boldsymbol{U}^\circ\boldsymbol{T}_U\boldsymbol{T}_{C1}=\boldsymbol{T}_U^T\boldsymbol{U}^\diamond\boldsymbol{T}_U,\quad \boldsymbol{U}^\bullet=\boldsymbol{Y}\boldsymbol{U}\boldsymbol{Y} \tag{67}$$

$$\boldsymbol{T}_{C1}\boldsymbol{T}_W^T\boldsymbol{W}^\circ\boldsymbol{T}_W\boldsymbol{T}_{C1}=\boldsymbol{T}_W^T\boldsymbol{W}^\diamond\boldsymbol{T}_W, \\ \boldsymbol{W}^\bullet=\text{diag}\left[\boldsymbol{Y}\cdots\boldsymbol{Y}\right]\boldsymbol{W}\text{diag}\left[\boldsymbol{Y}\cdots\boldsymbol{Y}\right] \tag{68}$$

and denoting $\boldsymbol{V}^{-1}=\boldsymbol{V}^\bullet$ then (6) implies

$$\boldsymbol{T}_{C1}\boldsymbol{T}_{V2}^T\boldsymbol{V}\boldsymbol{T}_{V2}\boldsymbol{T}_{C1}\leq\boldsymbol{T}_{V2}^T\boldsymbol{V}_2^\bullet\boldsymbol{T}_{V2} \tag{69}$$

Replacing the matrix $\boldsymbol{A}$ in (32) by the closed-loop system matrix $\boldsymbol{A}_c=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}$ results in

$$\boldsymbol{A}_c\boldsymbol{Y}=\boldsymbol{A}\boldsymbol{Y}-\boldsymbol{B}\boldsymbol{K}\boldsymbol{Y} \tag{70}$$

and with the notation $\boldsymbol{K}\boldsymbol{Y}=\boldsymbol{Z}$ (64) can be replaced by $\boldsymbol{T}_A^\bullet\boldsymbol{Y}^\diamond$.

Writing now compactly $\boldsymbol{P}^\bullet=\boldsymbol{T}_{C1}\boldsymbol{P}^\diamond\boldsymbol{T}_{C1}$ as given in (54), then (63) implies (53). This concludes the proof. ■

VI. Illustrative Example

To demonstrate the algorithm properties it was assumed that system is given by (1), (2), where $h=6$

$$\boldsymbol{A}=\begin{bmatrix}2.6 & 0.0 & -0.8\\ 1.2 & 0.2 & 0.0\\ 0.0 & -0.5 & 3.0\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{bmatrix}$$

$$\boldsymbol{A}_h=\begin{bmatrix}0.00 & 0.02 & 0.00\\ 0.00 & 0.00 & -1.00\\ -0.02 & 0.00 & 0.00\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}1 & 3\\ 2 & 1\\ 1 & 1\end{bmatrix}$$

Setting $m=3$ and solving (52), (53) with respect the LMI matrix variables $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{U}^\bullet$, $\boldsymbol{V}^\bullet$, and $\boldsymbol{W}^\bullet$ using Self-Dual-Minimization (SeDuMi) package [14] for Matlab [6], the gain matrix problem was solved as feasible giving

$$\boldsymbol{K}=\begin{bmatrix}-4.6371 & -2.8106 & 19.8292\\ 4.1418 & 1.9946 & -12.9338\end{bmatrix}$$

$$\boldsymbol{A}_c=\begin{bmatrix}-5.1883 & -3.1732 & 18.1722\\ 6.3324 & 3.8266 & -26.7247\\ 0.4953 & 0.3160 & -3.8954\end{bmatrix}$$

and the stable eigenvalue spectrum of the closed-loop system matrix $\text{eig}(\boldsymbol{A}_c)=\{-0.2110\ -0.9606\ -4.0855\}$.

To characterize the steady-state control properties the extended closed-loop system matrix $\boldsymbol{A}_{ce}=\boldsymbol{A}+\boldsymbol{A}_h-\boldsymbol{B}\boldsymbol{K}$ was computed, where

$$\boldsymbol{A}_{ce}=\begin{bmatrix}-5.1883 & -3.1532 & 18.1722\\ 6.3324 & 3.8266 & -27.7247\\ 0.4753 & 0.3160 & -3.8954\end{bmatrix}$$

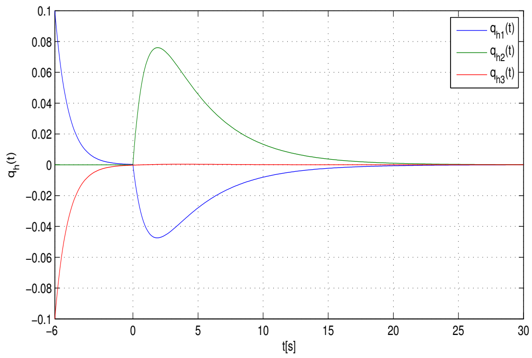

Fig. 1. Distributed delay state response of the system

This matrix eigenvalue spectrum is also stable since

$$\mathrm{eig}(\boldsymbol{A}_{ce}) = \{-0.2380 \;\; -1.1129 \;\; -3.9062\}$$

In the presented Fig. 1 the example is shown of the unforced closed-loop system state response, where the initial state was $\boldsymbol{q}_h^T(-6) = [0.1\ 0.0\ -0.1]$. It is possible to verify that closed-loop dynamic properties for this unstable autonomous time-delay system are less conservative.

VII. CONCLUDING REMARKS

Modified design conditions, explained with respect to special forms of structured matrix variables and based on an extended version of the Lyapunov-Krasovskii functional, are given in the paper. Obtained formulation is a convex LMI problem where the manipulation is accomplished in that manner that produces the closed-loop system asymptotical stability. Presented illustrative example confirms the effectiveness of proposed control design techniques. In particular, with the use of an extended version of Lyapunov-Krasovskii functional, it was shown how to adapt the standard approach to design optimal matrix parameters of state controller for systems with distributed time delays.

ACKNOWLEDGMENT

The work presented in this paper was supported by Grant Agency of Ministry of Education and Academy of Science of Slovak Republic VEGA under Grant No. 1/0256/11. This support is very gratefully acknowledged.

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