On the Grundy and b-chromatic numbers of a graph

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Abstract. The *Grundy number* of a graph G, denoted by $\Gamma(G)$, is the largest k such that G has a *greedy k-colouring*, that is a colouring with k colours obtained by applying the greedy algorithm according to some ordering of the vertices of G. Trivially $\Gamma(G) \leq \Delta(G) + 1$. In this paper, we show that deciding if $\Gamma(G) \leq \Delta(G)$ is NP-complete. We then show that deciding if $\Gamma(G) \geq |V(G)| - k$ is fixed parameter tractable with respect to the parameter k.

Keywords: Colouring, Online Colouring, Greedy Colouring, NP-completeness, Fixed Parameter Complexity.

1 Introduction

A *k-colouring* of a graph $G = (V, E)$ is a surjective mapping $c : V \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. A *k-colouring* may also be seen as a partition of the vertex set of G into k disjoint *stable sets* $S_i = \{v \mid c(v) = i\}$, $1 \leq i \leq k$. For convenience (and with a slight abuse of terminology), by colouring we mean either the mapping c or the partition $(S_1, \ldots, S_k)$. The elements of $\{1, \ldots, k\}$ are called *colours*. We may sometimes refer to S_i by its colour i. A graph is k-colourable if it admits a k-colouring. The *chromatic number* is $\chi(G) = min\{k \mid G \text{ is } k\text{-colourable}\}$.

The most basic and most widespread on-line algorithm producing colourings is the greedy algorithm or first-fit algorithm. A *greedy colouring* relative to a vertex ordering $\sigma = v_1 < v_2 < \cdots < v_n$ of $V(G)$ is obtained by colouring the vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest positive integer not already used on its lowered-indexed neighbours. A greedy colouring has the following property:

For every $j < i$, every vertex v in S_i has a neighbour in S_j. (P)

Otherwise v would have been coloured by an integer not greater than j. Conversely, a colouring satisfying Property (P) is a greedy colouring relative to any

vertex ordering in which the vertices of S_i precede those of S_j when $i < j$. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k-colouring. Hence the Grundy number and its ratio with the chromatic number measure how bad the greedy algorithm may perform on a graph. Easily, $\chi(G) \leq \Gamma(G) \leq \Delta(G)+1$.

Determining the chromatic number of a graph is well-known to be an NP-hard problem. Similarly, determining the Grundy number of a graph is NP-hard [16]. But deciding if the chromatic number of a graph is at most k is NP-complete [7] for $k \geq 3$, whereas Zaker [16] showed that for any fixed k, it is decidable in polynomial time if a given graph has Grundy number at most k. To show this, he proved that there is a finite number of graphs called *k-atoms* such that if $\Gamma(G) \geq k$ then G contains a k-atom as an induced subgraph.

A colouring of G is a *b-colouring* if every colour class has a *b*-vertex, that is a vertex with at least one neighbour in each other colour class. The *b-chromatic number* of a graph G, denoted $\chi_b(G)$, is the largest integer k such that G admits a *b*-colouring with k colours. Since an optimal proper colouring of G is trivially a *b*-colouring, and that every *b*-vertex has at most $\Delta(G)$ neighbours, we have $\chi(G) \leq \chi_b(G) \leq \Delta(G)+1$. Note that the Grundy and *b*-chromatic number may not be bounded one in function of the other. For example, the disjoint union of k stars of degree at least $k-1$ has *b*-chromatic number k, while its Grundy number is 2.

Deciding whether a graph admits a *b*-colouring with a given number of colours is an NP-complete problem, even for connected bipartite graphs [12] or chordal graphs [8].

Brooks [3] proved that, for any connected graph G, $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle. This implies that it can be checked in polynomial time if $\chi(G) \leq \Delta(G)$. Extensions of Brooks' Theorem have also been considered. A well-known conjecture of Borodin and Kostochka [2] states that every graph of maximal degree $\Delta \geq 9$ and chromatic number at least Δ has a Δ-clique. Reed [15] proved that this is true when Δ is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [1]. Further information about this problem can be found in the monograph of Jensen and Toft [11, Problem 4.8]. Generalisation of this problem has also been studied by Farzad, Molloy and Reed [6] and Molloy and Reed [13]. In particular, it is proved [13] that determining whether a graph with large constant maximum degree Δ is $(\Delta - q)$-colourable can be done in linear time if $(q + 1)(q + 2) \leq \Delta$. This threshold is optimal by a result of Emden-Weinert, Hougardy and Kreuter [5], since they proved that for any two constants Δ and $q \leq \Delta-3$ such that $(q+1)(q+2) > \Delta$, determining whether a graph of maximum degree Δ is $(\Delta - q)$-colourable is NP-complete.

On the opposite, Theorem 3 of [8] imply that it is NP-complete to decide if a given graph G satisfies $\chi_b(G) = \Delta(G)$. Thus there is no analog of Brooks' Theorem for *b*-colouring.

A natural question is then to ask if an analog of Brooks' Theorem exists for the Grundy number. One may ask if it is decidable in polynomial time whether

a graph G satisfies $\Gamma(G) = \Delta(G) + 1$. In Section 2, we answer this question in the negative by showing that this problem is NP-complete even when restricted to bipartite graphs. In particular, it implies that it is NP-hard to compute the Grundy number of a bipartite graph.

We then investigate some parameterized versions of the Grundy number problem and the b-chromatic number problem. For an introduction to parameterized algorithms and complexity, we refer the reader to [4] or [14]. Telle (See [4], Exercise 3.2.7) proved that the following problem is Fixed Parameter Tractable (FPT).

DUAL OF COLOURING
Instance: A graph G and an integer k.
Parameter: k.
Question: $\chi(G) \leq |V(G)| - k$?

In Section 3, we show that the analog for greedy colouring is also FPT.

DUAL OF GREEDY COLOURING
Instance: A graph G and an integer k.
Parameter: k.
Question: $\Gamma(G) \geq |V(G)| - k$?

In Section 4, we show that the analog for b-colouring is also FPT.

DUAL OF b-COLOURING
Instance: A graph G and an integer k.
Parameter: k.
Question: $\chi_b(G) \geq |V(G)| - k$?

In fact, Telle showed DUAL OF COLOURING is FPT by showing that it has a quadratic kernel. Being FPT is equivalent to having a kernel but not necessarily polynomial. Then, natural questions are the following.

Problem 1. Does DUAL OF GREEDY COLOURING have a polynomial kernel?

Problem 2. Does DUAL OF b-COLOURING have a polynomial kernel?

2   NP-hardness results

Before proving some complexity results, we need a preliminary lemma.

Lemma 1. *Let G be a graph and x a vertex of G. If there is a greedy colouring c such that x is coloured p, then, for any $1 \leq i \leq p$, there is a greedy colouring such that x is coloured i.*

Proof. For $1 \leq i \leq p$, let S_i be the set of vertices coloured i by c. Then for any $1 \leq i \leq p$, $(S_{p-i+1}, \ldots, S_p)$ is a greedy i-colouring of $G[\bigcup_{j=p-i+1}^{p} S_j]$ in which x is coloured i. This partial greedy colouring of G may be extended into a greedy colouring of G in which x is coloured i. $\square$

We now show that no Brooks type theorem exists for the Grundy number.

Theorem 1. *It is NP-complete to decide if a bipartite graph G satisfies $\Gamma(G) = \Delta(G) + 1$.*

Proof. The problem is trivially in NP. To show that it is also NP-complete, we present a reduction from 3-edge-colourability of 3-regular graphs, which is known to be NP-complete [9].

Let G be a 3-regular graph with $t - 3$ vertices. Set $V(G) = \{v_4, v_5, \ldots, v_t\}$ and $E(G) = \{e_1, \ldots, e_p\}$. Let I be the vertex-edge incidence graph of G, that is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which an edge of G is adjacent to its two endvertices. Also, let $M_{p,p}$ denote the graph obtained from the complete bipartite graph $K_{p,p}$ by removing a perfect matching. It can be easily checked that $\Gamma(M_{p,p}) = p$. We construct from I a new bipartite graph H as follows. For each vertex $e_i \in E(G)$, we add a copy $M_{3,3}(e_i)$ of $M_{3,3}$ and identify one of its vertices with e_i. We add a new vertex r adjacent to all the vertices of $V(G)$. We add copies $M_{1,1}^r$, $M_{2,2}^r$, $M_{3,3}^r$, $M_{t+1,t+1}^r$ of K_1, K_2, $M_{3,3}$, $M_{t+1,t+1}$ and we choose arbitrary vertices v_1, v_2, v_3, v_{t+1} respectively from each copy and add the edges $v_1 r$, $v_2 r$, $v_3 r$, $v_{t+1} r$. Finally, for every $4 \le i \le t$, we do the following: for every $4 \le j \le i - 1$, we add a copy $M_{j,j}^i$ of $M_{j,j}$, choose an arbitrary vertex v_j^i of it and add the edge $v_i v_j^i$. See Figure 1.

Observe that:

(i) $d_H(r) = t + 1$,

(ii) $d_H(v_i) = 1 + (i - 1) = i$, for $4 \le i \le t + 1$.

(iii) $d_H(e_j) = 4$, for $1 \le j \le p$, since e_i has two neighbours in I and two in $M_{3,3}(e_i)$.

(iv) $d_H(v_j^i) = j$, for $5 \le i \le t$ and $4 \le j < i$, since a vertex in $M_{j,j}^i$ has degree $j - 1$ and v_j^i is adjacent to v_i.

(v) $\Delta(H) = t + 1$ and the only vertices with degree $t + 1$ are r and v_{t+1}.

Let us show that $\Gamma(H) = \Delta(H) + 1 = t + 2$ if and only if G is 3-edge-colourable.

Assume first G has a 3-edge-colouring ϕ. By Lemma 1, for every $1 \le i \le p$, there is a greedy colouring of the copy of $M_{3,3}(e_i)$ associated with e_i where e_i is coloured $\phi(e_i)$. Then in I every vertex in $V(G)$ has one neighbour of each colour in $\{1, 2, 3\}$. There is a greedy colouring of $M_{j,j}^i$, $4 \le j < i \le t$ so that v_j^i is coloured j.

Then we greedily extend the union of these colourings to v_i, $4 \le i \le t$, so that v_i is coloured i. We also greedily colour $M_{1,1}^r$, $M_{2,2}^r$, $M_{3,3}^r$, $M_{t+1,t+1}^r$ in such a way that r has one neighbour coloured i, $i \in \{1, 2, 3, t + 1\}$. Finally, r has one neighbour of each colour j, $1 \le j \le t + 1$. So we colour it with $t + 2$.

Hence $\Gamma(H) \ge t + 2$ and so $\Gamma(H) = t + 2$ because $\Delta(H) = t + 1$.

Let us now show that if $\Gamma(H) = t + 2$ then G is 3-edge-colourable. Assume that c is a greedy $(t + 2)$-colouring of H.
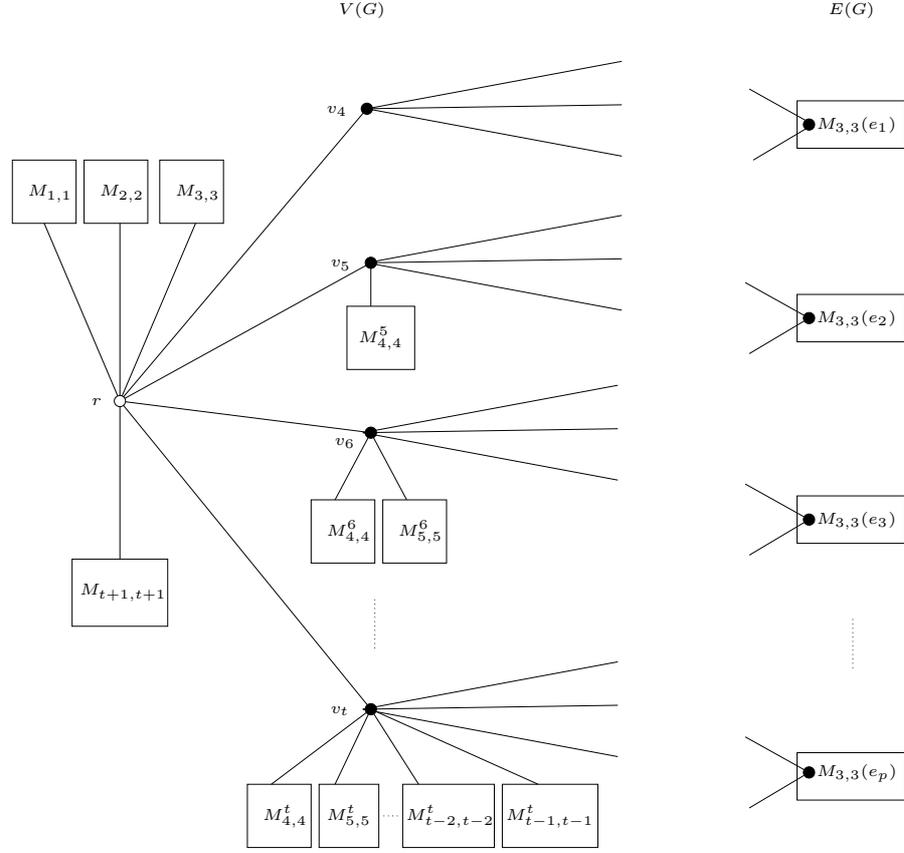
Fig. 1. The graph H of the reduction in Theorem 1

Claim 1 $\{c(r), c(v_{t+1})\} \subseteq \{t+1, t+2\}$

Proof. Let u be a vertex such that $c(u) = t+2$. Then, u must have one neighbour coloured with each of the other $t+1$ colours and then $d(u) \geq t+1$. Hence, by Observation (v), u is either r or v_{t+1}.

Case 1: $u = r$.
Then, $c(v_{t+1}) = t+1$, since the only neigbours of r with degree at least t are v_t and v_{t+1}, $d(v_t) = t$, and v_t is adjacent to r which is already coloured $t+2$.
Case 2: $u = v_{t+1}$. The only neighbour of v_{t+1} that could be coloured $t+1$ is r, since all its neighbours in $M_{t+1,t+1}^r$ have degree t and are adjacent to v_{t+1} which is coloured $t+2$. $\square$

Claim 2 *For* $1 \leq i \leq t$, $c(v_i) = i$.

Proof. By Claim 1, $\{c(r), c(v_{t+1})\} \subseteq \{t+1, t+2\}$. Since $d_H(r) = t+1$, r has one neighbour coloured i, for each $1 \leq i \leq t$. A neighbour of r which is coloured

t must have degree at least t. So, by Observation (ii), it must be v_t. And so on, by induction, we show that $c(v_i) = i$, for $1 \leq i \leq t$. $\square$

We now prove that c induces a proper 3-edge-colouring of G.

Consider vertex v_i, $4 \leq i \leq t$. By Claim 2, it is coloured i, and by Observation(ii) it has degree equal to $i + 1$. Since it is adjacent to r, which by Claim 1 has a colour greater than i, there are only $i - 1$ vertices remaining for the other $i - 1$ colours. So, v_i has exactly one neighbour coloured j, for each $1 \leq j \leq i - 1$. Hence the three edges incident to v_i in G, which are adjacent to v_i in I, have different colours. $\square$

As a corollary to Theorem 1, it is NP-hard to compute the Grundy number of a bipartite graph.

Corollary 1. *Given a bipartite graph G and an integer k, it is NP-complete to decide if $\Gamma(G) \geq k$.*

Theorem 1 also implies the following.

Corollary 2. *Let $k \geq 0$ be a fixed integer. It is NP-complete to decide if a bipartite graph G satisfies $\Gamma(G) \geq \Delta(G) + 1 - k$.*

Proof. We present a reduction from the problem of deciding if $\Gamma(G) = \Delta(G) + 1$ for a bipartite graph G, that we just proved to be NP-complete. Let G be a graph of maximum degree Δ and H the disjoint union of G and the star $S_{\Delta+k+1}$ on $\Delta + k + 1$ vertices ($S_{\Delta+k+1}$ has $\Delta + k$ edges incident to a vertex). Then $\Delta(H) = \Delta + k$ and $\Gamma(G) = \Gamma(H)$ because $\Gamma(S_{\Delta+k+1}) = 2$. Hence $\Gamma(H) \geq \Delta(H) + 1 - k$ if and only if $\Gamma(G) = \Delta(G) + 1$. $\square$

3   Dual of Greedy Colouring

The aim of this section is to prove the following theorem which shows that DUAL OF GREEDY COLOURING is FPT.

Theorem 2. DUAL OF GREEDY COLOURING *can be solved in time*
$O\left((2k)^{2k} \cdot |E| + 2^{2k}k^{3k+5/2}\right).$

A *vertex cover* in a graph G is a set $C \subseteq V(G)$ such that for every $e \in E(G)$, at least one of the endvertices of e is in C. A vertex cover is said to be *minimal* if there is no vertex cover $C' \subset C$. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set and such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

The proof of Theorem 2 may be outlined as follows. We first show that a graph $G = (V, E)$ has Grundy number at least $|V| - k$ if and only if its complement has a vertex cover with certain properties and in particular size at most $2k$. We then give an algorithm in $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$ that decides if a given minimal vertex cover of $\overline{G}$ is contained in a vertex cover having such properties.

There are at most 2^{2k} minimal vertex covers of size at most $2k$ and we can enumerate them in time $O\left(2^{2k} \cdot |V|\right)$ using a search tree (see for example Section

8.2 of [14]). Hence applying the above-mentioned algorithm for each minimal vertex cover yields an algorithm in time $O\left((2k)^{2k} \cdot |E| + 2^{2k}k^{3k+5/2}\right)$ for DUAL OF GREEDY COLOURING.

Lemma 2. *Let $G = (V, E)$ be a graph and $k \geq 0$ an integer. Then, $\Gamma(G) \geq |V| - k$ if and only if there is a vertex cover C of $\overline{G}$ such that $G[C]$ admits a greedy colouring $(C_1, C_2, \ldots, C_{k'})$ with the following properties:*

P1*: $|C| - k \leq k' \leq k$;*
P2*: $|C_i| \geq 2$, for every $1 \leq i \leq k'$;*
P3*: For each $v \in V \backslash C$ and for every $1 \leq i \leq k'$, there is $u \in C_i$ such that $uv \in E$.*

Proof. ($\Rightarrow$) Assume that $\Gamma(G) \geq |V| - k$ and consider a greedy $\Gamma(G)$-colouring c. Let C be the set of vertices that are in a colour class with more than one vertex. Then $V \backslash C$ is the set of vertices that are alone in their colour classes.

Let u and v be two vertices in $V \setminus C$. Without loss of generality we may assume that $c(u) > c(v)$. Then, as c is a greedy colouring, u has a neighbour coloured $c(v)$, which must be v. So $uv \in E$. Hence $V \backslash C$ is clique in G and so an independent set in $\overline{G}$. Consequently, C is a vertex cover in $\overline{G}$.

Let c' be the greedy colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ induced by c. Then, $|C_i| \geq 2$, for every $1 \leq i \leq k'$, so $|C| \geq 2k'$. By definition of C, we have $\Gamma(G) = |V| - |C| + k' \leq |V| - k'$. Since $\Gamma(G) \geq |V| - k$, we obtain $k \geq k' \geq |C| - k$.

Finally, let $v \in V \backslash C$ and $1 \leq i \leq k'$. If the colour of the vertices of C_i in c is smaller than $c(v)$, then v is adjacent to at least one vertex of C_i because c is greedy. If not then every vertex of C_i is adjacent to v because it is the sole vertex coloured $c(v)$. In both cases, v is adjacent to at least one vertex in C_i, so c' also has Property P3.

($\Leftarrow$) Let C be a vertex cover of $\overline{G}$ such that there is a greedy colouring $c' = \left(C_1, C_2, \ldots, C_{k'}\right)$ of $G[C]$ having Properties P1, P2 and P3. One can extend c' to the entire graph G by assigning $|V| - |C|$ distinct colours to the vertices of $V \backslash C$. As a consequence of P3 and the fact that $V \backslash C$ is an independent set in $\overline{G}$ and therefore a clique in G, the obtained colouring is greedy. Because of P1, it uses $k' + |V| - |C| \geq (|C| - k) + |V| - |C| = |V| - k$ colours. $\square$

Let C be a vertex cover of $\overline{G}$. A greedy colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ having the Properties P1, P2 and P3 of Lemma 2 is said to be *good*. C is *suitable* if $G[C]$ has a good greedy colouring. Observe that Property P1 implies that a suitable vertex cover has cardinality at most $2k$.

Proposition 1. *Let C be a suitable vertex cover of $\overline{G}$, $(C_1, C_2, \ldots, C_{k'})$ a good greedy colouring of $G[C]$ and $C_{min} \subseteq C$ a minimal vertex cover. Then for all $1 \leq i \leq k'$, $|C_i \setminus C_{min}| \leq 1$.*

Proof. Each colour class C_i, $1 \leq i \leq k'$, is an independent set of size at least 2 in G. So it is a clique of size at least 2 in $\overline{G}$. Since C_{min} is a vertex cover in $\overline{G}$, $|C_i \cap C_{min}| \geq |C_i| - 1$, so $|C_i \setminus C_{min}| \leq 1$. $\square$

Lemma 3. *Let k be an integer, $G = (V, E)$ a graph and C_{min} a minimal vertex cover of $\overline{G}$ of size at most $2k$. It can be determined in time $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$ if C_{min} is contained in a suitable vertex cover C.*

Proof. In order to determine if C_{min} is contained in a suitable vertex cover, we enumerate all possible proper colourings of $G[C_{min}]$ with k' colours, $|C_{min}| - k \le k' \le k$. For each of them, we then check in time $O(|E| + k^{k+5/2})$ if it can be extended into a good greedy colouring of a suitable vertex cover. There are at most $k^{|C_{min}|} \le k^{2k}$ proper colourings of C_{min} with at most k colours and they can be enumerated in time $O(k^{2k})$. Hence the running time of our algorithm is $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$.

Let us now detail an algorithm that, given a proper colouring $c = (C_1, C_2, \ldots, C_{k'})$ of $G[C_{min}]$, decides if it can be extended into a good greedy colouring of a suitable vertex cover in time $O(|E| + k^{k+5/2})$. By Proposition 1, for such an extension at most one vertex of $V \setminus C_{min}$ is added in each colour class.

If c is a good colouring of C_{min} then we are done. So we may assume that it is not. We say that a colour class S_i is *defective* with respect to a colouring $s = (S_1, \ldots, S_l)$ of $S \subseteq V$ if at least one of the following holds:

(i) $|S_i| < 2$;
(ii) For some $j > i$, there is $v \in S_j$ with no neighbour in S_i;
(iii) There is $v \in V \setminus S$ such that v has no neighbours in S_i.

Let S_i be a defective colour class with respect to s. An *i-candidate* with respect to s is a vertex $v \in V \setminus \bigcup_{j=1}^{l} S_j$ such that $S_i \cup \{v\}$ is an independent set which is not defective with respect to the colouring $(S_1, \ldots, S_{i-1}, S_i \cup \{v\}, S_{i+1}, \ldots, S_l)$. We denote by $X_s(j)$ the set of j-candidates with respect to s and D_s the set of defective colour classes with respect to s. If $|X_s(j)| \ge k$, we say that j is a colour class of *type 1*. Otherwise, we say that it is of *type 2*. It is easy to see that the set of defective colour classes of c and their candidates can be computed in time $O(|E|)$.

Clearly, if c can be extended into a good colouring, it means that we can place candidates to some of its colour classes and obtain a colouring without defective colour classes. Because of Proposition 1, we are only allowed to place at most one vertex in each colour class. As we will show later, the only defective colour classes that may not receive a candidate in the extension of c to a good colouring are those of type 2.

Claim 3 *Let $C \supseteq C_{min}$, s be a k'-colouring of C, i one of its defective colour classes and v an i-candidate. Let s' be the extension of s where we place v in colour class i. Then, for every colour class $j \ne i$, $X_{s'}(j) = X_s(j) \setminus \{v\}$.*

Proof. First, assume that j is not defective in s. If it does become defective in s' it is due to condition (ii), since it cannot satisfy (i) or (iii) after the insertion of v in colour i. But then, since j does not satisfy (ii) in s, v is the only vertex in i that may have no neighbours coloured j, which implies that j satisfies condition (iii) in s, a contradiction.

Now assume that j is defective in s. We first prove that $X_s(j) \backslash \{v\} \subseteq X_{s'}(j)$. In this case, again we have that (i) and (iii) remain unchanged after the insertion of v, in the sense that if a vertex different from v is a candidate for j in s because of one of these conditions, the same will happen in s'. Regarding condition (iii), the only thing that may change in s' is that v is now one of the vertices with no neighbours in j. Since v is not in C_{min}, it is adjacent to every j-candidate. Then, every vertex distinct from v that was a candidate for j in s remains a candidate for j in s'.

The converse, that is that every vertex in $X_{s'}(j)$ is also in $X_s(j) \backslash \{v\}$, is trivial. □

In particular, Claim 3 shows that the insertion of a candidate in a defective colour classes does not create a new defective colour class.

Claim 4 *Let $s = (S_1, \ldots, S_{k'})$ be a k'-colouring of C_{min} and assume it can be extended into a good colouring $s' = (S'_1, \ldots, S'_{k'})$ of a suitable vertex cover. If j is a defective colour class in s and $|S'_j \backslash S_j| = 0$, then j is of type 2 in s. Moreover, j is defective only because of (iii).*

Proof. Let j be such that $S'_j = S_j$. If j satisfies (i) then there is only one vertex coloured j in s', and thus s' cannot be a good colouring, a contradiction. If j satisfies (ii) then there is a vertex v coloured $j' > j$ with no neighbours in S_j. Since no vertices were added to S_j in s', vertex v also has no neighbours coloured j in s', a contradiction. Hence j can be defective only because of condition (iii). But then, if $|X_s(j)| \geq k$, as we add at most one vertex to each colour class when extending s to s', at least one vertex in $X_s(j)$ is not in any colour class of s'. Hence, there is a vertex that is not coloured in s' and has no neighbours with colour j, which implies that j is a defective colour class in s', a contradiction. □

In order to determine if c can be properly extended, in a first step we consider all possible extensions of the type 2 colour classes. For such a colour class of type 2, we can choose to add to it either one of its candidates or none. By Claim 4 this later case is possible if the colour class satisfies only (iii). There are k_1 defective colour classes of type 2, where $k_1 \leq k$. Moreover each of these colour classes has k_2 candidates with $k_2 < k$, and so has $k_2 + 1$ possible ways of extension: adding one of the k_2 candidates or adding none. Hence, we can enumerate all the possible extensions of the type 2 colour classes in time $O(k_1^{k_2+1}) = O(k^k)$. In the second step, for each possible extension, we check if the type 1 colour classes could be extended in order to obtain a good greedy colouring.

Let c' be one possible extension of c as considered in the last paragraph. If a colour class S_i of type 2 has not been extended, it may still be defective. If it is defective because it satisfies (i) or (ii), then it will remain defective after the second step in which we add some candidate to colour classes of type 1. Hence, we can stop, it will never lead to a good colouring. If it is defective because it satisfies (iii) (and only (iii)) then all the vertices $v \in V \setminus \bigcup_{j=1}^{l} S_j$ such that v has no neighbours in S_i must be placed into some type 1 colour class in any

extension to a good colouring. In particular, they need to be candidates of at least one colour class of type 1. We call such vertices v *necessary candidates*.

Let $D_1 \subseteq D_{c'}$ be the set of defective colour classes in c' that are of type 1 in c. Also let N be the set of necessary candidates in c' and C the vertex cover given by the vertices coloured in c'. Remember that a suitable vertex cover C must satisfy $|C| - k \leq k'$, and so if $|C| + |D_1| - k > k'$ there is no way of properly extending c', since by Claim 4 we need to place one candidate in each of the $|D_1|$ defective colour classes of type 1. The number of colour classes in c is at most k, and after a candidate is placed in a defective colour class, the colour class is no longer defective. So, since the type 1 colour classes have at least k candidates in c and because of Claim 3, there are at least $|D_{c'}|$ candidates for each of the $|D_1|$ defective colour classes of type 1 in c. As a consequence, there are enough candidates to place in each colour class in D_1. But we also have to ensure that every necessary candidate is placed in a defective colour class. This is equivalent to finding a matching in the bipartite graph H with vertex set $V(H) = D_1 \cup N$ and with edge set $E(H) = \{(i \in D_1, v \in N) \mid$ vertex v is an i-candidate$\}$ such that every vertex in N is saturated. This can be done in time $O\left((|V(H)| + |E(H)|)\sqrt{|V(H)|}\right) = O(k^{5/2})$ by the algorithm of Hopcroft and Karp [10].

If such a matching does not exist, then we cannot properly extend c' by adding candidates to the vertices in D_1, and so we may reject c'. If such a matching exists, since each type 1 colour class has more than $|D_1|$ candidates, we can greedily extend c' to a good colouring.

Hence one can check in time $O(|E| + k^{k+5/2})$ if a proper colouring of $G[C_{min}]$ can be extended into a good greedy colouring of G. □

We are now able to prove Theorem 2.

Proof (of Theorem 2).
Let G be an instance of the problem. To answer the question, we enumerate all minimal vertex covers of $\overline{G}$, and check, for each one, if it is contained in a suitable vertex cover. To enumerate all minimal vertex covers takes time $O(2^{2k})$. For each of these at most 2^{2k} minimal vertex covers, we check if it is contained in a suitable vertex cover. By Lemma 5, it can be done in $O\left(k^{2k} \cdot |E| + k^{3k+5/2}\right)$. Hence the total running time is $O\left((2k)^{2k} \cdot |E| + 2^{2k}k^{3k+5/2}\right)$. □

4 Dual of *b*-Colouring

In this section, we show that DUAL OF *b*-COLOURING is FPT.

Theorem 3. DUAL OF *b*-COLOURING *can be solved in time*
$O\left(2^{2k}3^k k^{2k}(k+1)!|V(G)|\right)$.

The proof is very similar to the one of previous section. We first state a lemma analogous to Lemma 2. Its proof is so similar to the one of Lemma 2 that we omit it here.

Lemma 4. *Let $G = (V, E)$ be a graph and $k \geq 0$ an integer. Then, $\chi_b(G) \geq |V| - k$ if and only if there is a vertex cover C of $\overline{G}$ such that $G[C]$ admits a b-colouring $(C_1, C_2, \ldots, C_{k'})$ with the following properties:*

P1: $|C| - k \leq k' \leq k$;
P2: $|C_i| \geq 2$, for every $1 \leq i \leq k'$;
P3: *For every $1 \leq i \leq k'$ there is a b-vertex $v_i \in C_i$ such that $uv_i \in E$ for all $u \in V \backslash C$.*

Let C be a vertex cover of $\overline{G}$. We say that a *b*-colouring $(C_1, C_2, \ldots, C_{k'})$ of $G[C]$ having the Properties P1, P2 and P3 of Lemma 4 is *good*. C is *suitable* if $G[C]$ has a good *b*-colouring. Property P1 implies that a suitable vertex cover has cardinality at most $2k$.

An analog of Proposition 1 can be easily proved.

Proposition 2. *Let C be a suitable vertex cover of $\overline{G}$, $(C_1, C_2, \ldots, C_{k'})$ a good b-colouring of $G[C]$ and $C_{min} \subseteq C$ a minimal vertex cover. Then for all $1 \leq i \leq k'$, $|C_i \setminus C_{min}| \leq 1$.*

Then in order to determine if $\chi_b(G) \geq |V| - k$ one may enumerate all minimal vertex covers of $\overline{G}$ with size at most $2k$ and verify, for each such vertex cover, if it is suitable or if it may be extended to a suitable vertex cover.

Lemma 5. *Let k be an integer, $G = (V, E)$ a graph and C_{min} a minimal vertex cover of $\overline{G}$ of size at most $2k$. It can be determined in time $O\left(3^k k^{2k}(k+1)!|V(G)|\right)$ if C_{min} is contained in a suitable vertex cover C.*

Proof. In order to determine if C_{min} is contained in a suitable vertex cover, we first enumerate all possible proper colourings of $G[C_{min}]$ with k' colours, $|C_{min}| - k \leq k' \leq k$. For each of them, we then check in time $O\left(3^k(k+1)!|V(G)|\right)$ if it can be extended into a good *b*-colouring of a suitable vertex cover. There are at most $k^{|C_{min}|} \leq k^{2k}$ proper colourings of C_{min} with at most k colours and they can be enumerated in time $O\left(k^{2k}\right)$. Hence our algorithm is $O\left(3^k k^{2k}(k+1)!|V(G)|\right)$.

Let us now detail an algorithm that, given a proper colouring $c = (C_1, C_2, \ldots, C_{k'})$ of $G[C_{min}]$, decides if it can be extended into a good *b*-colouring of a suitable vertex cover in FPT time. If c is a good *b*-colouring of C_{min} then we are done. Otherwise, we need to check if we can extend c to vertices in $V \backslash C_{min}$ and obtain a good *b*-colouring. By Proposition 2, for such an extension at most one vertex of $V \setminus C_{min}$ is added in each colour class.

In order to check if c can be extended to a good *b*-colouring, we must guess for each colour which vertex will be a *b*-vertex in such an extension. For each colour i such a *b*-vertex may either already be in C_i or be in $V \setminus C_{min}$. Hence we try all possible sets $B \subset C_{min}$ such that $|B \cap C_i| \leq 1$ for all $1 \leq i \leq k'$ and check if one can extend c into a good *b*-colouring *highlighting* B, such that all vertices of B are *b*-vertices. The number of possible choices of B is given by $p = \prod_{i=1}^{k'}(|C_i| + 1)$. Since $\sum_{i=1}^{k'} |C_i| = n$ and p is maximized when the $|C_i|$ are

all equal, we have $p \le \left(\dfrac{n+k'}{k'}\right)^{k'} \le \left(1+\dfrac{n}{k'}\right)^{k'}$. As $f(x) = \left(1+\dfrac{n}{x}\right)^x$ is an increasing function, we get that $\left(1+\dfrac{n}{k'}\right)^{k'} \le \left(1+\dfrac{n}{k}\right)^{k} \le \left(1+\dfrac{2k}{k}\right)^{k} = 3^k$.

For all i such that $C_i \cap B \ne \emptyset$, let F_i be the set of vertices of $V \setminus C_{min}$ which are not adjacent to the vertex b_i of $C_i \cap B$. In any extension of c to a good b-colouring highlighting B, all the vertices in F_i are to be moved to one of the colour classes because of (P3). The vertices of $F = \bigcup_{\{i|b_i \ne \emptyset\}} F_i$ are called *forced* vertices. If $|F| > k'$, c cannot be extended to a good b-colouring because at most one vertex of $V \setminus C_{min}$ may be added to each colour by Proposition 2. Hence we consider the case when $|F| \le k'$. We enumerate all possible ways of moving the forced vertices into different colour classes, every forced vertex v being moved to a colour class C such that $\{v\} \cup C$ is stable. There are at most $k'! \le k!$ such possibilities.

Let c^* be one of the so obtained colourings of $C^* = C \cup F$. If a vertex v is moved to a colour class C_i such that $C_i \cap B = \emptyset$, then it must be the b-vertex of this class because at most one vertex can be added to each colour class. Hence we have to look for a good b-colouring extending c^* and highlighting B^* the set containing all the vertices of B and the forced vertices moved to colour classes not intersecting B.

A colour class S_i is *defective* with respect to a colouring $s = (S_1, \ldots, S_l)$ of $S \subseteq V$ and a set $B \subset S$ if at least one of the following holds:

(i) $B \cap S_i = \emptyset$;
(ii) $|S_i| < 2$;
(iii) There is a vertex $b \in B \setminus S_i$ with no neighbour in S_i;

Let S_i be a defective colour class with respect to (s, B). An *i-candidate* with respect to (s, B) is a vertex $v \in V \setminus S$ such that $S_i \cup \{v\}$ is an independent set which is not defective with respect to $(S_1, \ldots, S_{i-1}, S_i \cup \{v\}, S_{i+1}, \ldots, S_l), B_i)$ where $B_i = B$ if $B \cap S_i \ne \emptyset$ and $B_i = B \cup \{v\}$ otherwise. We denote by $X_{s,B}(i)$ the set of i-candidates with respect to (s, B) and $D_{s,B}$ the set of defective colour classes with respect to (s, B).

Claim 5 *Let $C \supset C^*$, s be a k'-colouring of C, B a subset of S with at most one vertex in each colour class, i a defective colour class with respect to (s, B) and v an i-candidate. Let s' be the extension of s where we place v in colour class i. Then, for every colour class $j \ne i$, $X_{s',B_i}(j) = X_{s,B}(j) \setminus \{v\}$.*

Proof. If j is not defective with respect to (s, B), then it cannot become defective with repect to (s', B_i), since the insertion of v in colour i cannot cause j to satisfy one of the conditions (i), (ii) or (iii).

Then, we assume that j is defective in s. To see that $X_{s,B}(j) \setminus \{v\} \subseteq X_{s',B_i}(j)$, we have again that (i) (ii) and (iii) remain unchanged after the insertion of v, in the sense that if a vertex different from v is a j-candidate with respect to (s, B) because of one of these conditions, the same will happen with respect to (s', B_i).

Finally, it follows directly from the definition that every vertex in $X_{s',B_i}(j)$ is also in $X_{s,B}(j)\backslash\{v\}$. □

In particular, Claim 3 shows that the insertion of a candidate in a defective colour class does not create a new defective colour class.

Claim 6 *Let $s = (S_1, \ldots, S_{k'})$, be a k'-colouring of C^* and assume it can be extended into a good b-colouring $s' = (S'_1, \ldots, S'_{k'})$ highlighting B^*. If j is a defective colour class in s then $|S'_j \backslash S_j| = 1$.*

Proof. If j satisfies (i) for (s, B^*), then S'_j contains the b-vertex of colour j which is not in S_j. If j satisfies (ii), then since $|S'_j| \geq 2$ by Property P2, $|S'_j| > |S_j|$. If j satisfies (iii) then a vertex $b \in B \setminus S_j$ has no neighbours in S_j. But S'_j contains a neighbour of b since it highlghts B. Again $|S'_j| > |S_j|$.

In all cases S'_j has more vertices than S_j. So by Proposition 2, $|S'_j \backslash S_j| = 1$. □

Recall that if C' is suitable it must satisfy $|C'| - k \leq k'$, and so if $|C^*| + |D_{c^*,B^*}| - k > k'$ there is no way extend c^* into a good b-colouring. The number of colour classes in c^* is at most k', and after a candidate is placed in a defective colour class, the colour class is no longer defective. As a consequence, to decide if c^* can be extended to a good b-colouring highlighting B^* is equivalent to find a matching in the bipartite graph H with vertex set $V(H) = D_{c^*,B^*} \cup \{v \in X_{c^*,B^*}(i), 1 \leq i \leq k'\}$ and with edge set $E(H) = \{(i \in D_{c^*,B^*}, v \in V\backslash C^*) \mid \text{vertex } v \text{ is an } i\text{-candidate}\}$ such that every vertex in D_{c^*,B^*} is saturated. This can be done in time $O\left((|V(H)| + |E(H)|)\sqrt{|V(H)|}\right) = O\left(k|V(G)|\right)$, since $|V(H)| = O\left(|V(G)|\right)$ and $|E(H)| = O\left(k|V(H)|\right) = O\left(k|V(G)|\right)$ and because of the algorithm of Hopcroft and Karp [10].

If such a matching does not exist, then we cannot properly extend c^* by adding candidates to the colour classes, and we must reject c. If such a matching exists, we can move each saturated candidate to the matched defective colour class to obtain a good b-colouring highlighting B^*.

Hence one can check in time $O\left(3^k(k+1)!|V(G)|\right)$ if a proper colouring of $G[C_{min}]$ can be extended into a good b-colouring of G. □

Proof (of Theorem 3).

Let G be an instance of the problem. To answer the question, we enumerate all minimal vertex covers of $\overline{G}$, and check, for each one, if it is contained in a suitable vertex cover. To enumerate all minimal vertex covers takes time $O(2^{2k})$. For each of these at most 2^{2k} minimal vertex covers, we check if it is contained in a suitable vertex cover. By Lemma 5, it can be done in time $O\left(3^k k^{2k}(k+1)!|V(G)|\right)$. Hence the total running time is $O\left(2^{2k}3^k k^{2k}(k+1)!|V(G)|\right)$. □

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