

# A Guide to Algorithm Design: Paradigms, Methods, and Complexity Analysis

Anne Benoit, Yves Robert, Frédéric Vivien

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*Anne Benoit, Yves Robert, Frédéric Vivien*

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# ***A guide to algorithm design***

Paradigms, methods and complexity analysis

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