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# A Conditional Logical Framework <sup>\*</sup>

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**Abstract.** The *Conditional Logical Framework*  $LF_{\kappa}$  is a variant of the Harper-Honsell-Plotkin’s Edinburgh Logical Framework LF. It features a generalized form of  $\lambda$ -abstraction where  $\beta$ -reductions fire *under the condition* that the argument satisfies a *logical predicate*. The key idea is that the type system *memorizes* under what conditions and where reductions have yet to fire. Different notions of  $\beta$ -reductions corresponding to different predicates can be combined in  $LF_{\kappa}$ . The framework  $LF_{\kappa}$  subsumes, by simple instantiation, LF (in fact, it is also a subsystem of LF!), as well as a large class of new generalized conditional  $\lambda$ -calculi. These are appropriate to deal smoothly with the side-conditions of both Hilbert and Natural Deduction presentations of Modal Logics. We investigate and characterize the metatheoretical properties of the calculus underpinning  $LF_{\kappa}$ , such as subject reduction, confluence, strong normalization.

## 1 Introduction

The Edinburgh Logical Framework LF of [HHP93] was introduced both as a general theory of logics and as a metalanguage for a generic proof development environment. In this paper, we consider a variant of LF, called *Conditional Logical Framework*  $LF_{\kappa}$ , which allows to deal uniformly with logics featuring *side-conditions* on the application of inference rules, such as *Modal Logics*. We study the language theory of  $LF_{\kappa}$  and we provide proofs for subject reduction, confluence, and strong normalization. By way of example, we illustrate how special instances of  $LF_{\kappa}$  allow for smooth encodings of Modal Logics both in Hilbert and Natural Deduction style.

The motivation for introducing  $LF_{\kappa}$  is that the type system of LF is too coarse as to the “side conditions” that it can enforce on the application of rules. Rules being encoded as functions from proofs to proofs and rule application simply encoded as lambda application, there are only roundabout ways to encode provisos, even as simple as that appearing in a *rule of proof*. Recall that a rule of proof can be applied only to premises which do not depend on any assumption, as opposed to a *rule of derivation* which can be applied everywhere. Also rules which appear in many natural deduction presentations of Modal and Program Logics are very problematic in standard LF. Many such systems feature rules which can be applied only to premises which depend solely on assumptions of a particular shape [CH84], or whose derivation has been carried out using only certain sequences of rules. In general, Modal, Program, Linear or Relevance

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Logics appear to be encodable in LF only encoding a very heavy machinery, which completely rules out any natural Curry-Howard paradigm, see *e.g.* [AHMP98]. As we will see for Modal Logics,  $\text{LF}_\kappa$  allows for much simpler encodings of such rules, which open up promising generalizations of the proposition-as-types paradigm.

The idea underlying the Conditional Logical Framework  $\text{LF}_\kappa$  is inspired by the Honsell-Lenisa-Liquori's *General Logical Framework* GLF see [HLL07], where we proposed a uniform methodology for extending LF, which allows to deal with pattern matching and restricted  $\lambda$ -calculi. The key idea, there, is to separate two different notions that are conflated in the original LF. As already mentioned, much of the rigidity of LF arised from the fact that  $\beta$ -reduction can be applied always in full generality. One would like to fire a  $\beta$ -reduction under certain conditions on typed terms, but the type system is not rich enough to be able to express such restrictions smoothly. What we proposed in [HLL07] is to use as type of an application, in the term application rule, (O·Appl) below, not the type which is obtained by carrying out directly in the metalanguage the substitution of the argument in the type, but a new form of type which simply records the information that such a reduction can be carried out. An application of the Type Conversion Rule can then recover, if possible,  $\frac{\Gamma \vdash M : \Pi x:A.B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : (\lambda x:A.B) N}$  the usual effect of the application rule. This key idea leads to the following object application rule:

Once this move has been made, we have a means of annotating in a type the information that a reduction *is waiting to be carried out in the term*. If we take seriously this move, such a type need not be necessarily definitionally equal to the reduced one as in the case of LF. Without much hassle we have a principled and natural way of typing calculi featuring generalized or restricted forms of  $\beta$ -reduction, which *wait for some condition to be satisfied before they can fire*. Furthermore, such calculi can be used for underpinning new powerful Logical Frameworks, where all the extra complexity in terms can be naturally tamed using the expressive power of the new typing system. Once this program is carried out in a sufficiently modular form, we have a full-fledged Logical Framework.

More specifically, in  $\text{LF}_\kappa$  we consider a new form of  $\lambda$  and corresponding  $\Pi$  abstraction, *i.e.*  $\lambda_{\mathcal{P}}x:A.M$  and  $\Pi_{\mathcal{P}}x:A.M$ , where  $\mathcal{P}$  is a predicate, which ranges over a suitable set of predicates. The reduction  $(\lambda_{\mathcal{P}}x:A.M) N$  fires only if the predicate  $\mathcal{P}$  holds on  $N$ , and in this case the redex progresses, as usual, to  $M[N/x]$ . Therefore the final object application rule in  $\text{LF}_\kappa$  will be:

In this rule a type where a reduction is “stuck”, if the predicate  $\mathcal{P}$  is not true on  $N$ , is assigned to an object application. However, when we view this object as a subterm of another term, such reduction could become allowed in the future, after other reductions are performed in the term, which provide substitutions for  $N$ . In  $\text{LF}_\kappa$  more predicates can be combined.  $\text{LF}_\kappa$  subsumes standard LF, which is recovered by considering the trivial predicate that is constantly true.

Historically, the idea of introducing stuck-reduction in objects and types, in the setting of higher-order term rewriting systems with sophisticated pattern-matching capabilities, was first introduced in Cirstea-Kirchner-Liquori's Rho-cube [CKL01b], in order to design a hierarchy of type systems for the untyped Rewriting Calculus of [CKL01a], and then it was generalized to a more general framework of Pure Type

Systems with Patterns [BCKL03]. This typing protocol was essential to preserve the strong normalization of typable terms, as proved in [HLL07]. The idea underlying the Conditional Logical Framework  $LF_{\kappa}$  is the same exploited in [HLL07] for the General Logical Framework GLF. However, there is an important difference between the two frameworks in the definition of predicates. On one hand, predicates in [HLL07] are used both to determine whether  $\beta$ -reduction fires and to compute a substitution, while in the present paper they are used only to determine whether  $\beta$ -reduction fires. On the other hand, in [HLL07] predicates are defined on terms, while here they are defined on typed judgments. This adds extra complexity both in the definition of the system and in the study of its properties, but it greatly simplifies the treatment of Modal Logics and of other situations where conditions depending on types have to be expressed.

Apart from Modal Logics, we believe that our Conditional Logical Framework could also be very helpful in modeling dynamic and reactive systems: for example bio-inspired systems, where reactions of chemical processes take place only provided some extra structural or temporal conditions; or process algebras, where often no assumptions can be made about messages exchanged through the communication channels. Indeed, it could be the case that a redex, depending on the result of a communication, can remain stuck until a “good” message arrives from a given channel, firing in that case an appropriate reduction (this is a common situation in many protocols, where “bad” requests are ignored and “good ones” are served). Such dynamical (run-time) behaviour could be hardly captured by a rigid type discipline, where bad terms and hypotheses are ruled out *a priori*, see *e.g.* [NPP08].

In this paper we develop all the metatheory of  $LF_{\kappa}$ . In particular, we prove subject reduction, strong normalization, confluence; this latter under the sole assumption that the various predicate reductions nicely combine, *i.e.* no reduction can prevent a redex, which could fire, from firing after the reduction. Since  $\beta$ -reduction in  $LF_{\kappa}$  is defined only on typed terms, in order to prove subject reduction and confluence, we need to devise a new approach, alternative to the one in [HHP93]. Our approach is quite general, and in particular it yields alternative proofs for the original LF.

In conclusion, the work on  $LF_{\kappa}$  carried out in this paper is valuable in three ways. First, being  $LF_{\kappa}$  so general, the results in this paper potentially apply to a wide range of Logical Frameworks, therefore many fundamental results are proved only once and uniformly for all systems. Secondly, the  $LF_{\kappa}$  approach is useful in view of implementing a “telescope” of systems, since it provides relatively simple sufficient conditions to test whether a potential extension of the framework is safe. Thirdly,  $LF_{\kappa}$  can suggest appropriate extensions of the proposition-as-types paradigm to a wider class of logics.

*Synopsis.* In Section 2, we present the syntax of  $LF_{\kappa}$ , its type system, and the predicate reduction. In Section 3, we present instantiations of  $LF_{\kappa}$  to known as well as to new calculi, and we show how to encode smoothly Modal Logics. The  $LF_{\kappa}$ ’s metatheory is carried out in Section 4. Conclusions and directions for future work appear in Section 5. Proofs appear in a Web Appendix available at the author’s web pages.

## 2 The System

*Syntax.* In the following definition, we introduce the  $LF_{\kappa}$  pseudo-syntax for kinds, families, objects, signatures and contexts.

**Definition 1 (LF<sub>κ</sub> Pseudo-syntax)**

$\Sigma \in \mathcal{S}$	$\Sigma ::= \emptyset \mid \Sigma, a:K \mid \Sigma, f:A$	<i>Signatures</i>
$\Gamma, \Delta \in \mathcal{C}$	$\Gamma ::= \emptyset \mid \Gamma, x:A$	<i>Contexts</i>
$K \in \mathcal{K}$	$K ::= \text{Type} \mid \Pi_{\mathcal{P}}x:A.K \mid \lambda_{\mathcal{P}}x:A.K \mid K M$	<i>Kinds</i>
$A, B, C \in \mathcal{F}$	$A ::= a \mid \Pi_{\mathcal{P}}x:A.B \mid \lambda_{\mathcal{P}}x:A.B \mid A M$	<i>Families</i>
$M, N, Q \in \mathcal{O}$	$M ::= f \mid x \mid \lambda_{\mathcal{P}}x:A.M \mid M N$	<i>Objects</i>

where  $a, f$  are typed constants standing for fixed families and terms, respectively, and  $\mathcal{P}$  is a predicate ranging over a set of predicates, which will be specified below.

LF<sub>κ</sub> is parametric over a set of predicates of a suitable shape. Such predicates are defined on typing judgments, and will be discussed in the section introducing the type system.

*Notational conventions and auxiliary definitions.* Let “ $T$ ” range over any term in the calculus (kind, family, object). The abstractions  $\checkmark_{\mathcal{P}}x:A.T$  ( $\checkmark \in \{\lambda, \Pi\}$ ) bind the variable  $x$  in  $T$ . Domain  $\text{Dom}(\Gamma)$  and codomain  $\text{CoDom}(\Gamma)$  are defined as usual. Free  $\text{Fv}(T)$  and bound  $\text{Bv}(T)$  variables are defined as usual. As usual, we suppose that, in the context  $\Gamma, x:T$ , the variable  $x$  does not occur free in  $\Gamma$  and  $T$ . We work modulo  $\alpha$ -conversion and Barendregt’s hygiene condition.

*Type System.* LF<sub>κ</sub> involves type judgments of the following shape:

$\Sigma \text{ sig}$	$\Sigma$ is a valid signature
$\vdash_{\Sigma} \Gamma$	$\Gamma$ is a valid context in $\Sigma$
$\Gamma \vdash_{\Sigma} K$	$K$ is a kind in $\Gamma$ and $\Sigma$
$\Gamma \vdash_{\Sigma} A : K$	$A$ has kind $K$ in $\Gamma$ and $\Sigma$
$\Gamma \vdash_{\Sigma} M : A$	$M$ has type $A$ in $\Gamma$ and $\Sigma$
$\Gamma \vdash_{\Sigma} T \mapsto_{\beta} T' (: T'')$	$T$ reduces to $T'$ in $\Gamma, \Sigma$ (and $T''$ )
$\Gamma \vdash_{\Sigma} T =_{\beta} T' (: T'')$	$T$ converts to $T'$ in $\Gamma, \Sigma$ (and $T''$ )

The typing rules of LF<sub>κ</sub> are presented in Figure 1. As remarked in the introduction, rules (F·Appl) and (O·Appl) do not utilize metasubstitution as in standard LF, but rather introduce an explicit type redex. Rules (F·Conv) and (O·Conv) allow to recover the usual rules, if the reduction fires.

*Typed Operational Semantics.* The “type driven” operational semantics is presented in Figure 2, where the most important rule is (O·Red), the remaining ones being the contextual closure of  $\beta$ -reduction. For lack of space we omit similar rules for kinds and constructors. According to rule (O·Red), reduction is allowed only if the argument in the context satisfies the predicate  $\mathcal{P}$ . In this sense, reduction becomes “conditioned” by  $\mathcal{P}$ . In LF<sub>κ</sub>, we can combine more predicate reductions, *i.e.*, we can define and combine several predicates guarding  $\beta$ -reduction, whose shape is as follows. Each predicate is determined by a set  $\mathcal{A}$  of families (types), and the intended meaning is that it holds on

### Signature and Context rules

$$\begin{array}{c}
\frac{}{\emptyset \text{ sig}} \text{(S-Empty)} \quad \frac{\Sigma \text{ sig}}{\vdash_{\Sigma} \emptyset} \text{(C-Empty)} \\
\frac{\Sigma \text{ sig} \quad \vdash_{\Sigma} K \quad a \notin \text{Dom}(\Sigma)}{\Sigma, a:K \text{ sig}} \text{(S-Kind)} \\
\frac{\Sigma \text{ sig} \quad \vdash_{\Sigma} A : \text{Type} \quad f \notin \text{Dom}(\Sigma)}{\Sigma, f:A \text{ sig}} \text{(S-Type)} \\
\frac{\vdash_{\Sigma} \Gamma \quad \Gamma \vdash_{\Sigma} A : \text{Type} \quad x \notin \text{Dom}(\Gamma)}{\vdash_{\Sigma} \Gamma, x:A} \text{(C-Type)}
\end{array}
\quad
\begin{array}{c}
\frac{\Gamma, x:A \vdash_{\Sigma} B : \text{Type}}{\Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}}x:A.B : \text{Type}} \text{(F-Pi)} \\
\frac{\Gamma, x:A \vdash_{\Sigma} B : K}{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}}x:A.B : \Pi_{\mathcal{P}}x:A.K} \text{(F-Abs)} \\
\frac{\Gamma \vdash_{\Sigma} A : \Pi_{\mathcal{P}}x:B.K \quad \Gamma \vdash_{\Sigma} N : B}{\Gamma \vdash_{\Sigma} AN : (\lambda_{\mathcal{P}}x:B.K) N} \text{(F-AppI)} \\
\frac{\Gamma \vdash_{\Sigma} A : K' \quad \Gamma \vdash_{\Sigma} K \quad \Gamma \vdash_{\Sigma} K =_{\beta} K'}{\Gamma \vdash_{\Sigma} A : K} \text{(F-Conv)}
\end{array}$$

### Kind rules

$$\begin{array}{c}
\frac{\vdash_{\Sigma} \Gamma}{\Gamma \vdash_{\Sigma} \text{Type}} \text{(K-Type)} \\
\frac{\Gamma, x:A \vdash_{\Sigma} K}{\Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}}x:A.K} \text{(K-Pi)} \\
\frac{\Gamma, x:A \vdash_{\Sigma} K}{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}}x:A.K} \text{(K-Abs)} \\
\frac{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}}x:A.K \quad \Gamma \vdash_{\Sigma} N : A}{\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}}x:A.K) N} \text{(K-AppI)}
\end{array}$$

### Family rules

$$\frac{\vdash_{\Sigma} \Gamma \quad a:K \in \Sigma}{\Gamma \vdash_{\Sigma} a : K} \text{(F-Const)}$$

### Object rules

$$\begin{array}{c}
\frac{\vdash_{\Sigma} \Gamma \quad x:A \in \Gamma}{\Gamma \vdash_{\Sigma} x : A} \text{(O-Var)} \\
\frac{\vdash_{\Sigma} \Gamma \quad f:A \in \Sigma}{\Gamma \vdash_{\Sigma} f : A} \text{(O-Const)} \\
\frac{\Gamma, x:A \vdash_{\Sigma} M : B}{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}}x:A.M : \Pi_{\mathcal{P}}x:A.B} \text{(O-Abs)} \\
\frac{\Gamma \vdash_{\Sigma} M : \Pi_{\mathcal{P}}x:A.B \quad \Gamma \vdash_{\Sigma} N : A}{\Gamma \vdash_{\Sigma} MN : (\lambda_{\mathcal{P}}x:A.B) N} \text{(O-AppI)} \\
\frac{\Gamma \vdash_{\Sigma} M : A \quad \Gamma \vdash_{\Sigma} B : \text{Type} \quad \Gamma \vdash_{\Sigma} A =_{\beta} B : \text{Type}}{\Gamma \vdash_{\Sigma} M : B} \text{(O-Conv)}
\end{array}$$

**Fig. 1.**  $\text{LF}_K$  Type System

a typed judgment  $\Gamma \vdash_{\Sigma} M : A$  and a set of variables  $\mathcal{X} \subseteq \text{Dom}(\Gamma)$  if “ $\Gamma \vdash_{\Sigma} M : A$  is derivable and all the free variables in  $M$  which are in  $\mathcal{X}$  appear in subterms typable with a type in  $\mathcal{A}$ ”. This intuition is formally stated in the next definition.

### Definition 2 (Good families (types) and predicates)

Let  $\mathcal{A} \subseteq \mathcal{F}$  be a set of families. This induces a predicate  $\mathcal{P}_{\mathcal{A}}$  (denoted by  $\mathcal{P}$ , for simplicity), defined on typed judgments  $\Gamma \vdash M : A$  and sets  $\mathcal{X}$  such that  $\mathcal{X} \subseteq \text{Dom}(\Gamma)$ . The truth table of  $\mathcal{P}$  appears in Figure 3.

We call good a predicate  $\mathcal{P}$  defined as above, and good types the set of types in  $\mathcal{A}$  inducing it.

The following lemma states formally the intended meaning of our predicates:

### Lemma 1 ( $\mathcal{P}$ Satisfiability).

Given a predicate  $\mathcal{P} \in \mathcal{L}$  induced by a set of families (types)  $\mathcal{A}$ ,  $\mathcal{P}$  holds on a typed

$$\begin{array}{c}
\frac{\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}} x:A.M) N : C \quad \Gamma \vdash_{\Sigma} M[N/x] : C \quad \mathcal{P}(\text{Fv}(N); \Gamma \vdash_{\Sigma} N : A)}{\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}} x:A.M) N \mapsto_{\beta} M[N/x] : C} \text{(O-Red)} \\
\frac{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.M : \Pi_{\mathcal{P}} x:A.B \quad \Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.N : \Pi_{\mathcal{P}} x:A.B \quad \Gamma, x:A \vdash_{\Sigma} M \mapsto_{\beta} N : B}{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.M \mapsto_{\beta} \lambda_{\mathcal{P}} x:A.N : \Pi_{\mathcal{P}} x:A.B} \text{(O-}\lambda\text{-Red}_1) \\
\frac{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.M : C \quad \Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:B.M : C \quad \Gamma \vdash_{\Sigma} A \mapsto_{\beta} B : \text{Type}}{\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.M \mapsto_{\beta} \lambda_{\mathcal{P}} x:B.M : C} \text{(O-}\lambda\text{-Red}_2) \\
\frac{\Gamma \vdash_{\Sigma} M N : (\lambda_{\mathcal{P}} x:A.B) N \quad \Gamma \vdash_{\Sigma} P N : (\lambda_{\mathcal{P}} x:A.B) N \quad \Gamma \vdash_{\Sigma} M \mapsto_{\beta} P : \Pi_{\mathcal{P}} x:A.B}{\Gamma \vdash_{\Sigma} M N \mapsto_{\beta} P N : (\lambda_{\mathcal{P}} x:A.B) N} \text{(O-Appl-Red}_1) \\
\frac{\Gamma \vdash_{\Sigma} M N : (\lambda_{\mathcal{P}} x:A.B) N \quad \Gamma \vdash_{\Sigma} M P : (\lambda_{\mathcal{P}} x:A.B) N \quad \Gamma \vdash_{\Sigma} N \mapsto_{\beta} P : A}{\Gamma \vdash_{\Sigma} M N \mapsto_{\beta} M P : (\lambda_{\mathcal{P}} x:A.B) N} \text{(O-Appl-Red}_2) \\
\frac{\Gamma \vdash_{\Sigma} M \mapsto_{\beta} N : A \quad \Gamma \vdash_{\Sigma} A =_{\beta} B : \text{Type}}{\Gamma \vdash_{\Sigma} M \mapsto_{\beta} N : B} \text{(O-Conv-Red)}
\end{array}$$

**Fig. 2.**  $\text{LF}_{\kappa}$  Reduction (Object rules)

$$\begin{array}{c}
\frac{\Gamma \vdash_{\Sigma} M:A \quad A \in \mathcal{A}}{\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M : A)} \text{(O-Start}_1) \quad \frac{\Gamma \vdash_{\Sigma} M:A}{\mathcal{P}(\emptyset; \Gamma \vdash_{\Sigma} M : A)} \text{(O-Start}_2) \\
\frac{\mathcal{P}(\mathcal{X}; \Gamma, x:A \vdash_{\Sigma} M : B)}{\mathcal{P}(\mathcal{X} \setminus \{x\}; \Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}} x:A.M : \Pi_{\mathcal{P}} x:A.B)} \text{(O-Abs)} \\
\frac{\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M : \Pi_{\mathcal{P}} x:A.B) \quad \mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} N : A)}{\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M N : (\lambda_{\mathcal{P}} x:A.B) N)} \text{(O-Appl)} \\
\frac{\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M : A) \quad \Gamma \vdash_{\Sigma} B : \text{Type} \quad \Gamma \vdash_{\Sigma} A =_{\beta} B : \text{Type}}{\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M : B)} \text{(O-Conv)}
\end{array}$$

**Fig. 3.**  $\mathcal{P}$ 's truth table

judgment  $\Gamma \vdash_{\Sigma} M : B$  and a set of variables  $\mathcal{X} \subseteq \text{Dom}(\Gamma)$ , if  $\Gamma \vdash_{\Sigma} M : B$  is derivable and all the free variables in  $M$  which are in  $\mathcal{X}$  appear in subterms typable with a type in  $\mathcal{A}$ .

Hence, if we take  $\mathcal{X} = \text{Fv}(M)$ , then  $\mathcal{P}(\mathcal{X}; \Gamma \vdash_{\Sigma} M : A)$  will take into account exactly the free variables of  $M$ , according to the abovementioned intended meaning.

$$\begin{array}{c}
\frac{\Gamma \vdash_{\Sigma} A : K}{\Gamma \vdash_{\Sigma} A =_{\beta} A : K} \text{ (F.Refl.eq)} \\
\frac{\Gamma \vdash_{\Sigma} A =_{\beta} B : K \quad \Gamma \vdash_{\Sigma} B =_{\beta} C : K}{\Gamma \vdash_{\Sigma} A =_{\beta} C : K} \text{ (F.Trans.eq)} \\
\frac{\Gamma \vdash_{\Sigma} B =_{\beta} A : K}{\Gamma \vdash_{\Sigma} A =_{\beta} B : K} \text{ (F.Sym.eq)} \\
\frac{\Gamma \vdash_{\Sigma} A \mapsto_{\beta} B : K}{\Gamma \vdash_{\Sigma} A =_{\beta} B : K} \text{ (F.Red.eq)}
\end{array}$$

**Fig. 4.**  $\text{LF}_{\kappa}$  Definitional Equality (Family rules)

$$\begin{array}{ll}
\text{A}_1 : \phi \rightarrow (\psi \rightarrow \phi) & \text{K} : \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \\
\text{A}_2 : (\phi \rightarrow (\psi \rightarrow \xi)) \rightarrow (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \xi) & \text{4} : \Box\phi \rightarrow \Box\Box\phi \\
\text{A}_3 : (\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi) & \text{T} : \Box\phi \rightarrow \phi \\
\text{MP} : \frac{\phi \quad \phi \rightarrow \psi}{\psi} & \text{NEC} : \frac{\phi}{\Box\phi}
\end{array}$$

**Fig. 5.** Hilbert style rules for Modal Logic  $S_4$

Moreover, it is worth noticing that, once the “good families” are chosen, predicates are automatically defined as a consequence (look at the examples in the next section).

As far as definitional equality is concerned, due to lack of space, we give in Figure 4 only the rules on families, the ones for kinds and objects being similar. Notice that typing,  $\beta$ -reduction, and equality are mutually defined. Moreover,  $\beta$ -reduction is parametric over a (finite) set of good predicates, that is in  $\text{LF}_{\kappa}$  we can combine several good predicates at once.

Finally, notice that our approach is different from static approaches, where “bad” terms are ruled out *a priori* via rigid type disciplines. Namely, in our framework stuck redexes can become enabled in the future. Consider, *e.g.* a redex  $(\lambda_{\mathcal{P}_1} x:A.M) N$  which is stuck because a free variable  $y$  occurring into  $N$  does not satisfy the constraint imposed by predicate  $\mathcal{P}_1$ . Then, it could be the case that such redex is inserted into a context where  $y$  will be instantiated by a term  $P$ , by means of an outer (non-stuck) redex, like, *e.g.* in  $(\lambda_{\mathcal{P}_2} y:B.(\lambda_{\mathcal{P}_1} x:A.M) N) P$ . The resulting redex  $(\lambda_{\mathcal{P}_1} x:A[P/y].M[P/y]) N[P/y]$  could then fire since the constraint imposed by the predicate  $\mathcal{P}_1$  is satisfied by  $N[P/y]$ .

### 3 Instantiating $\text{LF}_{\kappa}$ to Modal Logics

The Conditional Logical Framework is quite expressive. By instantiating the set of predicates, we can recover various known and new interesting Logical Frameworks. The original LF can be recovered by considering the trivial predicate induced by the set  $\mathcal{A}$  of all families. More interesting instances of  $\text{LF}_{\kappa}$  are introduced below for providing smooth encodings of Modal Logics.

*Modal Logic in Hilbert style.* The expressive power of the Conditional Logical Framework allows to encode smoothly and uniformly both rules of proof as well as rules of derivation. We recall that the former are rules which apply only to premises which do not depend on any assumption, such as the rule of *necessitation* in Modal Logics, while



the latter apply to all premises, such as *modus ponens*. The idea is to use a conditioned  $\Pi$ -abstraction in rules of proof and a standard  $\Pi$ -abstraction in rules of derivation.

We shall not develop here the encodings of all the gamut of Modal Logics, in Hilbert style, which is extensively treated in [AHMP98]. By way of example, we shall only give the signature for classical  $S_4$  (see Figure 5) in Hilbert style (see Figure 6), which features necessitation (rule NEC in Figure 5) as a rule of proof. For notational convention in Figure 6 and in the rest of this section, we will denote by  $o^n$  the expression  $\underbrace{o \rightarrow o \rightarrow \dots \rightarrow o}_n$ . The target language of the encoding is the instance of  $\text{LF}_\kappa$ , obtained

by combining standard  $\beta$ -reduction with the  $\beta$ -reduction conditioned by the predicate  $\text{Closed}_o$  induced by the set  $\mathcal{A} = \{o\}$ . Intuitively,  $\text{Closed}_o(\text{Fv}(M); \Gamma \vdash_{S_4} M : \text{True}(\phi))$  holds iff “all free variables occurring in  $M$  belong to a subterm which can be typed in the derivation with  $o$ ”. This is precisely what is needed to encode it correctly, provided  $o$  is the type of propositions. Indeed, if all the free variables of a proof term satisfy such condition, it is clear, by inspection of the typing system’s object rules (see Figure 1), that there cannot be subterms of type  $\text{True}(\dots)$  containing free variables. Intuitively, this corresponds to the fact that the proof of the encoded modal formula does not depend on any assumptions. The following Adequacy Theorem can be proved in the standard way, using the properties of  $\text{LF}_\kappa$  in Section 4.

**Theorem 1 (Adequacy of the encoding of  $S_4$  - Syntax)**

Let  $\epsilon$  be an encoding function (induced by the signature in Figure 6) mapping object level formulae of  $S_4$  into the corresponding canonical terms<sup>1</sup> of  $\text{LF}_\kappa$  of type  $o$ . If  $\phi$  is a propositional modal formula with propositional free variables  $x_1, \dots, x_k$ , then the following judgment  $\Gamma \vdash_{S_4} \epsilon(\phi) : o$  is derivable, where  $\Gamma \equiv x_1:o, \dots, x_k:o$  and each  $x_i$  is a free propositional variable in  $\phi$ . Moreover, if we can derive in  $\text{LF}_\kappa$   $\Gamma \vdash_{S_4} M : o$  where  $\Gamma \equiv x_1:o, \dots, x_k:o$  and  $M$  is a canonical form, then there exists a propositional modal formula  $\phi$  with propositional free variables  $x_1, \dots, x_k$  such that  $M \equiv \epsilon(\phi)$ .

The proof amounts to a straightforward induction on the structure of  $\phi$  (first part) and on the structure of  $M$  (second part). After proving the adequacy of syntax, we can proceed with the more interesting theorems about the adequacy of the truth judgments.

**Theorem 2 (Adequacy of the encoding of  $S_4$  - Truth Judgment)**

$\phi_1, \dots, \phi_n \vdash_{S_4} \phi$  if and only if there exists a canonical form  $M$  such that

$$\Gamma, y_1:\text{True}(\epsilon(\phi_1)), \dots, y_n:\text{True}(\epsilon(\phi_n)) \vdash_{S_4} M : \text{True}(\epsilon(\phi))$$

where  $\Gamma \equiv x_1:o, \dots, x_k:o$  for each  $x_i$  free propositional variable in  $\phi_1, \dots, \phi_n, \phi$ .

*Classical Modal Logic  $S_4$  and  $S_5$  in Prawitz Style.* By varying the notion of good types in the general format of  $\text{LF}_\kappa$ , one can immediately generate Logical Frameworks which accommodate both classical Modal Logics  $S_4$  and  $S_5$  in Natural Deduction style introduced by Prawitz. Figure 7 shows common and specific rules of  $S_4$  and  $S_5$ .

<sup>1</sup> In this case, as in [HHP93], in stating the adequacy theorem it is sufficient to consider long  $\lambda\beta\eta$ -normal forms without stuck redexes as canonical forms. Namely, non-reducible terms with stuck redexes must contain free variables not belonging to subterms typable with  $o$ , and clearly such terms do not correspond to any  $S_4$ -formula.

Propositional Connectives and Judgments  
 $o : \text{Type} \quad \supset : o^3 \quad \neg : o^2 \quad \Box : o^2 \quad \text{True} : o \rightarrow \text{Type}$

Propositional Axioms  
 $A_1 : \Pi\phi:o. \Pi\psi:o. \text{True}(\phi \supset (\psi \supset \phi))$   
 $A_2 : \Pi\phi:o. \Pi\psi:o. \Pi\xi:o. \text{True}((\phi \supset (\psi \supset \xi)) \supset (\phi \supset \psi) \supset (\phi \supset \xi))$   
 $A_3 : \Pi\phi:o. \Pi\psi:o. \text{True}((\neg\psi \supset \neg\phi) \supset ((\neg\psi \supset \phi) \supset \psi))$

Modal Axioms  
 $K : \Pi\phi:o. \Pi\psi:o. \text{True}(\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi))$   
 $4 : \Pi\phi:o. \text{True}(\Box\phi \supset \Box\Box\phi)$   
 $T : \Pi\phi:o. \text{True}(\Box\phi \supset \phi)$

Rules  
 $MP : \Pi\phi:o. \Pi\psi:o. \text{True}(\phi) \rightarrow \text{True}(\phi \supset \psi) \rightarrow \text{True}(\psi)$   
 $NEC : \Pi\phi:o. \Pi_{\text{Closed}_o} x: \text{True}(\phi). \text{True}(\Box\phi)$

**Fig. 6.** The signature  $\Sigma_{S_4}$  for classic  $S_4$  Modal Logic in Hilbert style

**Modal Logic common rules in Natural Deduction style**

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} (\wedge I) \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} (\wedge E_1) \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} (\wedge E_2)$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} (\vee I_1) \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} (\vee I_2) \quad \frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \xi \quad \Gamma, \psi \vdash \xi}{\Gamma \vdash \xi} (\vee E)$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} (\rightarrow I) \quad \frac{\Gamma \vdash \phi \rightarrow \psi \quad \Gamma \vdash \phi}{\psi} (\rightarrow E)$$

$$\frac{\Gamma, \phi \vdash \neg\phi}{\Gamma \vdash \neg\phi} (\neg I) \quad \frac{\Gamma \vdash \neg\phi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi} (\neg E) \quad \frac{\Gamma, \neg\phi \vdash \phi}{\Gamma \vdash \phi} (\text{RAA})$$

**Specific rules for Modal Logic  $S_4$  in Natural Deduction style**

$$\frac{\Box\Gamma \vdash \phi}{\Box\Gamma \vdash \Box\phi} (\Box I) \quad \frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \phi} (\Box E)$$

**Specific rules for Modal Logic  $S_5$  in Natural Deduction style**

$$\frac{\Box\Gamma_0, \neg\Box\Gamma_1 \vdash \phi}{\Box\Gamma_0, \neg\Box\Gamma_1 \vdash \Box\phi} (\Box I) \quad \frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \phi} (\Box E)$$

**Fig. 7.** Modal Logic (common rules and  $S_{4,5}$  rules) in  $\text{LF}_K$

We combine again standard  $\beta$ -reduction with a suitable notion of  $\beta$ -reduction conditioned by a predicate *Boxed*. As in the previous case such predicate can be defined by fixing a suitable notion of good type. In the case of  $S_4$  a type is good if it is of the shape

Propositional Connectives and Judgments

$o : \text{Type}$      $\text{and} : o^3$      $\text{or} : o^3$      $\supset : o^3$      $\neg : o^2$      $\Box : o^2$      $\text{True} : o \rightarrow \text{Type}$

Propositional Rules

$\text{And}_I : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi) \rightarrow \text{True}(\psi) \rightarrow \text{True}(\phi \text{ and } \psi)$

$\text{And}_{E_1} : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi \text{ and } \psi) \rightarrow \text{True}(\phi)$

$\text{And}_{E_2} : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi \text{ and } \psi) \rightarrow \text{True}(\psi)$

$\text{Or}_{I_1} : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi) \rightarrow \text{True}(\phi \text{ or } \psi)$

$\text{Or}_{I_2} : \Pi \phi : o. \Pi \psi : o. \text{True}(\psi) \rightarrow \text{True}(\phi \text{ or } \psi)$

$\text{Or}_E : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi \text{ or } \psi) \rightarrow (\text{True}(\phi) \rightarrow \text{True}(\xi)) \rightarrow (\text{True}(\psi) \rightarrow \text{True}(\xi)) \rightarrow \text{True}(\xi)$

$\text{Imp}_I : \Pi \phi : o. \Pi \psi : o. (\text{True}(\phi) \rightarrow \text{True}(\psi)) \rightarrow \text{True}(\phi \supset \psi)$

$\text{Imp}_E : \Pi \phi : o. \Pi \psi : o. \text{True}(\phi \supset \psi) \rightarrow \text{True}(\phi) \rightarrow \text{True}(\psi)$

$\text{Neg}_I : \Pi \phi : o. (\text{True}(\phi) \rightarrow \text{True}(\neg \phi)) \rightarrow \text{True}(\neg \phi)$

$\text{Neg}_E : \Pi \phi : o. \Pi \psi : o. \text{True}(\neg \phi) \rightarrow \text{True}(\phi) \rightarrow \text{True}(\psi)$

$\text{RAA} : \Pi \phi : o. (\text{True}(\neg \phi) \rightarrow \text{True}(\phi)) \rightarrow \text{True}(\phi)$

Modal Rules

$\text{Box}_I : \Pi \phi : o. \Pi_{\text{Boxed } x} \text{True}(\phi). \text{True}(\Box \phi)$

$\text{Box}_E : \Pi \phi : o. \Pi x : \text{True}(\Box \phi). \text{True}(\phi)$

**Fig. 8.** The signature  $\Sigma_S$  for classic  $S_4$  or  $S_5$  Modal Logic in Natural Deduction style

$\text{True}(\Box A)$  for a suitable  $A$  or  $o$ . In the case of  $S_5$  a type is good if it is either of the shape  $\text{True}(\Box A)$  or  $\text{True}(\neg \Box A)$  or  $o$ . Again the intended meaning is that all occurrences of free variables appear in subterms having a  $\Box$ -type or within a syntactic type  $o$  in the case of  $S_4$ , and a  $\Box$ -type or  $\neg \Box$ -type or within a syntactic type  $o$  in the case of  $S_5$ .

Thus, *e.g.* for  $S_4$ , the encoding of the Natural Deduction ( $\Box$ ) rule of Prawitz (see Figure 7) can be rendered as  $\Pi \phi : o. \Pi_{\text{Boxed } x} \text{True}(\phi). \text{True}(\Box \phi)$ , where  $o : \text{Type}$  represents formulæ, while  $\text{True} : o \rightarrow \text{Type}$  and  $\Box : o \rightarrow o$ .

Quite a remarkable property of this signature is that it encodes a slightly more usable version of Natural Deduction  $S_4$  than the one originally introduced by Prawitz. Our formulation is precisely what is needed to achieve a normalization result in the logic which could not be done in the original system of Prawitz. Being able to refer to *boxed subterms*, rather than just boxed variables, is what *makes the difference*. Once again  $\text{LF}_\kappa$  encodings *improve presentations of logical systems!*

## 4 Properties of $\text{LF}_\kappa$

In this section, we study relevant properties of  $\text{LF}_\kappa$ . We show that, without any extra assumption on the predicates, the type system satisfies a list of basic properties, including the subderivation property, subject reduction and strong normalization. The latter follows easily from the strong normalization result for LF, see [HHP93]. Confluence and judgment decidability can be proved under the assumption that the various predicate reductions nicely combine, in the sense that no reduction can prevent a redex,

which could fire, from firing after the reduction. The difficulty in proving subject reduction and confluence for  $\text{LF}_\kappa$  lies in the fact that predicate  $\beta$ -reductions do not have corresponding untyped reductions, while standard proofs of subject reduction and confluence for dependent type systems are based on underlying untyped  $\beta$ -reductions (see e.g. [HHP93]). We provide an original technique, based solely on typed  $\beta$ -reductions, providing a fine analysis of the structure of terms which are  $\beta$ -equivalent to  $\Pi$ -terms.

In the following, we will denote by  $\Gamma \vdash_\Sigma \alpha$  any judgment defined in  $\text{LF}_\kappa$ . The proof of the following theorem is straightforward.

### Theorem 3 (Basic Properties)

#### Subderivation Property

1. Any derivation of  $\Gamma \vdash_\Sigma \alpha$  has subderivations of  $\Sigma$  sig and  $\vdash_\Sigma \Gamma$ .
2. Any derivation of  $\Sigma, a:K$  sig has subderivations of  $\Sigma$  sig and  $\vdash_\Sigma K$ .
3. Any derivation of  $\Sigma, f:A$  sig has subderivations of  $\Sigma$  sig and  $\vdash_\Sigma A : \text{Type}$ .
4. Any derivation of  $\vdash_\Sigma \Gamma, x:A$  has subderivations of  $\Sigma$  sig and  $\vdash_\Sigma A : \text{Type}$ .
5. Given a derivation of  $\Gamma \vdash_\Sigma \alpha$  and any subterm occurring in the subject of the judgment, there exists a derivation of a smaller length of a judgment having that subterm as a subject.
6. If  $\Gamma \vdash_\Sigma A : K$ , then  $\Gamma \vdash_\Sigma K$ .
7. If  $\Gamma \vdash_\Sigma M : A$ , then  $\Gamma \vdash_\Sigma A : \text{Type}$  if there are no stuck redexes in  $A$ .

#### Derivability of Weakening and Permutation

If  $\Gamma$  and  $\Delta$  are valid contexts, and every declaration occurring in  $\Gamma$  also occurs in  $\Delta$ , then  $\Gamma \vdash_\Sigma \alpha$  implies  $\Delta \vdash_\Sigma \alpha$ .

#### Transitivity

If  $\Gamma, x:A, \Delta \vdash_\Sigma \alpha$  and  $\Gamma \vdash_\Sigma M : A$ , then  $\Gamma, \Delta[M/x] \vdash_\Sigma \alpha[M/x]$ .

#### Convertibility of types in domains

1. For all  $\Gamma, x:A, \Delta \vdash_\Sigma \alpha$  and  $\Gamma, \Delta \vdash_\Sigma A =_\beta A' : K$ , then  $\Gamma, x:A', \Delta \vdash_\Sigma \alpha$ .
2. If  $\mathcal{P}(\mathcal{X}; \Gamma, x:A, \Delta \vdash_\Sigma M : B)$  holds and  $\Gamma, \Delta \vdash_\Sigma A =_\beta A' : K$ , then  $\mathcal{P}(\mathcal{X}; \Gamma, x:A', \Delta \vdash_\Sigma M : B)$  holds.

Strong normalization of  $\text{LF}_\kappa$  follows from the one of  $\text{LF}$ , since there is a trivial map of  $\text{LF}_\kappa$  in  $\text{LF}$ , which simply forgets about predicates. Thus, if there would be an infinite reduction in  $\text{LF}_\kappa$ , this would be mapped into an infinite reduction in  $\text{LF}$ .

### Theorem 4 (Strong Normalization)

1. If  $\Gamma \vdash_\Sigma K$ , then  $K \in \text{SN}^\mathcal{K}$ .
2. If  $\Gamma \vdash_\Sigma A : K$ , then  $A \in \text{SN}^\mathcal{F}$ .
3. If  $\Gamma \vdash_\Sigma M : A$ , then  $M \in \text{SN}^\mathcal{O}$ .

Where  $\text{SN}^{\{\mathcal{K}, \mathcal{F}, \mathcal{O}\}}$  denotes the set of strongly normalizing terms of kinds, families, and objects, respectively.

In the following we will denote by  $\Gamma \vdash_\Sigma A \rightleftharpoons_\beta B : K$  the fact that either  $\Gamma \vdash_\Sigma A \mapsto_\beta B : K$  or  $\Gamma \vdash_\Sigma B \mapsto_\beta A : K$  holds. Moreover, in the next results we will use a *measure* of the complexity of the proofs of judgments which takes into account all the rules applied in the derivation tree. More precisely, we have the following definition:

**Definition 3 (Measure of a derivation)**

Given a proof  $\mathcal{D}$  of the judgment  $\Gamma \vdash_{\Sigma} \alpha$ , we define the measure of  $\mathcal{D}$ , denoted by  $\#\mathcal{D}$ , as the number of all the rules applied in the derivation of  $\mathcal{D}$  itself.

The following lemma is easily proved by induction on  $\#\mathcal{D}$ .

**Lemma 2 (Reduction/Expansion).**

For any derivation  $\mathcal{D} : \Gamma \vdash_{\Sigma} A =_{\beta} B : K$ , either  $A \equiv B$  or there exist  $C_1, \dots, C_n$  ( $n \geq 0$ ) such that:

1. There exist  $\mathcal{D}_1 : \Gamma \vdash_{\Sigma} A \rightleftharpoons_{\beta} C_1 : K$  and  $\mathcal{D}_2 : \Gamma \vdash_{\Sigma} C_1 \rightleftharpoons_{\beta} C_2 : K \dots$  and  $\mathcal{D}_n : \Gamma \vdash_{\Sigma} C_{n-1} \rightleftharpoons_{\beta} C_n : K$  and  $\mathcal{D}_{n+1} : \Gamma \vdash_{\Sigma} C_n \rightleftharpoons_{\beta} B : K$  and, for all  $1 \leq i \leq n+1$ , we have  $\#\mathcal{D}_i < \#\mathcal{D}$ .
2. For any  $1 \leq i \leq n$ , we have that there exist  $\mathcal{D}'_1 : \Gamma \vdash_{\Sigma} A =_{\beta} C_i : K$  and  $\mathcal{D}'_2 : \Gamma \vdash_{\Sigma} C_i =_{\beta} B : K$  and  $\#\mathcal{D}'_1, \#\mathcal{D}'_2 < \#\mathcal{D}$ .

This lemma allows us to recover the structure of a term which is  $\beta$ -equivalent to a  $\Pi$ -term. The proof proceeds by induction on  $\#\mathcal{D}$ .

**Lemma 3 (Key lemma).**

1. If  $\mathcal{D} : \Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}x:A}.K =_{\beta} K'$  holds, then either  $\Pi_{\mathcal{P}x:A}.K \equiv K'$  or there are  $\mathcal{P}_1, \dots, \mathcal{P}_n$ , and  $D_1, \dots, D_n$ , and  $M_1, \dots, M_n$  ( $n \geq 0$ ), and  $K_A, \mathcal{D}_1, \mathcal{D}_2$  such that:
  - (a)  $K' \equiv ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot (\Pi_{\mathcal{P}x:A'}.K'')) M_n) \dots) M_1)$ .
  - (b)  $\mathcal{D}_1 : \Gamma \vdash_{\Sigma} A =_{\beta} ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot A') M_n) \dots) M_1) : K_A$ .
  - (c)  $\mathcal{D}_2 : \Gamma, x:A \vdash_{\Sigma} K =_{\beta} ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot K'') M_n) \dots) M_1)$ .
  - (d)  $\#\mathcal{D}_1, \#\mathcal{D}_2 < \#\mathcal{D}$ .
2. If  $\mathcal{D} : \Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}x:A}.B =_{\beta} C : K$  holds, then either  $\Pi_{\mathcal{P}x:A}.B \equiv C$  or there are  $\mathcal{P}_1, \dots, \mathcal{P}_n$ , and  $D_1, \dots, D_n$ , and  $M_1, \dots, M_n$  ( $n \geq 0$ ), and  $K_A, K_B$ , and  $\mathcal{D}_1, \mathcal{D}_2$  such that:
  - (a)  $C \equiv ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot (\Pi_{\mathcal{P}x:A'}.B')) M_n) \dots) M_1)$ .
  - (b)  $\mathcal{D}_1 : \Gamma \vdash_{\Sigma} A =_{\beta} ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot A') M_n) \dots) M_1) : K_A$ .
  - (c)  $\mathcal{D}_2 : \Gamma, x:A \vdash_{\Sigma} B =_{\beta} ((\lambda_{\mathcal{P}_1} y_1 : D_1 \dots ((\lambda_{\mathcal{P}_n} y_n : D_n \cdot B') M_n) \dots) M_1) : K_B$ .
  - (d)  $\#\mathcal{D}_1, \#\mathcal{D}_2 < \#\mathcal{D}$ .

**Corollary 1 ( $\Pi$ 's injectivity).**

1. If  $\Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}x:A}.K =_{\beta} \Pi_{\mathcal{P}x:A'}.K'$ , then  $\Gamma \vdash_{\Sigma} A =_{\beta} A' : K_A$  and  $\Gamma, x:A \vdash_{\Sigma} K =_{\beta} K'$ .
2. If  $\Gamma \vdash_{\Sigma} \Pi_{\mathcal{P}x:A}.B =_{\beta} \Pi_{\mathcal{P}x:A'}.B' : K$ , then  $\Gamma \vdash_{\Sigma} A =_{\beta} A' : K'$  and  $\Gamma, x:A \vdash_{\Sigma} B =_{\beta} B' : K''$ .

The proof of the following theorem uses the Key Lemma.

**Theorem 5 (Unicity, Abstraction and Subject Reduction)****Unicity of Types and Kinds**

1. If  $\Gamma \vdash_{\Sigma} A : K_1$  and  $\Gamma \vdash_{\Sigma} A : K_2$ , then  $\Gamma \vdash_{\Sigma} K_1 =_{\beta} K_2$ .
2. If  $\Gamma \vdash_{\Sigma} M : A_1$  and  $\Gamma \vdash_{\Sigma} M : A_2$ , then  $\Gamma \vdash_{\Sigma} A_1 =_{\beta} A_2 : K$ .

**Abstraction Typing**

1. If  $\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}x:A}.T : \Pi_{\mathcal{P}x:A'}.T'$ , then  $\Gamma \vdash_{\Sigma} A =_{\beta} A' : K$ .

2. If  $\Gamma \vdash_{\Sigma} \lambda_{\mathcal{P}}x:A.T : \Pi_{\mathcal{P}}x:A.T'$ , then  $\Gamma, x:A \vdash_{\Sigma} T : T'$ .

**Subject Reduction**

1. If  $\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}}x:A.K) N$ , then  $\Gamma \vdash_{\Sigma} K[N/x]$ .
2. If  $\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}}x:A.B) N : K$  and  $\mathcal{P}(\text{Fv}(N); \Gamma \vdash_{\Sigma} N : A)$  holds, then  $\Gamma \vdash_{\Sigma} B[N/x] : K$ .
3. If  $\Gamma \vdash_{\Sigma} (\lambda_{\mathcal{P}}x:A.M) N : C$  and  $\mathcal{P}(\text{Fv}(N); \Gamma \vdash_{\Sigma} N : A)$  holds, then  $\Gamma \vdash_{\Sigma} M[N/x] : C$ .

In the following, we consider notions of reduction for  $\text{LF}_{\kappa}$  that are *well-behaved* in the following sense:

1. a redex which can fire, can still fire after any  $\beta$ -reduction in its argument (possibly corresponding to a different predicate);
2. a redex which can fire, can still fire after application to its argument of a substitution coming from another reduction.

Formally:

**Definition 4 (Well behaved  $\beta$ -reduction)**

Assume that the  $\text{LF}_{\kappa}$   $\beta$ -reduction is determined by the set  $\mathbf{P}$  of good predicates. Then the  $\beta$ -reduction is well-behaved if, for all  $\mathcal{P}, \mathcal{P}' \in \mathbf{P}$ , the following two conditions are satisfied:

1. If  $\mathcal{P}(\text{Fv}(N); \Gamma \vdash_{\Sigma} N : A)$  holds and  $\Gamma \vdash_{\Sigma} N \mapsto_{\beta} N' : A$ , then  $\mathcal{P}(\text{Fv}(N'); \Gamma \vdash_{\Sigma} N' : A)$  holds.
2. If  $\mathcal{P}(\text{Fv}(N); \Gamma', y:A'; \Gamma \vdash_{\Sigma} N : A)$  and  $\mathcal{P}'(\text{Fv}(N'); \Gamma' \vdash_{\Sigma} N' : A')$  hold, then  $\mathcal{P}(\text{Fv}(N[N'/y]); \Gamma', \Gamma[N'/y] \vdash_{\Sigma} N[N'/y] : A[N'/y])$  holds.

Definition 4 above allows one to combine several notions of predicate reduction, provided the latter are all well-behaved.

Since  $\text{LF}_{\kappa}$  is strongly normalizing, in order to prove confluence of the system, by *Newman's Lemma*, it is sufficient to show that  $\text{LF}_{\kappa}$   $\beta$ -reduction is *locally confluent*, i.e. (in the case of objects) if  $\Gamma \vdash_{\Sigma} M_1 \mapsto_{\beta} M_2 : C$  and  $\Gamma \vdash_{\Sigma} M_1 \mapsto_{\beta} M_3 : C$ , then there exists  $M_4$  such that  $\Gamma \vdash_{\Sigma} M_2 \mapsto_{\beta} M_4 : C$  and  $\Gamma \vdash_{\Sigma} M_3 \mapsto_{\beta} M_4 : C$ . Under the hypothesis that  $\beta$ -reduction is well-behaved, using Theorem 5, we can prove that the reduction is locally confluent.

**Theorem 6 (Local Confluence)**

If  $\beta$ -reduction is well behaved, then it is locally confluent.

Finally, from Newman's Lemma, using Theorems 4 and 6, we have:

**Theorem 7 (Confluence)**

Assume  $\beta$ -reduction is well behaved. Then the relation  $\mapsto_{\beta}$  is confluent, i.e.:

1. If  $\Gamma \vdash_{\Sigma} K_1 \mapsto_{\beta} K_2$  and  $\Gamma \vdash_{\Sigma} K_1 \mapsto_{\beta} K_3$ , then there exists  $K_4$  such that  $\Gamma \vdash_{\Sigma} K_2 \mapsto_{\beta} K_4$  and  $\Gamma \vdash_{\Sigma} K_3 \mapsto_{\beta} K_4$ .
2. If  $\Gamma \vdash_{\Sigma} A_1 \mapsto_{\beta} A_2 : K$  and  $\Gamma \vdash_{\Sigma} A_1 \mapsto_{\beta} A_3 : K$ , then there exists  $A_4$  such that  $\Gamma \vdash_{\Sigma} A_2 \mapsto_{\beta} A_4 : K$  and  $\Gamma \vdash_{\Sigma} A_3 \mapsto_{\beta} A_4 : K$ .

3. If  $\Gamma \vdash_{\Sigma} M_1 \mapsto_{\beta} M_2 : C$  and  $\Gamma \vdash_{\Sigma} M_1 \mapsto_{\beta} M_3 : C$ , then there exists  $M_4$  such that  $\Gamma \vdash_{\Sigma} M_2 \mapsto_{\beta} M_4 : C$  and  $\Gamma \vdash_{\Sigma} M_3 \mapsto_{\beta} M_4 : C$ .

Judgements decidability show that  $\text{LF}_{\kappa}$  can be used as a framework for proof checking.

**Theorem 8 (Judgements decidability of  $\text{LF}_{\kappa}$ )**

If  $\mapsto_{\beta}$  is well-behaved, then it is decidable whether  $\Gamma \vdash_{\Sigma} \alpha$  is derivable.

The standard pattern of the proof applies, provided we take care that reductions are typed in computing the normal form of a type.

It is easy to show that, for all instances of  $\text{LF}_{\kappa}$  considered in Section 3, the corresponding  $\beta$ -reductions are well behaved, thus judgement decidability holds.

## 5 Conclusions and Directions for Future Work

In this paper, we have investigated the language theory of the Conditional Logical Framework  $\text{LF}_{\kappa}$ , which subsumes the Logical Framework  $\text{LF}$  of [HHP93], and generates new Logical Frameworks. These can feature a very broad spectrum of generalized typed (possibly by value)  $\beta$ -reductions, together with an expressive type system which records when such reductions have not yet fired. The key ingredient in the typing system is a decomposition of the standard term-application rule. A very interesting feature of our system is that it allows for dealing with values induced by the typing system, *i.e.* values which are determined by the typing system, through the notion of good predicates. We feel that our investigation of  $\text{LF}_{\kappa}$  is quite satisfactory: we have proved major metatheoretical results, such as strong normalization, subject reduction and confluence (this latter under a suitable assumption). For  $\text{LF}_{\kappa}$  we have achieved decidability, which legitimates it as a metalanguage for proof checking and interactive proof editing. We have shown how suitable instances of  $\text{LF}_{\kappa}$  provide smooth encodings of Modal Logics, compared with the heavy machinery needed when we work directly into  $\text{LF}$ , see *e.g.* [AHMP98]. Namely, the work of specifying the variable occurrence side-conditions is factored out once and for all into the framework.

Here is a list of comments and directions for future work.

- Some future efforts should be devoted to the task of investigating the structure of canonical forms including stuck redexes. Such analysis could clarify the rôle of stuck  $\beta$ -reductions and stuck terms in the activity of encoding object logics into  $\text{LF}_{\kappa}$ . Moreover, following the approach carried out in [WCPW02], we could benefit from a presentation of  $\text{LF}_{\kappa}$  based upon a clear characterization of canonical forms in order to avoid the notion of  $\beta$ -conversion and the related issues.
- We believe that our metalogical Framework has some considerable potential. In particular, it could be useful for modeling dynamic situations, where the static approach of rigid typed disciplines is not sufficient. We plan to carry out more experiments in the future, *e.g.* in the field of reactive systems, where the rôle of stuck redexes could be very helpful in modeling the dynamics of variables instantiations.
- Our results should scale up to systems corresponding to the full Calculus of Constructions [CH88].
- Is there an interesting Curry-Howard isomorphism for  $\text{LF}_{\kappa}$ , and for other systems blending rewriting facilities and higher order calculi?

- Investigate whether  $LF_{\kappa}$  could give sharp encodings of Relevance and Linear Logics. Is the notion of good predicate involved in the definition of  $LF_{\kappa}$  useful in this respect? Or do we need a different one?
- Compare with work on Deduction Modulo [DHK03].
- In [KKR90], Kirchner-Kirchner-Rusinowitch developed an *Algebraic Logical Framework* for first-order constrained deduction. Deduction rules and constraints are given for a first-order logic with equality. Enhancing  $LF_{\kappa}$  with constraints seems to be a perfect fit for a new race of metalanguages for proof checking and automatic theorem proving. Without going much into the details of our future research, the abstraction-term could, indeed, have the shape  $\lambda_{\mathcal{P}}\bar{x}; \mathcal{C}.M$ , where  $\mathcal{P}$  records the first-order formula,  $\bar{x}$  is a vector of variables occurring in the formula and  $\mathcal{C}$  are constraints over  $\bar{x}$ .
- Until now, the predicate states a condition that takes as *input* the argument and its type. It would be interesting to extend the framework with another predicate, say  $\mathcal{Q}$ , applied to the body of the function. The abstraction would then have the form  $\lambda_{\mathcal{P}}x:A.M^{\mathcal{Q}}$ . This extension would put conditions on the function *output*, so leading naturally to a framework for defining Program Logics *à la* Hoare-Floyd.
- Implement new proof assistants based on dependent type systems, like *e.g.* Coq, based on  $LF_{\kappa}$ .

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