Markov-modulated stochastic recursive equations with applications to delay-tolerant networks

Dieter Fiems, Eitan Altman

Markov-modulated stochastic recursive equations with applications to delay-tolerant networks

Dieter Fiems[a], Eitan Altman[b]

[a]*Department of Telecommunications and Information Processing, Ghent University, St-Pietersnieuwstraat 41, 9000 Gent, Belgium*
[b]*INRIA, BP93, 06902 Sophia Antipolis, France*

Abstract

We investigate a class of Markov-modulated stochastic recursive equations. This class includes multi-type branching processes with immigration as well as linear stochastic equations. Conditions are established for the existence of a stationary solution and expressions for the first two moments of this solution are found. Furthermore, the transient characteristics of the stochastic recursion are investigated: we obtain the first two moments of the transient solution as well. Finally, to illustrate our approach, the results are applied to the performance evaluation of packet forwarding in delay-tolerant mobile ad-hoc networks.

1. Introduction

We consider the sequence of random column vectors $X_n \in \mathbb{R}_+^M$ ($\mathbb{R}_+$ is the set of non-negative reals as usual), adhering to the stochastic recursive equation,

$$X_{n+1} = A_n(X_n, Y_n) + B_n(Y_n), \quad n \in \mathbb{Z}. \tag{1}$$

Here $Y = \{Y_n\}$ denotes a Markov chain, taking values on a finite state-space $\Theta = \{1, 2, \ldots, N\}$ whereas $A_n : \mathbb{R}_+^M \times \Theta \to \mathbb{R}_+^M$ and $B_n : \Theta \to \mathbb{R}_+^M$ denote vector-valued random processes. Moreover, $A_n(\cdot, i)$ are independent random processes for all $i \in \Theta$, $n \in \mathbb{Z}$ and further adhere to the following assumptions.

- For each $i \in \Theta$ and $n \in \mathbb{Z}$, $A_n(\cdot, i)$ has a divisibility property. Let $x = x^1 + x^2 + \ldots + x^k \in \mathbb{R}_+^M$, then $A_n(x, i)$ has the following representation,

$$A_n(x, i) = \sum_{\ell=1}^{k} \hat{A}_n^{(\ell)}(x^\ell, i), \tag{2}$$

whereby $\hat{A}_n^{(\ell)}(\cdot, i)$, $\ell = 1, \ldots, k$, are identically distributed, but not necessarily independent, with the same distribution as $A_n(\cdot, i)$.

URL: `Dieter.Fiems@UGent.be` (Dieter Fiems), `altman@sophia.inria.fr` (Eitan Altman)

- For each $i \in \Theta$ and $n \in \mathbb{Z}$, the mean of $A_n(\cdot, i)$ adheres to the following linearity property,

$$\mathrm{E}[A_n(x,i)] = \mathcal{A}_i^{(n)} x\,, \quad x \in \mathbb{R}_+^M, i \in \Theta\,, n \in \mathbb{Z}\,. \tag{3}$$

  Here $\{\mathcal{A}_i^{(n)}, i \in \Theta, n \in \mathbb{Z}\}$ is a set of fixed $M \times M$ matrices. Note that this implies $A_n(0,i) = 0$, almost surely.

- Finally, for each $i \in \Theta$, $n \in \mathbb{Z}$ and $x = [x_1, \ldots, x_M] \in \mathbb{R}_+^M$, the correlation matrix of $A_n(x,i)$ is assumed to have the following representation,

$$\mathrm{E}[A_n(x,i)A_n(x,i)'] = F_i^{(n)}(xx') + \sum_{\ell=1}^{M} x_\ell \Gamma_{i,\ell}^{(n)}\,. \tag{4}$$

  (Here and in the remainder, for some matrix R, R' denotes its transpose.) For each $i \in \Theta$ and $n \in \mathbb{Z}$, $F_i^{(n)}$ is a linear operator that maps $M \times M$ non-negative definite matrices on $M \times M$ non-negative definite matrices and satisfies $F_i^{(n)}(0) = 0$. Further, $\{\Gamma_{i,\ell}^{(n)}, i \in \Theta, \ell = 1, \ldots, M, n \in \mathbb{Z}\}$ is a set of fixed $M \times M$ matrices.

The class of stochastic recursive equations above include various types of stochastic processes: stochastic difference equations (also called stochastic auto-regressive processes) [12], branching processes in a discrete state space [4], branching processes in a continuous state space [3], as well as any linear combination of the preceding processes [1]. The main contribution of this work is the introduction of a random environment Y_n in the above class of recursive equations (which we assume for simplicity to be a function of the state of some finite-state Markov chain). The environment not only mitigates independence assumptions on the semi-linear processes, it also allows for introducing correlation between the semi-linear process and the immigration process. Once the theoretical foundations are established, various models of delay-tolerant ad-hoc networks are investigated whose dynamics can be described by the stochastic recursive equations at hand. In particular, the impact of the random environment on the performance of these networks is investigated.

Before proceeding to our main results, we survey some related literature. A special case of our framework are branching processes with a random environment. These have been well studied, both with and without immigration; see the survey [5] and the references therein. For example, conditions are presented for the extinction when the random environment is stationary ergodic. Further, the stability, strong law of large numbers and central limit theorems for multi-type branching processes with immigration in a random environment have been studied in [16, 21]. These processes find applications in very diverse fields, including biological systems and queueing theory. For example, McNamara et al. [18] consider an asexual species with non-overlapping generations. Individuals born some a year, reach maturity and reproduce one year later and then die. The number of individuals of the different genotypes in the consecutive years constitute a multi-type branching process. A numerical example is presented for which the expected population size has the same growth rate with and without a random environment. However, the probability that the population gets extinct is different in the two cases, being equal to one in the case of the random environment.

Prime examples in queueing theory where branching processes with immigration play a major role, include infinite server queues [8], processor sharing queues [13, 19], as well as various polling systems. Resing [20] already demonstrated that the numbers of customers in the different queues of a polling system at polling instants are described by a multi-type branching process with immigration. Similarly, station times — the time the server remains with a particular queue — at polling instants are described by such a branching process as well, albeit with a continuous state space [3].

Many of the above queueing models have natural extensions that involve random environments. A polling example that can be modelled in our framework is studied in [17]. Every time one of the queues empties, some parameters can change at random. Another application is a polling system in which the polling order of the queues is determined according to a Markov chain: if the nth station being polled is station i, then the probability that the next one to be polled is j is given by the transition probability p_{ij}. The infinite server queue with random environment has been studied recently in [7, 9]. These authors assume a framework of i.i.d. exponentially distributed interarrivals and i.i.d. exponentially distributed service times. Our approach allows, in contrast, for obtaining explicit expressions for the first and second moments in the more general setting of general stationary ergodic arrivals and general independent bounded service times, with a Markovian random environment.

We also mention some applications of stochastic linear difference equations in networking applications. Such recursions naturally occur in distributed power control [10, 15] as well as in AIMD (additive increase multiplicative decrease) protocols like TCP. Not only scenarios with a single TCP-session can be described by a stochastic linear difference equations [2]. Such equations can also describe the dynamics of scenarios with multiple TCP-sessions [6] . The random environment introduced here can then be used to mitigate independence assumptions in these linear recursions.

The remainder of this paper is organised as follows. In the next section, some additional notation for the stochastic recursive equation at hand is introduced. Section 3 is concerned with conditions for the existence of a stationary solution and expressions for the first two moments of this stationary solution. Here, we also focus on the special case of degenerate branching processes. Transient characteristics are the subject of Section 4. With the theoretical results established, Section 5 investigates various applications of our stationary and transient frameworks in the context of delay-tolerant mobile ad-hoc networks. Finally conclusions are drawn in Section 6.

2. Notation

Before proceeding to our main results, we introduce some additional notation. Let $p_{ij}^{(n)} = \Pr[Y_{n+1} = j | Y_n = i]$ denote the transition probability of the Markov chain Y_k at time n $(i, j \in \Theta)$ and let $P^{(n)} = [p_{ij}^{(n)}]$ denote the corresponding transition matrix. The probability that the Markov chain is in state i at slot n is denoted by $\pi_i^{(n)} = \Pr[Y_n = i]$.

For the immigration process B_n, the following notation is introduced for the first order vector $b_i^{(n)} \in \mathbb{R}_+^M$ and second order moment matrix $\mathcal{B}_{ij}^{(m,n)} \in \mathbb{R}_+^{M\times M}$:

$$b_i^{(n)} = \mathrm{E}[B_n(i)] \,, \quad \mathcal{B}_{ij}^{(m,n)} = \mathrm{E}[B_m(i) B_n(j)'] \,, \quad m \leq n \,.$$

In addition, the following block matrices $\hat{\mathcal{A}}^{(n)}$ and $\hat{\mathcal{B}}^{(m,n)} \in \mathbb{R}_+^{MN\times MN}$, the linear operator $\hat{F}^{(n)} : \mathbb{R}_+^{MN\times MN} \to \mathbb{R}_+^{MN\times MN}$ and the block vector $\hat{b}^{(n)} \in \mathbb{R}_+^{MN}$ are defined to simplify further notation,

$$\hat{\mathcal{A}}^{(n)} = \begin{bmatrix} \mathcal{A}_1^{(n)} p_{11}^{(n)} & \mathcal{A}_2^{(n)} p_{21}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{N1}^{(n)} \\ \mathcal{A}_1^{(n)} p_{12}^{(n)} & \mathcal{A}_2^{(n)} p_{22}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{N2}^{(n)} \\ \vdots & & \ddots & \vdots \\ \mathcal{A}_1^{(n)} p_{1N}^{(n)} & \mathcal{A}_2^{(n)} p_{2N}^{(n)} & \dots & \mathcal{A}_N^{(n)} p_{NN}^{(n)} \end{bmatrix} , \tag{5}$$

$$\hat{\mathcal{B}}^{(m,n)} = \sum_{i\in\Theta} \pi_i^{(m)} \begin{bmatrix} \mathcal{B}_{i1}^{(m,n)} p_{i1}^{(m)} & \mathcal{B}_{i2}^{(m,n)} p_{i1}^{(m)} & \dots & \mathcal{B}_{iN}^{(m,n)} p_{i1}^{(m)} \\ \mathcal{B}_{i1}^{(m,n)} p_{i2}^{(m)} & \mathcal{B}_{i2}^{(m,n)} p_{i2}^{(m)} & \dots & \mathcal{B}_{iN}^{(m,n)} p_{i2}^{(m)} \\ \vdots & & \ddots & \\ \mathcal{B}_{i1}^{(m,n)} p_{iN}^{(m)} & \mathcal{B}_{i2}^{(m,n)} p_{iN}^{(m)} & \dots & \mathcal{B}_{iN}^{(m,n)} p_{iN}^{(m)} \end{bmatrix} , \tag{6}$$

$$\hat{F}^{(n)}\left(\begin{bmatrix} x_1x_1' & x_2x_2' & \dots & x_Nx_N' \end{bmatrix}\right) = \\ \sum_{i\in\theta} \begin{bmatrix} p_{i1} F_i^{(n)}(x_ix_i') & p_{i2} F_i^{(n)}(x_ix_i') & \dots & p_{iN} F_i^{(n)}(x_ix_i') \end{bmatrix} , \tag{7}$$

$$\hat{b}^{(n)} = \sum_{i\in\Theta} \pi_i^{(n)} \begin{bmatrix} p_{i1}^{(n)} b_i^{(n)} \\ p_{i2}^{(n)} b_i^{(n)} \\ \vdots \\ p_{iN}^{(n)} b_i^{(n)} \end{bmatrix} , \tag{8}$$

for $m \leq n \in \mathbb{Z}$. Recall that $p_{ij}^{(n)}$ is the one-step transition probability at time n and note that the probabilities $p_{ij}^{(n)}$ in (5) are transposed with respect to the transition matrix $P^{(n)}$.

3. Stationary analysis

In this section, we investigate the existence of a stationary solution of the recursion (1) in a stationary ergodic framework, and obtain expressions for the first two moments of this solution. For this, we make the following additional assumptions.

- The process $\{B_n, n \in \mathbb{Z}\}$ is stationary ergodic.
- The Markov chain Y_n is stationary ergodic.
- The processes A_n are identically distributed.

In view of these assumptions, we may simplify notation as follows: $\mathcal{A}_i^{(n)} = \mathcal{A}_i$, $b_i^{(n)} = b_i$, $\hat{b}^{(n)} = \hat{b}$, $\mathcal{B}_{ij}^{(m,n)} = \mathcal{B}_{ij}^{(n-m)}$, $p_{ij}^{(n)} = p_{ij}$, $P^{(n)} = P$, $\pi_i^{(n)} = \pi_i$, $\hat{\mathcal{A}}^{(n)} = \hat{\mathcal{A}}$, $\hat{F}^{(n)} = \hat{F}$ and $\hat{\mathcal{B}}^{(m,n)} = \hat{\mathcal{B}}^{(n-m)}$. For ease of notation, the following operator will prove useful: for any $x \in \mathbb{R}_+^M$, let $\bigotimes_{l=m}^n A_l(x, Y_l) = x$ for $n < m$ whereas, for $n \geq m$, this operator is defined by the following recursion,

$$\bigotimes_{l=m}^{n} A_l(x, Y_l) = A_n\left(\bigotimes_{l=m}^{n-1} A_l(x, Y_l), Y_n\right).$$

The operator above can be applied likewise on $\hat{A}_n^{(\ell)}$ for each ℓ, see equation (2). We now state the stability theorem.

Theorem 1. *Assume that (i) $b_i < \infty$ component-wise for all $i \in \Theta$; and (ii) that all the eigenvalues of the matrix $\hat{\mathcal{A}}$ are within the open unit disk. Then, there exists a unique stationary solution X_n^*, distributed like,*

$$X_n^* =_d \sum_{\ell=0}^{\infty} \bigotimes_{k=n-\ell}^{n-1} \hat{A}_k^{(n-\ell)}(B_{n-\ell-1}(Y_{n-\ell-1}), Y_k), \tag{9}$$

for $n \in \mathbb{Z}$. The sum on the right side of the former expression converges absolutely almost surely. Furthermore, one can construct a probability space such that $\lim_{n\to\infty} \|X_n - X_n^*\| = 0$, almost surely, for any initial value X_0.

Proof. The proof follows a standard Loynes scheme. We define on the same probability space, the sequence of processes $X^{[\ell]} = \{X_n^{[\ell]}; n \in \mathbb{Z}, n \geq -\ell\}$, with initial state zero $X_{-\ell}^{[\ell]} = 0$ and governed by the recursive equation (1). One easily verifies that for fixed n, the sequence $X_n^{[\ell]}$ is monotone increasing (component-wise) in ℓ. Hence, the limit $\lim_{\ell\to\infty} X_n^{[\ell]} \triangleq X_n^*$ is well defined; the right-hand side of (9) follows by consecutively applying the recursion (1), taking into account the divisibility (2). We now show that $X_n^* < \infty$, almost surely.

In view of the recursion (1), we find,

$$\mathrm{E}[X_{n+1}^{[\ell]} 1\{Y_{n+1} = j\}] = \sum_{i\in\Theta}\left(\mathcal{A}_i\, \mathrm{E}[X_n^{[\ell]} 1\{Y_n = i\}] p_{ij} + b_i \pi_i p_{ij}\right),$$

for all $j \in \Theta$. Here $1\{\}$ is the indicator function which equals 1 if its argument is true and 0 if this is not the case. The former system of equations can be expressed in matrix notation as,

$$\mu_{n+1}^{[\ell]} = \hat{\mathcal{A}} \mu_n^{[\ell]} + \hat{b}\,.$$

$\mu_n^{[\ell]}$ denotes a (block) column vector with elements $\mu_n^{[\ell]}(i) \triangleq \mathrm{E}[X_n^{[\ell]} 1\{Y_n = i\}]$ for $i \in \Theta$. Since $\mu_{-\ell}^{[\ell]} = 0$ and by induction on n we find,

$$\mu_n^{[\ell]} = \sum_{k=0}^{n+\ell-1} \hat{\mathcal{A}}^k \hat{b}\,,$$

a geometric series which converges to $(\mathcal{I} - \hat{\mathcal{A}})^{-1}\hat{b} < \infty$, component-wise for $\ell \to \infty$ ($\mathcal{I}$ is the identity matrix). Here, we invoked the fact that $\hat{\mathcal{A}}$ has no eigenvalues outside of the open unit disk. Since $\lim_{\ell\to\infty} \mathrm{E}[X_n^{[\ell]}] < \infty$, we conclude that $\mathrm{E}[X_n^*]$ is finite by the Monotone Convergence Theorem which in turn implies that X_n^* is finite almost surely.

Consider now, the process $\{X_n; n \in \mathbb{N}\}$, governed by the recursion (1), with initial state X_0. Consecutively applying the recursion yields,

$$X_n = \bigotimes_{k=0}^{n-1} \hat{A}_k^{(0)}(X_0, Y_k) + \sum_{\ell=0}^{n-1} \bigotimes_{k=n-\ell}^{n-1} \hat{A}_k^{(n-\ell)}(B_{n-\ell-1}(Y_{n-\ell-1}), Y_k)\,,$$

for $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Hence, we have,

$$\begin{aligned}\|X_n - X_n^*\| &= \left\| \bigotimes_{k=0}^{n-1} \hat{A}_k^{(0)}(X_0, Y_k) - \bigotimes_{k=0}^{n-1} \hat{A}_k^{(0)}(X_0^*, Y_k) \right\| \\ &\leq \|Q_n(X_0)\| + \|Q_n(X_0^*)\|\,, \end{aligned} \tag{10}$$

with

$$Q_n(x) = \bigotimes_{k=0}^{n-1} \hat{A}_k^{(0)}(x, Y_k)\,, \quad n \in \mathbb{N}^*\,.$$

Note that $\|Q_n(x)\|$ is a super-martingale. We have that $\mathrm{E}[\|Q_n(x)\| | Q_{n-1}(x)] = \mathrm{E}[\|A_n(Q_{n-1}(x))\| | Q_{n-1}(x)] \leq \|Q_{n-1}(x)\|$ since the spectral radius of $\hat{\mathcal{A}}$ is smaller than one. We now show that both terms on the right-hand side of (10) converge to 0 almost surely for $n \to \infty$.

Let $\hat{\mu}_n(x)$ denote the block column vector whose ith element is equal to $\mathrm{E}\left[Q_n(x)\, 1\{Y_n = i\}\right]$, $i \in \Theta$. By conditioning on the state of the Markov chain Y_n and in view of the linearity in the mean (3), we find,

$$\hat{\mu}_n(x) = \hat{\mathcal{A}}\, \hat{\mu}_{n-1}(x) = \hat{\mathcal{A}}^n \hat{\mu}_0(x)\,, \quad n \in \mathbb{N}^*\,,$$

where $\hat{\mu}_0(x)$ denotes a block column vector whose ith element equals $\pi_i x$. Since $\hat{\mathcal{A}}$ has no eigenvalues outside the open unit disk, we have $\mu_n(x) \to 0$ (component-wise) for all $x \in \mathbb{R}_+^M$, which implies $\mathrm{E}[\|Q_n(x)\|] \to 0$ for $n \to \infty$ and $\sup_n \mathrm{E}[\|Q_n(x)\|] < \infty$. By Doob's martingale convergence theorem, $\|Q_n(x)\|$ converges a.s. Convergence to 0 then immediately follows from convergence of the means and Fatou's lemma, which in turn implies $\|X_n - X_n^*\| \to 0$, see (10). $\square$

Remark 1. The stability condition of Theorem 1 is sufficient but not necessary. Indeed, for linear recursions, a tighter stability condition was already established in [12]. In contrast, for multi-type Galton-Watson processes the condition is necessary as well [4]. It is tempting to investigate simplified stability conditions, e.g. in terms of average growth rate. However, for branching processes, McNamara et al. already showed that the extinction probability does depend on the environment process and that averaging over the different environment states does not yield conditions for extinction [18].

Having established sufficient stability conditions, we now focus on expressions for the first and second moments of X_0^*, conditioned on the state of the Markov chain Y_0. To facilitate notation, let Ψ denote the set of block row vectors with K blocks, each block being an $M \times M$ nonnegative definite matrix. Further, let $\hat{F}^n(x) = \hat{F}(\hat{F}^{n-1}(x))$ with $\hat{F}^1(x) = \hat{F}(x)$ for all $x \in \Psi$. Note that $\hat{F}^n(x) \neq \hat{F}^{(n)}(x)$, the latter being the time-dependent linear operator, see section 2.

Let μ, the conditional first moment vector, be the block column vector with elements $\mu_i \triangleq \mathrm{E}[X_0^* 1\{Y_0 = i\}]$, $i \in \Theta$. Analogously, let Ω, the conditional second moment matrix, denote the block row vector with elements $\Omega_i \triangleq \mathrm{E}[X_0^*(X_0^*)' 1\{Y_0 = i\}]$, $i \in \Theta$. The following theorem provides expressions for these vectors.

Theorem 2. *Assume that the stability conditions of Theorem 1 are satisfied. The conditional first moment vector is then given by,*

$$\mu = (\mathcal{I} - \hat{\mathcal{A}})^{-1} \hat{b} \,. \tag{11}$$

Under the additional assumption that the second moments of $B_0(i)$ are finite, $i \in \Theta$, and that

$$\lim_{n\to\infty} \|\hat{F}^n(x)\| = 0$$

for all $x \in \Psi$, the elements Ω_i of the conditional second moment matrix of X_0^* are the unique solution of the system of equations,

$$\Omega_j = \sum_{i\in\Theta} \Big(F_i(\Omega_i) + \sum_{\ell=1}^{M} \mu_i^{(\ell)} \Gamma_{i,\ell} + \mathcal{B}_{ii}^{(0)} \pi_i + \mathcal{A}_i \Lambda_i + \Lambda_i' \mathcal{A}_i' \Big) p_{ij} \,, \tag{12}$$

$j \in \Theta$, where Λ_i denotes the ith diagonal (block) element of $\sum_{k=0}^{\infty} \hat{\mathcal{A}}^k \hat{\mathcal{B}}^{(k+1)}$ and with $\mu_i^{(\ell)}$ the ℓth element of μ_i.

Proof. Taking the expectations in (1) immediately yields the following system of equations,

$$\begin{aligned}
\mu_j &= \mathrm{E}[X_1^* 1\{Y_1 = j\}] \\
&= \sum_{i\in\Theta} \mathrm{E}[(A_0(X_0^*, i) + B_0(i)) 1\{Y_0 = i, Y_1 = j\}] \\
&= \sum_{i\in\Theta} \mathcal{A}_i \mu_i p_{ij} + \sum_{i\in\Theta} b_i p_{ij} \pi_i \,, \qquad j \in \Theta \,.
\end{aligned}$$

In block matrix notation, this system of equations is equivalent to,

$$\mu = \hat{\mathcal{A}}\mu + \hat{b}\,,$$

which implies (11). Notice that $\mathcal{I}-\hat{\mathcal{A}}$ is non-singular since $\hat{\mathcal{A}}$ has only eigenvalues within the unit disk.

For the second moment, we first focus on the set of matrices $\Phi_{ij}(n)$,

$$\Phi_{ij}(n) = \mathrm{E}[X_0^* B_n(j)' 1\{Y_0 = i\}]\,, \quad i, j \in \Theta, n \in \mathbb{N}\,. \tag{13}$$

In view of the recursion (1) this matrix satisfies,

$$\begin{aligned}
\Phi_{ij}(n) &= \mathrm{E}[X_1^* B_{n+1}(j)' 1\{Y_1 = i\}] \\
&= \sum_{\ell\in\Theta} \mathrm{E}[(A_0(X_0^*, \ell) + B_0(\ell)) B_{n+1}(j)' 1\{Y_0 = \ell\}] p_{\ell i} \\
&= \sum_{\ell\in\Theta} \mathcal{A}_\ell \Phi_{\ell j}(n+1) p_{\ell i} + \mathcal{B}_{\ell j}^{(n+1)} \pi_\ell p_{\ell i}\,,
\end{aligned}$$

for $i, j \in \Theta$, $n \in \mathbb{N}$. In matrix notation, this system of equations reads,

$$\Phi(n) = \hat{\mathcal{A}}\Phi(n+1) + \hat{\mathcal{B}}^{(n+1)} = \sum_{k=0}^{\infty} \hat{\mathcal{A}}^k \hat{\mathcal{B}}^{(n+k+1)}\,, \tag{14}$$

where $\Phi(n)$ denotes the block matrix with elements $\Phi_{ij}(n)$, $i, j \in \Theta$. Notice that the sum in (14) converges to a finite-valued matrix since (i) the finiteness of the second moments of $B_0(i)$ ($i \in \Theta$) implies that the elements of $\hat{\mathcal{B}}^{(n)}$ are uniformly bounded and since (ii) the eigenvalues of $\hat{\mathcal{A}}$ are within the unit disk.

With the expression of $\Phi(n)$ at hand, and in view of the recursion (1), we immediately find,

$$\begin{aligned}
\Omega_j &= \mathrm{E}[X_1^* (X_1^*)' 1\{Y_1 = j\}] \\
&= \sum_{i\in\Theta} \mathrm{E}[(A_0(X_0^*, i) + B_0(i))(A_0(X_0^*, i) + B_0(i))' 1\{Y_0 = i, Y_1 = j\}] \\
&= \sum_{i\in\Theta} \left(F_i(\Omega_i) + \sum_{\ell=1}^{M} \mu_i^{(\ell)} \Gamma_{i,\ell} \right) p_{ij} + \sum_{i\in\Theta} \mathcal{B}_{ii}^{(0)} \pi_i p_{ij} \\
&\qquad + \sum_{i\in\Theta} \mathcal{A}_i \Phi_{ii}(0) p_{ij} + \sum_{i\in\Theta} \Phi_{ii}(0)' \mathcal{A}_i' p_{ij}\,,
\end{aligned}$$

which corresponds to (12). Here $\mu_i^{(\ell)}$ denotes the ℓth element of the vector μ_i as defined on page 7.

To show uniqueness of the solution, let $\hat{\Omega}$ denote a second solution, then,

$$\Omega_j - \hat{\Omega}_j = \sum_{i\in\Theta} F_i(\Omega_i) p_{ij} - \sum_{i\in\Theta} F_i(\hat{\Omega}_i) p_{ij} = \sum_{i\in\Theta} F_i(\Omega_i - \hat{\Omega}_i) p_{ij}\,,$$

and therefore,

$$\Omega - \hat{\Omega} = \hat{F}(\Omega - \hat{\Omega})$$

Since $\|\hat{F}^n(\Omega)\| \to 0$ for $n \to \infty$, repeated application of the former equation shows that $\|\Omega_l - \hat{\Omega}_l\| = 0$ which proves uniqueness. $\square$

In view of the proof of Theorem 1, it is clear that X_n converges to 0 almost surely for $|\hat{\mathcal{A}}| < 1$ and in the absence of immigration. For $|\hat{\mathcal{A}}| = 1$, the case of degenerate branching offers a non-trivial stationary solution. A multi-type (Galton-Watson) branching process is degenerate if every individual in a generation has exactly 1 offspring in the next generation. As such, the total number of individuals remains constant over the consecutive generations, only their distribution over the different types changes. Hence, a degenerate branching process in a random environment is a finite Markov chain such that existence of and convergence to a stationary process is guaranteed if the chain is ergodic. A sufficient condition for ergodicity of the branching process is the following: there exists a state $i \in \Theta$ such that $\mathcal{A}_i$ is the transition matrix of an ergodic Markov chain and such that $p_{ij} > 0$ for all $j \in \Theta$. Expressions for the moments of this stationary process are given in the following theorem.

Theorem 3. *Let $\{A_n\}$ be a sequence of degenerate (Galton-Watson) branching processes and let $B_n = 0$ a.s. Moreover, assume that there are K individuals and that there exists an environment state $\iota \in \Theta$ such that $\mathcal{A}_\iota$ is the transition matrix of an ergodic Markov chain and such that $p_{\iota j} > 0$ for all $j \in \Theta$. Then, the conditional first moment vector is the unique solution of,*

$$\mu = \hat{\mathcal{A}}\mu\,, \quad e'\mu = K\,. \tag{15}$$

The conditional second moment matrix is the unique solution of the system of equations,

$$\Omega_j = \sum_{i\in\Theta} \left(\mathcal{A}_i\Omega_i\mathcal{A}_i' + \sum_{\ell=1}^{M} \mu_i^{(\ell)}\Gamma_i^{(\ell)}\right)p_{ij}\,, \quad \sum_{i\in\Theta} e'\Omega_i e = K^2\,. \tag{16}$$

Proof. Taking expectations in (1) yields the first equation of (15), while the second follows from the degeneracy of the branching process. By the degeneracy and by the existence of the environment state ι, $\hat{\mathcal{A}}$ is the transition matrix of an ergodic Markov chain, which implies uniqueness of μ. Plugging equation (1) into $\mathrm{E}[A(x,j)A(x,j)']$ yields the first equation of (16) while the second equation is implied by the degeneracy of the branching process. To show uniqueness, let Ω and $\tilde{\Omega}$ denote two solutions of (16). We then have,

$$\Omega_j - \tilde{\Omega}_j = \sum_{i\in\Theta} \mathcal{A}_i \left(\Omega_i - \tilde{\Omega}_i\right) \mathcal{A}_i' p_{ij}$$

Let $\mathrm{vec}(X)$ denote the column vector which consists of the consecutive columns of the matrix X. Applying vectorisation on the system of equations above yields,

$$\mathrm{vec}(\Omega) - \mathrm{vec}(\tilde{\Omega}) = \widehat{\mathcal{A}}(\mathrm{vec}(\Omega) - \mathrm{vec}(\tilde{\Omega}))\,, \tag{17}$$

with,

$$\widehat{\mathcal{A}} = \left[\mathcal{A}_j \otimes \mathcal{A}_j p_{ji}\right]_{i,j\in\Theta}\,,$$

whereby $\otimes$ denotes the standard Kronecker product. Again, by the existence of the environment state ι and by degeneracy, the matrix $\widehat{\mathcal{A}}$ is the transition matrix of an ergodic Markov chain. Hence $\lim_{n\to\infty}\widehat{A}^i$ is the product of a column vector γ and a row vector of ones. Since $e'\,\mathrm{vec}\,(\Omega) = K^2$, repeated application of (17) yields $\mathrm{vec}\,(\Omega) - \mathrm{vec}\,(\tilde{\Omega}) = \gamma e'(\mathrm{vec}\,(\Omega) - \mathrm{vec}\,(\tilde{\Omega})) = \gamma(K^2 - K^2) = 0$ which implies uniqueness of Ω. $\square$

4. Transient analysis

The previous section focused on the existence and the first moments of the steady state process. We now consider its transient analysis. In contrast to the steady-state analysis, few additional assumptions are required. That is, apart from the assumptions introduced in section 1, we only assume that the initial vector X_0 and the initial Markov state Y_0 are independent from the processes A_n and B_n.

Let $\mu(n)$ denote the conditional first moment vector at slot n. That is, $\mu(n)$ is the column vector whose ith (block) element equals $\mu_i(n) = \mathrm{E}[X_n 1\{Y_n = i\}]$ and let $\mu_i^{(\ell)}(n)$ denote the ℓth element of this vector. Analogously, let $\Omega(n)$ denote the conditional second order moment (block) row vector whose ith element equals $\Omega_i(n) = \mathrm{E}[X_n X_n' 1\{Y_n = i\}]$. Expressions for these block vectors are given in the Theorem below.

Theorem 4. *The first moment vector at slot $n+1$ can be obtained recursively by,*

$$\mu(n+1) = \hat{\mathcal{A}}^{(n)}\mu(n) + \hat{b}^{(n)}\,. \tag{18}$$

The second moment vector at slot $n+1$ can be obtained by the following recursion

$$\begin{aligned}\Omega_j(n+1) = &\sum_{i\in\Theta}\left(F_i^{(n)}(\Omega_i(n)) + \sum_{\ell=1}^{M}\mu_i^{(\ell)}(n)\Gamma_{i,\ell}^{(n)}\right)p_{ij}^{(n)}\\ &+\sum_{i\in\Theta}\mathcal{B}_{ii}^{(n,n)}\pi_i^{(n)}p_{ij}^{(n)} + \sum_{i\in\Theta}\mathcal{A}_i^{(n)}\Phi_i^{(n)}p_{ij}^{(n)}\\ &+\sum_{i\in\Theta}(\Phi_i^{(n)})'(\mathcal{A}_i^{(n)})'p_{ij}^{(n)}\,,\end{aligned} \tag{19}$$

with $\Phi_i^{(n)}$ the ith diagonal (block) element of

$$\Phi^{(n)} = \hat{\mathcal{A}}^{(n-1)}\hat{\mathcal{A}}^{(n-2)}\cdots\hat{\mathcal{A}}^{(0)}\Upsilon^{(n)} + \sum_{k=0}^{n-1}\hat{\mathcal{A}}^{(n-1)}\hat{\mathcal{A}}^{(n-2)}\cdots\hat{\mathcal{A}}^{(k+1)}\hat{\mathcal{B}}^{(k)}\,, \tag{20}$$

and where $\Upsilon^{(n)}$ is a block matrix with elements $\Upsilon_{ij}^{(n)} = \mu_i(0)(b_j^{(n)})'$, $i,j\in\Theta$.

Proof. Taking expectations in (1) immediately yields the following system of equations,

$$\mu_j(n+1) = \sum_{i\in\Theta}\mathcal{A}_i^{(n)}\mu_k(n)p_{ij}^{(n)} + \sum_{i\in\Theta}b_i^{(n)}p_{ij}^{(n)}\pi_i^{(n)}\,,\quad j\in\Theta\,.$$

which in turn leads to (18). Proceeding analogously, we find (19) for $\Omega_j(n)$,

$$\begin{aligned}
\Omega_j(n+1) = &\mathrm{E}[X_{n+1}(X_{n+1})'1\{Y_{n+1}=j\}] \\
= &\sum_{i\in\Theta}\left(F_i^{(n)}(\Omega_i(n)) + \sum_{\ell=1}^{M}\mu_i^{(\ell)}(n)\Gamma_{i,\ell}^{(n)}\right)p_{ij}^{(n)} \\
&+ \sum_{i\in\Theta}\mathcal{B}_{ii}^{(n,n)}\pi_i^{(n)}p_{ij}^{(n)} \\
&+ \sum_{i\in\Theta}\mathcal{A}_i^{(n)}\Phi_{ii}^{(n,n)}p_{ij}^{(n)} + \sum_{i\in\Theta}(\Phi_{ii}^{(n,n)})'(\mathcal{A}_i^{(n)})'p_{ij}^{(n)},
\end{aligned}$$

with,

$$\Phi_{ij}^{(m,n)} = \mathrm{E}[X_m B_n(j)'1\{Y_m = i\}]\,.$$

For the latter expectation, we further have the following recursion,

$$\begin{aligned}
\Phi_{ij}^{(m,n)} &= \sum_{\ell\in\Theta}\mathrm{E}[(A_{m-1}(X_{m-1},\ell) + B_{m-1}(\ell))B_n(j)'1\{Y_{m-1}=\ell, Y_m = i\}] \\
&= \sum_{\ell\in\Theta}\mathcal{A}_\ell^{(m-1)}\Phi_{\ell j}^{(m-1,n)}p_{\ell i}^{(m-1)} + \mathcal{B}_{\ell j}^{(m-1,n)}\pi_\ell^{(m-1)}p_{\ell i}^{(m-1)}\,, \qquad (21)
\end{aligned}$$

whereas the independence of the initial state yields,

$$\Phi_{ij}^{(0,n)} = \mathrm{E}[X_0 B_n(j)'1\{Y_0 = i\}] = \mu_i(0)(b_j^{(n)})'\,.$$

The system of equations (21), can be rewritten in block matrix notation,

$$\Phi^{(m,n)} = \hat{\mathcal{A}}^{(m-1)}\Phi^{(m-1,n)} + \hat{\mathcal{B}}^{(m-1)}$$

Introducing $\Upsilon^{(n)} = \Phi^{(0,n)}$ and $\Phi^{(n)} = \Phi^{(n,n)}$ and solving the recursion then yields (20), which completes the proof. □

5. Applications in Delay Tolerant Mobile Ad-hoc Networks

With the theory established, we now focus on some applications in the context of packet-forwarding in delay tolerant mobile ad-hoc networks. The first two examples concern models with a fixed number of nodes. The last two examples are concerned with models where the number of nodes vary during time. In all examples, we focus on packet dissemination, not on packet delivery probability or packet delivery time.

5.1. Fixed number of mobile nodes

As a first example, we consider a network that consists of N mobile nodes. Some fixed node wishes to send a packet to a destination node. As connectivity is assumed to be low, the source makes use of the mobility of other mobiles that serve as relays. Whenever the source is within the transmission range of another node, it transmits a packet to that node. Whenever a node with a copy of a packet is within the transmission range of the destination, it transmits the packet to it.

Remark 2. In this example as well as in all the following ones, we restrict to the two-hop routing scheme [11]; we do not consider epidemic routing, i.e. a relay node that receives a packet from the source does not relay it further to other intermediate nodes.

Time is discrete and it is assumed that at each time n, each node has a probability $p_i^{(n)} \geq p > 0$ to meet any other node. This probability also depends on the state $i \in \Theta$ of a modulating Markov chain. This modulation chain (the environment) thus affects all nodes in the same way. The transition matrix of this chain at time n is denoted by $P^{(n)} = [p_{ij}^{(n)}]_{i,j\in\Theta}$, in accordance with the notation introduced before.

The validity of (a continuous time version of) this model without the random environment has been discussed in [14], and its accuracy has been shown for a number of mobility models (Random Walker, Random Direction, Random Waypoint). The random environment enables us to further capture the fluctuations in time of the connectivity probability. For example, the channel quality can significantly fluctuate during rain storms.

Let $\xi_n^{(\ell)}$ be the indicator that the ℓth node without a copy of the packet receives the packet at time n, and let Y_n denote the state of the modulating chain. Given Y_n, we assume that $\{\xi_n^{(\ell)}\}_\ell$ is a sequence of independent and identically Bernoulli distributed random variables with $\mathrm{E}[\xi_n^{(1)}|Y_n = i] = p_i^{(n)}$, $i \in \Theta$.

Let X_n be the number of nodes with a copy of the packet at time n, then the following recursion can be established,

$$X_{n+1} = X_n + \sum_{\ell=1}^{N-X_n} \xi_n^{(\ell)} .$$

The framework established in the preceding sections cannot be used directly for this recursion as the divisibility property is not satisfied. A simple change of variables, $\tilde{X}_n = N - X_n$, however yields a recursion which is captured by our framework,

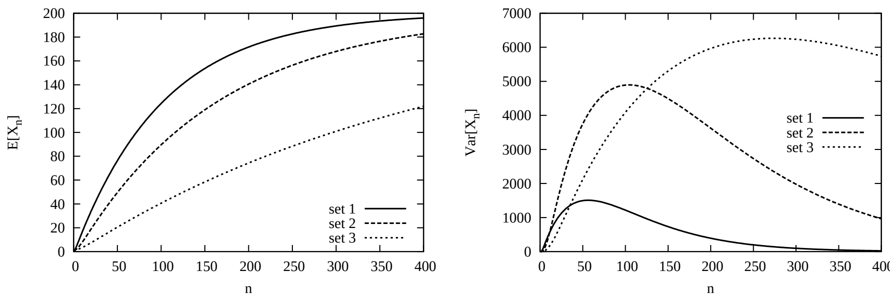
$$\tilde{X}_{n+1} = \tilde{X}_n - \sum_{\ell=1}^{\tilde{X}_n} \xi_n^{(\ell)} = \sum_{\ell=1}^{\tilde{X}_n} (1 - \xi_n^{(\ell)}) = A_n(\tilde{X}_n, Y_n) + B_n .$$

where

$$A_n(x, y) = \sum_{\ell=1}^{x} (1 - \xi_n^{(\ell)}) , \quad B_n = 0 .$$

Here, the dependence of $\xi_n^{(\ell)}$ on the state of the environment y is implicit. Notice that $\tilde{X}_n$ denotes the number of nodes that do not have a copy of the packet at time n. In view of the recursion for $\tilde{X}_n$, we obviously find $\hat{\mathcal{B}}^{(n)} = 0$ and $\hat{b}^{(n)} = 0$ for all n. Further, we have,

$$\mathcal{A}_i^{(n)} = 1 - p_i^{(n)} , \quad F_i^{(n)}(x^2) = (1 - p_i^{(n)})^2 x^2 , \quad \Gamma_{i,1}^{(n)} = (1 - p_i^{(n)}) p_i^{(n)} ,$$



(a) mean

(b) variance

Figure 1: Mean and variance of the number of nodes that have the packet as a function of time for various parameter settings.

for $i \in \Theta$.

The steady state behaviour is trivial, $\tilde{X}_n$ converges to 0 and X_n converges to N. Hence, we here focus on the transient behaviour. For this, we assume that the Markov chain is a time-homogeneous 2-state chain; state 1 (state 2) corresponding to low (high) interference levels. Let $1-\alpha$ and $1-\beta$ denote the transition probabilities from state 1 to state 2 and from state 2 to state 1, respectively and let p_i denote the probability that a node receives the packet in state i ($i = 1, 2$). For ease of notation, let σ denote the (long-term) fraction of slots that the interference level is low (the chain is in state 1) and let the interference cycle time τ denote the mean length of a high and a low interference period,

$$\sigma = \frac{1-\beta}{2-\alpha-\beta}\,, \quad \tau = \frac{1}{1-\alpha} + \frac{1}{1-\beta}\,. \tag{22}$$

In Figure 1, the time-evolution of the mean and the variance of the number of nodes that have the packet is depicted for different parameter sets; for all sets, we have $N = 200$ nodes, $p_1 = 0.5\%$, $p_2 = 10\%$ and at time 0 the interference level is low. Further, the following transition probabilities are chosen: $\alpha = 17/18$ and $\beta = 1/2$ for set 1, $\alpha = 89/90$ and $\beta = 9/10$ for set 2 and $\alpha = 449/450$ and $\beta = 49/50$ for set 3. The fraction of time σ that the interference level is high is equal for all three sets. However, in comparison with set 1, the mean length of the interference cycle time τ is 5 times (25 times) longer for set 2 (set 3). It is readily observed that the lengths of the interference cycle times have a huge impact on the performance. Longer periods yield a slower, more variable spreading of the packet among the nodes.

5.2. Packet discarding

We now move to some models where the steady state behaviour is non-trivial. In order to avoid packets to remain forever at nodes, it has been suggested to use expiration timers for packets [22]. Assume that each node uses a geometrically distributed initial value for the timer. The timer is initiated when a packet is received for relaying and the packet is discarded when the timer expires. Let q denote the probability that the timer does not expire at the end of a slot.

Retaining the assumptions and notation of the previous example, the number of nodes with the packet (excluding the source node) at consecutive slots are related as follows,

$$X_{n+1} = \sum_{\ell=1}^{X_n} \zeta_n^{(\ell)} + \sum_{\ell=1}^{N-X_n} \xi_n^{(\ell)} .$$

Here, $\zeta_n^{(\ell)}$ are i.i.d. indicators that equal 0 if the ℓth copy of a packet is discarded at time n. Note that for this model, the source node is not included in X to ensure that the source does not discard the packet.

As in the previous example, this model is not directly covered by our framework. However, let $W_n = X_n$ and $Z_n = N - X_n$, then we may rewrite the equation as follows,

$$\begin{aligned} W_{n+1} &= \sum_{\ell=1}^{W_n} \zeta_n^{(\ell)} + \sum_{\ell=1}^{Z_n} \xi_n^{(\ell)} , \\ Z_{n+1} &= N - \sum_{\ell=1}^{W_n} \zeta_n^{(\ell)} - \sum_{\ell=1}^{Z_n} \xi_n^{(\ell)} = \sum_{\ell=1}^{W_n} (1-\zeta_n^{(\ell)}) + \sum_{\ell=1}^{Z_n} (1-\xi_n^{(\ell)}) . \end{aligned}$$

In vector notation this set of equations can be written as follows,

$$\begin{bmatrix} W_{n+1} \\ Z_{n+1} \end{bmatrix} = A_n \left(\begin{bmatrix} W_n \\ Z_n \end{bmatrix}, Y_n \right) + B_n ,$$

with,

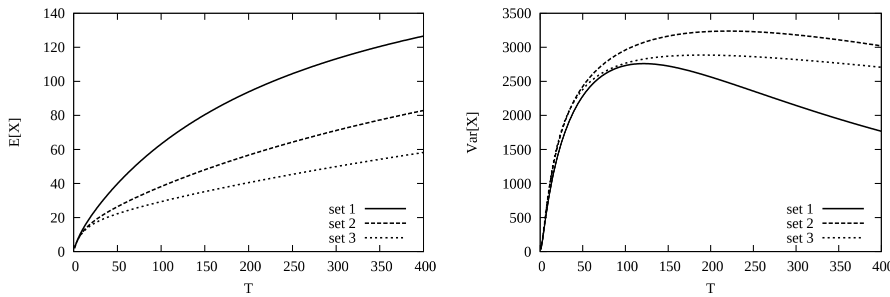
$$A_n \left(\begin{bmatrix} w \\ z \end{bmatrix}, y \right) = \begin{bmatrix} 0 \\ N \end{bmatrix} + \left(\sum_{\ell=1}^{w} \zeta_n^{(\ell)} + \sum_{\ell=1}^{z} \xi_n^{(\ell)} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} , \quad B_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

where the dependence of $\xi_n^{(\ell)}$ on the state of the environment y is implicit.

This shows that our framework for degenerate branching processes is applicable, see Theorem 3. Assuming time homogeneity, we get the following matrices,

$$\begin{aligned} \mathcal{A}_i &= \begin{bmatrix} q & p_i \\ 1-q & 1-p_i \end{bmatrix} , \\ \Gamma_{i,1} &= q(1-q) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} , \\ \Gamma_{i,2} &= p_i(1-p_i) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \end{aligned}$$

Figure 2 depicts the mean and variance of the number of nodes that have the packet (in steady state) vs. the mean packet discarding time $T = (1-q)^{-1}$ for



(a) mean (b) variance

Figure 2: Mean and variance at stationary regime of the number of nodes that have the packet as a function of the mean discarding time for various parameter settings.

the same parameter sets as in Figure 1. Obviously, as nodes keep the packets longer, the mean number of nodes that have the packet increases. In contrast, the variance first increases with the packet discarding time and then decreases again. A number of different effects are at play here. Longer discarding times imply that the packet is transmitted less as the number of nodes in the system is constant such that the number of nodes without the packet decreases. As such, longer discarding times may or may not increase the variance of the number of nodes in the system: on the one hand, the variance of the discarding times increases, on the other hand the variance of the arrival process — the number of nodes receiving the packet — decreases. In addition, positive correlation in the interference process translates into positive correlation into this arrival process which increases the variance since arrivals are more clustered.

5.3. Variations in the total number of mobiles

We again retain the assumptions of subsection 5.1 but now account for variations in the number of nodes: new nodes arrive, may or may not receive the packet after some time and then leave the system. In contrast to the preceding example, there is no packet discarding.

Let W_n denote the number of nodes that have the packet at time slot n and let Z_n denote the number of nodes that do not have the packet. We have the following recursion,

$$\begin{aligned} W_{n+1} &= \sum_{j=1}^{W_n} \zeta_{n,1}^{(j)} + \sum_{j=1}^{Z_n} \zeta_{n,2}^{(j)} \nu_n^{(j)} \\ Z_{n+1} &= \sum_{j=1}^{Z_n} \zeta_{n,2}^{(j)} (1 - \nu_n^{(j)}) + C_n \end{aligned} \tag{23}$$

Here $\zeta_{n,1}^{(j)}$ is the indicator that the jth node that has the packet remains in the system at slot n, $\zeta_{n,2}^{(j)}$ is the indicator that the jth node that does not have the packet remains in the system at slot n and $\nu_n^{(j)}$ is the indicator that the jth node that does not have the packet, receives the packet at slot n. Finally, C_n denotes the number of new nodes that arrive during slot n.

We make the following assumptions. The indicators $\zeta_{n,1}^{(j)}$ and $\zeta_{n,2}^{(j)}$ constitute doubly indexed sequences of independent Bernoulli distributed random variables; let q denote the probability that a node does not leave the system. A node then remains in the system for $T = (1-q)^{-1}$ slots on average. Further, the indicators $\nu_n^{(i)}$ are Bernoulli distributed random variables whose distributions depend on the interference level (the state of the Markov chain) during slot n. Let p_1 (p_2) denote the probability that a node receives the packet if the Markov chain is in state 1 (state 2). Finally, the sequence of new nodes C_n is stationary ergodic. Under these assumptions, the recursion (23) clearly fits the framework; let $X_n \triangleq [W_n, Z_n]'$. Adhering to the notation of the framework of Sections 2 to 4, the following matrices and vectors characterise the recursion,

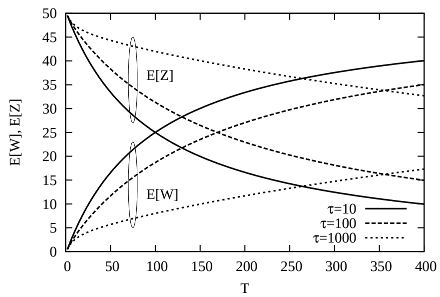
$$
\begin{aligned}
P &= \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}, \quad \mathcal{A}_i = \begin{bmatrix} q & qp_i \\ 0 & q(1-p_i) \end{bmatrix}, \\
\Gamma_{i,1} &= \begin{bmatrix} q(1-q) & 0 \\ 0 & 0 \end{bmatrix}, \\
\Gamma_{i,2} &= \begin{bmatrix} qp_i(1-qp_i) & -q^2(1-p_i)p_i \\ -q^2(1-p_i)p_i & q(1-p_i)(1-q(1-p_i)) \end{bmatrix},
\end{aligned}
$$

and $F^{(i)}(xx') = \mathcal{A}_i xx' \mathcal{A}_i'$, with $x = [w, z]'$ and for $i = 1, 2$. Moreover, we have $B_n = [0\, C_n]'$, independent of the environment, such that for $i = 1, 2$,

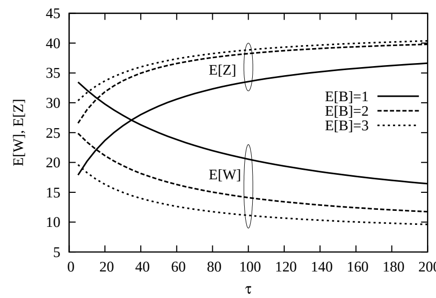
$$
b_i = \begin{bmatrix} 0 \\ \mathrm{E}[C_0] \end{bmatrix}, \quad \mathcal{B}_{ij}^{(k)} = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{E}[C_0 C_k] \end{bmatrix}.
$$

We focus on the steady-state behaviour. Figure 3 depicts the mean number of nodes $\mathrm{E}[W]$ that have the packet and the mean number of nodes $\mathrm{E}[Z]$ that do not have the packet in steady-state. In Figure 3(a), these means are plotted vs. the mean residence time T of the nodes for different values of the interference cycle time τ as defined in (22). The mean number of nodes in the system is equal to 50; the mean number of arrivals $\mathrm{E}[B]$ in a slot is reduced for increasing mean residence times. If there is no interference (state 1), a node receives the packet with probability $p_1 = 0.1$ whereas no transmission is possible during periods of high interference ($p_2 = 0$). For all curves, the interference level is low during $\sigma = 10\%$ of the slots. It is readily observed that the mean residence time of a node has a considerable impact on $\mathrm{E}[W]$, even though the arrival rate of new nodes is reduced to maintain the mean number of nodes in the network. Obviously, if nodes remain longer, they carry the packet for a longer time which explains the increase in the mean number of nodes that carry the packet. Further, increasing interference cycle times yield decreasing $\mathrm{E}[W]$.

This is confirmed by Figure 3(b) where $\mathrm{E}[Z]$ and $\mathrm{E}[W]$ are depicted vs. the interference cycle time τ for various values of the mean number of arrivals $\mathrm{E}[B]$.



(a) mean vs. residence time



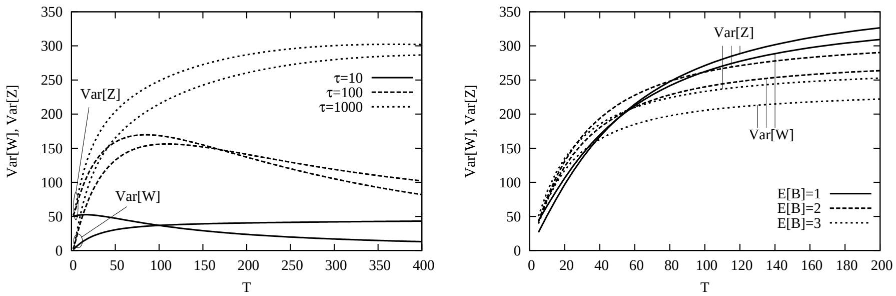
(b) mean vs. interference cycle time

Figure 3: Mean number of nodes that have the packet and of the number of nodes that do not have the packet vs. the mean residence time for various values of the mean interference cycle time (a) and vs. the mean interference cycle time for various values of the mean number of arrivals in a slot (b).

Here, the interference level is low during $\sigma = 20\%$ of the slots and a node receives the packet in these slots with probability $p_1 = 0.2$ (and there is no transmission during the other slots, $p_2 = 0$). Moreover, various values of the mean number of arrivals are assumed as depicted and the mean residence time is chosen such that the mean number of nodes in the system is equal to 50.

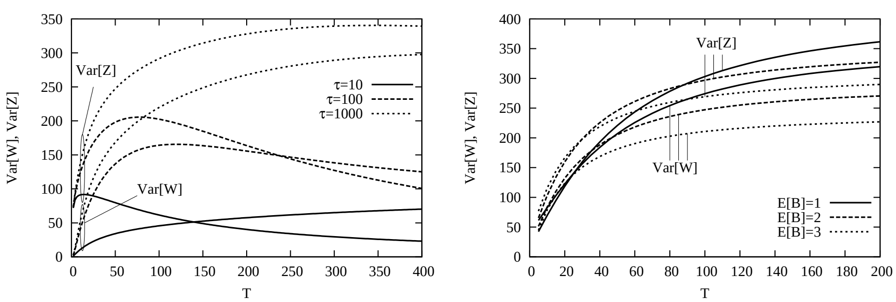
Note that the mean number of nodes with, and without the packet does not depend on the second order moments of the number of arrivals during the consecutive slots in accordance with equation (11). Hence, second order moments of B_n have not been mentioned in the preceding examples. The variance however does depend on these moments. In Figures 4 and 5, the variance of the number of nodes with and without the packet are depicted versus the mean residence time (a) and versus the mean interference cycle time (b) for the same parameters as in Figure 3. Additionally, the number of new arrivals are assumed to constitute a series of Poisson distributed random variables; this means that the variance of the number of new arrivals equals the mean number of new arrivals. These variables constitute an independent sequence in Figure 4 whereas their autocorrelation function $\rho(n)$ has a geometric decay — $\rho(n) = 1/2^n$ — in Figure 5. Amongst others, discrete autoregressive processes of order 1 have such an autocorrelation function.

From these figures, it is readily observed that the variance of the number of nodes with (without) the packet is affected by the mean residence time, the mean interference cycle time and by the correlation of the number of new arrivals. For increasing values of the residence time, the variance of the number of nodes with (without) the packet first increases and then again decreases. A similar behaviour was already observed in Figure 2. The mean number of nodes in the system being constant, the variance of the residence times increases while the variance of the (Poisson-distributed) number of new arrivals decreases. In addition, positive correlation in the arrival process increases the variance since arrivals are more clustered. Finally, increasing the interference cycle time means



(a) variance vs. residence time (b) variance vs. interference cycle time

Figure 4: Variance of the number of nodes that have the packet and of the number of nodes that do not have the packet vs. the mean residence time for various values of the mean interference cycle time (a) and vs. the mean interference cycle time for various values of the mean number of arrivals in a slot (b). There is no arrival correlation.



(a) variance vs. residence time (b) variance vs. interference cycle time

Figure 5: Variance of the number of nodes that have the packet and of the number of nodes that do not have the packet vs. the mean residence time for various values of the mean interference cycle time (a) and vs. the mean interference cycle time for various values of the mean number of arrivals in a slot (b). The autocorrelation function of the new arrivals decays geometrically.

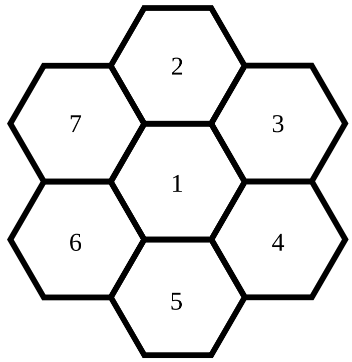


Figure 6: Spatial grid of the nodes

that there are longer periods with packet transmissions as well as longer periods without transmissions. Longer interference cycle times thus imply higher variances.

5.4. Mobility of the source and the nodes

As a more advanced application of our framework, we now consider a scenario where a number of mobile nodes and a source node move through a grid, for example the spatial grid depicted in Figure 6. In each of the regions of the grid, new nodes arrive according to a stationary ergodic process which then travel through the grid until they leave. If a node is in the same region as the source, the node receives the packet with a fixed (possibly region-dependent) probability. In contrast to the preceding examples, the environment now corresponds to the position of the source node in the grid. In the remainder, let N denote the number of regions.

Let $W_n(k)$ denote the number of nodes in region k at time n that have the packet and let $Z_n(k)$ denote the number of nodes in region k that do not have the packet. Further let W_n and Z_n denote the column vectors with elements $W_n(k)$ and $Z_n(k)$, respectively. Let Y_n denote the region where the source node resides at time n and let $C_n^{(k)}$ denote the number of new nodes that arrive in region k at time n; C_n is the column vector with elements $C_n^{(k)}$. We then have the following recursion,

$$\begin{aligned} W_{n+1} &= \sum_{i=1}^{N} \sum_{j=1}^{W_n(i)} \zeta_{n,1}^{(i,j)} + \sum_{i=1}^{N} \sum_{j=1}^{Z_n(i)} \zeta_{n,2}^{(i,j)} \nu_n^{(i,j)} \\ Z_{n+1} &= \sum_{i=1}^{N} \sum_{j=1}^{Z_n(i)} \zeta_{n,2}^{(i,j)} (1 - \nu_n^{(i,j)}) + C_n \end{aligned} \tag{24}$$

Here $\zeta_{n,1}^{(i,j)}$ is a column vector of indicators; its kth element is the indicator that the jth node in region i that has the packet at time n moves to region k. The indicator vector $\zeta_{n,2}^{(i,j)}$ is defined likewise. Its kth element is the indicator that the jth node in region i that does not have the packet at time n moves to region k. Further, $\nu_n^{(i,j)}$ denotes the indicator that the jth node in region i that does not have the packet at time n, receives the packet. Notice that some of the packets may leave the grid as not all packets necessarily move to any of the regions.

We make the following assumptions on these indicators. The indicators $\nu_n^{(i,j)}$ are Bernoulli distributed random variables whose distributions only depends on the position Y_n of the source node. Further, a node moves from one region to another with a fixed probability. Hence, the vectors $\zeta_{n,1}^{(i,j)}$ and $\zeta_{n,2}^{(i,j)}$ constitute triply indexed sequences of independent and identically distributed random vectors. Finally, new nodes arrive according to a stationary ergodic process. We then have the following representation of recursion (24),

$$\begin{bmatrix} W_{n+1} \\ Z_{n+1} \end{bmatrix} = A_n \left(\begin{bmatrix} W_n \\ Z_n \end{bmatrix}, Y_n \right) + \begin{bmatrix} 0 \\ C_n \end{bmatrix}, \tag{25}$$

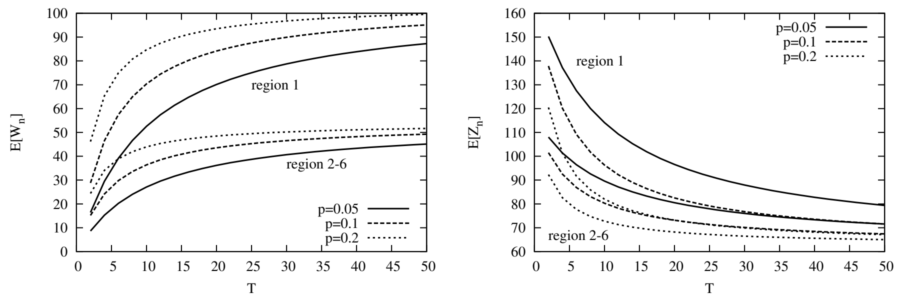
where the sequence A_n is independent and identically distributed, adhering to assumptions (2) through (4) such that our framework is applicable. Moreover, we have,

$$b_i = \begin{bmatrix} 0 \\ \mathrm{E}[C_0] \end{bmatrix}, \quad \mathcal{B}_{ij}^{(k)} = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{E}[C_0 C_k'] \end{bmatrix},$$

where the zero in the first expression is a colum vector of length N and where the zeroes in the second expression are $N \times N$ matrices.

To limit the number of parameters involved, we introduce some additional assumptions. All nodes move in the grid depicted in Figure 6. Each node remains in the same region during $T = (1-\alpha)^{-1}$ time slots on average and then moves to any of the neighbouring regions with probability $1/6$, or, if in the outer regions, outside of the grid with probabilty $1/2$. T is referred to as the mean residence time of the nodes. Analogously, the source node remains in the same region for T slots on average and then moves to any of the neighbouring regions with equal probability. The source node never leaves the grid. A node that does not have the packet which is in the same region as the source node, receives the packet with probability p. Under these assumptions, the system parameters are determined in the appendix.

In Figure 7 the mean number of nodes with (a) and without (b) the packet in the different regions is depicted vs. the mean residence time T in a region for different values of the transmission probability p. The number of new arrivals in the different regions scales with the residence times of the nodes: $\mathrm{E}\, C^{(i)} = 50/T$ for $i = 1, 2, \ldots, 7$ such that the total number of nodes in the different regions remains constant. First, notice that by symmetry, the characteristics of regions 2 to 7 are the same. Further, it is clear that longer residence times imply that more nodes receive the packet. Clearly, nodes do not only remain longer in a region but also longer in the grid. Hence, the probability that they receive the packet increases. Since the number of new arrivals is scaled with the residence



(a) Nodes with the packet

(b) Nodes without the packet

Figure 7: Mean number of nodes with the packet (a) and without the packet (b) vs. the mean residence time in a region

times, the mean number of nodes without the packet decreases for increasing mean residence times; see Figure 7(b).

6. Conclusions

This paper provides expressions for the first two moments — both transient moments and steady state moments — for stochastic recursive equations which encompasses both linear stochastic recursive equations and multi-type branching processes with immigration in a random environment. The immigration term in these recursions is assumed to be stationary ergodic, whereas the random environment is taken to be Markovian. With the theory established, various examples in the context of delay-tolerant ad-hoc networks are developed: we find transient moments for a fixed number of mobile nodes in a Markovian environment and steady state moments for a fixed number of nodes with packet discarding after the expiration of a timer, for a variable number of mobile nodes in a Markovian environment and for a variable number of mobile nodes moving in a grid.

Acknowledgement

The first author is a postdoctoral fellow with the Research Foundation, Flanders (F.W.O.-Vlaanderen), Belgium.

References

[1] E. Altman. Semi-linear stochastic difference equations. *Discrete Event Dynamic Systems*, 19:115-136, 2008.

[2] E. Altman, C. Barakat, and V. M. Ramos Ramos. Analysis of AIMD protocols over paths with variable delay. In: *Proceedings of IEEE Infocom 2004*, Hong-Kong, March 2004.

[3] E. Altman and D. Fiems. Expected waiting time in polling systems with correlated vacations. *Queueing Systems*, 56(3–4):241–253, 2007.

[4] K. Athreya and P. Ney. *Branching Processes*. Springer, 1972.

[5] K. Athreya and A. Vidyashankar. *Handbook of Statistics*, Volume 19, Chapter 2: Branching Processes. Elsevier Science B.V., 2001.

[6] F. Baccelli and D. Hong. AIMD, fairness and fractal scaling of TCP traffic. In: *Proceedings of INFOCOM 2002*, pp. 229–238, 2002.

[7] B. D'Auria. M/M/∞ queues in semi-Markovian random environment. *Queueing Systems*, 58(3):221–237, 2008.

[8] I. Eliazar. On the discrete-time $G/GI/\infty$ queue. Technical report, Holon Institute of Technology, 2006.

[9] G. Falin. The M/M/∞ queue in a random environment. *Queueing Systems*, 58(1):65–76, 2008.

[10] G. J. Foschini and Z. Miljanic. A simple distributed autonomous power control algorithm and its convergence. *IEEE Transactions on Vehicular Technology*, 42(4):641–646, 1993.

[11] M. Garetto, P. Giaccone, and E. Leonardi. On the effectiveness of the 2-hop routing strategy in mobile ad hoc networks. In *Proceedings of the IEEE International Conference on Communications (ICC '07)*, 2007.

[12] P. Glasserman and D. Yao. Stochastic vector difference equations with stationary coefficients. *Journal of Applied Probability*, 32(4):851–866, 1995.

[13] S. A. Grishechkin. On a relation between processor sharing queues and Crump-Mode-Jagers branching processes. *Advances in Applied Probability*, 24:653–698, 1992.

[14] R. Groenevelt and P. Nain. Message delay in MANETs. In *Proceedings of ACM SIGMETRICS*, pages 412–413, Banff, June 2005.

[15] R. M. Karthik and J. Kuri. Convergence of Power Control in a Random Channel Environment. In: *Proceedings of IEEE GLOBECOM 2008*, pp. 1–5, 2008.

[16] E. Key. Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. *Annals of Probability*, 15:344–353, 1987.

[17] I. MacPhee, M. Menshikov, D. Petritis, and S. Popov. A Markov chain model of a polling system with parameter regeneration. *Annals of Applied Probability*, 17(5–6):1447–1473, 2007.

[18] J. McNamara, A. Houston, and E. Collins. Optimality models in behavioral biology. *SIAM Review*, 43(3):413–466, 2001.

[19] R. Núñez Queija. *Processor-Sharing Models for Integrated-Services Networks*. PhD thesis, Eindhoven University of Technology, 2000.

[20] J. Resing. Polling systems and multi-type branching processes. *Queueing Systems*, 13:409–426, 1993.

[21] A. Roitershtein. A note on multitype branching processes with immigration in a random environment. *Annals of Probability*, 35(4):1573–1592, 2007.

[22] T. Small and Z. Haas. Resource and performance tradeoffs in delay-tolerant wireless networks. In *Proceedings of the ACM SIGCOMM workshop on Delay Tolerant Networks*, Philadelphia, August 2005.

A. Moments for the mobility application

In this appendix, we calculate the moments of the branching process of the mobility application of Section 5.4. Under the assumptions of Section 5.4, the transition matrix of the Markov chain Y is given by the matrix P below.

$$P = \begin{bmatrix} \alpha & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} \\ \frac{\overline{\alpha}}{3} & \alpha & \frac{\overline{\alpha}}{3} & 0 & 0 & 0 & \frac{\overline{\alpha}}{3} \\ \frac{\overline{\alpha}}{3} & \frac{\overline{\alpha}}{3} & \alpha & \frac{\overline{\alpha}}{3} & 0 & 0 & 0 \\ \frac{\overline{\alpha}}{3} & 0 & \frac{\overline{\alpha}}{3} & \alpha & \frac{\overline{\alpha}}{3} & 0 & 0 \\ \frac{\overline{\alpha}}{3} & 0 & 0 & \frac{\overline{\alpha}}{3} & \alpha & \frac{\overline{\alpha}}{3} & 0 \\ \frac{\overline{\alpha}}{3} & 0 & 0 & 0 & \frac{\overline{\alpha}}{3} & \alpha & \frac{\overline{\alpha}}{3} \\ \frac{\overline{\alpha}}{3} & \frac{\overline{\alpha}}{3} & 0 & 0 & 0 & \frac{\overline{\alpha}}{3} & \alpha \end{bmatrix}$$

Here, $\overline{\alpha} = 1 - \alpha$ is introduced for ease of notation. Further, we introduce the auxiliary sub-stochastic matrix Q, its entries being the probabilities that a node moves from one region to the other,

$$Q = \begin{bmatrix} \alpha & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} \\ \frac{\overline{\alpha}}{6} & \alpha & \frac{\overline{\alpha}}{6} & 0 & 0 & 0 & \frac{\overline{\alpha}}{6} \\ \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & \alpha & \frac{\overline{\alpha}}{6} & 0 & 0 & 0 \\ \frac{\overline{\alpha}}{6} & 0 & \frac{\overline{\alpha}}{6} & \alpha & \frac{\overline{\alpha}}{6} & 0 & 0 \\ \frac{\overline{\alpha}}{6} & 0 & 0 & \frac{\overline{\alpha}}{6} & \alpha & \frac{\overline{\alpha}}{6} & 0 \\ \frac{\overline{\alpha}}{6} & 0 & 0 & 0 & \frac{\overline{\alpha}}{6} & \alpha & \frac{\overline{\alpha}}{6} \\ \frac{\overline{\alpha}}{6} & \frac{\overline{\alpha}}{6} & 0 & 0 & 0 & \frac{\overline{\alpha}}{6} & \alpha \end{bmatrix}$$

Given the matrix Q above, it is easily shown that the 14×14 matrices $\mathcal{A}_k$ which correspond with the process $A_n()$ of equation (25), are given by,

$$\mathcal{A}_k = \begin{bmatrix} Q & QS_k \\ 0 & Q(I - S_k) \end{bmatrix} \tag{26}$$

where S_k denotes a 7×7 matrix of zeroes except for its kth diagonal element which equals p.

For the second moment, it is necessary to determine the linear operators $F^{(i)}$ and the matrices $\Gamma_{i,j}$, $i = 1, \ldots, 7$, $j = 1, \ldots, 14$. These immediately follow from the following expressions for the second order (cross) moments,

$$\begin{aligned}
&\mathrm{E}[Z_{n+1}Z'_{n+1}|Z_n, Y_n = i, C_n = 0] \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} Z_n(k)Z_n(\ell)\,\mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(\ell,1)}\right]' \\
&\quad + \sum_{k=1}^{N} Z_n(k)\left(\mathrm{E}\left[\zeta_{n,2}^{(k,1)}(\zeta_{n,2}^{(k,1)})'\right] - \mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]'\right) \\
&\quad - p\sum_{k=1}^{N} Z_n(i)Z_n(k)\left(\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]' + \mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]'\right) \\
&\quad + p(2-p)Z_n(i)\,\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]' \\
&\quad - pZ_n(i)E\left[\zeta_{n,2}^{(i,1)}(\zeta_{n,2}^{(i,1)})'\right] + p^2 Z_n(i)^2\,\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]',
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}[X_{n+1}X'_{n+1}|X_n, Z_n, Y_n = i] \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} X_n(k)X_n(\ell)\,\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,1}^{(\ell,1)}\right]'
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{N} X_n(k)\left(\mathrm{E}\left[\zeta_{n,1}^{(k,1)}(\zeta_{n,1}^{(k,1)})'\right]-\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]'\right)\\
&+p^2 Z_n(i)^2\,\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]'+p(1-p)Z_n(i)\,\mathrm{E}\left[\zeta_{n,2}^{(i,1)}(\zeta_{n,2}^{(i,1)})'\right]\\
&+p\sum_{k=1}^{N} X_n(k)Z_n(i)\left(\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]'+\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]'\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}[X_{n+1}Z'_{n+1}|X_n,Z_n,Y_n=i,C_n=0]\\
&=\sum_{k=1}^{N}\sum_{\ell=1}^{N} X_n(k)Z_n(\ell)\,\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(\ell,1)}\right]'\\
&\quad-p\sum_{k=1}^{N} X_n(k)Z_n(i)\,\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]\mathrm{E}\left[\zeta_{n,2}^{(i,1)}\right]'.
\end{aligned}
$$

In the former expressions, the column vector $\mathrm{E}\left[\zeta_{n,1}^{(k,1)}\right]=\mathrm{E}\left[\zeta_{n,2}^{(k,1)}\right]$ is equal to the kth column of the matrix P whereas the matrix $E\left[\zeta_{n,1}^{(i,1)}(\zeta_{n,1}^{(i,1)})'\right]=E\left[\zeta_{n,2}^{(i,1)}(\zeta_{n,2}^{(i,1)})'\right]$ is a diagonal matrix whose diagonal entries equal the entries of the ith column of the matrix P.