Admission control to an M/M/1 queue with partial information

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Abstract

We consider both a cooperative as well as non-cooperative admission into an M/M/1 queue. The only information available is a signal that sais whether the queue size is smaller than some L or not. We first compute the globally optimal and the Nash equilibrium stationary policy as a function of L. We compare the performance to that of full information and of no information on the queue size. We identify the L that optimizes the equilibrium performance.

1 Introduction

This paper is devoted to revisitting the problem of whether an arrival should queue or not in an M/M/1 queue. This is perhaps the first problem to be studied in optimal control of queues, going back to the seminal paper of Pinhas Naor [1]. Naor considered an M/M/1 queue, in which a controller has to decide whether arrivals should enter a queue or not. The objective was to minimize a weighted difference between the average expected waiting time of those that enter, and the acceptance rate of customers. Naor then considers the individual optimal threshold (which can be viewed as a Nash equilibrium in a non-cooperative game between the players) and shows that it is also of a threshold type with a threshold $L' > L$. Thus under individual optimality, arrivals that join the queue wait longer in average. Finally, he showed that there exists some toll such that if it is imposed on arrivals for joining the queue then the threshold value of the individually optimal policy can be made to agree with the social optimal one. Since this seminal work of Naor there has been a huge amount of research that extend the model. More general interarrival and service times have been considered, more general networks, other objective functions and other queueing disciplines, see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein.

In the original work of Naor, the decision maker(s) have full state information when entering the system. However, the fact that a threshold policy is optimal implies that for optimally controlling arrivals we only need partial information - we need a signal to indicate whether the queue exceeds or not the threshold value L. The fact that this much simpler information structure is sufficient for obtaining the same performance as in the full information case motivates us to study the performance of threshold policy and related optimization issues.

We first consider the socially optimal control policy for a given (non-necessarily optimal) threshold value L. When L is chosen non-optimally then the optimal policy for the partial information problem does not anymore coincide with the policy with full information.

We then study the individual optimization problem with the same partial information: a signal (red) if the queue length exceeds some value L and a green signal otherwise.

For both the social and the individual optimization problems we show that the following structure holds: either whenever the signal is green all arrivals are accepteed with probability 1, or whenever the signal is red all arrivals are rejected with probability 1.

We note that by using this signalling approach instead of providing full state information, users cannot choose any threshold policy with parameter different than L. Thus, in the individual optimisation case, one could hope that by determining the signalling according to the value L that optimizes the socially optimal problem (in case of full information), one would obtain the socially optimal performance. We show that this is not the case, and determine the value L for which the reaction of the users optimizes the system performance. We compare this to the performance in case of full information and to that of no information.

2 The model

Assume an M/M/1 queue where the admission rate is $\underline{\lambda}$ for $i \geq L$ and is otherwize $\overline{\lambda}$. Let μ be the service rate and set $\overline{\rho} = \overline{\lambda}/\mu$ and $\underline{\rho} = \underline{\lambda}/\mu$. We shall make the standard stability assumption that $\underline{\rho} < 1$. The balance equations are given

$$\mu\pi(i+1, L) = \lambda\pi(i, L)$$

where $\lambda = \underline{\lambda}$ for $i > L$ and is otherwize given by $\lambda = \overline{\lambda}$. The solution of these equations give

$$\pi(i, L) = \pi(0, L)\overline{\rho}^i$$

for $i \leq L$ and otherwize

$$\pi(i, L) = \pi(L, L)\underline{\rho}^{i-L}. \tag{1}$$

Hence

$$\begin{aligned} \pi(0, L) &= \frac{1}{\sum_{i=0}^{L-1} \overline{\rho}^i + \overline{\rho}^L \sum_{i=0}^{\infty} \underline{\rho}^i} \\ &= \frac{1}{\frac{1-\overline{\rho}^L}{1-\overline{\rho}} + \frac{\overline{\rho}^L}{1-\underline{\rho}}} \end{aligned}$$

Thus

$$\pi(L,L) = \frac{1-\underline{\rho}}{1-\left(\frac{(1-\underline{\rho})(1-\overline{\rho}^{-L})}{1-\overline{\rho}}\right)} \tag{2}$$

Assume that an arrival receives the information on whether the size of the queue exceeds $L-1$ or not. If it does not exceeds we shall say that it receives a "green" signal denoted by G, and otherwise a red one (R). The conditional state probabilities given the signals are denoted by

$$\pi(i,L|R) = (1-\underline{\rho})\underline{\rho}^{i-L}$$

for $i \geq L$, and is otherwise zero. The conditional tail distribution is

$$P(I > n|R) = \underline{\rho}^{n+1-L}$$

for $n \geq L$, and is otherwise 1. Thus

$$E(I|R) = (L-1) + \frac{1}{(1-\underline{\rho})} \tag{3}$$

For a green light we have:

$$\pi(i,L|G) = \frac{1-\overline{\rho}}{1-\overline{\rho}^L}\overline{\rho}^i$$

for $0 \leq i < L$ and is otherwise zero. Hence the tail probabilities are

$$P(I > n|G) = \frac{\overline{\rho}^{n+1} - \overline{\rho}^L)}{1-\overline{\rho}^L}$$

for $n < L$, and is otherwise 0. Hence

$$E(I|G) = \frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right) \tag{4}$$

3 The partially observed control problem

We assume that $\nu < \underline{\lambda}$ is the rate of some uncontrolled Poisson flow. In addition there is an independent Poisson arrival flow of intensity ζ. We restrict to stationary policies, i.e. policies that are only function of the observation. A policy is thus a set of two probabilities: $Q(s)$ where s is either R or G. $Q(s)$ is the probability of accepting an arrival when the signal is s. For a given policy, we obtain the framework of the previous section with

$$\underline{\lambda} = \nu + \zeta Q(R), \quad \overline{\lambda} = \nu + \zeta Q(G).$$

Our goal is to minimize over q

$$J_{\mathbf{q}} = E_{\mathbf{q}}[I] - \gamma T_{acc}(\mathbf{q}) = \sum_{s=G,R} P_{\mathbf{q}}(s)(E_{\mathbf{q}}[I|s] - \gamma T_{acc}(\mathbf{q}))$$

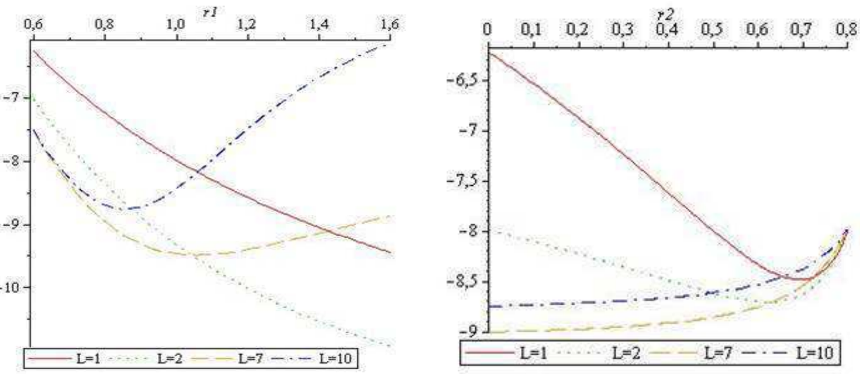

Figure 1: The optimal policy for several values of L

where

$$T_{acc} = \overline{\lambda} * P(G) + \underline{\lambda} * P(R) = \mu[P(R)(\underline{\rho} - \overline{\rho}) + \overline{\rho}]$$

and $P(R) = P(I \geq L)$ is given by

$$P(R) = \pi(L, L)\frac{1}{1-\underline{\rho}} \tag{5}$$

$$E[I] = E[I|R] * P(R) + E[I|G] * P(G) = (E[I|R] - E[I|G]) * P(R) + E[I|G],$$

$$= (((L-1) + \frac{1}{\underline{\rho}^L(1-\underline{\rho})}) - (\frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right))) \times \frac{(1-\overline{\rho})\overline{\rho}^L}{(1-\underline{\rho}) + \overline{\rho}^L(\underline{\rho} - \overline{\rho})} +$$

$$\vdash \left(\frac{1}{1-\overline{\rho}^L}\left(\frac{(\overline{\rho}^L - \overline{\rho})}{\overline{\rho}-1} - (L-1)\overline{\rho}^L\right)\right)$$

The expression obtained for $J_{\mathbf{q}}$ is lower semi-continuous in the policy $\mathbf{Q} = \{Q(s), s = G, R\}$. Hence a minimizing policy $\mathbf{q}^*$ exists.

Theorem 1 *If $\underline{\rho} \geq 1$ then for any L and any q, $E[I]$ is infinite.*

Proof. The expected queue length for any L and q is bounded from below at any time by the one obtained by using $q = 0$. We thus have an M/M/1 queue with a workload $\rho \geq 1$ which is known to have infinite expectation. ■

3.1 The structure of optimal policies

Figure 1 shows the values of the two component of the vector ρ corresponding to the optimal policy. We assume that ν and λ are such that $\overline{\rho} = 0.8$ and $\underline{\rho} = 0.3$. We further took $\mu = 1$, $\gamma = 15$, for 4 different values of the threshold L.

We observe the following structure: for any L, the optimal vector ρ satisfies the following property: whenever the minimum cost is achieved at an interior point for one of the components of ρ, then it is achieved on the boundary for the other component. More precisely, the optimal ρ satisfies either $\rho_2 = (r2) = \underline{\rho}$ or $\rho_1 = (r1) = \overline{\rho}$. We shall next prove this structure for the partially observable control problem.

Theorem 2 *Assume that $0 < \nu/\mu < 1$. Then there is a unique optimal stationary strategy and it has the following property: either $q^*(G) = 1$ or $q^*(R) = 0$.*

Proof. Let q be optimal. We first show that $\alpha > 0$ where $\alpha := \mu - (\nu + q(R)\zeta)$. Indeed, if it were not the case then we would have $\underline{\rho} \geq 1$ so by the previous theorem, the queue length and hence the cost would be infinite. But then q cannot be optimal since the cost can be made finite by choosing $q(R) = 0$.

Assume that an optimal policy q does not have the structure stated in the Theorem. This would imply that $q(R)$ can be further decreased and $q(G)$ increased. In particular, one can perturb q in that way so that T_{acc} is unchanged. More precisely, note first that T_{acc} is monotone increasing in both $q(R)$ and in $q(G)$. Hence

$$T_{acc}(1, q(R)) \geq T_{acc}(q) \geq T_{acc}(q(G), 0),$$

Hence, if $T_{acc}(1,0) < T_{acc}(q)$ then there is some $q_2(R)$ such that $T_{acc}(1, q_2(R)) = T_{acc}$, otherwise there is some $q_2(G)$ such that $T_{acc}(0, q_2(G)) = T_{acc}$.

We have $P_{q_2}(I = 0) = 1 - T_{acc}(q_2) = 1 - T_{acc}(q)$ (e.g. from Little's Theorem). From rate balance arguments it follows that

$$P_{q2}(I = i) = (1 - T_{acc})(q_2)\overline{\rho}_2^i \qquad \text{for } i \leq L. \tag{6}$$

Hence

$$P_{q2}(I \geq i) < P_q(I \leq i) \tag{7}$$

for $i \geq L$. Thus

$$P_{q2}(R) < P_q(R).$$

By combining this with (1) it follows that

$$P_{q2}(I \geq i) = P_{q2}(R)\underline{\rho}(q2)^{i-L} \leq P_{q2}(R)\underline{\rho}(q2)^{i-L} \leq= P_{q2}(I \geq i)$$

Hence (7) holds for all i. Taking the sum over i we thus obtain that

$$E_{q2}[I] < E_q[I].$$

Since T_{acc} are the same under q and q_2, it follows that $J_{q2} < J_q$. Hence q is not optimal, which contradicts the assumption in the beginning of the proof. This establishes the structure of optimal policies.■

3.2 Optimizing the signal

Here we briefly discuss the case of choosing L so as to minimize $J_{\mathbf{q}}$ not only with respect to $\mathbf{q}$ but also with respect to the value L of the threshold.

To that end we make we first consider the problem of minimizing J over all stationary policies in case that full state information is available. This is a Markov decision process and an optimal policy is known to exists within the pure stationary policies. Moreover, a direct extension of the proof in [1] can be used to show that the structure of the optimal policy is of a threshold type: accept all arrivals as long as the state is below a threshold and reject all controlled arrivals otherwize. Note however that this policy makes use only of the information available also in our cases, i.e. of whether the state exceeds L or not.

We conclude that the problem of optimizing $J_{\mathbf{q}}$ over both L and $\mathbf{q}$ has an optimal pure threhold policy i.e. with $Q(R) = 0$ and $Q(G) = 1$, or in other words $\mathbf{q} = (1, 0)$.

The optimal L for our problem can therefore be computed by minimizing $J_{\mathbf{q}}$ over pure threshold policies. In Figure (**??**) We compute this optimal L for $\mu = 1$, $\eta = 0.01$, $\lambda = 0.98$ and $\gamma = 20$. and obtain $L = 5$,

4 The game problem

We again assume that there is some uncontrolled flow ν and a flow of strategic players with intensity ζ. All users receive the signal G or R as before, and we restrict to policies as in the control case.

Assume that an arrival has a reward $\psi > 0$ for being processed in the queue, and a waiting cost of $E[W|s]$ where W is the waiting time. Note that $E[W|s] = E[I|s]/\mu$.

Let $Y(P)$, where $P = P(s)$, $s = R, G$ be the set of best responses of an individual if all the rest use $P(s), s = R, G$, and the system is in the corresponding steady state.

Then q is an equilibrium strategy if and only if $q \in Y(q)$. Note that the cost $J(q, P)$ corresponding to a strategy q of a player, when all others play P satisfies the following in order to be a best response to P: for each s, if $q(s)$ is not pure (is not 0 or 1) then at s, any other probability q' is also a best response.

The cost for a user for entering when the signal is s given that the strategy of other users is $\mathbf{q} = (q_G, q_R)$ is given by

$$V_{\mathbf{q}}(s) = E_{\mathbf{q}}[W|s] - \gamma = E_{\mathbf{q}}[I|s]/\mu - \gamma \tag{8}$$

It is zero if it does not enter. Here, $E_{\mathbf{q}}[I, s]$ are given by

$$E_{\mathbf{q}}(I|R) = (L - 1) + \frac{1}{(1 - \rho)} \tag{9}$$

where $\mathbf{q} = (1, q_R)$ and where $\rho = (\nu + \zeta q_R)/\mu$, and

$$E_{\mathbf{q}}(I|G) = \frac{1}{1-\rho^L}\left(\frac{(\rho^L - \rho)}{\rho - 1} - (L-1)\rho^L\right) \tag{10}$$

where $\mathbf{q} = (q_G, 0)$ and where $\rho = (\nu + \zeta q_G)/\mu$. (the derivations of the above are as in (3) and (4), respectively.

4.1 Structure of equilibrium

The following gives the structure of equilibria policies.

Theorem 3 *(1) The equilibrium policy is to enter for any signal if and only if* $V_{(1,1)}(R) \leq 0$
(2) The equilibrium is of the form $q = (1, q_{R)}$ *where* $q_R \in (0,1)$ *if and only if* $V_{(1,1)}(R) > 0 > V_{(1,0)}(R)$*. In this case, the equilibrium is given by the* $q = (1, q_R)$ *where* q_R *is the solution of* $V_{(1,q_R)} = 0$ *where* V_q *is given in (8).*
(3) The equilibrium is of the form $q = (q_G, 0)$ *where* $q_G \in (0,1]$ *if and only if* $V_{(1,0)}(G) \geq 0$*. In this case, the equilibrium is given by the* $\mathbf{q} = (q_G, 0)$ *where* q_G *is the solution of* $V_{(q_G,0)} = 0$ *and where* $V_{\mathbf{q}}$ *is given in (8).*

Proof. Follows directly from continuity of the expected queue length with respect to q and from the fact that $V(q_G, q_R)$ is strictly monotone increasing in both arguments. Then continuity is a consequence of ergodicity which turns out to be uniform in ρ, as established in the Appendix, and from the fact that the Lyapunov function is exponential, and from the fact that $V(q_G, q_R)$ is strictly monotone increasing in both arguments.

4.2 Numerical Examples

We consider here as an example the parameters $\gamma = 20$, $\mu = 1$, $\lambda = 0.98$ and $\zeta = 0.01$. For all L condition (1) of Theorem 3 does not hold, so (1,1) is not an equilibrium. condition (2) of the Theorem holds for $L \leq 20$. In that case, the equilibrium is given by $(1, q_R)$ where q_R is given in Fig 2. The value at equilibrium is given in Figure **??** for the case of the signal G and is otherwize zero for all $L \leq 20$.

4.3 Optimizing the signal

We are interested here in finding the L for which the induced equilibrium gives the best system performance. We plot the system performance j at equilibrium as a function of L in Figure **??**.

The optimal L is seen to equal 20 and the corresponding performance measres at equilibrium are are $J = -14.13$ and $T_{acc} = 0.83$.

If we take the $L = 5$ which we had computed for optimizing the system performance, and use it in the game setting, we obtain $T_{acc} = 0.95$ and $J^* =$

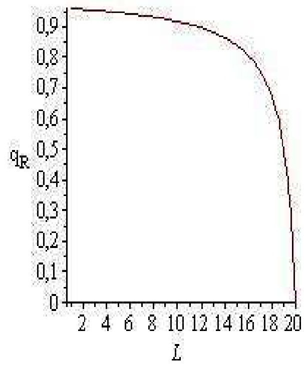

Figure 2: Equilibrium action of q_R as a function of L

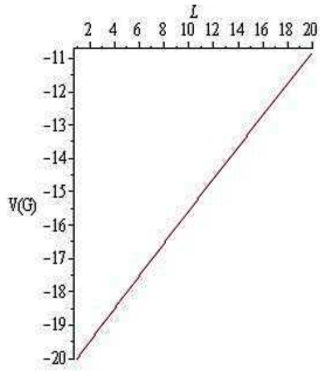

Figure 3: Equilibrium value V_G for signal G as a function of L. We used case (2) in Theorem 3 and the results are therefore valid only for $L \leq 20$.

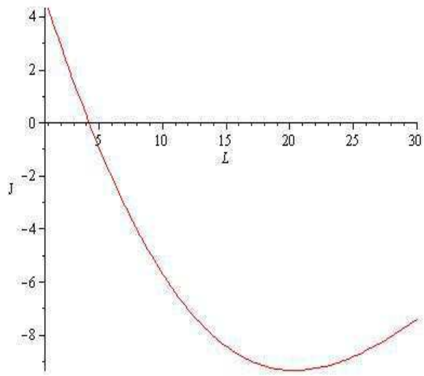

Figure 4: The social value J at equilibrium as a function of L

-3.49. which indeed gives a much worse performance than the performance under the $L = 20$.

$V_{(1,1)}(R) = E_{(1,1)}(I|R) = \gamma\ \overline{\rho} = 0.99$ and $\underline{\rho} = 0.01\ \rho = 0.01 + q0.98$

In the case of no information $J = 1$.

For finding the optimal threshold for the game problem we have to find the L that:

$$min_L[J_{(1,q(L))}(L)] = min_L[E[I](L) - \gamma T_{acc}(1, q(L))]$$

$V_{(1,1)}(R)$ is positif for every L, so the equilibrium is not on that case 1.

$V_{(1,0)}(R) < 0$ for $L < 20$ in between $L = 1$ to $L = 19$ equilibrium will have the form ib case 2, for those parameters V is linear on L with minumum on 1.

The equilibrium is of the form of case 3, i.e., $V)(1,0)(G) > 0$ for $L > 44$ with minumum $L = 45$ and $V = 0.31$

The L which minimizes V is $L = 1$, for which $V_{(1,0)}(R) = -20$

5 Appendix: Uniform f-ergodicity and the continuity of the Markov chain

Consider the Markov chain embedded at each transition at the queue. Thus for $I \geq \max(L, 1)$, with probability β the event is a departure and otherwise it is an arrival, where

$$\beta := \frac{\mu}{\mu + \nu + q(R)\zeta}.$$

Note $\alpha > 0$ implies that $\beta > 1/2$.

Define $f(i) = \exp(\gamma i)$, for any $I \geq \max(L, 1)$,

$$E[f(I_{t+1}) - f(I_t)|I_t = i] = \beta \exp[\gamma(i-1)] + (1-\beta)\exp[\gamma(i+1)] - \exp(\gamma i)$$

$$= f(i)\Delta \quad \text{where} \quad \Delta = \beta z^{-1} + (1-\beta)z - 1$$

and where $z := \exp(-\gamma)$. Note that $\Delta = 0$ at

$$z_{1,2} = \frac{1 \pm \sqrt{1 - 4\beta(1-\beta)}}{2(1-\beta)} = \{1, \frac{\beta}{1-\beta}\}$$

Thus $\Delta < 0$ for all γ in the interval $\left(0, \log\left(\frac{\beta}{1-\beta}\right)\right)$ (which is nonempty since we showed that $1 > \beta > 1/2$). We conclude that for any γ in that interval, f is a Lyapunov function and the Markov chain is γ-geometrically ergodic.

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