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A stochastic HJB equation for optimal control of forward-backward SDEs

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Abstract

We study optimal stochastic control problems of general coupled systems of forward-backward stochastic differential equations with jumps. By means of the Itô-Ventzell formula the system is transformed to a controlled backward stochastic partial differential equation (BSPDE) with jumps. Using a comparison principle for such BSPDEs we obtain a general stochastic Hamilton-Jacobi- Bellman (HJB) equation for such control problems. In the classical Markovian case with optimal control of jump diffusions, the equation reduces to the classical HJB equation.

The results are applied to study risk minimization in financial markets.

1 Introduction

{intro}

In classical theory of stochastic control of systems described by a stochastic differential equations (SDEs) there are two important solution methods:

(i) Dynamic programming, which leads to the classical Hamilton-Jacobi-Bellman (HJB) equation. This is a deterministic non-linear partial differential equation (PDE) in the (unknown) value function for the problem.

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(ii) The maximum principle, which involves the maximization of the Hamiltonian and an associated backward stochastic differential equation (BSDE) in the (unknown) adjoint processes.

Dynamic programming is a very efficient solution method, but it only works if the system is Markovian. The maximum principle, on the other hand, works also in non-Markovian settings, but the drawback is that it leads to a complicated coupled system of forward-backward SDEs (FBSDEs) with constraints, and this system is difficult to solve in general.

In view of this it is natural to ask if there is an extension of the HJB approach to non-Markovian systems. The answer has been known to be yes for some time, at least in some cases. See [P], where a stochastic HJB equation is proved for non-Markovian, Brownian motion driven SDEs, in which the diffusion coefficient does not depend on the control.
The purpose of this paper is to show that the answer is yes also in a more general context. More precisely, and more generally, we give a method for solving optimal control problems for general non-Markovian systems of *forward-backward* stochastic differential equations (FBSDEs) by means of a *stochastic* HJB equation, which is a backward stochastic partial differential equations (BSPDEs). Our underlying models are Itô-Lévy processes (with jumps). If the system is a Markovian SDE, then our stochastic HJB equation becomes deterministic and coincides with the classical HJB equation.
In the last part of the paper we illustrate our theory by studying some applications to finance. In particular we, apply our results to study a problem of risk minimization in a financial market.

In[P] a stochastic version of the classical HJB equation is studied and existence and uniqueness is proved for this type of SPDEs. However, there it is assumed that the control does not enter the diffusion coefficient.

The relation between FBSDEs and B(S)PDEs has been known for several years. See, for example, [MPY] for the Markovian case (which leads to a deterministic backward PDE). For the more general, possibly non-Markovian case, see e.g. the recent paper [MYZ] and the references therein.

For papers on optimal control of general SDEs and associated stochastic HJB equations, see e.g. [P] and [BM] and the references therein. None of the above papers deal with jumps.

The novelty of our paper lies in the application of this connection studied in [MYZ] to optimal control of FBSDEs and in the extension to jump models.

2 Optimal control of FBSDEs

{sec2}

We refer to [ØS1] for information about stochastic calculus and control for jump diffusions.

Consider the following controlled coupled FBSDE:
The forward equation in $X(t)$ has the form

$$
\begin{cases}
dX(t) &= \alpha(t, X(t), Y(t), Z(t), K(t,\cdot), u(t, X(t)))dt \\
&+\beta(t, X(t), Y(t), Z(t), K(t,\cdot), u(t, X(t)))dB(t) \\
&+ \int_{\mathbb{R}} \gamma(t, X(t), Y(t), Z(t), K(t,\cdot), u(t, X(t)), \zeta)\tilde{N}(dt, d\zeta) \ ; \ t \in [0,T] \\
X(0) &= x \in \mathbb{R}
\end{cases} \tag{2.1}
$$

{eq1.1}

and the backward equation in $Y(t), Z(t), K(t,\zeta)$ has the form

$$
\begin{cases}
dY(t) &= -g(t, X(t), Y(t), Z(t), K(t,\cdot), u(t, X(t)))dt + Z(t)dB(t) \\
&+ \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt, d\zeta) \ ; \ t \in [0,T] \\
Y(T) &= h(X(T)).
\end{cases} \tag{2.2}
$$

{eq1.2}

Here $B(t) = B(t,\omega)$ and $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt \ ; \ t \in [0,T], \omega \in \Omega, \zeta \in \mathbb{R}_0 := \mathbb{R} - \{0\}$ is a Brownian motion and an (independent) compensated Poisson random measure, respectively, on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}, P)$.

The given functions

$$
\begin{aligned}
\alpha(t,x,y,z,k,u,\omega) &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times V \times \Omega \to \mathbb{R} \\
\beta(t,x,y,z,k,u,\omega) &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times V \times \Omega \to \mathbb{R} \\
\gamma(t,x,y,z,k,u,\zeta,\omega) &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times V \times \mathbb{R}_0 \times \Omega \to \mathbb{R} \\
g(t,x,y,z,k,u,\omega) &: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times V \times \Omega \to \mathbb{R}
\end{aligned}
$$

are assumed to be $\mathbb{F}$-predictable for each x, y, z, k, u. $\mathcal{R}$ denotes the set of functions $k(\zeta) : \mathbb{R}_0 \to \mathbb{R}$ and V is a given set of admissible control values $u(t,x,\omega)$, where $u(t) = u(t, X(t), \omega)$ is our control process. The function $h(x,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ is assumed to be $\mathcal{F}_T$-measurable for each x.

We let $\mathcal{A}$ denote the set of admissible control processes $u(t)$, to be specified further below.

We want to find $\hat{u} \in \mathcal{A}$ such that

$$
\sup_{u\in\mathcal{A}} Y^u(0) = Y^{\hat{u}}(0). \tag{2.3}
$$

{eq1.3}

To this end, let us first recall the following extension of the Itô-Ventzell formula:

{th5.1}

Theorem 2.1 (The Itô-Ventzell formula with jumps) *Suppose $y(t,x)$ solves the SPDE*

$$
\begin{aligned}
dy(t,x) = A(y(\cdot), z(\cdot), k(\cdot))(t,x)dt + z(t,x)dB(t) \\
\int_{\mathbb{R}} k(t,x,\zeta)\tilde{N}(d, d\zeta) \ ; \ t \geq 0
\end{aligned} \tag{2.4}
$$

{eq5.3}

for some partial integro-differential operator A acting on x, and $X(t)$ satisfies an equation of the form

$$
dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \theta(t,\zeta)\tilde{N}(dt, d\zeta) \ ; \ t \geq 0. \tag{2.5}
$$

{eq5.4}

for given $\mathbb{F}$ *-predictable processes* α, β, θ*. Put*

$$Y(t) = y(t, X(t)) \; ; \; t \geq 0. \tag{2.6}$$ {eq5.5}

Then

$$\begin{aligned} dY(t) = &A(y(\cdot), z(\cdot), k(\cdot))(t, X(t))dt + z(t, X(t))dB(t) + \int_{\mathbb{R}} k(t, X(t), \zeta)\tilde{N}(dt, d\zeta) \\ &+ y'(t, X(t))[\alpha(t)dt + \beta(t)dB(t)] + \frac{1}{2}y''(t, X(t))\beta^2(t)dt \\ &+ \int_{\mathbb{R}} \{y(t, X(t) + \theta(t, \zeta)) - y(t, X(t)) - y'(t, X(t))\theta(t, \zeta)\}\nu(d\zeta)dt \\ &+ \int_{\mathbb{R}} \{y(t, X(t^-) + \theta(t, \zeta)) - y(t, X(t^-))\}\tilde{N}(dt, d\zeta) \\ &+ z'(t, X(t))\beta(t)dt \\ &+ \int_{\mathbb{R}} \{k(t, X(t^-) + \theta(t, \zeta))\}\tilde{N}(dt, d\zeta) \\ &+ \int_{\mathbb{R}} \{k(t, X(t^-) + \theta(t, \zeta)) - k(t, X(t^-))\}\nu(d\zeta)dt, \end{aligned} \tag{2.7}$$

where $y'(t, x) = \dfrac{\partial y}{\partial x}(t, x)$ *etc.*

Proof. See [ØZ] and the references therein. □

We now return to problem (2.3). First we try to write the solution $Y(t)$ of (2.2) on the form

$$Y(t) = y(t, X(t)) \tag{2.8}$$ {eq1.4}

for some random field $y(t, x) = y(t, x, \omega)$ which, together with $z(t, x)$ and $k(t, x, \zeta)$, satisfies a BSPDE of the form

$$\begin{cases} dy(t, x) &= A(y(\cdot), z(\cdot), k(\cdot))(t, x)dt + z(t, x)dB(t) \\ &\quad + \int_{\mathbb{R}} k(t, x, \zeta)\tilde{N}(dt, d\zeta) \; ; \; t \in [0, T] \\ y(T, x) &= h(x), \end{cases} \tag{2.9}$$ {eq1.5}

for some partial integro-differential operator A acting on x. By the Itô-Ventzell formula,

$$
\begin{aligned}
dY(t) = & A(y(\cdot), z(\cdot), k(\cdot))(t, X(t))dt \\
& \quad + z(t, X(t))dB(t) + \int_{\mathbb{R}} k(t, X(t), \zeta)\tilde{N}(dt, d\zeta) \\
& + y'(t, X(t))[\alpha(t)dt + \beta(t)dB(t)] + \frac{1}{2}y''(t, X(t))\beta^2(t)dt \\
& + \int_{\mathbb{R}} \{y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) - y'(t, X(t))\gamma(t, \zeta)\}\nu(d\zeta)dt \\
& + \int_{\mathbb{R}} \{y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t))\}\tilde{N}(dt, d\zeta) \\
& + z'(t, X(t))\beta(t)dt \\
& + \int_{\mathbb{R}} \{k(t, X(t) + \gamma(t, \zeta), \zeta) - k(t, X(t), \zeta)\}\nu(d\zeta)dt \\
& + \int_{\mathbb{R}} k(t, X(t^-) + \gamma(t, \zeta), \zeta)\tilde{N}(dt, d\zeta),
\end{aligned}
\tag{2.10}
$$
{eq1.6}

where we have used the shorthand notation

$$\alpha(t) = \alpha(t, X(t), Y(t), Z(t), K(t, \cdot), u(t)) \text{ etc.}$$

Rearranging the terms we see that

$$
\begin{aligned}
dY(t) = & [A(y(\cdot), z(\cdot), k(\cdot))(t, X(t)) + y'(t, X(t))\alpha(t) + \frac{1}{2}y''(t, X(t))\beta^2(t) \\
& + \int_{\mathbb{R}} \{y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) - y'(t, X(t))\gamma(t, \zeta)\}\nu(d\zeta)dt \\
& + z'(t, X(t))\beta(t) \\
& + \int_{\mathbb{R}} \{k(t, X(t) + \gamma(t, \zeta), \zeta) - k(t, X(t), \zeta)\}\nu(d\zeta)]dt \\
& + [z(t, X(t)) + y'(t, X(t))\beta(t)]dB(t) \\
& + \int_{\mathbb{R}} \{y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) \\
& + k(t, X(t) + \gamma(t, \zeta), \zeta)\}\tilde{N}(dt, d\zeta).
\end{aligned}
\tag{2.11}
$$
{eq1.7}

Comparing (2.11) with (2.2) we deduce that

$$\begin{aligned}
&A(y(\cdot), z(\cdot), k(\cdot))(t, X(t)) \\
&\quad = -[g(t, X(t), Y(t), Z(t), K(t, \cdot), u(t, X(t))) + y'(t, X(t))\alpha(t) + \frac{1}{2}y''(t, X(t))\beta^2(t) \\
&\quad + \int_{\mathbb{R}} \{y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) - y'(t, X(t))\gamma(t, \zeta)\}\nu(d\zeta) \\
&\quad + z'(t, X(t))\beta(t) + \int_{\mathbb{R}} \{k(t, X(t) + \gamma(t, \zeta), \zeta) - k(t, X(t), \zeta)\}\nu(d\zeta)] \qquad (2.12) \quad \{eq1.8\}
\end{aligned}$$

$$Z(t) = z(t, X(t)) + y'(t, X(t))\beta(t) \qquad (2.13) \quad \{eq1.9\}$$

$$K(t, \zeta) = y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) + k(t, X(t) + \gamma(t, \zeta), \zeta). \qquad (2.14) \quad \{eq1.10\}$$

In the following we will use the shorthand notations

$$\beta(t) := \beta(t, x, y(t, x), z(t, x), k(t, x, \cdot)) \text{ etc}, \qquad (2.15)$$

$$\tilde{z}(t, x) := z(t, x) + y'(t, x)\beta(t) \qquad (2.16)$$

$$\tilde{k}(t, x, \zeta) := y(t, x + \gamma(t, \zeta)) - y(t, x) + k(t, x + \gamma(t, \zeta), \zeta). \qquad (2.17)$$

We summarize what we have proved as follows:

{th2.1}

Theorem 2.2 *Suppose that* $(y(t, x), z(t, x), k(t, x, \cdot))$ *satisfies the BSPDE*

$$\begin{cases} dy(t, x) &= -G_u(t, x)dt + z(t, x)dB(t) + \int_{\mathbb{R}} k(t, x, \zeta)\tilde{N}(dt, d\zeta) \\ y(T, x) &= h(x) \end{cases} \qquad (2.18) \quad \{eq1.11\}$$

with

$$\begin{aligned}
G_u(t, x) := {}& g(t, x, y(t, x), \tilde{z}(t, x), \tilde{k}(t, x, \cdot), u(t, x)) \\
& + y'(t, x)\alpha(t) + \frac{1}{2}y''(t, x)\beta^2(t) + z'(t, x)\beta(t) \\
& + \int_{\mathbb{R}} \{y(t, x + \gamma(t, \zeta)) - y(t, x) - y'(t, x)\gamma(t, \zeta)\}\nu(d\zeta) \\
& + \int_{\mathbb{R}} \{k(t, x + \gamma(t, \zeta), \zeta) - k(t, x, \zeta)\}\nu(d\zeta), \qquad (2.19) \quad \{eq1.12\}
\end{aligned}$$

Then $(Y(t), Z(t), K(t, \zeta))$*, given by*

$$Y(t) := y(t, X(t)), \qquad (2.20)$$

$$Z(t) := z(t, X(t)) + y'(t, X(t))\beta(t), \qquad (2.21)$$

$$K(t, \zeta) := y(t, X(t) + \gamma(t, \zeta)) - y(t, X(t)) + k(t, X(t) + \gamma(t, \zeta), \zeta), \qquad (2.22)$$

is a solution of the FBSDE system (2.1)-(2.2).

Definition 2.3 *We say that the driver* $G_u(t,x)$ *given by (2.19) satisfies the comparison principle if the corresponding BSPDE* (2.18) *satisfies the comparison principle with respect to* u*. In other words, for all* $u_1, u_2 \in \mathcal{A}$ *and all* $\mathcal{F}_T$*-measurable* h_1, h_2 *with corresponding solutions* $y_1(t,x), y_2(t,x)$*, respectively, of (2.18) such that*

$$G_{u_1}(t,x) \le G_{u_2}(t,x)$$

for all $t,x \in [0,T] \times \mathbb{R}$ *and*

$$h_1(x) \le h_2(x)$$

for all $x \in \mathbb{R}$*, we have*

$$y_1(t,x) \le y_2(t,x)$$

for all $t,x \in [0,T] \times \mathbb{R}$.

Sufficient conditions for the validity of comparison principles for BSPDEs can be found in [MYZ] and [ØSZ2].

For example, from Theorem 2.13 in [MYZ] we get:

{th2.3}

Theorem 2.4 *Assume that the following holds:*

- $N = K = 0$*, i.e. there are no jumps*
- *The coefficients* α*,* β*, and* g *are* $\mathbb{F}$ *- progressively measurable for each fixed* (x,y,z) *and* h *is* $\mathcal{F}_T$ *- measurable for each fixed* x
- α, β, g, h *are uniformly Lipschitz - continuous in* (x,y,z)
- α *and* β *are bounded and*

$$E[\int_0^T g^2(t,0,0,0)dt + h^2(0)] < \infty \tag{2.23}$$

- $\alpha(t,x,y,z,u)$ *does not depend on* z

Then $G_u(t,x)$ *satisfies the comparison principle.*

From the above we deduce the following result, which may be regarded as a *stochastic HJB equation* for optimal control of possibly non-Markovian FBSDEs.

{th1.1}

Theorem 2.5 *Suppose that* $G_u(t,x)$ *satisfies the comparison principle. Moreover, suppose that for all* t,x,ω *there exists a maximizer* $u = \hat{u}(t,x) = \hat{u}(y,y',y'',z,z',k)(t,x,\omega)$ *of the function* $u \to G_u(t,x)$*.*

Suppose the system (2.18) *with* $u = \hat{u}$ *has a unique solution* $(\hat{y}(t,x), \hat{z}(t,x), \hat{k}(t,x,\cdot))$ *and that* $\hat{u}(t,X(t)) \in \mathcal{A}$*. Then* $\hat{u}(t,X(t))$ *is an optimal control for the problem* (2.3)*, with optimal value*

$$sup_{u\in\mathcal{A}} Y^u(0) = Y^{\hat{u}}(0) = \hat{y}(0,x). \tag{2.24}$$ {eq1.13}

Note that in this general non-Markovian setting the classical *value function* from the dynamic programming is replaced by the solution $\hat{y}(t,x)$ of the BSDE (2.18) for $u = \hat{u}$. See Example 3.1 below for more details.

3 Applications

{sec4}

We now illustrate Theorem 2.5 by looking at some examples and applications. First we consider the classical Merton problem:

{exa2.1}

Example 3.1 *[Maximizing expected utility from terminal wealth]* Consider a financial market consisting of two investment possibilities, as follows:
(i) A risk free investment, with unit price

$$S_0(t) := 1 \ ; \ t \in [0,T]. \tag{3.1}$$ {eq3.1}

(ii) a risky investment, with unit price

$$dS_1(t) = S_1(t)[b(t)dt + \sigma(t)dB(t)] \ ; \ t \in [0,T]. \tag{3.2}$$ {eq3.2}

Let $u(t, X(t))$ be a *portfolio*, representing the *amount* invested in the risky asset at time t. If we assume that u is self-financing, then the corresponding wealth $X(t)$ at time t is given by the stochastic differential equation

$$\begin{cases} dX(t) &= u(t,X(t))[b(t)dt + \sigma(t)dB(t)] \ ; \ t \in [0,T] \\ X(0) &= x > 0 \end{cases} \tag{3.3}$$ {eq3.3}

Let $(Y(t), Z(t))$ be the solution of the BSDE

$$\begin{cases} dY(t) &= Z(t)dB(t) \ ; \ t \in [0,T] \\ Y(T) &= U(X(T)) \ ; \end{cases} \tag{3.4}$$ {eq2.2}

where $U(x) = U(x,\omega)$ is a utility function, possibly random. Then

$$Y(0) = E[U(X(T))].$$

In this case we get, from (2.19),

$$G_u(t,x) = y'(t,x)ub(t) + \frac{1}{2}y''(t,x)u^2\sigma^2(t,x) + z'(t,x)u\sigma(t) \tag{3.5}$$ {eq2.3}

which is maximal when

$$u = \hat{u}(t,x) = -\frac{y'(t,x)b(t) + z'(t,x)\sigma(t)}{y''(t,x)\sigma^2(t)}. \tag{3.6}$$ {eq2.4}

Substituting this into $G_{\hat{u}}(t,x)$ we obtain

$$G_{\hat{u}}(t,x) = -\frac{(y'(t,x)b(t) + z'(t,x)\sigma(t))^2}{2y''(t,x)\sigma^2(t)}. \tag{3.7}$$ {eq2.5}

Hence the BSPDE for $y(t,x)$ gets the form

$$\begin{cases} dy(t,x) &= \dfrac{(y'(t,x)b(t)+z'(t,x)\sigma(t))^2}{2y''(t,x)\sigma^2(t)}dt + z(t,x)dB(t)\ ;\ t\in[0,T] \\ y(T,x) &= U(x). \end{cases} \tag{3.8}$$ {eq2.6}

If b,σ and U are *deterministic*, we can choose $z(t,x)=0$ and this leads to the following PDE for $y(t,x)$:

$$\frac{\partial y}{\partial t}(t,x) - \frac{y'(t,x)^2 b^2(t)}{2y''(t,x)\sigma^2(t)} = 0\ ;\ t\in[0,T]. \tag{3.9}$$ {eq2.7}

This is the classical Merton PDE for the value function, usually obtained by dynamic programming and the HJB equation.

Hence we may regard (3.6)-(3.8) as a generalization of the classical Merton solution (3.9) to the case with stochastic $b(t),\sigma(t)$ and $U(x)$, where classical dynamic programming cannot be used. Thus we see that the Markovian case corresponds to the special case when $\hat{z}=0$ of the BSDE (3.9). Therefore $\hat{y}(s,x)$ is a stochastic generalization of the value function

$$\varphi(s,x) := \sup_{u\in\mathcal{A}} E[U(X_{s,x}^{(u)}(T))] \tag{3.10}$$ {eq5.23}

where

$$\begin{cases} dX_{s,x}(t) = u(t)[b_0(t)dt + \sigma_0(t)dB(t)]\ ;\ t\geq s \\ X_{s,x}(s) = x \end{cases} \tag{3.11}$$ {eq5.24}

Compare with the use of the classical HJB:

$$\begin{cases} \dfrac{\partial\varphi}{\partial s}(s,x) + \max\limits_v \left\{\dfrac{1}{2}v^2\sigma_0^2(s)\varphi''(s,x) + vb_0(s)\varphi'(s,x)\right\} &= 0\ ;\ s<T \\ \varphi(T,x) &= U(x). \end{cases} \tag{3.12}$$ {eq3.10}

The maximum is attained at

$$v = \hat{u}(s,x) = -\frac{b_0(s)\varphi'(s,x)}{\varphi''(s,x)\sigma_0^2(s)} \tag{3.13}$$ {eq5.26}

Substituted into (3.12) this gives the HJB equation

$$\frac{\partial\varphi}{\partial s}(s,x) - \frac{\varphi'(s,x)^2 b_0^2(s)}{\varphi''(s,x)\sigma_0^2(s)} = 0, \tag{3.14}$$ {eq5.27}

which is identical to (3.9).

{exa5.6}

Example 3.2 (Risk minimizing portfolios) Now suppose $X(t)=X_x^{(u)}(t)$ is as in (3.3), while $(Y(t),Z(t))$ is given by the BSDE

$$\begin{cases} dY(t) = -\left(-\dfrac{1}{2}Z^2(t)\right)dt + Z(t)dB(t) \\ Y(T) = X(T). \end{cases} \tag{3.15}$$ {eq5.28}

Note that $g(z) = -\frac{1}{2}z^2$ is concave.

We want to minimize $-Y(0) = \rho(X(T))$, where $\rho(X(T))$ is the *risk* of $X(T)$ with respect to the driver $g(z) = \frac{1}{2}z^2$. See e.g. [QS], [R] for more information about the representation of risk measures via BSDEs. Here

$$\begin{aligned} G_u(t,x) = &-\frac{1}{2}(z(t,x) + y'(t,x)u\sigma(t))^2 + y'(t,x)ub(t) \\ &+ \frac{1}{2}y''(t,x)u^2\sigma^2(t) + z'(t,x)u\sigma(t), \end{aligned} \tag{3.16}$$ {eq5.29}

which is minimal when $u = \hat{u}(t,x)$ satisfies

$$\hat{u}(t,x) = -\frac{z(t,x)y'(t,x)\sigma(t) - y'(t,x)b(t) - z'(t,x)\sigma(t)}{((y'(t,x))^2 - y''(t,x))\sigma^2(t)}. \tag{3.17}$$ {eq5.30}

This gives

$$G_{\hat{u}}(t,x) = -\frac{1}{2}\hat{z}^2(t,x) + \frac{(\hat{z}(t,x)\hat{y}'(t,x)\sigma(t) - \hat{y}'(t,x)b(t) - \hat{z}'(t,x)\sigma(t))^2}{2((\hat{y}'(t,x))^2 - \hat{y}''(t,x))\sigma^2(t)}. \tag{3.18}$$ {eq5.31}

and hence $(\hat{y}(t,x), \hat{z}(t,x))$ solves the BSPDE

$$\begin{cases} d\hat{y}(t,x) = -G_{\hat{u}}(t,x)dt + \hat{z}(t,x)dB(t)\ ;\ 0 \le t \le T \\ \hat{y}(T,x) = x. \end{cases} \tag{3.19}$$ {eq5.32}

Let us try to choose $\hat{z}(t,x) = 0$.
Then (3.19) reduces to the PDE

$$\begin{cases} \dfrac{\partial \hat{y}(t,x)}{\partial t} = -\dfrac{(\hat{y}'(t,x)b(t))^2}{2((\hat{y}'(t,x))^2 - \hat{y}''(t,x))\sigma^2(t)}\ ;\ 0 \le t \le T \\ \hat{y}(T,x) = x. \end{cases} \tag{3.20}$$ {eq5.33}

We try a solution of the form

$$\hat{y}(t,x) = x + a(t), \tag{3.21}$$ {eq5.34}

where $a(t)$ is deterministic. Substituted into (3.20) this gives

$$\begin{cases} a'(t) = -\dfrac{1}{2}\left(\dfrac{b(t)}{\sigma(t)}\right)^2\ ;\ 0 \le t \le T \\ a(T) = 0 \end{cases} \tag{3.22}$$ {eq5.35}

which gives

$$a(t) = \int_t^T \frac{1}{2}\left(\frac{b(s)}{\sigma(s)}\right)^2 ds\ ;\ 0 \le t \le T.$$

With this choice of $a(t)$ we see that (3.20) is satisfied and we conclude that the minimal risk is

$$\rho_{min}(X(T)) = -Y^{(\hat{u})}(0) = -\hat{y}(0,x) = -x - \int_0^T \frac{1}{2}\left(\frac{b(s)}{\sigma(t)}\right)^2 ds \qquad (3.23) \quad \{\texttt{eq5.36}\}$$

Hence by (3.17) the optimal (risk minimizing) portfolio is

$$\hat{u}(t, X(t)) = \frac{b(t)}{\sigma^2(t)}. \qquad (3.24) \quad \{\texttt{eq5.37}\}$$

Remark 3.1 It is interesting to note that (3.23) *can be interpreted by means of* entropy *as follows:*
Recall that in general the entropy of a measure Q with respect to the measure P is defined by

$$H(Q \mid P) := E\left[\frac{dQ}{dP}\ln\frac{dQ}{dP}\right]$$

.

Define

$$\Gamma(t) = \exp\left(-\int_0^t \frac{b(s)}{\sigma(s)}dB(s) - \frac{1}{2}\int_0^t (\frac{b(s)}{\sigma(s)})^2 ds\right). \qquad (3.25)$$

Then

$$d\Gamma(t) = \Gamma(t)\left[-\frac{b(t)}{\sigma(t)}dB(t)\right] \quad ; \; \Gamma(0) = 1. \qquad (3.26)$$

By the Itô formula we have

$$\begin{aligned} d(\Gamma(t)\ln\Gamma(t)) &= \Gamma(t)\left[-\frac{b(t)}{\sigma(t)}dB(t) - \frac{1}{2}\left(\frac{b(t)}{\sigma(t)}\right)^2 dt\right] \\ &\quad + (\ln\Gamma(t))\Gamma(t)\left(-\frac{b(t)}{\sigma(t)}dB(t)\right) + \Gamma(t)\left(-\frac{b(t)}{\sigma(t)}\right)\left(-\frac{b(t)}{\sigma(t)}\right)dt. \end{aligned}$$

Hence, if we define the measure $Q_\Gamma(\omega)$ by

$$dQ_\Gamma(\omega) := \Gamma(T)dP(\omega) \qquad (3.27)$$

we get

$$\begin{aligned} E\left[\frac{dQ_\Gamma}{dP}\ln\frac{dQ_\Gamma}{dP}\right] &= E[\Gamma(T)\ln\Gamma(T)] \\ &= E\left[\int_0^T \Gamma(t)\frac{1}{2}\left(\frac{b(t)}{\sigma(t)}\right)^2 dt\right] = \frac{1}{2}\int_0^T \left(\frac{b(t)}{\sigma(t)}\right)^2 dt, \end{aligned}$$

which proves that (3.23) *can be written*

$$\rho_{min}(X(T)) = -x - H(Q_\Gamma \mid P) \qquad (3.28)$$

Note that Q_Γ is the unique equivalent martingale measure *for the market* (3.1),(3.2).
Thus we conclude that the negative of the minimal risk is equal to the initial wealth x plus the entropy of the equivalent martingale measure.
It is natural to ask what the corresponding result would be in the incomplete market case.

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