Real numbers, chaos, and the principle of a bounded density of information

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Abstract. Two definitions of the notion of a chaotic transformation are compared: sensitivity to initial conditions and sensitivity to perturbations. Only the later is compatible with the idea that information has a finite density.

1 The notion of information in Physics

Information is not a new notion in Physics, as Ludwig Boltzmann already defined the entropy of a system as the logarithm of the number of microscopic states corresponding to its macroscopic state, that is, in modern terms, as the amount of information necessary to describe its microscopic state when its macroscopic state is known.

This definition presupposes that the number of microscopic states corresponding to a macroscopic state is finite, an hypothesis that would only be clarified later by quantum theory, and still in a very partial way.

After Bolzmann, this idea of a bound on the number of possible states of a given system, that is on the amount of information contained in such a system, or equivalently on the density of information in the Universe has slowly emerged. It is, for instance, one of the hypotheses assumed by Robin Gandy [5] in his "proof" of the physical Church-Turing thesis. It is also a thesis proposed by Jacob Bekenstein [2] in his investigation of back hole entropy. Bekenstein even proposes a bound, that is, unfortunately, not a bound on the amount of information contained in a system, but on the ratio between this amount of information and the energy of the system.

2 Physics without real numbers

This hypothesis of a bound on the density of information in Universe, is however inconsistent with the most common formulations of Physics, for instance Newtonian theory.

In Newtonian theory, just like information travels at an infinite velocity because the motion of a mass induces an instantaneous modification of the gravitational field in the whole Universe, an object as simple as a pencil contains an infinite amount of information, because its length is a real number, containing an infinite number of digits.

This idea that a magnitude, such as the length of a pencil, is a real number comes from an idealization of the process of measurement. The measure of the length of a segment, for instance the length of a pencil, is defined as the number of times a yardstick fits in this segment. More precisely, this natural number is a lower approximation of the length of the segment, with an accuracy which is the length of the yardstick—or twice this length, if the last fit is uncertain. A more precise measure is obtained by dividing this yardstick in ten equal parts, and counting the number of tenths of yardsticks that fit in the segment. Dividing again this tenth of yardstick in ten parts, an even more precise measure is obtained, and so on. The result of each individual measurement is thus a rational number, and only the hypothetical possibility to repeat this process indefinitely leads to the idea that the measured magnitude, per se, is the limit of an bounded increasing sequence of rational numbers, that is a real number.

In theory, the fact that the length of a pencil is a real number permits to record an infinite amount of information by sharpening the pencil to give it definite length. However, this idea is inconsistent with a principle, I will call the *caliper principle*, that, although it does not exactly have the status of a fundamental principle of Physics, is used as if it were one: the principle that a measuring instrument yields only an approximation of the measured magnitude, and that it is therefore impossible, except according to this idealization, to measure more than the first digits of a physical magnitude. Historically, this caliper principle might be one of the first formulations of the idea of a bounded density of information in the Universe, even if it only prevents the access to an infinite amount of information and not the existence of this infinite amount of information itself. According to this principle, this idealization of the process of measurement is a fiction. This suggests the idea, reminiscent of Pythagoras' views, that Physics could be formulated with rational numbers only.

We can therefore wonder why real numbers have been invented and, moreover, used in Physics. A hypothesis is that the invention of real numbers is one of the many situations, where the complexity of an object is increased, so that it can be apprehended more easily [3, 4]. Let us illustrate this idea with an example.

If we restrict to rational numbers, the parabola of equation $y = x^2 - 2$ does not intersect the x-axis, because there exists no rational number whose square is equal to 2. But, because the continuous function $x \mapsto x^2 - 2$ takes a negative value at 1.4 and a positive one at 1.5, it can be proved that, for all positive rational numbers ε, there exists a rational number x such that the absolute value of this function at x is smaller than ε. It is even possible to build a bounded and increasing sequence of rational numbers 1.4, 1.41, 1.414, ... such that the image of this sequence by the function $x \mapsto x^2 - 2$ goes to 0 at infinity. We have here two relatively complex formulations of the intermediate value theorem. But postulating a limit $\sqrt{2}$ to this sequence permits to give a much simpler formulation to this theorem: there exists a number whose image by this function is 0.

Thus, according to the caliper principle, real numbers area fictions that permit to apprehend the Universe more easily, but there is no reason to think that physical magnitudes, per se, are real numbers.

3 The status of the principle of a bounded density of information

If we have very few reasons to believe that physical magnitudes, per se, are real numbers, that is that they contain an infinite amount of information, we must admit that we have also few reasons to believe that, on the opposite, the density of information in the Universe is bounded.

This thesis is a hypothesis.

However this thesis is not a metaphysical hypothesis, that would only depend on the way we decide to describe the Universe—that we could decide to describe in a discrete or continuous way—and not of the properties of the Universe itself. Indeed, once a bound is fixed, the principle of a bounded density of information is a falsifiable statement: it is sufficient to record $b + 1$ bits of information in a system included in a sphere of radius 1m and to read it back to refute the thesis that the amount of information contained in a sphere of radius 1m is bounded by b, that is that the number of states of such a system is bounded by 2^b.

In the same way, it would be sufficient to transmit some information faster than light to refute the thesis that the velocity of propagation of information is bounded by the speed of light.

These two theses have a similar status. The only difference is that the bound is known in one case and not in the other.

4 Sensitivity to initial conditions

The introduction of real numbers in Physics has had the advantage to simplify the way we apprehend the Universe. But postulating that real numbers are more than just fictions, and that physical magnitudes are, per se, real numbers, has a remarkable consequence: it becomes possible, for a physical transformation, to be sensitive to initial conditions. It becomes possible, for instance, for the flap of a butterfly's wings in Brazil to set off a tornado in Texas.

An example of a process that is sensitive to initial conditions is the baker's transformation b, that maps each real number x, between 0 and 1, to the real number, also between 0 and 1, $2x$ when x is between 0 and $1/2$ and $2 - 2x$ when it is between $1/2$ and 1. This transformation is sensitive to initial conditions, because its iteration magnifies, step by step, small differences between two initial values x and x'. For instance, this transformation iterated on the two initial values 0 and $a/2^N$, that can be made as close as desired by taking N large enough, will lead in N steps to 0 and a respectively. Iterating this transformation progressively unfolds an infinite amount of information present in the initial state of the system.

The definition of the notion of sensitivity to initial conditions assumes that the initial values x and x' are slightly different, but that the processes applied to these values are rigorously identical: the flap of a butterfly's wings can modify the "initial" state of the atmosphere, but the butterflies must stop flapping their wings during the later evolution of the atmosphere.

The existence of transformations that are sensitive to initial conditions and unfold, step by step, an infinite amount of information present in the initial state of the system seems to be inconsistent with the principle of a bounded density of information. But more annoying is that the existence of non perturbed evolutions assumed by the definition of sensitivity to initial conditions is inconsistent with the much weaker and more consensual caliper principle, that no measurement can give, say, more that twenty relevant digits.

Indeed, is we assume physically possible to apply a perfect baker's transformation to a physical magnitude, and we have a measuring instrument I_1 that permits to measure this magnitude with an accuracy 2^{-10}, that is three decimal digits, it becomes possible to build an other measuring instrument I_2 that permits to measure some magnitudes with an accuracy 2^{-100}, that is thirty decimal digits.

This instrument just iterates the baker's transformation 100 times on the magnitude x to be measured and measures with the first instrument the 101 results $x = s_0, ..., s_{100}$ of these iterations. If one of the 101 measures yields a result that it between $1/2 - 2^{-10}$ and $1/2 + 2^{-10}$, so that we cannot decide if the measured magnitude is smaller or larger of $1/2$, the global measurement with the instrument I_2 fails. Otherwise, it is possible to determine whether each of these magnitudes is smaller or larger than $1/2$ and, in this case, the magnitude x is can be determined with an accuracy 2^{-102}. Indeed, it is easy to prove, by induction on i, that knowing if each of the terms $s_0, ..., s_{100}$ is smaller or greater than $1/2$ is sufficient to determine s_i with an accuracy $2^{-(102-i)}$. For $i = 100$, if s_{100} is smaller than $1/2$, then $1/4$ is an approximation of s_i, and if it is larger, $3/4$ is an approximation. In both cases, the accuracy is $1/4 = 2^{-2}$. Otherwise, by induction hypothesis, we know s_{i+1} with an accuracy $2^{-(102-(i+1))}$. If s_i is smaller than $1/2$ then $s_{i+1} = 2s_i$, that is $s_i = s_{i+1}/2$, and if it is larger, $s_{i+1} = 2 - 2s_i$, that is $s_i = 1 - s_{i+1}/2$. In both cases we obtain s_i with an accuracy $2^{-(102-(i+1))}/2 = 2^{-(102-i)}$. Thus, at the end, we obtain x with an accuracy 2^{-102}.

The success rate of the instrument I_2 is larger than eighty percent. To prove this, we prove that the measurement always succeeds when the number x does not have a sequence of 9 identical digits among the 110 first digits of its binary development. Indeed, the baker's transformation acts on the binary development of a number z in the following way: if the first digit of the number z is a zero, that is if $z \leq 1/2$, then it shifts all digits to the left, that is multiplies it by 2, if its is a one, that is if $z \geq 1/2$, then it replaces each one by a zero and each zero by a one, that is takes the opposite and adds 1, and shifts all the digits to the left, that is multiplies it by 2. Thus, applying the baker's transformation 100 times to the initial state x, that does not contain a sequence of 9 identical digits in the first 110 digits of its binary development, yields a sequence of 101 numbers s_0, ..., s_{100}, such that no element of this sequence contains a sequence of 9 identical digits in the first 10 digits of its binary development. Note that a number z such that $1/2 - 2^{-10} = 0.0111111111 < z \leq 1/2 = 0.011111111...$ has a sequence of 9 ones in its first 10 digits and that a number z such that $1/2 = 0.1000000000... \leq z < 1/2 + 2^{-10} = 0.1000000001$ has a sequence of 9 zeros in its first 10 digits. Thus, as none of the s_0, ..., s_{100} has a sequence of 9 identical digits in its first 10 digits, none is in the grey area of numbers between $1/2 - 2^{-10}$ and $1/2 + 2^{-10}$. Finally, the probability for a real number not to have a sequence 9 identical digits among the 110 first digits of its binary development is 0.815..., as the number k_l of sequences of l digits not containing a sequence of 9 identical digits verifies the induction relation: $k_0 = 1$, for $l+1 \leq 8$: $k_{l+1} = 2k_l$, $k_9 = 2k_8 - 2$, and for $l + 1 \geq 10$: $k_{l+1} = 2k_l - k_{l-8}$, from which we get $k_{110} = 1.058...\ 10^{33}$ and $k_{110}/2^{110} = 0.815...$

Thus, the success rate of I_2 is larger than or equal to 0.815... and the mere existence of a magnitude x for which this measurement can be performed is sufficient to contradict the caliper principle.

5 Sensitivity to perturbations

Thus, a consequence of the caliper principle is that the a physical dynamical system must always be slightly perturbed. The course of its evolution is defined neither by

$$s_0 = x$$

$$s_{i+1} = f(s_i)$$

nor by

$$s_0 = x + p_0$$

$$s_{i+1} = f(s_i)$$

where p_0 would be a modification of the initial state x, but by

$$s_0 = x + p_0$$

$$s_{i+1} = f(s_i) + p_{i+1}$$

where $p_0, p_1, ...$ is a *perturbation sequence.*

It thus becomes difficult to distinguish, in the causes of a tornado in Texas, the role of a modification of the “initial” state of the atmosphere, due to a flap of a butterfly’s wings, from later perturbations due, for instance, to other flaps of butterflies’ wings.

A transformation f that is sensitive to perturbations is sensitive to initial conditions, as the sequence $p_0, 0, 0, ...$ is a perturbation sequence. *Shadowing* lemmas show that for many dynamical systems, the converse also holds: if a system is sensitive to initial conditions, then it is also sensitive to perturbations:

the information brought by perturbations during the evolution of the system can be aggregated in its initial state, to produce almost the same evolution.

For instance, consider N iterations $s_0, ..., s_N$ of the baker's transformation, perturbed by a sequence $p_0, p_1, ...$, such that for all i, $|p_i| \leq \varepsilon$ and s_i is between 0 and 1. Then it is easy to prove, by decreasing induction on i, that for all i, there exists a element s'_i, such that $|s'_i - s_i| \leq \varepsilon$ and the non perturbed evolution starting at step i with s'_i yields at s_N at step N. For $i = N$ just take $s'_N = s_N$. Assume the property holds at $i+1$. From $|s_{i+1} - s'_{i+1}| \leq \varepsilon$, $s_{i+1} = b(s_i) + p_i$, and $|p_i| \leq \varepsilon$, we get $|s'_{i+1} - b(s_i)| \leq 2\varepsilon$. If s_i is smaller than $1/2$, we let $s'_i = s'_{i+1}/2$. As s'_{i+1} is smaller than 1, s'_i is smaller than $1/2$ and $b(s'_i) = 2s'_i = s'_{i+1}$. Then, from $|s'_{i+1} - b(s_i)| \leq 2\varepsilon$, we get $|2s'_i - 2s_i| \leq 2\varepsilon$ and $|s'_i - s_i| \leq \varepsilon$. And if s_i is larger than $1/2$, we let $s'_i = 1 - s'_{i+1}/2$. As s'_{i+1} is smaller than 1, s'_i is larger than $1/2$ and $b(s'_i) = 2 - 2s'_i = s'_{i+1}$. Then, from $|s'_{i+1} - b(s_i)| \leq 2\varepsilon$, we get $|(2 - 2s'_i) - (2 - 2s_i)| \leq 2\varepsilon$ and $|s'_i - s_i| \leq \varepsilon$. Thus, in both cases we have $b(s'_i) = s'_{i+1}$ and $|s'_i - s_i| \leq \varepsilon$.

Thus, there exists an initial value s'_0 such that $|s'_0 - x| \leq 2\varepsilon$ and the non perturbed evolution from s'_0 yields s_N after N steps.

This equivalence between sensitivity to initial conditions and sensitivity to perturbations explains why the definition of a chaotic transformation speaks only about sensitivity to initial conditions and not about sensitivity to perturbations brought during the evolution of the system.

But this aggregation, in the initial state, of an unbounded amount of information, brought during the evolution of the system is not possible for systems where the amount of information is bounded. For example, the baker's transformation on the finite set $\{0, 0.1, ..., 0.9, 1\}$ is not sensitive to initial conditions: by modifying the initial value 0 of a quantity less than or equal to 0.1, it is possible to reach the values 0, 0.1, 0.2, 0.4 and 0.8, but not, unlike in the continuous case, the values 0.9 and 1, for instance—of course this example has only an didactic value, the size of a cell of a discrete physical system would be rather on the order of magnitude of Planck's length, and the number of states rather on the order of magnitude 10^{35} than 10.

In contrast, this transformation is sensitive to perturbations: it is possible to reach all the values a in the set $\{0, 0.1, ..., 0.9, 1\}$ starting from 0 and perturbing the system of a quantity at most 0.1 at each step—of course, a smaller perturbation would not mean anything. For instance, the value 0.9 is obtained by the sequence: 0, 0.1, 0.2, 0.4, 0.9. More generally, if $a = p\ 0.1$ where p is a natural number, we let N be a natural number such that $p/2^N = 0$, where $/$ is the integer division, and $s_i = (p/2^{N-i})\ 0.1$. We have $s_0 = 0$, $s_N = a$ and for all i between 0 and $n - 1$, $s_i \leq 1/2$, $s_{i+1} = 2s_i$ or $s_{i+1} = 2s_i + 0.1$. Thus $s_{i+1} = b(s_i) + p_i$, with $|p_i| \leq 0.1$.

This transformation is sensitive to perturbations but not to initial conditions, which shows that these two conditions are not equivalent in this case.

6 The definition of the notion of a chaotic transformation

Thus, there are at least two different reasons for the perturbed baker's transformation to produce the value 0.9 after four iterations, starting from 0. One is that the initial value was not 0, but $0.9/16 = 0.05625$, an other is that at the second and fourth iteration, a perturbation of 0.1 has been brought.

The first explanation—sensitivity to initial conditions—postulates a non perturbed evolution that is inconsistent with the caliper principle and the existence of a punctual cornucopia that provides an infinite amount of information not accessible to measurement, but appearing during the evolution of the system. And it provides no explanation why this evolution cannot itself be considered as a measurement.

In contrast, the second—sensitivity to perturbations—does not assume the existence of a non perturbed transformation, remains possible even if we assume that information has a bounded density, and locates the source of the information that appears during the evolution in the environment of the system, with which it always interacts. Sensitivity to perturbations seems therefore to be a good alternative to sensitivity to initial conditions, when defining the notion of a chaotic transformation.

This clarification of the definition of the notion of a chaotic transformation is one of the benefits of using a notion of information, and a principle of a bounded density of information in Sciences, such as Physics, besides its obvious use in Computer Science.

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References

1. P. Arrighi and G. Dowek. The principle of a finite density of information. H. Zenil, *Irreducibility and Computational Equivalence: Wolfram Science 10 Years After the Publication of A New Kind of Science*, 2012.
2. J. D. Bekenstein. Universal upper bound to entropy-to-energy ratio for bounded systems. *Phys. Rev. D*, 23:287–298, 1981.
3. A. Berthoz, La Simplexité, Odile Jacob, 2009.
4. G. Dowek, La notion de nombre réel : une solution simplexe ?, to appear.
5. R. Gandy. Church's thesis and principles for mechanisms. *The Kleene Symposium*, North-Holland, 1980.