Fixed-time consensus algorithm for multi-agent systems with integrator dynamics

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Abstract: The paper addresses the problem of exact average-consensus reaching in a prescribed time. The communication topology is assumed to be defined by a weighted undirected graph and the agents are represented by integrators. A nonlinear control protocol, which ensures a finite-time convergence, is proposed. With the designed protocol, any prescribed convergence time can be guaranteed regardless of the initial conditions.

Keywords: multi-agent systems, consensus, fixed-time convergence.

1. INTRODUCTION

Consensus or agreement problem is a key problem in decentralized cooperative control of multi-agent systems Olfati-Saber et al. (2007), Ren & Cao (2011). This is partly due to many important applications in control of spacecrafts, mobile robots, UAVs, sensor networks and in other fields such as optimization. In the leaderless consensus problem it is required that all nodes-agents to converge to a common value which is not prespecified in advance (minimize the disagreement between the agents in the system to zero).

In Olfati-Saber & Murray (2004), a classical continuous linear consensus control protocol for networks of integrators was studied. It was shown that the second smallest eigenvalue of interaction graph Laplacian, called algebraic connectivity, determines the convergence rate of consensus algorithms. Evidently, the protocols that provide high convergence rate are more preferred in applications. In Xiao & Boyd (2004), Shafi et al. (2011) the problems of vertex-edge weight design to provide the desired spectra of graph Laplacians are considered. It should be noted that by maximizing the algebraic connectivity we obtain a better performance of linear algorithms but still reach an agreement just asymptotically. Moreover, the convergence time essentially depends on the initial conditions of the agents. On the other hand, in numerous practical applications it is required that the transient processes have to be finished in a prescribed time.

The *main objective* of this paper is to design an average-consensus control protocol which provides the *finite-time* convergence property to the multi-agent system; on top of that, *any guaranteed settling time has to be be specified in advance regardless of the initial positions of the agents.*

The theory of finite-time stability and stabilization problems has been a subject of intensive research in recent years; e.g., see Haimo (1986), Bhat & Bernstein (2000), Moulay & Perruquetti (2006), Orlov (2009), Polyakov & Poznyak (2012). For use of finite-time control ideology in consensus problems see Cortés (2006), Hui et al. (2010), Wang & Xiao (2010), Xiao et al. (2009). Finite-time stability analysis usually exploits the theory of non-smooth Lyapunov functions and involves such concepts as weak and strong stability, differential inclusions, generalized gradients and derivatives Roxin (1966), Hui et al. (2010).

Obviously, there is a great need in finite-time consensus algorithms; moreover algorithms that guarantee any predefined convergence time *regardless of the initial conditions of the agents* are most desired. The corresponding modification of the *finite-time stability* was called *fixed-time stability* (see, Polyakov (2012)). Fixed-time algorithms can be also found in Andrieu et al. (2008) and Cruz-Zavala et al. (2011). In Parsegov et al. (2012), a fixed-time control protocol for a specific formation control problem was designed.

Polynomial state feedback control systems have attracted considerable attention in nonlinear control Allgöwer (2006). This class of control systems appears in models of a wide range of applications such as chemical processes, electronic circuits and mechatronics, biological systems, etc.

This paper presents new consensus control protocols of a polynomial type which guarantee fixed-time convergence to a common value.

The paper is organized as follows. The next section presents the notations used in the paper. The section 3 discusses the problem statement and the basic assumptions. After then some preliminaries are given. The main theoretical result is formulated and proven in Section 5.

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Finally, the numerical simulations results and conclusions are presented.

2. NOTATIONS

- $\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.
- Denote by $D^*\varphi(t)$ the upper right-hand Dini derivative of the function $\varphi : \mathbb{R} \to \mathbb{R}$ at the point $t \in \mathbb{R}$, i.e.
$$D^*\varphi(t) := \limsup_{h\to+0} \frac{\varphi(t+h) - \varphi(t)}{h}.$$
- Let us introduce the power operation, which preserves the sign of the number,
$$s^{[k]} := \mathrm{sign}(s)|s|^k, \tag{1}$$
where $s \in \mathbb{R}$ and $\mathrm{sign}(s)$ is the sign function
$$\mathrm{sign}(s) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$$
In this case, $(-2)^{[2]} = -4$.
If $A \in \mathbb{R}^{n\times m}$ then the introduced power operation $A^{[p]}$ is understood componentwise. For example,
$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad A^{[2]} = \begin{bmatrix} 1 & -4 \\ 9 & 0 \end{bmatrix}.$$
- $\lambda_*(P)$ is the minimum *positive* eigenvalue of the symmetric matrix $P \in \mathbb{R}^{n\times n} : P = P^\top$.
- For $z \in \mathbb{R}^n$ and $p \in \mathbb{R}_+$ we denote
$$\|z\|_p := \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}}. \tag{2}$$
- Denote a unit vector by $\mathbf{1}_n = [1, 1, \ldots, 1]^\top \in \mathbb{R}^n$.
- If $\nu = (\nu_1, ..., \nu_n)^\top \in \mathbb{R}^n$ then the notation $\mathrm{diag}(\nu)$ is introduced for the diagonal matrix with the elements $\nu_i \in \mathbb{R}, i = 1, 2, ..., n$.

3. PROBLEM STATEMENT

We consider a group of n numbered mobile agents. Let their positions at time $t \geq 0$ be denoted by $x_i(t) \in \mathbb{R}$, $i = 1, 2, ..., n$. The dynamic model of each agent is described by a simple integrator:
$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, n, \tag{3}$$
where $u_i \in \mathbb{R}$ is the state feedback, called control protocol, to be designed based on the information received by agent i from its neighbors, $x = [x_1, x_2, \ldots, x_n]^\top$.

The objective in this paper is to design a feedback control protocol u_i, which

- solves the average-consensus problem in a fixed time for all initial conditions, i.e.
$$\exists T_{\max} \in \mathbb{R}_+ \ : \ x_i(t) = x^*, \quad t > T_{\max},$$
where $i = 1, 2, ..., n$ and $x^* := (1/n) \sum\limits_{i=1}^{n} x_i(0)$;
- exploits only the local information about the distances of the agent from its neighbors according to the communication topology, i.e.
$$u_i = \sum_{j=1}^{n} \phi_{ij}\Big(x_j - x_i\Big), \ i = 1, 2, \ldots, n, \tag{4}$$
where ϕ_{ij} are continuous functions of distances for all i, j and $\phi_{ij} = 0$ if the corresponding agents are not neighboring.

Below we consider some helpful notions, definitions and auxiliary lemmas needed for further discussion.

4. PRELIMINARIES

4.1 Graphs and Linear Consensus Protocol

Algebraic graph theory plays an important role in the analysis of consensus problems. Each agent of multi-agent system communicates with some of the agents according to the communication topology (structure of the system). Such a structure can be represented by a (generally) directed graph. In this work, we consider graphs which are undirected and do not contain self-edges.

We use weighted undirected graph $\mathcal{G}(A)$ for representing the communication topology, where $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$ is a weighted *adjacency matrix*. Each agent in the multi-agent system is associated with a vertex in the graph and a_{ij} is a weight of the information channel represented by the edge $\{i, j\}$. If $a_{ij} = 0$ it means that there is no edge between the corresponding vertices-agents. Undirected topology means the neighboring agents receive the same information about the distance between them.

Notice that under the assumptions above, a well-known linear control protocol Olfati-Saber & Murray (2004), Chebotarev & Agaev (2009) of the form
$$u_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i), \ i = 1, 2, \ldots, n, \tag{5}$$
does solve the average-consensus problem; however the agents reach consensus asymptotically and the settling time depends on the initial conditions.

The dynamics of multi-agent system (3) under control protocol (5) can be rewritten in a vector form
$$\dot{x} = -\mathcal{L}x, \tag{6}$$
where $\mathcal{L}$ is a symmetric matrix $n \times n$ named *the Laplacian of graph* $\mathcal{G}(A)$: $\mathcal{L} = diag(A\mathbf{1}_n) - A$.

The matrix $\mathcal{L}$ of an undirected graph has some properties that are important in analysis of consensus problems (see Olfati-Saber & Murray (2004), Chebotarev & Agaev (2009), Ren & Cao (2011)):

- $\mathcal{L}$ has at least one zero eigenvalue with associated eigenvector $\mathbf{1}_n$.
- $\mathcal{L}$ has a simple zero eigenvalue if and only if the corresponding graph is connected.
- $x^\top \mathcal{L}x = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2$; the semipositive definiteness means that all other eigenvalues are positive.
- For a connected graph the second smallest eigenvalue $\lambda_*(\mathcal{L}) = \lambda_2(\mathcal{L})$ is called the algebraic connectivity of $\mathcal{L}$. It quantifies the convergence rate of consensus algorithms.
- The algebraic connectivity equals
$$\min_{\|x\|\neq 0,\ \mathbf{1}_n^\top x = 0} \frac{x^\top \mathcal{L}x}{\|x\|^2} = \lambda_*(\mathcal{L}) > 0, \tag{7}$$
and therefore $\sum\limits_{i=1}^{n} x_i = 0$ implies $x^\top \mathcal{L}x \geq \lambda_*(\mathcal{L})x^\top x$.

4.2 Fixed-time Convergence

Consider the following system:

$$\dot{z} = g(t, z), \quad z(0) = z_0, \tag{8}$$

where $z \in \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a possibly discontinuous nonlinear function. In this case, the solutions of (8) are understood in the sense of Filippov (1988). Assume that the origin is an equilibrium point of system (8).

Definition 1. (Bhat & Bernstein (2000)). The origin is said to be a *globally finite-time stable* equilibrium point for system (8) if it is globally asymptotically stable and any solution $z(t, z_0)$ of (8) attains it in finite time, i.e., $z(t, z_0) = 0 \; \forall\, t \geq T(z_0)$, where $T \colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the *settling-time function*.

For example, any solution of the system $\dot{z} = -z^{\frac{1}{3}}, z \in \mathbb{R}$, converges to the origin in the finite time $T(z_0) := \frac{2}{3}\sqrt[3]{|z_0|^2}$.

Definition 2. (Polyakov (2012)). The origin is said to be a *fixed-time stable* equilibrium point of system (8) if it is globally finite-time stable and the settling-time function $T(z_0)$ is bounded, i.e. there exists $T_{\max} > 0\colon\ T(z_0) \leq T_{\max}\ \forall z_0 \in \mathbb{R}^n$.

The system

$$\dot{z} = -z^{\left[\frac{1}{2}\right]} - z^{[2]}, \quad z \in \mathbb{R}, \quad z(0) = z_0$$

is fixed-time stable, since its solution has the form

$$z(t,z_0) = \begin{cases} \operatorname{sign}(z_0)\tan^2\left(\dfrac{T(z_0) - t}{2}\right) & 0 \leq t \leq T(z_0), \\ 0 & t > T(z_0), \end{cases}$$

where $T(z_0) = 2\arctan(\sqrt{|z_0|}) \leq \pi$.

Lemma 3. (Polyakov (2012)). If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

(1) $V(z) = 0 \Leftrightarrow z = 0$;
(2) any solution $z(t)$ of (8) satisfies the inequality $D^*V(z(t)) \leq -(\alpha V^p(z(t)) + \beta V^q(z(t)))^k$ for some $\alpha, \beta, p, q, k > 0$: $pk < 1$, $qk > 1$,

then the origin is globally fixed-time stable for system (8) and the following estimate holds:

$$T(z_0) \leq \frac{1}{\alpha^k(1 - pk)} + \frac{1}{\beta^k(qk - 1)}, \quad \forall z_0 \in \mathbb{R}^n.$$

This lemma together with its refinement given below are the cornerstones for the design of the nonlinear fixed-time average-consensus control protocol.

Consider the case where the constants p and q are of the form $p = 1 - \frac{1}{2\gamma}$ and $q = 1 + \frac{1}{2\gamma}$, $\gamma > 1$.

Lemma 4. (Parsegov et al. (2012)). If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that

(1) $V(z) = 0 \Leftrightarrow z = 0$;
(2) any solution $z(t)$ of (8) satisfies the inequality $D^*V(z(t)) \leq -\alpha V^p(z(t)) - \beta V^q(z(t))$ for some $\alpha, \beta > 0$, $p = 1 - \frac{1}{2\gamma}$, $q = 1 + \frac{1}{2\gamma}$, $\gamma > 1$,

then the origin is globally fixed-time stable for system (8) and the following estimate of the settling time function holds:

$$T(z_0) \leq T_{\max} := \frac{\pi\gamma}{\sqrt{\alpha\beta}} \quad \forall z_0 \in \mathbb{R}^n.$$

The last lemma provides more exact estimate of the settling time (see, Parsegov et al. (2012) for the details).

5. FIXED-TIME CONSENSUS CONTROL PROTOCOL

Consider now the situation where every edge $\{i, j\}$ is associated with some function ϕ_{ij} that satisfy

$$\phi_{ij}\Big(x_j - x_i\Big) = -\phi_{ji}\Big(x_i - x_j\Big), \tag{9}$$

for two neighboring agents and $\phi_{ij} \equiv 0$ if there is no edge between the vertices i and j. In Olfati-Saber & Murray (2004), the functions ϕ_{ij} are referred to as *action functions.*

Recall $s^{[k]} := \operatorname{sign}(s)|s|^k$ (see, the section with notations).

Choose the following action functions meeting the condition (9)

$$\phi_{ij} = \alpha(a_{ij}(x_j - x_i))^{[\mu]} + \beta(a_{ij}(x_j - x_i))^{[\nu]}, \tag{10}$$

where $\alpha, \beta \in \mathbb{R}_+$, $\mu \in (0, 1)$, $\nu > 1$ are control parameters and $a_{ij} = a_{ji} \geq 0$, $i, j = 1, 2, \ldots, n$ are elements of the weighted adjacency matrix A.

Theorem 5. Consider system (3) with a connected communication topology, i.e. the graph $\mathcal{G}(A)$ is connected. Given the control protocol

$$u_i = \sum_{j=1}^{n} \phi_{ij}\Big(x_j - x_i\Big), \quad i = 1, 2, \ldots, n, \tag{11}$$

the system (3) solves the average-consensus problem in a finite time, which is globally bounded by $T^1_{\max}$:

$$T^1_{\max} = \frac{2}{\bar{\alpha}(1 - \mu)} + \frac{2}{\bar{\beta}(\nu - 1)}, \quad \forall x_0 \in \mathbb{R}^n, \tag{12}$$

$$\bar{\alpha} = \alpha 2^{\mu}\left(\lambda_*(\mathcal{L}_\mu)\right)^{\frac{\mu+1}{2}}, \quad \bar{\beta} = \beta 2^{\nu} n^{\frac{1-\nu}{2}}\left(\lambda_*(\mathcal{L}_\nu)\right)^{\frac{\nu+1}{2}},$$

where $\alpha, \beta \in \mathbb{R}_+$, $\mu \in (0, 1)$ and $\nu > 1$ are control parameters, $\mathcal{L}_\mu$ and $\mathcal{L}_\nu$ are the Laplacians of the graphs $\mathcal{G}\left(A^{\left[\frac{2\mu}{\mu+1}\right]}\right)$ and $\mathcal{G}\left(A^{\left[\frac{2\nu}{\nu+1}\right]}\right)$, respectively.

Proof. The proof uses on some ideas introduced in Olfati-Saber & Murray (2004) for nonlinear action functions φ_{ij} of general form.

Denote

$$x^* := (1/n)\sum_{i=1}^{n} x_i(0).$$

Introduce a vector $\delta = [\delta_1, \delta_2, \ldots, \delta_n]^\top$ called disagreement, $x_i(t) = x^* + \delta_i(t)$. It is easy to see that $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $x_j - x_i = \delta_j - \delta_i$, for all $i, j = 1, 2, \ldots, n$.

Since the equality (9) holds, then

$$\sum_{i=1}^{n} \dot{x}_i(t) = 0. \tag{13}$$

Hence $(1/n)\sum_{i=1}^n x_i(t) = (1/n)\sum_{i=1}^n x_i(0) = x^*$ and

$$\sum_{i=1}^{n} \delta_i(t) = 0, \quad \forall t > 0. \tag{14}$$

Therefore, average-consensus problem will be implied by stability of the following system

$$\dot{\delta}_i = \sum_{j=1}^{n} \phi_{ij}\Big(\delta_j - \delta_i\Big), \quad i = 1, 2, \ldots, n. \tag{15}$$

Introduce the following Lyapunov function candidate

$$V(\delta) = 0.5\delta^\top\delta = 0.5\sum_{i=1}^{n}\delta_i^2. \tag{16}$$

Its total derivative is

$$\dot V(\delta) = \sum_{i=1}^{n}\delta_i\dot\delta_i = \sum_{i=1}^{n}\delta_i\sum_{j=1}^{n}\phi_{ij}\Big(\delta_j-\delta_i\Big).$$

Taking into account (9) we derive

$$\begin{aligned}\dot V(\delta) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\phi_{ij}\Big(\delta_j-\delta_i\Big)\\ &= 0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big)+0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j\phi_{ji}\Big(\delta_i\text{-}\delta_j\Big)\\ &= 0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big)-0.5\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_j\phi_{ij}\Big(\delta_j\text{-}\delta_i\Big)\\ &= -0.5\sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_j-\delta_i)\phi_{ij}\Big(\delta_j-\delta_i\Big). \end{aligned}\tag{17}$$

Since $s = \mathrm{sign}(s)|s|$ then from (10) and (17) we can easily obtain

$$\begin{aligned}\dot V(\delta) &= \tfrac{1}{2}\sum_{i,j=1}^{n}(\delta_i-\delta_j)\left(\alpha a_{ij}^{\mu}(\delta_j-\delta_i)^{[\mu]}+\beta a_{ij}^{\nu}(\delta_j-\delta_i)^{[\nu]}\right)\\ &= -\tfrac{1}{2}\sum_{i,j=1}^{n}\left(\alpha a_{ij}^{\mu}|\delta_j-\delta_i|^{\mu+1}+\beta a_{ij}^{\nu}|\delta_j-\delta_i|^{\nu+1}\right)\le 0.\end{aligned}$$

Since $1<\mu+1<2,\ \ \nu+1>2$, then using the norm equivalence property

$$\|z\|_l \le \|z\|_r \le n^{\frac{1}{r}-\frac{1}{l}}\|z\|_l$$

for any $z\in\mathbb{R}^n$ and $l>r>0$ ($\|\cdot\|_p$ is defined by (2)), we estimate $\dot V(\delta)$ as follows:

$$\begin{aligned}\dot V(\delta) \le{}& -0.5\alpha\left(\sum_{i,j=1}^{n}a_{ij}^{\frac{2\mu}{\mu+1}}(\delta_j-\delta_i)^2\right)^{\frac{\mu+1}{2}}\\ &-0.5\beta n^{\frac{1-\nu}{2}}\left(\sum_{i,j=1}^{n}a_{ij}^{\frac{2\nu}{\nu+1}}(\delta_j-\delta_i)^2\right)^{\frac{\nu+1}{2}}.\end{aligned}$$

The equality (7) implies that for $\sum_{i=1}^{n}\delta_i=0$ we have

$$\sum_{i,j=1}^{n}a_{ij}^{\frac{2\mu}{\mu+1}}(\delta_j-\delta_i)^2 = 2\delta^\top\mathcal{L}_\mu\delta \ge 4\lambda_*(\mathcal{L}_\mu)V(\delta), \tag{18}$$

$$\sum_{i,j=1}^{n}a_{ij}^{\frac{2\nu}{\nu+1}}(\delta_j-\delta_i)^2 = 2\delta^\top\mathcal{L}_\nu\delta \ge 4\lambda_*(\mathcal{L}_\nu)V(\delta), \tag{19}$$

where $\mathcal{L}_\mu$ and $\mathcal{L}_\nu$ are the Laplacians of a weighted graphs $\mathcal{G}\left(A^{\left[\frac{2\mu}{\mu+1}\right]}\right)$ and $\mathcal{G}\left(A^{\left[\frac{2\nu}{\nu+1}\right]}\right)$, respectively.

Taking into account (18), (19) and (14) we get

$$\begin{aligned}\dot V(\delta(t)) \le{}& -2^{\mu}\alpha(\lambda_*(\mathcal{L}_\mu))^{\frac{\mu+1}{2}}V^{\frac{\mu+1}{2}}\\ &-2^{\nu}\beta n^{\frac{1-\nu}{2}}(\lambda_*(\mathcal{L}_\nu))^{\frac{\nu+1}{2}}V^{\frac{\nu+1}{2}}.\end{aligned}$$

Introduce the following notations: $p=\frac{\mu+1}{2}$, $q=\frac{\nu+1}{2}$, $\bar\alpha = 2^{2p-1}\alpha(\lambda_*(\mathcal{L}_\mu))^p$, $\bar\beta = 2^{2q-1}\beta n^{1-q}(\lambda_*(\mathcal{L}_\nu))^q$.

Then the total derivative of the Lyapunov function calculated along the trajectories of (15) satisfies the following inequality

$$\begin{gathered}\dot V(\delta(t)) \le -\bar\alpha V^p(\delta(t)) - \bar\beta V^q(\delta(t)),\\ \bar\alpha,\bar\beta>0,\quad 0<p<1,\quad q>1,\end{gathered}\tag{20}$$

which by Lemma 3 immediately implies fixed-time stability of the origin of the system (15) with the settling time estimate (12).

The theorem presents quite a conservative settling time estimate, since its proof is based on the results of Lemma 3. A more accurate estimate can be derived with the use of the Lemma 4 as formulated by the next corollary.

Corollary 6. If, under the conditions of Theorem 5, the parameters μ and ν of protocol (11) are chosen as $\mu = 1-\frac{1}{\gamma}$ $\nu = 1+\frac{1}{\gamma}$, $\gamma>1$, then the settling time can be estimated by the following value

$$T_{\max}^2 := \frac{\pi\gamma n^{\frac{1}{4\gamma}}}{2\sqrt{\alpha\beta}\lambda_*^{\frac{1}{2}-\frac{1}{4\gamma}}(\mathcal{L}_\mu)\lambda_*^{\frac{1}{2}+\frac{1}{4\gamma}}(\mathcal{L}_\nu)}. \tag{21}$$

The proof of this corollary immediately follows from inequality (20) and Lemma 4.

The *fixed-time* control protocols have an advantage in convergence rate with respect to both finite-time and linear protocols even in the case real-life applications, which usually deal with the practical stability concept. Due to system uncertainties, exogenous disturbances, delays and measurement noises the practical realization of a control protocol provides only convergence to a zone (a neighborhood of the origin). So, in practice both finite-time and linear protocols have similar qualitative property, namely, they provide finite-time convergence to the neighborhood of the origin. Moreover, in practical applications finite-time algorithms are only locally faster than linear ones. In contrast to this, the *fixed-time* control guarantees convergence to the same zone in a fixed time, which can be prescribed independently of initial conditions. It is faster than finite-time and linear protocols globally and locally. More detailed analysis of the practical fixed-time stability is given in Polyakov (2012).

5.1 Example.

To demonstrate the efficiency of the proposed fixed-time consensus protocol we consider the multi-agent system which consists of 6 agents with integrator dynamics (3) and the interaction topology represented by the following undirected graph

For the same initial conditions

$$x(0) = x_0 := [350, 100, 200, 250, 400, 500]^\top$$

Fig. 1 presents the results of simulations for the linear and the proposed nonlinear control protocols. The values of the

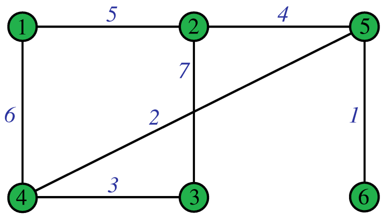

Fig. 1. Structure of the system

parameters of the fixed-time control law (11) were chosen as

$$\gamma = 1.1 \Rightarrow \mu \approx 0.091, \nu \approx 1.909, \quad \alpha = \beta = 1.$$

in order to compare them.

Fig. 2. Trajectories of the system under the linear and nonlinear protocols

The obtained estimates $T^1_{\max}$ and $T^2_{\max}$ confirm the theoretical conclusions of Theorem 5 and Corollary 6 showing fixed-time convergence to average consensus in the nonlinear case. The settling time is seen to be less than the theoretical estimates:

$$T^1_{\max} \approx 3.438, \quad T^2_{\max} \approx 2.548, \tag{22}$$

real convergence time is about $T(x_0) \approx 1.647$. It can be also seen that the second estimate gives much more precise result.

6. CONCLUSIONS

The contribution of the paper is the following:

- a nonlinear control protocol to solve an average-consensus problem is developed;
- it is proved that the guaranteed settling time of the system can be specified in advance regardless of the initial conditions of the agents (fixed-time convergence);
- the two different estimates for the settling time are obtained;
- as opposed to the known finite-time consensus control protocols the designed fixed-time control protocol guaranties the convergence rate faster than the linear one globally;
- the prespecified settling time is shown to be independent of the dimension of the agents.

On the other hand there is one drawback: we considered only the case of undirected topology. This problem is subjected for future research.

The theoretical results were successfully tested through several numerical experiments. The fixed-time stability framework applied in the paper looks promising in different multi-agent problems.

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