Generalized Nevanlinna-Pick interpolation on the boundary. Application to impedance matching

Laurent Baratchart, Martine Olivi, Fabien Seyfert

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L. Baratchart[1], M. Olivi[1] and F. Seyfert[1]

Abstract— **In this work we study a generalized Nevanlinna Pick interpolation problem, where transmission zero locations are imposed. Unlike in other variant of this problem considered by T.T. Giorgiou et al. the interpolation points are chosen on the boundary of the analyticity domain: that is, in our framework, on the real axis. This problem is motivated by important questions in electronic and microwave system design, and it relates to the broadband matching theories of Youla and Helton. An existence and uniqueness theorem is proved. The constructive proof is based on continuation techniques.**

I. INTRODUCTION

Communication systems like multiplexers, routers, power dividers, couplers, antenna receptor chains, are often realized by plugging elementary components together. Among these components, N-port junctions and filters are the most usual. Multiplexers are for example realized by plugging $N-1$ filters, one per channel, on a N-port junction. Filters can therefore be considered as the elementary two-port components, present in most telecommunication devices.

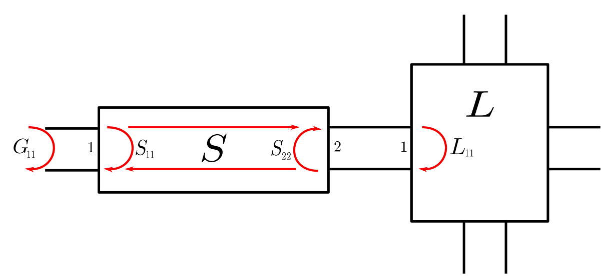

Fig. 1. Filter plugged on a system with reflexion coefficient L_{11}

When plugging filters on an existing system, a recurring question is to determine which frequencies will carry energy into the system and which frequencies will be rejected. In the setting of Figure 1, the system, that can be seen as the filter's load, is characterized by its reflection coefficient L_{11} while the effect of the filter is determined by its 2×2 scattering matrix S. If the filter is considered lossless, that is if

$$S(\omega)^*S(\omega) = Id, \quad \omega \in \mathbb{R}, \tag{1}$$

then the reflection coefficient G_{11} of the overall system (at each frequency ω) is computed as:

$$\begin{aligned} G_{11} &= S_{11} + \frac{S_{12}S_{21}L_{11}}{1-S_{22}L_{11}} \\ &= \frac{S_{11}-L_{11}\det(S)}{1-S_{22}L_{11}} \\ &= \det(S)\frac{S_{22}^*-L_{11}}{1-S_{22}L_{11}}. \end{aligned} \tag{2}$$

Here and below, a pair of lower indices indicates the corresponding entry of a 2×2 matrix. A matching frequency is by definition a frequency ω for which $G_{11}(\omega)=0$, which in view of equation (2), amounts to (if $|L_{11}(\omega)|<1$)

$$S_{22}(\omega) = \overline{L_{11}(\omega)}. \tag{3}$$

On the other hand a stopping frequency ω is characterised by $|G_{11}(\omega)|=1$, and therefore (if $|L_{11}(\omega)|<1$) equivalent to $|S_{22}(\omega)|=1$, that is:

$$S_{12}(\omega) = S_{21}(\omega) = 0. \tag{4}$$

The problem of synthesizing a filter, or a matching network, such that G_{11} is lowest possible on a whole frequency band is a very old one. When the filter is assumed finite-dimensional, it gave rise to the matching theory of Fano and Youla [4]. Provided a rational model is given for the load, this theory provides a parametrization of all possible reflection coefficients G_{11} that can be realized: it accounts, in terms of transmission zeros, for the fact that the rational model of the filter can be "extracted" from the response G. However, the unavailability until now of means to derive matching characteristics from this parametrization, at least when the load is of degree greater than one, partly explains its low impact in practice. The necessity to derive a rational model of the load, and to infer its transmission zeros, might also have contributed to poor dissemination among engineers. Among system manufacturers, brute force optimization along with inherent uncertainties is often used instead. Another approach was proposed by J. Helton [8] in an infinite dimensional setting: the problem is solved there, by reformulating the matching problem into an H^∞ approximation problem, for which Nehary's Hankel operator approach delivers an elegant solution. The convexity gained here for the matching problem, which eventually allows the derivation of an optimal solution, comes however with a price: the infinite order of the derived filter makes it hardly realizable in practice.

Hereafter we propose an intermediate approach where a finite dimensional filter response gets synthesized by imposing matching and stopping frequencies with respect to a frequency varying load.

II. AN INTERPOLATION PROBLEM

For simplicity of notation, we consider here the scattering matrix of a filter as a function of real frequencies. This choice differs from the conventional one, where the transfer function is defined on the imaginary axis rather than the real line. In this framework, assuming the filter is finite dimensional and stable, the scattering matrix is rational with poles in the open upper half-plane $\mathbb{C}^+$. The scattering matrix of a lossless filter is then inner in the lower half plane $\mathbb{C}^-$, which means that it satisfies (1) and is analytic, thus contractive, in $\mathbb{C}^-$.

We denote by $\mathbb{D}$ the open unit disk. For any complex matrix M, M^t denotes its conjugate and M^* its transpose conjugate. For a rational matrix function $F(s)$, we define $F^*(s)$ by

$$F^*(s) = F(\bar{s})^*, \quad s \in \mathbb{C}.$$

If p is a polynomial, then p^* is a polynomial too, of the same degree, and its roots are conjugate with respect to the roots of p.

All rational 2×2 inner matrices S, of McMillan degree N, such that $\lim_{s \to \infty} S(s) = Id$ can be parametrized as follow (Belevitch's form [2]):

$$S = \frac{1}{q} \begin{bmatrix} p^* & (-1)^{N+1} n \\ n^* & (-1)^N p \end{bmatrix} \tag{5}$$

where p, q are monic complex polynomials of degree N, n is a complex polynomial of degree at most $N-1$, and q is computed from p and n as the unique monic stable spectral factor satisfying the Feldtkeller equation:

$$qq^* = pp^* + nn^*. \tag{6}$$

If $\{x_1 \ldots x_N\}$ is a set of frequencies, the matching character of the filter at these frequencies, with respect to a load with reflection coefficient L_{11} amounts to:

$$\frac{p}{q}(x_k) = \overline{L_{11}(x_k)} \overset{def}{=} \gamma_k \tag{7}$$

We will suppose here that $|\gamma_k| < 1$, as the matching problem on a fully reflecting load known at discrete frequencies is not well defined (the expression in (2) is of the form $0/0$ in this case). We assume also it is given a set of $N-1$ stopping frequencies (possibly some at ∞), distinct from the x_k's: in view of (4) this prescribes the roots of the transmission polynomial n of the filter. In addition, we suppose that the leading term of n is given. Under this hypothesis we consider the following matching problem $\mathscr{P}$:

Problem $\mathscr{P}$: Given
- N distinct real frequencies $(x_1, x_2 \ldots x_N)$,
- N interpolation conditions $(\gamma_1, \gamma_2 \ldots \gamma_N)$ in $\mathbb{D}^N$,
- a complex polynomial n of degree $N-1$, such that $n(x_k) \neq 0$ for $k = 1, \ldots, N$,

to find a pair (p, q) of monic complex polynomials of degree N such that,

$$\begin{cases} \frac{p}{q}(x_k) & = & \gamma_k, \quad \text{for } k = 1, .., N \\ qq^* - pp^* & = & nn^* \end{cases} \tag{8}$$

and q is strictly stable (i.e. has all its roots in the open upper half-plane $\mathbb{C}^+$).

This is a variant of the classical Nevanlinna-Pick interpolation problem with degree constraint. In the latter, interpolation points belong to the stability domain while in problem $\mathscr{P}$ they belong to the boundary. The classical problem was first solved in [7] using a topological approach which is constructive. A huge literature exists on this problem, on theoretical developments as well as applications (see [3] and related papers). To our knowledge, problem $\mathscr{P}$ was never addressed. Although it can be viewed as a limit case of the standard problem, techniques used to solve the latter are not applicable here, and new tools are required.

III. MATCHING THEOREM.

We state below the main result of this paper.

Theorem 3.1: If n has all its roots off the real line, then $\mathscr{P}$ has one, and only one, solution.

The proof of the theorem relies on the local invertibility of an evaluation map, that we define in what follows.

Let the polynomial n be given and suppose that p is a monic polynomial of degree N. The polynomial $P(s) = p(s)p^*(s) + n(s)n^*(s)$ is self-reciprocal: $P = P^*$, and it satisfies $P(\omega) > 0$ for all real ω. The Feldkeller equation (6) thus associates with p, a unique monic and stable polynomial $q = q(p)$ by spectral factorization. Here stable means that its roots belong to the *closed* half-plane. But real roots of q are necessarily also shared by p and n, and corresponds to a drop of degree in p/q. Since we assumed that n has no real roots, this situation does not occur and q is in fact strictly stable. It is therefore legitimate to define an evaluation map ψ by

$$\psi: \quad p \to \begin{pmatrix} p(x_1)/q(x_1) \\ \vdots \\ p(x_N)/q(x_N) \end{pmatrix} \tag{9}$$

where $q = q(p)$. The following proposition holds,

Proposition 3.1: Let P_N be the set of all monic polynomials of degree N in $\mathbb{C}^N[X]$, and consider $p \in P_N$. Then:

- the map ψ is well-defined and differentiable on a neighborhood U of p in P_N,
- $D\psi$ has full rank at p.

Proof: The kernel of the differential of ψ is determined by following equations:

$$p(x_j)dq(x_j) - q(x_j)dp(x_j) = 0, \quad j = 1,\dots,N. \tag{10}$$

in which the polynomials dp and dq are related by the relation obtained differentiating (6)

$$pdp^* + p^*dp = qdq^* + q^*dq. \tag{11}$$

Step 1. Let x_j such that $p(x_j) \neq 0$. Computing (11) at x_j, and using the strict contractivity of p/q at this point we get that,

$$\frac{dp}{p}(x_j) = \frac{dq}{q}(x_j),$$

and is pure imaginary. Thus, for $j = 1,\dots,N$, we have

$$\begin{array}{l} dp(x_j) = i\alpha_j\, p(x_j) \\ dq(x_j) = i\alpha_j\, q(x_j) \end{array} \tag{12}$$

for some $\alpha_j \in \mathbb{R}$. Now if $p(x_j) = 0$, then(10) yields that $dp(x_j) = 0$ which ensures that (12) still holds.

Step 2. Considering that degree of dp and dq is at most $N-1$, the N interpolation conditions (12) allow us to write:

$$dp(t) = \sum_{j=1}^{N} L_{x_j}(t)p(x_j)\alpha_j i \tag{13}$$

$$dq(t) = \sum_{j=1}^{N} L_{x_j}(t)q(x_j)\alpha_j i \tag{14}$$

where the L_{x_j} are the Lagrange interpolation polynomials of the set $(x_1,\dots x_N)$.

Using the above parametrization of dp and dq, equation (11) can be rewritten as

$$\sum_{j=1}^{N} \beta_j(p_{x_j} - q_{x_j}) = 0 \tag{15}$$

where β_j is a real number,

$$\beta_j = \frac{\alpha_j}{\prod_{k\neq j}(x_j - x_k)},$$

and p_{x_j} and q_{x_j} are two families of polynomials with real coefficients and degree at most $N-1$, defined by

$$\begin{array}{ll} p_{x_j}(t) = \dfrac{1}{i}\dfrac{p(t)p^*(x_j) - p^*(t)p(x_j)}{(t - x_j)}, & j = 1,\dots,N \\ q_{x_j}(t) = \dfrac{1}{i}\dfrac{q(t)q^*(x_j) - q^*(t)q(x_j)}{(t - x_j)}, & j = 1,\dots,N. \end{array} \tag{16}$$

Equation (15) is equivalent to the following linear system of equations with unknowns β_j:

$$\sum_{j=1}^{N} \beta_j(q_{x_j}(x_k) - p_{x_j}(x_k)) = 0, k = 1\dots N. \tag{17}$$

The matrix $A = [a_{k,j}]$ of this system is given by

$$\begin{array}{rcl} a_{k,j} & = & 2\Re\left(\dfrac{(q(x_k)q^*(x_j) - p(x_k)p^*(x_j))}{i(x_k - x_i)}\right), \quad k \neq j \\ a_{k,k} & = & 2\Re\left(\dfrac{q'(x_k)q^*(x_k) - p(x_k)'p^*(x_k)}{i}\right). \end{array}$$

By inspection, the matrix $A = [a_{k,j}]$ is symmetric. Our objective is now to prove that A is positive definite using classical Nevanlinna-Pick theory.

Step 3. Let r be a stable spectral factor such that $rr^* = nn^*$, and consider the inner 2×2 extension S of p/q, namely:

$$\check{S} = \frac{1}{q}\begin{pmatrix} p & -r^* \\ r & p^* \end{pmatrix}. \tag{18}$$

Consider the Pick matrix $L = [L_{k,j}]$ associated with the tangential interpolation problem for $\check{S}$, at the points x_j, $j = 1,\dots,N$, in direction $\begin{bmatrix}1 & 0\end{bmatrix}^t$ which corresponds to the first column of $\check{S}$. This is an interpolation problem on the boundary and the Pick matrix can be computed as the limit of a classical Pick matrix, when the interpolation points tend non-tangentially to the boundary. The diagonal elements tend to the angular derivatives of $\check{S}$ at the interpolation points. For details on boundary interpolation see [1, chap. 21]. In our case, we get

$$L_{k,j} = \frac{1 - \frac{p}{q}(x_k)\frac{p^*}{q^*}(x_j) - \frac{r}{q}(x_k)\frac{r^*}{q^*}(x_j)}{i(x_j - x_k)}, \quad k \neq j \tag{19}$$

$$L_{k,k} = \frac{q'(x_k)q^*(x_k) - p'(x_k)p^*(x_k) - r'(x_k)r^*(x_k)}{i\,q(x_k)q^*(x_k)}. \tag{20}$$

Then consider the Pick matrix H associated with the tangential interpolation problem for $\check{S}^t$, at the points x_j, $j = 1,\dots,N$, in direction $\begin{bmatrix}1 & 0\end{bmatrix}^t$. It corresponds to interpolation for the first row of $\check{S}$. The Pick matrix H is given by,

$$H_{k,j} = \frac{1 - \frac{p}{q}(x_k)\frac{p^*}{q^*}(x_j) - \frac{r^*}{q}(x_k)\frac{r}{q^*}(x_j)}{i(x_j - x_k)}, \quad k \neq j \tag{21}$$

$$H_{k,k} = \frac{q'(x_k)q^*(x_k) - p'(x_k)p^*(x_k) - (r^*)'(x_k)r(x_k)}{i\,q(x_k)q^*(x_k)}. \tag{22}$$

Now, let Λ be the diagonal (non singular) matrix defined by,

$$\Lambda = \mathrm{diag}(q(x_1),\dots,q(x_N)).$$

It can be verified that

$$A = \Lambda L \Lambda^* + \Lambda^* H^t \Lambda. \tag{23}$$

Step 4. We prove that H and thus A is positive definite. This is ensured by the following theorem

Theorem 3.2: Let F a $m \times p$ inner rational matrix (analytic in $\mathbb{C}^-$). Let $\xi \in \mathbb{C}^p$ be a unit vector, and suppose that the column $F(s)\xi$ has McMillan degree N. Let $x_1, \cdots, x_N$ be N distinct real points. Then, the Pick matrix P given by

- $k \neq j, \quad P_{k,j} = \frac{1-\xi^* F(x_k)^* F(x_j)\xi}{i(x_j - x_k)}$
- $k = j, \quad P_{k,k} = \xi^* F(x_k)^* F(x_k)' \xi,$

which corresponds to the interpolation problem along ξ of the matrix F, is strictly positive.

The proof, which relies on a limiting process and on the reproducing kernel approach developped in [5] is omitted. As the first column of $\check{S}^t$ has McMillan degree N (as r^* and q have no common factor), Theorem 3.2 applies, which concludes the proof of Proposition 3.1. ■

Now, as the polynomial n has no roots on the real line, ψ defines a local homomorphism from $P_N \to \mathbb{D}^N$. The fiber $\psi^{-1}(0)$ is a singleton, namely the monic polynomial of degree N having as roots the $x_k's$.

Recall that a local homeomorphism from a topological space X to a topological space Y is called proper if the preimage of every compact set of Y is compact. The remark that:

$$|p(x_k)|^2 = \frac{|\gamma_k|^2}{1-|\gamma_k|^2}|n(x_k)|^2$$

shows that the preimage of every compact set of $\mathbb{D}^N$ is bounded in P_N and therefore compact, so that ψ is proper. We then have

Theorem 3.3: Let X and Y be two open sets in $\mathbb{R}^n$, with Y connected. Let $\phi : X \to Y$ be a proper local homomorphism. Suppose that there exists $y_0 \in Y$ such that the fiber $\phi^{-1}(y_0)$ is a singleton $\{x_0\}$, then ϕ is a global homomorphism from X onto Y.

Proof: As ϕ is proper and a local homeorphism, ϕ is a covering map (see [6, Th.4.22]). The path connectedness of Y implies that all fibers $\phi^{-1}(y)$, with $y \in Y$, have the same cardinality. From $\phi^{-1}(y_0) = x_0$ we conclude that every element $y \in Y$ has one, and only one preimage. ■

An application of Theorem 3.3 to ψ completes the proof of Theorem 3.1.

Remark 3.1: The case where n has roots on the real line is of considerable practical interest. To extend our result to this case, the idea is to restrict the definition domain of ψ to the set of polynomials p that have no common real roots with n and to adapt the rest of the proof. We leave this point for future works.

IV. CONCLUSION

In this paper, we considered a variant of the Nevanlinna-Pick interpolation problem with degree constraint. We have proved that existence and uniqueness of the solution still holds when interpolation takes place on the boundary of the stability domain. Moreover, our proof is constructive. Such techniques should be useful for microwave antenna and multiplexer design.

In fact, the design of a multiplexer requires solving a more complex of simultaneously matching several filters connected to a common access. The following heuristics can be used:

- matching frequencies are chosen as the reflection zeros of a single filter
- rejection frequencies are chosen so as to minimize cross-talk between channels
- each filter is adapted alternatively, solving for $\mathscr{P}$.

Although preliminary numerical results obtained by this method seem promising, the existence of a fixed point has not been proven yet. This remains an open issue whose solution would have far-reaching implications in the area of microwave multi-port design.

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