The stochastic porous media equation in ^{d}

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Stochastic Porous Media Equations in $\mathbb{R}^d$
Equations de milieux poreux dans $\mathbb{R}^d$

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Abstract

Existence and uniqueness of solutions to the stochastic porous media equation $dX - \Delta\psi(X)dt = XdW$ in $\mathbb{R}^d$ are studied. Here, W is a Wiener process, ψ is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ such that $\psi(r) \le C|r|^m$, $\forall r \in \mathbb{R}$. In this general case, the dimension is restricted to $d \ge 3$, the main reason being the absence of a convenient multiplier result in the space $\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d);\ |\xi|(\mathcal{F}\varphi)(\xi) \in L^2(\mathbb{R}^d)\}$, for $d \le 2$. When ψ is Lipschitz, the well-posedness, however, holds for all dimensions on the classical Sobolev space $H^{-1}(\mathbb{R}^d)$. If $\psi(r)r \ge \rho|r|^{m+1}$ and $m = \frac{d-2}{d+2}$, we prove the finite time extinction with strictly positive probability.

Résumé

Nous étudions existence et unicité pour les solutions d'une équation de milieux poreux $dX - \Delta\psi(X)dt = XdW$ dans $\mathbb{R}^d$. Ici W est un processus de Wiener, ψ est un graphe maximal monotone dans $\mathbb{R}\times\mathbb{R}$ tel que $\psi(r) \le C|r|^m$, $\forall r \in \mathbb{R}$. Dans ce contexte général, la dimension est restreinte à $d \ge 3$, essentiellement compte tenu de l'absence d'un résultat adéquat de multiplication dans l'espace $\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d);\ |\xi|(\mathcal{F}\varphi)(\xi) \in L^2(\mathbb{R}^d)\}$, pour $d \le 2$. Lorsque ψ est Lipschitz, le problème est néanmoins bien posé pour toute dimension dans l'espace de Sobolev classique $H^{-1}(\mathbb{R}^d)$. Si $\psi(r)r \ge \rho|r|^{m+1}$ et $m = \frac{d-2}{d+2}$, nous prouvons une propriété d' extinction en temps fini avec probabilité strictement positive.

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1. Introduction

Consider the stochastic porous media equation

$$\begin{aligned} &dX - \Delta\psi(X)dt = XdW \text{ in } (0,T)\times\mathbb{R}^d,\\ &X(0) = x \text{ on } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where ψ is a monotonically nondecreasing function on $\mathbb{R}$ (eventually multivalued) and $W(t)$ is a Wiener process of the form

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$$W(t) = \sum_{k=1}^{\infty} \mu_k e_k \beta_k(t),\ t \geq 0. \tag{1.2}$$

Here $\{\beta_k\}_{k=1}^{\infty}$ are independent Brownian motions on a stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, $\mu_k \in \mathbb{R}$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in $H^{-1}(\mathbb{R}^d)$ or $\mathcal{H}^{-1}$ (see (2.2) below) to be made precise later on.

On bounded domains $\mathcal{O} \subset \mathbb{R}^d$ with Dirichlet homogeneous boundary conditions, equation (1.1) was studied in [3], [4], [5], under general assumptions on $\psi : \mathbb{R} \to \overline{\mathbb{R}}$ (namely, maximal monotone multivalued graph with polynomial growth, or even more general growth conditions in [4]). It should be said, however, that there is a principial difference between bounded and unbounded domains, mainly due to the multiplier problem in Sobolev spaces on $\mathbb{R}^d$. If $d \geq 3$ and $\mathcal{O} = \mathbb{R}^d$, existence and uniqueness of solutions to (1.1) was proved in [21] (see, also, [23]) in a general setting which covers the case $\mathcal{O} = \mathbb{R}^d$ (see Theorems 3.9, Proposition 3.1 and Example 3.4 in [21]). However, it should be said that in [21] ψ is assumed continuous, such that $r\psi(r) \to \infty$ as $r \to \infty$, which we do not need in this paper.

We study the existence and uniqueness of (1.1) under two different sets of conditions requiring a different functional approach. The first one, which will be presented in Section 3, assumes that ψ is monotonically nondecreasing and Lipschitz. The state space for (1.1) is, in this case, $H^{-1}(\mathbb{R}^d)$, that is, the dual of the classical Sobolev space $H^1(\mathbb{R}^d)$. In spite of the apparent lack of generality (ψ Lipschitz), it should be mentioned that there are physical models described by such an equation as, for instance, the two phase Stefan transition problem perturbed by a stochastic Gaussian noise [2]; moreover, in this latter case there is no restriction on the dimension d.

The second case, which will be studied in Section 4, is that where ψ is a maximal monotone multivalued function with at most polynomial growth. An important physical problem covered by this case is the self-organized criticality model

$$dX - \Delta H(X - X_c)dt = (X - X_c)dW, \tag{1.3}$$

where H is the Heaviside function and X_c is the critical state (see [5], [6], [8]). More generally, this equation with discontinuous ψ covers the stochastic nonlinear diffusion equation with singular diffusivity $D(u) = \psi'(u)$.

It should be mentioned that, in this second case, the solution $X(t)$ to (1.1) is defined in a certain distribution space $\mathcal{H}^{-1}$ (see (2.2) below) on $\mathbb{R}^d$ and the existence is obtained for $d \geq 3$ only, as in the case of continuous ψ in [21]. The case $1 \leq d \leq 2$ remains open due to the absence of a multiplier rule in the norm $\|\cdot\|_{\mathcal{H}^{-1}}$ (see Lemma 4.1 below).

In Section 5, we prove the finite time extinction of the solution X to (1.1) with strictly positive probability under the assumption that $\psi(r)r \geq \rho|r|^{m+1}$ and $m = \frac{d-2}{d+2}\cdot$

Finally, we would like to comment on one type of noise. Existence and uniqueness can be proved with $g(t, X(t))$ by replacing $X(t)$ under (more or less the usual) abstract conditions on σ (see, e.g., [21], [23]). The main reason why in this paper we restrict ourselves to linear multiplicative noise is that first we want to be concrete, second the latter case is somehow generic (just think of taking the Taylor expansion of $\sigma(t, \cdot)$ up to first order), and third for this type of noise we prove finite time extinction in Section 5.

2. Preliminaries

To begin with, let us briefly recall a few definitions pertaining distribution spaces on $\mathbb{R}^d$, whose classical Euclidean norm will be denoted by $|\cdot|$.

Denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of all temperate distributions on $\mathbb{R}^d$ (see, e.g., [18]) and by $\mathcal{H}$ the space

$$\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d);\ \xi \mapsto |\xi|\mathcal{F}(\varphi)(\xi) \in L^2(\mathbb{R}^d)\}, \tag{2.1}$$

where $\mathcal{F}(\varphi)$ is the Fourier transform of φ. We denote by $L^2(\mathbb{R}^d)$ the space of square integrable functions on $\mathbb{R}^d$ with norm $|\cdot|_2$ and scalar product $\langle\cdot,\cdot\rangle_2$. In general $|\cdot|_p$ will denote the norm of $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d;\mathbb{R}^d)$, $1 \le p \le \infty$. The dual space $\mathcal{H}^{-1}$ of $\mathcal{H}$ is given by

$$\mathcal{H}^{-1} = \{\eta \in \mathcal{S}'(\mathbb{R}^d);\ \xi \mapsto \mathcal{F}(\eta)(\xi)|\xi|^{-1} \in L^2(\mathbb{R}^d)\}. \tag{2.2}$$

The duality between $\mathcal{H}$ and $\mathcal{H}^{-1}$ is denoted by $\langle\cdot,\cdot\rangle$ and is given by

$$\langle\varphi,\eta\rangle = \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(\xi)\overline{\mathcal{F}(\eta)}(\xi)d\xi \tag{2.3}$$

and the norm of $\mathcal{H}$ denoted by $\|\cdot\|_1$ is given by

$$\|\varphi\|_1 = \left(\int_{\mathbb{R}^d} |\mathcal{F}(\varphi)(\xi)|^2|\xi|^2 d\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^d} |\nabla\varphi|^2 d\xi\right)^{\frac{1}{2}}. \tag{2.4}$$

The norm of $\mathcal{H}^{-1}$, denoted by $\|\cdot\|_{-1}$ is given by

$$\|\eta\|_{-1} = \left(\int_{\mathbb{R}^d} |\xi|^{-2}|\mathcal{F}(\eta)(\xi)|^2 d\xi\right)^{\frac{1}{2}} = \left(\left\langle(-\Delta)^{-1}\eta,\eta\right\rangle\right)^{\frac{1}{2}}. \tag{2.5}$$

(We note that the operator $-\Delta$ is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}^{-1}$.) The scalar product of $\mathcal{H}^{-1}$ is given by

$$\langle\eta_1,\eta_2\rangle_{-1} = \left\langle(-\Delta)^{-1}\eta_1,\eta_2\right\rangle. \tag{2.6}$$

As regards the relationship of $\mathcal{H}$ with the space $L^p(\mathbb{R}^d)$ of p-summable functions on $\mathbb{R}^d$, we have the following.

Lemma 2.1. Let $d \ge 3$. Then we have

$$\mathcal{H} \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d) \tag{2.7}$$

algebraically and topologically.

Indeed, by the Sobolev embedding theorem (see, e.g., [15], p. 278), we have

$$|\varphi|_{\frac{2d}{d-2}} \le C|\nabla\varphi|_2,\ \forall\varphi \in C_0^\infty(\mathbb{R}^d),$$

and, by density, this implies (2.7), as claimed.

It should be mentioned that (2.7) is no longer true for $1 \le d \le 2$. However, by duality, we have

$$L^{\frac{2d}{d+2}}(\mathbb{R}^d) \subset \mathcal{H}^{-1},\ \forall d \ge 3. \tag{2.8}$$

Denote by $H^1(\mathbb{R}^d)$ the Sobolev space

$$\begin{aligned} H^1(\mathbb{R}^d) &= \{u \in L^2(\mathbb{R}^d);\ \nabla u \in L^2(\mathbb{R}^d)\} \\ &= \{u \in L^2(\mathbb{R}^d);\ \xi \mapsto \mathcal{F}(u)(\xi)(1+|\xi|^2)^{\frac{1}{2}} \in L^2(\mathbb{R}^d)\} \end{aligned}$$

with norm

$$|u|_{H^1(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (u^2 + |\nabla u|^2)d\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2(1+|\xi|^2)d\xi\right)^{\frac{1}{2}}$$

and by $H^{-1}(\mathbb{R}^d)$ its dual, that is,

$$H^{-1}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d);\ \mathcal{F}(u)(\xi)(1+|\xi|^2)^{-\frac{1}{2}} \in L^2(\mathbb{R}^d)\}.$$

The norm of $H^{-1}(\mathbb{R}^d)$ is denoted by $|\cdot|_{-1}$ and its scalar product by $\langle\cdot,\cdot\rangle_{-1}$. We have the continuous and dense embeddings

$$H^1(\mathbb{R}^d) \subset \mathcal{H},\ \mathcal{H}^{-1} \subset H^{-1}(\mathbb{R}^d).$$

It should be emphasized, however, that $\mathcal{H}$ is not a subspace of $L^2(\mathbb{R}^d)$ and so $L^2(\mathbb{R}^d)$ is not the pivot space in the duality $\langle\cdot,\cdot\rangle$ given by (2.3).

Given a Banach space Y, we denote by $L^p(0,T;Y)$ the space of all Y-valued p-integrable functions on $(0,T)$ and by $C([0,T];Y)$ the space of continuous Y-valued functions on $[0,T]$. For two Hilbert spaces H_1, H_2 let $L(H_1,H_2)$ and $L_2(H_1,H_2)$ denote the set of all bounded linear and Hilbert-Schmidt operators, respectively. We refer to [17], [20] for definitions and basic results pertaining infinite dimensional stochastic processes.

3. Equation (1.1) with the Lipschitzian ψ

Consider here equation (1.1) under the following conditions.

(i) $\psi : \mathbb{R} \to \mathbb{R}$ is monotonically nondecreasing, Lipschitz such that $\psi(0) = 0$.

(ii) W is a Wiener process as in (1.2), where $e_k \in H^1(\mathbb{R}^d)$, such that

$$C_\infty^2 := 36 \sum_{k=1}^{\infty} \mu_k^2 (|\nabla e_k|_\infty^2 + |e_k|_\infty^2 + 1) < \infty, \tag{3.1}$$

and $\{e_k\}$ is an orthonormal basis in $H^{-1}(\mathbb{R}^d)$.

We insert the factor 36 for convenience here to avoid additional large numerical constants in subsequent estimates.

Remark 3.1. By Lemma 4.1 below, $|\nabla e_k|_\infty$ in (3.1) can be replaced by $|\nabla e_k|_d$, and all the results in this section remain true.

Definition 3.2. Let $x \in H^{-1}(\mathbb{R}^d)$. A continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted process $X : [0,T] \to H^{-1}(\mathbb{R}^d)$ is called strong solution to (1.1) if the following conditions hold:

$$X \in L^2(\Omega; C([0,T]; H^{-1}(\mathbb{R}^d))) \cap L^2([0,T]\times\Omega; L^2(\mathbb{R}^d)) \tag{3.2}$$

$$\int_0^\bullet \psi(X(s))ds \in C([0,T]; H^1(\mathbb{R}^d)),\ \mathbb{P}\text{-a.s.} \tag{3.3}$$

$$X(t) - \Delta \int_0^t \psi(X(s))ds = x + \int_0^t X(s)dW(s),\ \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{3.4}$$

Remark 3.3. The stochastic (Itô-) integral in (3.4) is the standard one from [17] or [20]. In fact, in the terminology of these references, W is a Q-Wiener process W^Q on H^{-1}, where $Q : H^{-1} \to H^{-1}$ is the symmetric trace class operator defined by

$$Qh := \sum_{k=1}^{\infty} \mu_k^2 \langle e_k, h\rangle_{-1}\, e_k,\ h \in H^{-1}.$$

For $x \in H^{-1}$, define $\sigma(x) : Q^{1/2}H^{-1} \to H^{-1}$ by

$$\sigma(x)(Q^{1/2}h) = \sum_{k=1}^{\infty}(\mu_k \langle e_k, h\rangle_{-1}\, e_k \cdot x),\ h \in H. \tag{3.5}$$

By (3.1), each e_k is an H^{-1}-multiplier such that

$$|e_k \cdot x|_{-1} \le 2\,(|e_k|_\infty + |\nabla e_k|_\infty)\,|x|_{-1},\ x \in H^{-1}. \tag{3.6}$$

Hence, for all $x \in H^{-1},\ h \in H^{-1}$,

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu_k \langle e_k, h\rangle_{-1}\, e_k x|_{-1} &\le \left(\sum_{k=1}^{\infty} \mu_k^2 |e_k x|_{-1}^2\right)^{1/2} |h|_{-1}\\ &\le 2C_\infty |x|_{-1}|h|_{-1}\\ &= 2C_\infty |x|_{-1}|Q^{1/2}h|_{Q^{1/2}H^{-1}},\end{aligned}$$

and thus $\sigma(x)$ is well-defined and an element in $L(Q^{1/2}H^{-1}, H^{-1})$. Moreover, for $x \in H^{-1}$, by (3.5), (3.6),

$$\begin{aligned}\|\sigma(x)\|^2_{L_2(Q^{1/2}H^{-1},H^{-1})} &= \sum_{k=1}^{\infty} |\sigma(x)(Q^{1/2}e_k)|_{-1}^2 = \sum_{k=1}^{\infty} |\mu_k e_k x|_{-1}^2\\ &= \sum_{k=1}^{\infty} \mu_k^2 |e_k x|_{-1}^2 \le C_\infty^2 |x|_{-1}^2.\end{aligned} \tag{3.7}$$

Since $\{Q^{1/2}e_k \mid k \in \mathbb{N}\}$ is an orthonormal basis of $Q^{1/2}H^{-1}$, it follows that $\sigma(x) \in L_2(Q^{1/2}H^{-1}, H^{-1})$ and the map $x \mapsto \sigma(x)$ is linear and continuous (hence Lipschitz) from H^{-1} to $L_2(Q^{1/2}H^{-1}, H^{-1})$. Hence (e.g., according to [20, Section 2.3])

$$\int_0^t X(s)dW(s) := \int_0^t \sigma(X(s))dW^Q(s),\ t \in [0,T],$$

is well-defined as a continuous H^{-1}-valued martingale and by Itô's isometry and (3.7)

$$\begin{aligned}\mathbb{E}\left|\int_0^t X(s)dW(s)\right|_{-1}^2 &= \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E}\int_0^t |X(s)e_k|_{-1}^2 ds\\ &\le C_\infty^2 \mathbb{E}\int_0^t |X(s)|_{-1}^2 ds,\ t \in [0,T].\end{aligned} \tag{3.8}$$

Furthermore, it follows that

$$\begin{aligned}\int_0^t X(s)dW(s) &= \sum_{k=1}^{\infty} \int_0^t \sigma(X(s))(Q^{1/2}e_k)d\beta_k(s)\\ &= \sum_{k=1}^{\infty} \int_0^t \mu_k e_k X(s) d\beta_k(s),\ t \in [0,T],\end{aligned} \tag{3.9}$$

where the series converges in $L^2(\Omega; C([0,T]; H^{-1}))$.

In fact, $\int_0^\bullet X(s)dW(s)$ is a continuous L^2-valued martingale, because $X \in L^2([0,T] \times \Omega; L^2(\mathbb{R}^d))$ and, analogously to (3.7), we get

$$\|\sigma(x)\|^2_{L_2(Q^{1/2}H^{-1},L^2)} \le C_\infty^2 |x|_2^2,\ \ x \in L^2(\mathbb{R}^d).$$

In particular, by Itô's isometry,

$$\mathbb{E}\left|\int_0^t X(s)dW(s)\right|_2^2 \le C_\infty^2 \mathbb{E}\int_0^t |X(s)|_2^2 ds,\ t\in[0,T].$$

Furthermore, the series in (3.8) even converges in $L^2(\Omega; C([0,T]; L^2(\mathbb{R}^d)))$.

We shall use the facts presented in this remark throughout this paper without further notice.

Theorem 3.4. Let $d \ge 1$ and $x \in L^2(\mathbb{R}^d)$. Then, under assumptions (i), (ii), there is a unique strong solution to equation (1.1). This solution satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X(t)|_2^2\right] \le 2|x|_2^2 e^{3C_\infty^2 t}.$$

In particular, $X \in L^2(\Omega; L^\infty([0,T]; L^2(\mathbb{R}^d)))$. Assume further that

$$\psi(r)r \ge \alpha r^2,\ \forall r\in\mathbb{R}, \tag{3.10}$$

where $\alpha > 0$. Then, there is a unique strong solution X to (1.1) for all $x \in H^{-1}(\mathbb{R}^d)$.

Proof of Theorem 3.4. We approximate (1.1) by

$$\begin{aligned} &dX + (\nu-\Delta)\psi(X)dt = XdW(t),\ t\in(0,T),\\ &X(0) = x \ \text{ on } \ \mathbb{R}^d, \end{aligned} \tag{3.11}$$

where $\nu \in (0,1)$. We have the following.

Lemma 3.5. Assume that ψ is as in assumption (i). Let $x \in L^2(\mathbb{R}^d)$. Then, there is a unique $(\mathcal{F}_t)_{t\ge0}$-adapted solution $X = X^\nu$ to (3.11) in the following strong sense:

$$X^\nu \in L^2(\Omega, C([0,T]; H^{-1}(\mathbb{R}^d))) \cap L^2([0,T]\times\Omega; L^2(\mathbb{R}^d)), \tag{3.12}$$

and $\mathbb{P}$-a.s.

$$X^\nu(t) = x + (\Delta-\nu)\int_0^t \psi(X^\nu(s))ds + \int_0^t X^\nu(s)dW(s),\ t\in[0,T]. \tag{3.13}$$

In addition, for all $\nu \in (0,1)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^\nu(t)|_2^2\right] \le 2|x|_2^2 e^{3C_\infty^2 T}. \tag{3.14}$$

If, moreover, ψ satisfies (3.10), then for each $x \in H^{-1}(\mathbb{R}^d)$ there is a unique solution X^ν satisfying (3.12), (3.13).

Proof of Lemma 3.5. Let us start with the second part of the assertion, i.e., we assume that ψ satisfies (3.10) and that $x \in H^{-1}(\mathbb{R}^d)$. Then the standard theory (see, e.g., [20, Sections 4.1 and 4.2]) applies to ensure that there exists a unique solution X^ν taking value in $H^{-1}(\mathbb{R}^d)$ satisfying (3.12), (3.13) above. Indeed, it is easy to check that (H1)–(H4) from [20, Section 4.1] are satisfied with $V := L^2(\mathbb{R}^d)$, $H := H^{-1}(\mathbb{R}^d)$, $Au := (\Delta-\nu)(\psi(u))$, $u \in V$, and $H^{-1}(\mathbb{R}^d)$ is equipped with the equivalent norm

$$|\eta|_{-1,\nu} := \left\langle \eta, (\nu-\Delta)^{-1}\eta\right\rangle^{1/2},\ \eta\in H^{-1}(\mathbb{R}^d),$$

(in which case, we also write H_ν^{-1}). Here, as before, we use $\langle\cdot,\cdot\rangle$ also to denote the dualization between $H^1(\mathbb{R}^d)$ and $H^{-1}(\mathbb{R}^d)$. For details, we refer to the calculations in [20, Example 4.1.11], which because $p = 2$ go through when the bounded domain Ω there is replaced by $\mathbb{R}^d$. Hence [20, Theorem 4.2.4] applies to give the above solution X^ν.

In the case when ψ does not satisfy (3.10), the above conditions (H1), (H2), (H4) from [20] still hold, but (H3) not in general. Therefore, we replace ψ by $\psi+\lambda I$, $\lambda \in (0,1)$, and thus consider $A_\lambda(u) := (\Delta-\nu)(\psi(u)+\lambda u)$, $u \in V := L^2(\mathbb{R}^d)$ and, as above, by [20, Theorem 4.2.4], obtain a solution X_λ^ν, satisfying (3.12), (3.13), to

$$\begin{aligned}&dX_\lambda^\nu(t) + (\nu-\Delta)(\psi(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t))dt = X_\lambda^\nu(t)dW(t),\ t \in [0,T],\\ &X_\lambda^\nu(0) = x \in H^{-1}(\mathbb{R}^d).\end{aligned} \tag{3.15}$$

In particular, by (3.12),

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda^\nu(t)|_{-1}^2\right] < \infty. \tag{3.16}$$

We want to let $\lambda \to 0$ to obtain a solution to (3.11). To this end, in this case (i.e., without assuming (3.10)), we assume from now on that $x \in L^2(\mathbb{R}^d)$. The reason is that we need the following.

Claim 1. *We have $X_\lambda^\nu \in L^2([0,T]\times\Omega; H^1(\mathbb{R}^d))$ and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda^\nu(t)|_2^2\right] + 4\lambda\mathbb{E}\int_0^T |\nabla X_\lambda^\nu(s)|_2^2 ds \le 2|x|_2^2 e^{3C_\infty^2 T},$$

$$\textit{for all } \nu, \lambda \in (0,1).$$

Furthermore, X_λ^ν has continuous sample paths in $L^2(\mathbb{R}^d)$, $\mathbb{P}$-a.s.

Proof of Claim 1. We know that

$$X_\lambda^\nu(t) = x + (\Delta-\nu)\int_0^t (\psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s))ds + \int_0^t X_\lambda^\nu(s)dW(s),\ t \in [0,T]. \tag{3.17}$$

Let $\alpha \in (\nu,\infty)$. Recalling that $(\alpha-\Delta)^{-\frac12} : H^{-1}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and applying this operator to the above equation, we find

$$\begin{aligned}&(\alpha-\Delta)^{-\frac12}X_\lambda^\nu(t)\\ &= (\alpha-\Delta)^{-\frac12}x + \int_0^t (\Delta-\nu)(\alpha-\Delta)^{-\frac12}(\psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s))ds\\ &+ \int_0^t (\alpha-\Delta)^{-\frac12}\sigma(X_\lambda^\nu(s))Q^{1/2}dW(s),\ t \in [0,T].\end{aligned} \tag{3.18}$$

Applying Itô's formula (see, e.g., [20, Theorem 4.2.5] with $H = L^2(\mathbb{R}^d)$) to $|(\alpha-\Delta)^{-\frac12}X_\lambda^\nu(t)|_2^2$, we obtain, for $t \in [0,T]$,

$$
\begin{aligned}
&|(\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(t)|_2^2 = |(\alpha-\Delta)^{-\frac{1}{2}}x|_2^2\\
&+2\int_0^t \left\langle (\Delta-\nu)(\alpha-\Delta)^{-\frac{1}{2}}\psi(X_\lambda^\nu(s)),(\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(s)\right\rangle ds\\
&-2\lambda\int_0^t (|\nabla((\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(s))|_2^2+\nu|(\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(s)|_2^2)ds\\
&+\int_0^t \|(\alpha-\Delta)^{-\frac{1}{2}}\sigma(X_\lambda^\nu(s))Q^{1/2}\|^2_{L_2(H^{-1},L^2)}ds\\
&+2\int_0^t \left\langle (\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(s),(\alpha-\Delta)^{-\frac{1}{2}}\sigma(X_\lambda^\nu(s))Q^{1/2}dW(s)\right\rangle_2.
\end{aligned}
\tag{3.19}
$$

But, for $f\in L^2(\mathbb{R}^d)$, we have

$$(\alpha-\Delta)^{-\frac{1}{2}}(\Delta-\nu)(\alpha-\Delta)^{-\frac{1}{2}}f=(P-I)f,$$

where

$$P:=(\alpha-\nu)(\alpha-\Delta)^{-1}.$$

For the Green function g_α of $(\alpha-\Delta)$, we then have, for $f\in L^2(\mathbb{R}^d)$,

$$Pf=(\alpha-\nu)\int_{\mathbb{R}^d} f(\xi)g_\alpha(\cdot,\xi)d\xi.$$

Hence, by [23, Lemma 5.1], the integrand of the second term on the right-hand side of (3.19) with $f:=X_\lambda^\nu(s)$ ($\in L^2(\mathbb{R}^d)$ for ds-a.e. $s\in[0,T]$) can be rewritten as

$$
\begin{aligned}
\langle\psi(f),(P-I)f\rangle_2 \;=\; & -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\psi(f(\widetilde{\xi}))-\psi(f(\xi))][f(\widetilde{\xi})-f(\xi)]g_\alpha(\xi,\widetilde{\xi})d\widetilde{\xi}\,d\xi\\
&-\int_{\mathbb{R}^d}(1-P1(\xi))\cdot\psi(f(\xi))f(\xi)d\xi.
\end{aligned}
$$

Since ψ is monotone, $\psi(0)=0$ and $P1\le 1$, we deduce that

$$\langle\psi(f),(P-I)f\rangle\le 0.$$

Hence, after a multiplication by α, (3.19) implies that, for all $t\in[0,T]$ (see Remark 3.3),

$$
\begin{aligned}
&\alpha|(\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(t)|_2^2+2\lambda\int_0^t|\nabla(\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}}X_\lambda^\nu(s))|_2^2ds\\
&\le \alpha|(\alpha-\Delta)^{-\frac{1}{2}}x|_2^2+\int_0^t\sum_{k=1}^\infty\mu_k^2\left\langle\alpha(\alpha-\Delta)^{-1}(e_kX_\lambda^\nu(s)),e_kX_\lambda^\nu(s)\right\rangle_2ds\\
&+2\int_0^t\left\langle\alpha(\alpha-\Delta)^{-1}X_\lambda^\nu(s),\sigma(X_\lambda^\nu(s))Q^{1/2}dW(s)\right\rangle_2.
\end{aligned}
$$

Hence, by the Burkholder–Davis–Gundy (BDG) inequality (with $p = 1$) and since $\alpha(\alpha - \Delta)^{-1}$ is a contraction on $L^2(\mathbb{R}^d)$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} X^\nu_\lambda(s)|_2^2\right] \\
&\qquad +2\lambda\mathbb{E}\int_0^t |\nabla(\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} X^\nu_\lambda(s))|_2^2 ds \\
&\qquad \le |\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} x|_2^2 + C_\infty^2 \mathbb{E}\int_0^t |X^\nu_\lambda(s)|_2^2 ds \\
&\qquad +6\mathbb{E}\left(\int_0^t \sum_{k=1}^\infty \mu_k^2 \left\langle \alpha(\alpha-\Delta)^{-1} X^\nu_\lambda(s), e_k X^\nu_\lambda(s)\right\rangle_2^2 ds\right)^{1/2}.
\end{aligned} \tag{3.20}$$

The latter term can be estimated by

$$\begin{aligned}
&C_\infty \mathbb{E}\left[\sup_{s\in[0,t]} |\alpha(\alpha-\Delta)^{-1} X^\nu_\lambda(s)|_2 \left(\int_0^t |X^\nu_\lambda(s)|_2^2 ds\right)^{1/2}\right] \\
&\le \frac{1}{2}\,\mathbb{E}\left\{\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} X^\nu_\lambda(s)|_2^2\right] + \frac{1}{2}\,C_\infty^2 \mathbb{E}\int_0^t |X^\nu_\lambda(s)|_2^2 ds,
\end{aligned} \tag{3.21}$$

where we used that $\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}}$ is a contraction on $L^2(\mathbb{R}^d)$. Note that the first summand on the right-hand side is finite by (3.16), since the norm $|\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}}\cdot|_2$ is equivalent to $|\cdot|_{-1}$. Hence, we can subtract this term after substituting (3.21) into (3.20) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} X^\nu_\lambda(s)|_2^2\right] \\
&\qquad +4\lambda\mathbb{E}\int_0^t |\nabla(\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} X^\nu_\lambda(s))|_2^2 ds \\
&\qquad \le 2|\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}} x|_2^2 + 3C_\infty^2 \mathbb{E}\int_0^t |X^\nu_\lambda(s)|_2^2 ds, \ t\in[0,T].
\end{aligned} \tag{3.22}$$

Obviously, the quantity under the $\sup\limits_{s\in[0,t]}$ on the left-hand side of (3.22) is increasing in α. So, by the monotone convergence theorem, we may let $\alpha \to \infty$ in (3.22) and then, except for its last part, Claim 1 immediately follows by Gronwall's lemma, since $\sqrt{\alpha}(\alpha-\Delta)^{-\frac{1}{2}}$ is a contraction in $L^2(\mathbb{R}^d)$ and $x \in L^2(\mathbb{R}^d)$. The last part of Claim 1 then immediately follows from [19, Theorem 2.1]. ∎

Applying Itô's formula to $|X^\nu_\lambda(t) - X^\nu_{\lambda'}(t)|^2_{-1,\nu}$ (see [20, Theorem 4.2.5]), it follows from (3.17) that, for $\lambda, \lambda' \in (0,1)$ and $t\in[0,T]$,

$$\begin{aligned}
&|X^\nu_\lambda(t) - X^\nu_{\lambda'}(t)|^2_{-1,\nu} \\
&\quad +2\int_0^t \left\langle \psi(X^\nu_\lambda) - \psi(X^\nu_{\lambda'}) + (\lambda X^\nu_\lambda - \lambda' X^\nu_{\lambda'}), X^\nu_\lambda - X^\nu_{\lambda'}\right\rangle_2 ds \\
&\quad = \int_0^t \|\sigma(X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)\|^2_{L_2(Q^{1/2}H^{-1}, H^{-1}_\nu)} ds \\
&\quad +2\int_0^t \left\langle X^\nu_\lambda(s) - X^\nu_{\lambda'}(s), \sigma(X^\nu_\lambda(s) - X^\nu_{\lambda'}(s)) dW^Q(s)\right\rangle_{-1,\nu}.
\end{aligned} \tag{3.23}$$

Our assumption (i) on ψ implies that

$$(\psi(r)-\psi(r'))(r-r') \geq (\mathrm{L}ip\,\psi+1)^{-1}|\psi(r)-\psi(r')|^2, \text{ for } r, r' \in \mathbb{R},$$

where $\mathrm{L}ip\,\psi$ is the Lipschitz constant of ψ. Hence (3.23), (3.7) and the BDG inequality (for $p=1$ imply that, for all $t\in[0,T]$)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)|_{-1,\nu}^2\right]\\
&+2(\mathrm{L}ip\,\psi+1)^{-1}\mathbb{E}\int_0^t|\psi(X_\lambda^\nu(s))-\psi(X_{\lambda'}^\nu(s))|_2^2ds\\
&\leq 2(\lambda+\lambda')\mathbb{E}\int_0^t(|X_\lambda^\nu(s)|_2^2+|X_{\lambda'}^\nu(s)|_2^2)ds+C_\infty^2\int_0^t|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)|_{-1,\nu}^2ds\\
&+2\mathbb{E}\left(\int_0^t\sum_{k=1}^\infty\mu_k^2\left\langle X_\lambda^\nu(s)-X_{\lambda'}^\nu(s),(X_\lambda^\nu(s)-X_{\lambda'}^\nu(s))e_k\right\rangle_{-1,\nu}^2ds\right)^{1/2}.
\end{aligned}$$

By (3.7) and Young's inequality, the latter term is dominated by

$$\frac{1}{2}\,\mathbb{E}\left[\sup_{s\in[0,t]}|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)|_{-1,\nu}^2\right]+\frac{1}{2}\,C_\infty^2\mathbb{E}\int_0^t|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)|_{-1,\nu}^2ds.$$

Hence, because of $x\in L^2(\mathbb{R}^d)$ and Claim 1, we may now apply Gronwall's lemma to obtain that, for some constant C independent of λ', λ (and ν),

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_\lambda^\nu(t)-X_{\lambda'}^\nu(t)|_{-1,\nu}^2\right]+\mathbb{E}\int_0^T|\psi(X_\lambda^\nu(s))-\psi(X_{\lambda'}^\nu(s))|_2^2ds\leq C(\lambda+\lambda'). \tag{3.24}$$

Hence there exists an $(\mathcal{F}_t)$-adapted continuous H^{-1}-valued process $X^\nu=(X^\nu(t))_{t\in[0,T]}$ such that $X^\nu\in L^2(\Omega;C([0,T];H^{-1}))$. Now, by Claim 1, it follows that

$$X^\nu\in L^2([0,T]\times\Omega;L^2(\mathbb{R}^d)).$$

Claim 2. *X^ν satisfies equation* (3.13) (*i.e., we can pass to the limit in* (3.17) *as* $\lambda\to 0$).

Proof of Claim 2. We already know that

$$X_\lambda^\nu\longrightarrow X^\nu \text{ and } \int_0^\bullet X_\lambda^\nu(s)dW(s)\longrightarrow\int_0^\bullet X^\nu(s)dW(s)$$

in $L^2(\Omega;C([0,T];H^{-1}))$ as $\lambda\to 0$ (for the second convergence see the above argument using (3.7) and the BDG inequality). So, by (3.17) it follows that

$$\int_0^\bullet(\psi(X_\lambda^\nu(s))+\lambda X_\lambda^\nu(s)))ds,\ \lambda>0,$$

converges as $\lambda\to 0$ to an element in $L^2(\Omega;C([0,T];H^1)$. But, by (3.24) and Claim 1, it follows that

$$\int_0^\bullet(\psi(X_\lambda^\nu(s))+\lambda X_\lambda^\nu(s))ds\longrightarrow\int_0^\bullet\psi(X^\nu(s))ds \tag{3.25}$$

as $\lambda\to 0$ in $L^2(\Omega;L^2([0,T];L^2(\mathbb{R}^d)))$. Hence Claim 2 is proved. ■

Now, (3.14) follows from Claim 1 by lower semicontinuity. This completes the proof of Lemma 3.5. ■

Proof of Theorem 3.4 (continued). We are going to use Lemma 3.5 and let $\nu \to 0$. The arguments are similar to those in the proof of Lemma 3.5. So, we shall not repeat all the details.

Now, we rewrite (3.11) as

$$dX^\nu + (I-\Delta)\psi(X^\nu)dt = (1-\nu)\psi(X^\nu)dt + X^\nu dW(t) \tag{3.26}$$

and apply Itô's formula to $\varphi(x) = \frac{1}{2}|x|_{-1}^2$ (see, e.g., [20, Theorem 4.2.5]). We get, for $x \in H^{-1}$, by (3.8) and after taking expectation,

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|X^\nu(t)|_{-1}^2 + \mathbb{E}\int_0^t\int_{\mathbb{R}^d}\psi(X^\nu(s))X^\nu(s)d\xi\,ds \\
&\quad= \frac{1}{2}\,|x|_{-1}^2 + (1-\nu)\mathbb{E}\int_0^t \langle\psi(X^\nu(s)),X^\nu(s)\rangle_{-1}\,ds \\
&\quad+\frac{1}{2}\,\mathbb{E}\int_0^t\sum_{k=1}^\infty \mu_k^2|X^\nu e_k|_{-1}^2 ds \\
&\quad\le \frac{1}{2}\,|x|_{-1}^2 + \mathbb{E}\int_0^t |\psi(X^\nu)|_{-1}|X^\nu|_{-1}ds \\
&\quad+\frac{1}{2}C_\infty^2\mathbb{E}\int_0^t |X^\nu(s)|_{-1}^2 ds,\ \forall t\in[0,T].
\end{aligned}$$

Recalling that $|\cdot|_{-1} \le |\cdot|_2$, we get, via Young's and Gronwall's inequalities, for some $C \in (0,\infty)$ that

$$\mathbb{E}|X^\nu(t)|_{-1}^2 + \frac{\alpha}{2}\,\mathbb{E}\int_0^T |X^\nu(s)|_2^2 ds \le C|x|_{-1}^2,\ t\in[0,T],\ \nu\in(0,1), \tag{3.27}$$

because, by assumption (i), $\psi(r)r \ge \widetilde{\alpha}|\psi(r)|^2$, $\forall r \in \mathbb{R}$, with $\widetilde{\alpha} := (\mathrm{L}ip\,\psi + 1)^{-1}$. Here we set $\alpha = 0$ if (3.10) does not hold.

Now, by a similar calculus, for $X^\nu - X^{\nu'}$ we get

$$\begin{aligned}
&|X^\nu(t) - X^{\nu'}(t)|_{-1}^2 + 2\int_0^t\int_{\mathbb{R}^d}(\psi(X^\nu)-\psi(X^{\nu'}))(X^\nu - X^{\nu'})d\xi\,ds \\
&\le C\int_0^t \left\langle \psi(X^\nu)-\psi(X^{\nu'}), X^\nu - X^{\nu'}\right\rangle_{-1} ds \\
&+C\int_0^t (\nu|\psi(X^\nu)|_2 + \nu'|\psi(X^{\nu'})|_2)|X^\nu - X^{\nu'}|_{-1}ds \\
&+C\int_0^t |X^\nu - X^{\nu'}|_{-1}^2 ds + \sum_{k=1}^\infty\int_0^t \mu_k\left\langle (X^\nu - X^{\nu'}), e_k(X^\nu - X^{\nu'})\right\rangle_{-1} d\beta_k, \\
&\hspace{25em} t\in[0,T].
\end{aligned}$$

Taking into account that, by assumption (i),

$$(\psi(x)-\psi(y))(x-y) \ge \widetilde{\alpha}|\psi(x)-\psi(y)|^2,\ \forall x,y\in\mathbb{R}^d,$$

we get, for all $\nu, \nu' > 0$,

$$
\begin{aligned}
&|X^\nu(t) - X^{\nu'}(t)|_{-1}^2 + \tilde{\alpha} \int_0^t |\psi(X^\nu(s)) - \psi(X^{\nu'}(s))|_2^2 ds \\
&\le C_1 \int_0^t |X^\nu(s) - X^{\nu'}(s)|_{-1}^2 ds + \frac{\tilde{\alpha}}{2} \int_0^t |\psi(X^\nu(s)) - \psi(X^{\nu'}(s))|_2^2 ds \\
&+C_2(\nu + \nu') \int_0^t (|\psi(X^\nu(s))|_2^2 + |\psi(X^{\nu'}(s))|_2^2) ds \\
&+ \sum_{k=1}^\infty \int_0^t \mu_k \langle (X^\nu(s) - X^{\nu'}(s)), e_k(X^\nu(s) - X^{\nu'}(s))\rangle_{-1} d\beta_k(s),\ t \in [0, T].
\end{aligned}
$$

So, similarly to showing (3.24) in the proof of Lemma 3.5, by (3.14), if $x \in L^2(\mathbb{R}^d)$, and by (3.27), if $x \in H^{-1}(\mathbb{R}^d)$ and ψ satisfies (3.10), by the Burkholder-Davis-Gundy inequality, for $p = 1$, we get, for all $\nu, \nu' \in (0, 1)$,

$$
\mathbb{E} \sup_{t\in[0,T]} |X^\nu(t) - X^{\nu'}(t)|_{-1}^2 + \mathbb{E} \int_0^T |\psi(X^\nu(s)) - \psi(X^{\nu'}(s))|_2^2 ds \le C(\nu + \nu').
$$

The remaining part of the proof is now exactly the same as the last part of the proof of Lemma 3.5. ■

Remark 3.6. Theorem 3.4 is a basic tool for the probabilistic (double) representation of equation (1.1), which holds when ψ is Lipschitz, as it is proved in [10]. If (1.1) is not perturbed by noise, and ψ is possibly discontinuous, its probabilistic representation was performed in [14], [9], [10] with extensions and numerical simulations located in [11], [12].

4. Equation (1.1) for maximal monotone functions ψ with polynomial growth

In this section, we assume $d \ge 3$ and we shall study the existence for equation (1.1) under the following assumptions:

(j) $\psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph such that $0 \in \psi(0)$ and

$$
\sup\{|\eta|;\ \eta \in \psi(r)\} \le C(1 + |r|^m),\ \forall r \in \mathbb{R}, \tag{4.1}
$$

where $1 \le m < \infty$.

(jj) $W(t) = \sum_{k=1}^\infty \mu_k e_k \beta_k(t),\ t \ge 0$, where $\{\beta_k\}_{k=1}^\infty$ are independent Brownian motions on a stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, $\mu_k \in \mathbb{R}$, and $e_k \in C^1(\mathbb{R}^d) \cap \mathcal{H}^{-1}$ are such that $\{e_k\}$ is an orthonormal basis in $\mathcal{H}^{-1}$ and

$$
\sum_{k=1}^\infty \mu_k^2 (|e_k|_\infty^2 + |\nabla e_k|_d^2 + 1) < \infty. \tag{4.2}
$$

The existence of $\{e_k\}$ as in (jj) is ensured by the following lemma.

Lemma 4.1. Let $d \ge 3$ and let $e \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$ be such that $\nabla e \in L^d(\mathbb{R}^d; \mathbb{R}^d)$. Then

$$
\|xe\|_{-1} \le \|x\|_{-1}(|e|_\infty + C|\nabla e|_d),\ \forall x \in \mathcal{H}^{-1}, \tag{4.3}
$$

where C is independent of x and e.

Proof. We have

$$\|xe\|_{-1} = \sup\{\langle x, e\varphi\rangle \,;\, \|\varphi\|_1 \le 1\} \le \|x\|_{-1} \sup\{\|e\varphi\|_1 ; \|\varphi\|_1 \le 1\}. \tag{4.4}$$

On the other hand, by Lemma 2.1 we have, for all $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \|e\varphi\|_1 &\le |e\nabla\varphi + \varphi\nabla e|_2 \le |e\nabla\varphi|_2 + |\varphi\nabla e|_2 \\ &\le |e|_\infty |\nabla\varphi|_2 + |\varphi|_p |\nabla e|_d \le |e|_\infty \|\varphi\|_1 + C\|\varphi\|_1 |\nabla e|_d, \end{aligned}$$

where $p = \frac{2d}{d-2}\cdot$ Then, by (4.4), (4.3) follows, as claimed. □

Remark 4.2. (i) It should be mentioned that, for $d = 2$, Lemma 4.1 fails and this is the main reason our treatment of equation (1.1) under assumptions (j), (jj) is constrained to $d \ge 3$.

(ii) We note that Remark 3.3 with the rôle of $H^{-1}(\mathbb{R}^d)$ replaced by $\mathcal{H}^{-1}$ remains true in all its parts under condition (jj) above. We shall use this below without further notice.

We denote by $j : \mathbb{R} \to \mathbb{R}$ the potential associated with ψ, that is, a continuous convex function on $\mathbb{R}$ such that $\partial j = \psi$, i.e.,

$$j(r) \le \zeta(r - \overline{r}) + j(\overline{r}),\ \forall \zeta \in \psi(r),\ r, \bar{r} \in \mathbb{R}.$$

Definition 4.3. Let $x \in \mathcal{H}^{-1}$ and $p := \max(2, 2m)$. An $\mathcal{H}^{-1}$-valued adapted process $X = X(t)$ is called strong solution to (1.1) if the following conditions hold:

$$X \text{ is } \mathcal{H}^{-1}\text{-valued continuous on } [0,T], \mathbb{P}\text{-a.s.}, \tag{4.5}$$

$$X \in L^p(\Omega \times (0,T) \times \mathbb{R}^d). \tag{4.6}$$

There is $\eta \in L^{\frac{p}{m}}(\Omega \times (0,T) \times \mathbb{R}^d)$ such that

$$\eta \in \psi(X),\ dt \otimes \mathbb{P} \otimes d\xi\ \text{ – a.e. on } (0,T) \times \Omega \times \mathbb{R}^d \tag{4.7}$$

and $\mathbb{P}$-a.s.

$$\begin{aligned} X(t) = x + \Delta \int_0^t \eta(s)ds + \sum_{k=1}^\infty \mu_k \int_0^t X(s)e_k d\beta_k(s) \\ \text{in } \mathcal{D}'(\mathbb{R}^d),\quad t \in [0,T]. \end{aligned} \tag{4.8}$$

Here $\mathcal{D}'(\mathbb{R}^d)$ is the standard space of distributions on $\mathbb{R}^d$.

Theorem 4.4 below is the main existence result for equation (1.1).

Theorem 4.4. Assume that $d \ge 3$ and that

$$x \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap \mathcal{H}^{-1},\quad p := \max(2, 2m).$$

Then, under assumptions (j), (jj), there is a unique solution X to (1.1) such that

$$X \in L^2(\Omega; C([0,T]; \mathcal{H}^{-1})). \tag{4.9}$$

Moreover, if $x \ge 0$, a.e. in $\mathbb{R}^d$, then $X \ge 0$, a.e. on $(0,T) \times \mathbb{R}^d \times \Omega$.

Theorem 4.4 is applicable to a large class of nonlinearities $\psi : \mathbb{R} \to 2^{\mathbb{R}}$ and, in particular, to

$$\psi(r) = \rho H(r) + \alpha r,\ \forall r \in \mathbb{R}, \psi(r) = \rho H(r - r_c)r,$$

where $\rho > 0$, $\alpha, r_c \geq 0$, which models the dynamics of self-organized criticality (see [5], [6], [8]). Here H is the Heaviside function.

As mentioned earlier, Theorem 4.4 can be compared most closely to the main existence result of [21]. But there are, however, a few notable differences as we explain below. The function ψ arising in [21] is monotonically increasing, continuous and are assumed to satisfy a growth condition of the form $N(r) \leq r\psi(r) \leq C(N(r) + 1)r$, $\forall r \in \mathbb{R}$, where N is a smooth and Δ_2-regular Young function defining the Orlicz class L_N. In contrast to this, here ψ is any maximal monotone graph (multivalued) with arbitrary polynomial growth.

Proof of Theorem 4.4. Consider the approximating equation

$$\begin{aligned}&dX_\lambda - \Delta(\psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = X_\lambda dW,\ t \in (0, T),\\ &X_\lambda(0) = x,\end{aligned} \tag{4.10}$$

where $\psi_\lambda = \frac{1}{\lambda}(1 - (1 + \lambda\psi)^{-1}),\ \lambda > 0$. We note that $\psi_\lambda = \partial j_\lambda$, where (see, e.g., [1])

$$j_\lambda(r) = \inf\left\{\frac{|r - \bar{r}|^2}{2\lambda} + j(\bar{r});\ \bar{r} \in \mathbb{R}\right\},\ \forall r \in \mathbb{R}.$$

We have the following result.

Lemma 4.5. Let $x \in \mathcal{H}^{-1} \cap L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $p := 2m$, $d \geq 3$. Then (4.10) has a unique solution

$$X_\lambda \in L^2(\Omega; C([0, T]; \mathcal{H}^{-1})) \cap L^\infty([0, T]; L^p(\Omega \times \mathbb{R}^d)). \tag{4.11}$$

Moreover, for all $\lambda, \mu > 0$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \|(X_\lambda(t) - X_\mu(t))\|_{-1}^2 \leq C(\lambda + \mu) \tag{4.12}$$

$$\mathbb{E}|X_\lambda(t)|_p^p \leq C|x|_p^p,\ \forall t \in [0, T], \tag{4.13}$$

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\psi_\lambda(X_\lambda)|^{\frac{p}{m}} dt\, d\xi \leq C|x|_p^p,\ \forall \lambda > 0, \tag{4.14}$$

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \|X_\lambda(t)\|_{-1}^2\right] \leq C\|x\|_{-1}^2,\ \forall \lambda > 0, \tag{4.15}$$

where C is independent of λ, μ.

Proof. We consider for each fixed λ the equation (see (3.11))

$$\begin{aligned}&dX_\lambda^\nu + (\nu - \Delta)(\psi_\lambda(X_\lambda^\nu) + \lambda X_\lambda^\nu)dt = X_\lambda^\nu dW\\ &X_\lambda^\nu(0) = x,\end{aligned} \tag{4.16}$$

where $\nu > 0$. Let $x \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \cap \mathcal{H}^{-1}$. By Claim 1 in the proof of Lemma 3.5, (4.16) has a unique solution $X_\lambda^\nu \in L^2(\Omega; L^\infty([0, T]; L^2(\mathbb{R}^d))) \cap L^2(\Omega \times [0, T]; H^1(\mathbb{R}^d))$ with continuous sample paths in $L^2(\mathbb{R}^d)$.

As seen in the proof of Theorem 3.4, we have, for $\nu \to 0$,

$$
\begin{aligned}
X_\lambda^\nu \to X_\lambda \quad & \text{strongly in } L^2(\Omega; C([0,T]; H^{-1}(\mathbb{R}^d))) \\
& \text{weak-star in } L^2(\Omega; L^\infty([0,T]; L^2(\mathbb{R}^d))), \\
& \text{and, by (3.14), along a subsequence also,}
\end{aligned}
$$

where X_λ is the solution to (4.10). It remains to be shown that X_λ satisfies (4.11)–(4.15). In order to explain the ideas, we apply first (formally) Itô's formula to (4.16) for the function $\varphi(x) = \frac{1}{p}\,|x|_p^p$. We obtain

$$
\begin{aligned}
&\frac{1}{p}\,\mathbb{E}|X_\lambda^\nu(t)|_p^p + \mathbb{E}\int_0^t \int_{\mathbb{R}^d} (\nu-\Delta)(\psi_\lambda(X_\lambda^\nu) + \lambda X_\lambda^\nu)|X_\lambda^\nu|^{p-2} X_\lambda^\nu ds\, d\xi \\
&\quad = \frac{1}{p}\,|x|_p^p + \frac{p-1}{2}\,\mathbb{E}\int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^\infty \mu_k^2 |X_\lambda^\nu e_k|^2 |X_\lambda^\nu|^{p-2} dt\, d\xi.
\end{aligned}
\tag{4.17}
$$

Taking into account that $X_\lambda^\nu, \psi_\lambda(X_\lambda^\nu) \in L^2(0,T; H^1(\mathbb{R}^d))$, $\mathbb{P}$-a.s., by Claim 1 in the proof of Lemma 3.5, we have

$$
\int_0^t \int_{\mathbb{R}^d} (\nu-\Delta)(\psi_\lambda(X_\lambda^\nu) + \lambda X_\lambda^\nu)|X_\lambda^\nu|^{p-2} X_\lambda^\nu ds\, d\xi \geq \lambda(p-1)\int_0^t \int_{\mathbb{R}^d} |\nabla X_\lambda^\nu|^2 |X_\lambda^\nu|^{p-2} d\xi\, ds,
$$

and by (4.2) we have

$$
\mathbb{E}\int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^\infty \mu_k^2 |X_\lambda^\nu e_k|^2 |X_\lambda^\nu|^{p-2} ds\, d\xi \leq C_\infty \mathbb{E}\int_0^t \int_{\mathbb{R}^d} |X_\lambda^\nu|^p d\xi\, ds < \infty.
$$

Then, we obtain by (4.17) via Gronwall's lemma

$$
\mathbb{E}|X_\lambda^\nu(t)|_p^p \leq C|x|_p^p,\ t \in (0,T), \tag{4.18}
$$

and, by (4.1),

$$
\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |\psi_\lambda(X_\lambda^\nu)|^{\frac{p}{m}} dt\, d\xi \leq C|x|_p^p,\ t \in [0,T]. \tag{4.19}
$$

It should be said, however, that the above argument is formal, because the function φ is not of class C^2 on $L^2(\mathbb{R}^d)$ and we do not know a priori if the integral in the left side of (4.17) makes sense, that is, whether $|X_\lambda^\nu|^{p-2} X_\lambda^\nu \in L^2(0,T; L^2(\Omega; H^1(\mathbb{R}^d)))$. To make it rigorous, we approximate X_λ^ν by a sequence $\{X_\lambda^{\nu,\varepsilon}\}$ of solutions to the equation

$$
\begin{aligned}
& dX_\lambda^{\nu,\varepsilon} + A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon})dt = X_\lambda^{\nu,\varepsilon} dW, \\
& X_\lambda^{\nu,\varepsilon}(0) = x.
\end{aligned}
\tag{4.20}
$$

Here, $A_\lambda^{\nu,\varepsilon} = \frac{1}{\varepsilon}\,(I - (I + \varepsilon A_\lambda^\nu)^{-1}),\ \varepsilon \in (0,1)$, is the Yosida approximation of the operator $A_\lambda^\nu x = (\nu - \Delta)(\psi_\lambda(x) + \lambda x)$, $\forall x \in D(A_\lambda^\nu) = H^1(\mathbb{R}^d)$. We set $J_\varepsilon = (I + \varepsilon A_\lambda^\nu)^{-1}$ and note that J_ε is Lipschitz in $H = H^{-1}(\mathbb{R}^d)$ as well as in all $L^q(\mathbb{R}^d)$ for $1 < q < \infty$. Moreover, we have

$$
|J_\varepsilon(x)|_q \leq |x|_q,\ \forall x \in L^q(\mathbb{R}^d), \tag{4.21}
$$

see [3], Lemma 3.1. Since $A_\lambda^{\nu,\varepsilon}$ is Lipschitz in H, equation (3.1) has a unique adapted solution $X_\lambda^{\nu,\varepsilon} \in L^2(\Omega; C([0,T]; H)$ and by Itô's formula we have

$$
\frac{1}{2}\,\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{-1}^2 \leq \frac{1}{2}\,|x|_{-1}^2 + C_1 \sum_{k=1}^\infty \mu_k^2 \mathbb{E}\int_0^t |X_\lambda^{\nu,\varepsilon}(s)e_k|_{-1}^2 ds,
$$

which, by virtue of (jj), yields

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{-1}^2 \le C_2|x|_{-1}^2,\ \forall \varepsilon > 0,\ x \in H. \tag{4.22}$$

Similarly, since $A_\lambda^{\nu,\varepsilon}$ is Lipschitz in $L^2(\mathbb{R}^d)$ (see Lemma 4.6 below), we have also that $X_\lambda^{\nu,\varepsilon} \in L^2(\Omega; C([0,T]; L^2(\mathbb{R}^2)))$ and, again by Itô's formula applied to the function $|X_\lambda^{\nu,\varepsilon}(t)|_2^2$, we obtain

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 \le \frac{1}{2}\,|x|_2^2 + C_3\sum_{k=1}^\infty \mu_k^2 \mathbb{E}\int_0^\infty |X_\lambda^{\nu,\varepsilon}(s)e_k|_2^2 ds,$$

which yields, by virtue of (jj),

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 \le C_4|x|_2^2,\ \forall t \in [0,T]. \tag{4.23}$$

Claim 1. *For $p \in [2,\infty)$ and $x \in L^p(\mathbb{R}^d)$, we have that $X_\lambda^{\nu,\varepsilon} \in L_W^\infty([0,T]; L^p(\Omega; L^p(\mathbb{R}^d)) \cap L^2(\Omega; L^2(\mathbb{R}^d)))$, where here and below the subscript W refers to $(\mathcal{F}_t)$-adapted processes.*

Proof. For $R > 0$, consider the set

$$\begin{aligned}\mathcal{K}_R \;\; = \;\; & \{X \in L_W^\infty([0,T]; L^p(\Omega; L^p(\mathbb{R}^d)) \cap L^2(\Omega; L^2(\mathbb{R}^d))),\\ & e^{-p\alpha t}\mathbb{E}|X(t)|_p^p \le R^p,\ e^{-2\alpha t}\mathbb{E}|X(t)|_2^2 \le R^2, t \in [0,T]\}.\end{aligned}$$

Since, by (4.20), $X_\lambda^{\nu,\varepsilon}$ is a fixed point of the map

$$X \xrightarrow{F} e^{-\frac{t}{\varepsilon}}X + \frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}} J_\varepsilon(X(s))ds + \int_0^t e^{-\frac{(t-s)}{\varepsilon}} X(s)dW(s),$$

obtained by iteration in $C_W([0,T]; L^2(\Omega; H \cap L^2(\mathbb{R}^d)))$, it suffices to show that F leaves the set $\mathcal{K}_R$ invariant for $R > 0$ large enough. By (4.21), we have

$$\begin{aligned}&\left(e^{-p\alpha t}\mathbb{E}\left|e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}} J_\varepsilon(X(s))ds\right|_p^p\right)^{\frac{1}{p}}\\ &\qquad \le e^{-(\frac{1}{\varepsilon}+\alpha)t}|x|_p + e^{-\alpha t}\int_0^t \frac{1}{\varepsilon}e^{-\frac{(t-s)}{\varepsilon}}(\mathbb{E}|X(s)|_p^p)^{\frac{1}{p}}ds\\ &\qquad \le e^{-(\frac{1}{\varepsilon}+\alpha)t}|x|_p + \frac{R}{1+\alpha\varepsilon},\end{aligned} \tag{4.24}$$

and, similarly, that

$$\left(e^{-2\alpha t}\mathbb{E}\left|e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{(t-s)}{\varepsilon}} J_\varepsilon(X(s))ds\right|_2^2\right)^{\frac{1}{2}} \le e^{-(\frac{1}{\varepsilon}+\alpha)t}|x|_2 + \frac{R}{1+\alpha\varepsilon}. \tag{4.25}$$

Now, we set

$$Y(t) = \int_0^t e^{-\frac{(t-s)}{\varepsilon}} X(s)dW(s),\ t \ge 0.$$

We have

$$\begin{aligned}&dY + \frac{1}{\varepsilon}\,Y\,dt = X\,dW,\ \ t \ge 0,\\ &Y(0) = 0.\end{aligned}$$

Equivalently,

$$d(e^{\frac{t}{\varepsilon}}Y(t)) = e^{\frac{t}{\varepsilon}}X(t)dW(t),\ t>0;\ Y(0)=0.$$

By Lemma 5.1 in [19], it follows that $e^{\frac{t}{\varepsilon}}Y$ is an $L^p(\mathbb{R}^d)$-valued $(\mathcal{F}_t)$-adapted continuous process on $[0,\infty)$ and

$$\mathbb{E}|e^{\frac{t}{\varepsilon}}Y(t)|_p^p = \frac{1}{2}\,p(p-1)\sum_{k=1}^{\infty}\mu_k^2\mathbb{E}\int_0^t\int_{\mathbb{R}^d}|e^{\frac{s}{\varepsilon}}Y(s)|^{p-2}|e^{\frac{s}{\varepsilon}}X(s)e_k|^2ds.$$

This yields via Hypothesis (jj)

$$\mathbb{E}|e^{\frac{t}{\varepsilon}}Y(t)|_p^p \le \frac{1}{2}\,(p-1)\mathbb{E}\int_0^t|e^{\frac{s}{\varepsilon}}Y(s)|_p^pds + C\mathbb{E}\int_0^t|e^{\frac{s}{\varepsilon}}X(s)|_p^pds,\ \forall t\in[0,T],$$

and, therefore,

$$\mathbb{E}|Y(t)|_p^p \le C_1e^{-(\alpha+\frac{1}{\varepsilon})pt}\mathbb{E}\int_0^t|e^{\frac{s}{\varepsilon}}X(s)|_p^pds \le \frac{R^pe^{-p\alpha t}\varepsilon C_1}{p(1+\varepsilon\alpha)},\ \forall t\in[0,T].$$

Similarly, we get

$$e^{-2\alpha t}\mathbb{E}|Y(t)|_2^2 \le \frac{R^2\varepsilon C_1}{2(1+\varepsilon\alpha)},\ \forall t\in[0,T].$$

Then, by formulae (4.24), (4.25), we infer that, for α large enough and $R>2(|x|_p+|x|_2)$, F leaves $\mathcal{K}_R$ invariant, which proves Claim 1. □

Claim 2. *We have, for all $p\in[2,\infty)$ and $x\in L^p(\mathbb{R}^d)$, that there exists $C_p\in(0,\infty)$ such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]}\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_p^p \le C_p\ \text{ for all } \varepsilon,\lambda,\nu\in(0,1). \tag{4.26}$$

Proof. Again invoking Lemma 5.1 in [19], we have by (4.20) that $X_\lambda^{\nu,\varepsilon}$ satisfies

$$\begin{aligned}\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_p^p &= |x|_p^p - p\,\mathbb{E}\int_0^t\int_{\mathbb{R}^d}A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon})X_\lambda^{\nu,\varepsilon}|X_\lambda^{\nu,\varepsilon}|^{p-2}d\xi\,ds\\ &+p(p-1)\sum_{k=1}^{\infty}\mu_k^2\mathbb{E}\int_0^t\int_{\mathbb{R}^d}|X_\lambda^{\nu,\varepsilon}|^{p-2}|X_\lambda^{\nu,\varepsilon}e_k|^2d\xi\,ds.\end{aligned} \tag{4.27}$$

On the other hand, $A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}) = \frac{1}{\varepsilon}\,(X_\lambda^{\nu,\varepsilon} - J_\varepsilon(X_\lambda^{\nu,\varepsilon}))$ and so we have

$$\int_{\mathbb{R}^d}A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon})X_\lambda^{\nu,\varepsilon}|X_\lambda^{\nu,\varepsilon}|^{p-2}d\xi = \frac{1}{\varepsilon}\int_{\mathbb{R}^d}|X_\lambda^{\nu,\varepsilon}|^pd\xi - \frac{1}{\varepsilon}\int_{\mathbb{R}^r}J_\varepsilon(X_\lambda^{\nu,\varepsilon})|X_\lambda^{\nu,\varepsilon}|^{p-2}X_\lambda^{\nu,\varepsilon}d\xi.$$

Recalling (4.21), we get, via the Hölder inequality,

$$\int_{\mathbb{R}^d}A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon})X_\lambda^{\nu,\varepsilon}|X_\lambda^{\nu,\varepsilon}|^{p-2}d\xi \ge 0,$$

and so, by (4.27) and Hypothesis (jj), we obtain, via Gronwall's lemma, estimate (4.26), as claimed. □

Claim 3. *We have, for $\varepsilon \to 0$,*

$$X_\lambda^{\nu,\varepsilon} \longrightarrow X_\lambda^\nu \text{ strongly in } L_W^\infty([0,T]; L^2(\Omega; H))$$

and weakly in $L^\infty([0,T]; L^p(\Omega; L^p(\mathbb{R}^d)) \cap L^2(\Omega; L^2(\mathbb{R}^d)))$.*

Proof. For simplicity, we write X_ε instead of $X_\lambda^{\nu,\varepsilon}$ and X instead of X_λ^ν. Also, we set $\gamma(r) \equiv \psi_\lambda(r) + \lambda r$.

Subtracting equations (4.20) and (4.16), we get via Itô's formula and because $A_\lambda^{\nu,\varepsilon}$ is monotone on H

$$\begin{aligned}\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t) - X(t)|_{-1,\nu}^2 + \mathbb{E}\int_0^t \int_{\mathbb{R}^d} (\gamma(J_\varepsilon(X)) - \gamma(X))(X_\varepsilon - X) d\xi\, ds \\ \leq C\mathbb{E}\int_0^t |X_\varepsilon(s) - X(s)|_{-1,\nu}^2 ds,\end{aligned}$$

and hence, by Gronwall's lemma, we obtain

$$\mathbb{E}|X_\varepsilon(t) - X(t)|_{-1,\nu}^2 \leq C\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\gamma(J_\varepsilon(X)) - \gamma(X)||X_\varepsilon - X| d\xi\, ds. \tag{4.28}$$

On the other hand, it follows by (4.21) that

$$\int_{\Omega\times[0,T]\times\mathbb{R}^d} |J_\varepsilon(X)|^2 \mathbb{P}(d\omega) dt\, d\xi \leq \int_{\Omega\times[0,T]\times\mathbb{R}^d} |X|^2 \mathbb{P}(d\omega) dt\, d\xi,$$

while, for $\varepsilon \to 0$,

$$J_\varepsilon(y) \longrightarrow y \text{ in } H^{-1}, \ \ \forall y \in H^{-1},$$

(because $A_\lambda^{\nu,\varepsilon}$ is maximal monotone in $H^{-1}(\mathbb{R}^d)$) and so, $J_\varepsilon(X(t,\omega)) \longrightarrow X(t,\omega)$ in $H^{-1}(\mathbb{R}^d)$ for all $(t,\omega) \in (0,T) \times \Omega$. Hence, as $\varepsilon \to 0$,

$$J_\varepsilon(X) \longrightarrow X \text{ weakly in } L^2(\Omega \times [0,T] \times \mathbb{R}^d), \tag{4.29}$$

and, according to the inequality above, this implies that, for $\varepsilon \to 0$,

$$|J_\varepsilon(X)|_{L^2((0,T)\times\Omega\times\mathbb{R}^d)} \longrightarrow |X|_{L^2((0,T)\times\Omega\times\mathbb{R}^d)}.$$

Hence, $J_\varepsilon(X) \longrightarrow X$ strongly in $L^2(\Omega\times[0,T]\times\mathbb{R}^d)$ as $\varepsilon \to 0$. Now, taking into account that γ is Lipschitz, we conclude by (4.28), (4.29) and by estimates (4.23), (4.26) that Claim 3 is true. □

Now, we can complete the proof of Lemma 4.5. Namely, letting first $\varepsilon \to 0$ and then $\nu \to \infty$ in (4.26), we get (4.13) and hence (4.14) as desired.

Now, let us prove (4.12) and (4.15). Arguing as in the proof of Theorem 3.4, we obtain

$$\begin{aligned}&\frac{1}{2}\,|X_\lambda^\nu(t)|_{-1,\nu}^2 + \int_0^t \int_{\mathbb{R}^d} (\psi_\lambda(X_\lambda^\nu) + \lambda X_\lambda^\nu) X_\lambda^\nu d\xi\, ds \\ &\quad = \frac{1}{2}\,|x|_{-1,\nu}^2 + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^\infty \mu_k^2 |X_\lambda^\nu e_k|_{-1,\nu}^2 d\xi\, ds \\ &\quad + \int_0^t \left\langle X_\lambda^\nu, X_\lambda^\nu dW \right\rangle_{-1,\nu} ds.\end{aligned} \tag{4.30}$$

Keeping in mind that, by (4.3), $|X^\nu_\lambda e_k|_{-1,\nu} \le C|X^\nu_\lambda|_{-1,\nu}(|e_k|_\infty + |\nabla e_k|_d)$, where C is independent of ν, we obtain by the Burkholder-Davis-Gundy inequality for $p = 1$ (cf. the proof of Theorem 3.4)

$$\mathbb{E} \sup_{t\in[0,T]} |X^\nu_\lambda(t)|^2_{-1,\nu} + \lambda\mathbb{E} \int_0^T |X^\nu_\lambda|^2_2 ds \le C|x|^2_{-1,\nu}.$$

Taking into account that

$$\lim_{\nu\to 0} |y|_{-1,\nu} = \|y\|_{-1},\ \forall y \in \mathcal{H}^{-1},$$

we obtain, as in Theorem 3.4 (see the part following (3.26)), that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|^2_{-1}\right] + \lambda\mathbb{E}\int_0^T |X_\lambda(t)|^2_2 dt \le C\|x\|^2_{-1},\ \forall \lambda > 0, \tag{4.31}$$

where C is independent of λ. In particular, (4.15) holds.

Completely similarly, one proves (4.12). Namely, we have

$$d(X^\nu_\lambda - X^\nu_\mu) + (\nu - \Delta)(\psi_\lambda(X^\nu_\lambda) + \lambda X^\nu_\lambda - \psi_\mu(X^\nu_\mu) - \mu X^\nu_\mu)dt = (X^\nu_\lambda - X^\nu_\mu)dW$$

and again proceeding as in the proof of Theorem 3.4, we obtain as above that

$$\begin{aligned}
&\frac{1}{2}|X^\nu_\lambda(t) - X^\nu_\mu(t)|^2_{-1,\nu}\\
&\quad + \int_0^t \int_{\mathbb{R}^d} (\psi_\lambda(X^\nu_\lambda) + \lambda X^\nu_\lambda - \psi_\mu(X^\nu_\mu) - \mu X^\nu_\mu)(X^\nu_\lambda - X^\nu_\mu)d\xi\, ds\\
&\quad = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^\infty \mu_k^2 |(X^\nu_\lambda - X^\nu_\mu)e_k|^2_{-1,\nu} ds\\
&\quad + \int_0^t \left\langle X^\nu_\lambda - X^\nu_\mu, (X^\nu_\lambda - X^\nu_\mu)dW\right\rangle_{-1,\nu},\ t \in [0,T].
\end{aligned}$$

Then, applying once again the Burkholder-Davis-Gundy inequality for $p = 1$, and the fact that, by Hypothesis (j), $|\psi_\lambda(r)| \le C|r|^m,\ \forall \in \mathbb{R}$ with C independent of λ, we get, proceeding as in the proof of Theorem 3.4, that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^\nu_\lambda(t) - X^\nu_\mu(t)|^2_{-1}\right] \le C(\lambda + \mu),$$

where C is independent of ν, λ, μ. (For details, we refer to the proof of (3.10), (3.14) in [5]). Letting $\nu \to 0$ as in the previous case, we obtain (4.12), as claimed. This completes the proof of Lemma 4.5. □

Above we have used the lemma below.

Lemma 4.6. $A^{\nu,\varepsilon}_\lambda$ is Lipschitz in $L^2(\mathbb{R}^d)$.

Proof. It suffices to check that J_ε is Lipschitz in $L^2(\mathbb{R}^d)$. We set $\gamma(r) = \psi_\lambda(r) + \lambda r$. We have, for $x, \bar{x} \in L^2(\mathbb{R}^d)$,

$$J_\varepsilon(x) - J_\varepsilon(\bar{x}) - \varepsilon\Delta(\gamma(J_\varepsilon(x)) - \gamma(J_\varepsilon(\bar{x}))) = x - \bar{x}.$$

Multiplying by $\gamma(J_\varepsilon(x)) - \gamma(J_\varepsilon(\bar{x}))$ in $L^2(\mathbb{R}^d)$, we get

$$\langle J_\varepsilon(x) - J_\varepsilon(\bar{x}), \gamma(J_\varepsilon(x)) - \gamma(J_\varepsilon(\bar{x}))\rangle_2 \le |\gamma(J_\varepsilon(x)) - \gamma(J_\varepsilon(\bar{x}))|_2 |x - \bar{x}|_2.$$

Taking into account that $(\gamma(r)-\gamma(\bar{r}))(r-\bar{r}) \geq L|r-\bar{r}|$, $\forall r, \bar{r} \in \mathbb{R}$, and that γ is Lipschitz, we get

$$|J_\varepsilon(x)-J_\varepsilon(\bar{x})|_2 \leq C|x-\bar{x}|_2,$$

as claimed. □

Proof of Theorem 4.4 (continued). By (4.12)-(4.15), it follows that there is a process $X \in L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d))$ such that, for $\lambda \to 0$,

$$\begin{array}{rcll} X_\lambda & \to & X & \text{weak-star in } L^\infty([0,T];L^p(\Omega\times\mathbb{R}^d)) \\ \lambda X_\lambda & \to & 0 & \text{strongly in } L^2([0,T];L^2(\Omega\times\mathbb{R}^d)) \\ \psi_\lambda(X_\lambda) & \to & \eta & \text{weakly in } L^{\frac{p}{m}}([0,T]\times\Omega\times\mathbb{R}^d) \\ X_\lambda & \to & X & \text{strongly in } L^2(\Omega;C([0,T];\mathcal{H}^{-1})). \end{array} \tag{4.32}$$

It remains to be shown that X is a solution to (1.1) in the sense of Definition 4.3.

By (4.10) and (4.32), we see that

$$\begin{aligned} &dX-\Delta\eta dt = XdW,\ t\in(0,T) \\ &X(0)=x. \end{aligned} \tag{4.33}$$

To prove that $\eta \in \psi(X)$, a.e. in $\Omega\times(0,T)\times\mathbb{R}^d$, it suffices to show that, for each $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\limsup_{\lambda\to 0}\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2\psi_\lambda(X_\lambda)X_\lambda dt\,d\xi \leq \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2\eta X\,d\xi\,dt. \tag{4.34}$$

Indeed, we have by convexity of j_λ

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2\psi_\lambda(X_\lambda)(X_\lambda-Z)d\xi\,dt \geq \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2(j_\lambda(X_\lambda)-j_\lambda(Z))d\xi\,dt,$$
$$\forall Z\in L^p((0,T)\times\Omega\times\mathbb{R}^d),$$

and so, by (4.32) and (4.34), we see that

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2(\eta(X-Z))dtd\xi \geq \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2(j(X)-j(Z))d\xi\,dt,$$
$$\forall Z\in L^p((0,T)\times\Omega\times\mathbb{R}^d),$$

because, for $\lambda\to 0$, $j_\lambda(Z)\to j(Z)$, and $j_\lambda(X_\lambda)\to j(X)$, a.e. and thus, by Fatou's lemma

$$\liminf_{\lambda\to 0}\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2 j_\lambda(X_\lambda)d\xi\,dt \geq \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2 j(X)d\xi\,dt.$$

Now, we take $\varphi\in C_0^\infty(\mathbb{R}^d)$ to be non-negative, such that $\varphi=1$ on B_N and $\varphi=0$, outside B_{N+1} where for a given $N\in\mathbb{N}$, B_N is the closed ball of $\mathbb{R}^d$ with radius N. We get

$$\begin{aligned} &\mathbb{E}\int_0^T\int_{B_{N+1}}\varphi^2(\eta(X-Z))d\xi\,dt \geq \mathbb{E}\int_0^T\int_{\mathbb{R}^d}\varphi^2(j(X)-j(Z))d\xi\,dt, \\ &\qquad\qquad\qquad\forall Z\in L^p((0,T)\times\Omega\times\mathbb{R}^d). \end{aligned} \tag{4.35}$$

This yields

$$\mathbb{E}\int_0^T\int_{B_{N+1}}\varphi^2\eta(X-Z)d\xi\,dt \geq \mathbb{E}\int_0^T\int_{B_{N+1}}\varphi^2\zeta(X-Z)d\xi\,dt, \tag{4.36}$$

for all $Z \in L^p((0,T)\times\Omega\times B_{N+1})$ and $\zeta \in L^{p'}((0,T)\times\Omega\times B_{N+1})$ such that $\zeta \in \psi(Z)$, a.e. in $(0,T)\times\Omega\times B_{N+1}$.

We denote by $\widetilde{\psi} : L^p((0,T)\times\Omega\times B_{N+1}) \to L^{p'}((0,T)\times\Omega\times B_{N+1})$ the realization of the mapping ψ in $L^p((0,T)\times\Omega\times B_{N+1})$, that is,

$$\widetilde{\psi}(Z) = \left\{\zeta \in L^{p'}((0,T)\times\Omega\times B_{N+1}),\ \zeta \in \psi(Z),\ \text{a.e.}\right\}.$$

Since $\frac{m}{p} \leq p'$ with $\frac{1}{p'} = 1 - \frac{1}{p}$, by virtue of assumption (j), $\widetilde{\psi}$ is maximal monotone in $L^p((0,T)\times\Omega\times B_{N+1}) \times L^{p'}((0,T)\times\Omega\times B_{N+1})$, and so, the equation

$$J(Z) + \widetilde{\psi}(Z) \ni J(X) + \eta, \tag{4.37}$$

where $J(Z) = |Z|^{p-2}Z$, has a unique solution (Z,η) (see, e.g., [1], p. 31).

If, in (4.36), we take Z the solution to (4.37), we obtain that

$$\mathbb{E}\int_0^T\int_{B_{N+1}}\varphi^2(J(X)-J(Z))(X-Z)dtd\xi \leq 0.$$

Then, choosing $\alpha = \frac{2}{p}$, yields

$$\mathbb{E}\int_0^T\int_{B_{N+1}}\left(|\varphi^\alpha X|^{p-2}\varphi^\alpha X - |\varphi^\alpha Z|^{p-2}\varphi^\alpha Z\right)(\varphi^\alpha X - \varphi^\alpha Z)dtd\xi \leq 0.$$

Consequently, this gives

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^d}(J(\varphi^\alpha X) - J(\varphi^\alpha Z))(\varphi^\alpha X - \varphi^\alpha Z)dtd\xi \leq 0. \tag{4.38}$$

On the other hand, we have

$$J(\varphi^\alpha X) - J(\varphi^\alpha Z) = (p-1)|\lambda\varphi^\alpha X + (1-\lambda)\varphi^\alpha Z|^{p-2}(X-Z),$$

for some $\lambda = \lambda(X,Z) \in [0,1]$. Substituting into (4.38) yields

$$|\varphi^\alpha(X-Z)|^2 = 0 \ \text{ a.e. in } (0,T)\times\Omega\times B_{N+1},$$

Hence, $X = Z$ on $(0,T)\times\Omega\times B_N$.

Coming back to (4.37), this gives $\eta \in \psi(X)$, $dtdPd\xi$, a.e., because N is arbitrary.

To prove (4.34), we use the Itô formula in (4.16) to $x \to \frac{1}{2}\|\varphi x\|^2_{-1}$ to get, as in (4.30),

$$\begin{aligned}\frac{1}{2}\,\mathbb{E}\|\varphi X^\nu_\lambda(t)\|^2_{-1} &+ \mathbb{E}\int_0^t\left\langle(-\Delta)^{-1}(\nu-\Delta)(\psi_\lambda(X^\nu_\lambda) + \lambda X^\nu_\lambda, \varphi^2 X^\nu_\lambda)\right\rangle ds\\ &\leq \frac{1}{2}\,\|\varphi x\|^2_{-1} + \frac{1}{2}\,\mathbb{E}\int_0^t\sum_{k=1}^\infty \mu_k^2\|\varphi X^\nu_\lambda e_k\|^2_{-1}ds.\end{aligned}$$

Then, letting $\nu \to 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\|\varphi X_\lambda(t)\|_{-1}^2 + \mathbb{E}\int_0^t \left\langle \psi_\lambda(X_\lambda) + \lambda X_\lambda, \varphi^2 X_\lambda \right\rangle_2 ds \\
&\qquad \le \frac{1}{2}\,\|\varphi x\|_{-1}^2 + \frac{1}{2}\,\mathbb{E}\int_0^t \sum_{k=1}^\infty \mu_k^2 \|\varphi X_\lambda e_k\|_{-1}^2 ds.
\end{aligned} \tag{4.39}$$

On the other hand, by (4.33) we get similarly

$$\frac{1}{2}\,\mathbb{E}\|\varphi X(t)\|_{-1}^2 + \mathbb{E}\int_0^t \left\langle \eta(s), \varphi^2 X \right\rangle_2 ds = \frac{1}{2}\,\|\varphi x\|_{-1}^2 + \frac{1}{2}\,\mathbb{E}\int_0^1 \sum_{k=1}^\infty \mu_k^2 \|\varphi X e_k\|_{-1}^2,\ t \in [0,T].$$

Comparing with (4.39), we obtain (4.34), as claimed.

If $x \ge 0$, a.e. in $\mathbb{R}^d$, it follows that $X \ge 0$, a.e. in in $\Omega \times (0,T) \times \mathbb{R}^d$. To prove this, one applies Itô's formula in (4.16) to the function $x \to |x^-|_2^2$ and get $(X_\lambda^\nu)^- = 0$, a.e. in $\Omega \times (0,T) \times \mathbb{R}^d$. Then, for $\nu \to 0$, we obtain the desired result. This completes the existence proof for $x \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \cap \mathcal{H}^{-1}$.

Uniqueness. If X_1, X_2 are two solutions, we have

$$\begin{aligned}
&d(X_1 - X_2) - \Delta(\eta_1 - \eta_2)dt = (X_1 - X_2)dW,\ t \in (0,T),\\
&(X_1 - X_2)(0) = 0,
\end{aligned}$$

where $\eta_i \in \psi(X_i),\ i = 1,2$, a.e. in $\Omega \times (0,T) \times \mathbb{R}^d$.

Applying again, as above (that is, via the approximating device) Itô's formula in $\mathcal{H}^{-1}$ to $\frac{1}{2}\,\|\varphi(X_1 - X_2)\|_{-1}^2$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$, we get that

$$\begin{aligned}
&\frac{1}{2}\,d\|\varphi(X_1 - X_2)\|_{-1}^2 - \langle \Delta(\eta_1 - \eta_2), \varphi(X_1 - X_2)\rangle_{-1}\\
&= \frac{1}{2}\sum_{k=1}^\infty \mu_k^2 \|\varphi(X_1 - X_2)e_k\|_{-1}^2 dt + \langle (X_1 - X_2), \varphi(X_1 - X_2)dW\rangle_{-1} = 0.
\end{aligned}$$

Note that, since $\eta_1 - \eta_2 \in L^{\frac{p}{m}}(\Omega \times (0,T) \times \mathbb{R}^d)$, we have

$$-\mathbb{E}\int_0^T \langle \Delta(\eta_1 - \eta_2), \varphi(X_1 - X_2)\rangle_{-1}\, dt = \mathbb{E}\int_0^T \int_{\mathbb{R}^d} (\eta_1 - \eta_2), \varphi(X_1 - X_2) dt\, d\xi \ge 0,$$

and, therefore,

$$\mathbb{E}\|\varphi(X_1(t) - X_2(t))\|_{-1}^2 \le C \int_0^t \mathbb{E}\|\varphi(X_1 - X_2)\|_{-1}^2 ds,\ \forall t \in [0,T],$$

and, since φ was arbitrary in $C_0^\infty(\mathbb{R}^d)$, we get $X_1 \equiv X_2$, as claimed.

Remark 4.7. The self-organized criticality model (1.3), that is, $\psi(r) \equiv H(r) =$ Heaviside function, which is not covered by Theorem 4.4 for $1 \le d \le 2$, can, however, be treated in the special case

$$W(t) = \sum_{j=1}^N \mu_j \beta_j(t),\ \mu_j \in \mathbb{R},$$

(i.e., spatially independent noise) via the rescaling transformation $X = e^W Y$, which reduces it to the random parabolic equation

$$\frac{\partial}{\partial t} Y - e^{-W}\Delta\psi(Y) + \frac{1}{2}\sum_{j=1}^{N}\mu_j^2 Y = 0.$$

By approximating W by a smooth $W_\varepsilon \in C^1([0,T];\mathbb{R})$ and letting $\varepsilon \to 0$, after some calculation one concludes that the latter equation has a unique strong solution Y. We omit the details, but refer to [7] for a related treatment.

5. The finite time extinction

Assume here that ψ satisfies condition (j) of the beginning of Section 4 and that W is of the form (jj). Moreover, one assumes that

$$\psi(r)r \geq \rho|r|^{m+1},\ \forall r \in \mathbb{R}, \tag{5.1}$$

where m is as in Hypothesis (j).

Theorem 5.1. Let $d \geq 3$ and $m = \frac{d-2}{d+2}$. Let $x \in L^{m+1}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap \mathcal{H}^{-1}$ and let $X = X(t);\ t \in [0,T]$, be the solution to (1.1) given by Theorem 4.4. We set

$$\tau = \inf\{t \geq 0;\ \|X(t,\cdot)\|_{-1} = 0\}. \tag{5.2}$$

Then, for every $t > 0$,

$$X(t) = 0,\ \forall t \geq \tau, \tag{5.3}$$

and

$$\mathbb{P}[\tau \leq t] \geq 1 - \|x\|_{-1}^{1-m}\frac{C^*}{\rho\gamma^{m+1}(1 - e^{-C^*(1-m)t})}. \tag{5.4}$$

where $\gamma^{-1} = \sup\{\|u\|_{-1}|u|_{m+1}^{-1};\ u \in L^{m+1}\}$ and $C^* > 0$ is independent of the initial condition x.

Proof. We follow the arguments of [6]. The basic inequality is

$$\begin{aligned}
&\|X(t)\|_{-1}^{1-m} + \rho(1-m)\gamma^{m+1}\int_r^t \mathbf{1}_{[\|X(s)\|_{-1}>0]}ds \\
&\quad\leq \|X(r)\|_{-1}^{1-m} + C^*(1-m)\int_r^t \|X(s)\|_{-1}^{1-m}ds \\
&\quad+(1-m)\int_r^t \left\langle \|X(s)\|_{-1}^{(m+1)}X(s), X(s)dW(s)\right\rangle_{-1}, \\
&\qquad\qquad\qquad\mathbb{P}\text{-a.s.},\ 0 < r < t < \infty,
\end{aligned} \tag{5.5}$$

where C^* is a suitable constant. (We note that, by virtue of (2.8), $\gamma^{-1} < \infty$.) To get (5.5), we apply the Itô formula in (4.10) to the semimartingale $\|X_\lambda(t)\|_{-1}^2$ and to the function $\varphi_\varepsilon(r) = (r + \varepsilon^2)^{\frac{1-m}{2}}$, $r > -\varepsilon^2$, where X_λ is the solution to (4.10).

We have

$$
\begin{aligned}
& d\varphi_\varepsilon(\|X_\lambda(t)\|_{-1}^2) \\
& +(1-m)(\|X_\lambda(t)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}\left\langle X_\lambda(t), \psi_\lambda(X_\lambda(t))+\lambda X_\lambda(t)\right\rangle_2 dt \\
& =\frac{1}{2}\sum_{k=1}^{\infty}\mu_k^2\left[\frac{(1-m)\|X_\lambda(t)e_k\|_{-1}^2}{(\|X_\lambda(t)\|_{-1}^2+\varepsilon^2)^{\frac{m+1}{2}}}-(1-m^2)\frac{\|X_\lambda(t)e_k\|_{-1}^2\|X_\lambda(t)\|_{-1}^2}{(\|X_\lambda(t)\|_{-1}^2+\varepsilon^2)^{\frac{m+1}{2}}}\right]dt \\
& +2\left\langle\varphi_\varepsilon'(\|X_\lambda(t)\|_{-1}^2)X_\lambda(t), X_\lambda(t)dW(t)\right\rangle.
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \varphi_\varepsilon(\|X_\lambda(t)\|_{-1}^2)+\rho(1-m)\int_r^t(\|X_\lambda(s)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}\int_{\mathbb{R}^d}|X_\lambda|^{m+1}ds\,d\xi \\
& \quad\le\varphi_\varepsilon(\|X_\lambda(r)\|_{-1}^2)+C^*\int_r^t\|X_\lambda(s)\|_{-1}^2(\|X_\lambda(s)\|_{-1}^2+\varepsilon^2)^{-\frac{1+m}{2}}ds \\
& \quad+2\int_r^t\left\langle\varphi_\varepsilon'(\|X_\lambda(s)\|_{-1}^2)X_\lambda(s), X_\lambda(s)dW(s)\right\rangle_{-1}.
\end{aligned}
$$

Now, letting $\lambda\to 0$, we obtain that X satisfies the estimate

$$
\begin{aligned}
& \varphi_\varepsilon(\|X(t)\|_{-1}^2)+\rho(1-m)\int_r^t\left((\|X(s)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}\int_{\mathbb{R}^d}|X(s,\xi)|^{m+1}d\xi\right)ds \\
& \le\varphi_\varepsilon(\|X(t)\|_{-1}^2)+C^*\int_r^t\|X(s)\|_{-1}^2(\|X(s)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}ds \\
& +2\int_r^t\left\langle\varphi_\varepsilon'(\|X(s)\|_{-1}^2)X(s), X(s)dW(s)\right\rangle_{-1}.
\end{aligned}
\tag{5.6}
$$

Here, we have used the fact that, by Lemma 4.5, for $\lambda\to 0$,

$$X_\lambda\to X \text{ in } \mathcal{H}^{-1},$$

and, by (4.32) it follows, via Fatou's lemma,

$$\liminf_{\lambda\to 0}\int_{\mathbb{R}^d}|X_\lambda|^{m+1}d\xi\ge\int_{\mathbb{R}^d}|X|^{m+1}d\xi,$$

and

$$
\begin{aligned}
& (\|X(t)\|_{-1}^2+\varepsilon^2)^{\frac{1-m}{2}}+\rho(1-m)\gamma^{m+1}\int_r^t(\|X(s)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}\|X(s)\|_{-1}^{m+1}ds \\
& \quad\le(\|X(r)|_{-1}^2+\varepsilon^2)^{\frac{1-m}{2}}+C^*\int_r^t\|X(s)\|_{-1}^2(\|X(s)\|_{-1}^2+\varepsilon^2)^{-\frac{m+1}{2}}ds \\
& \quad+2\int_r^t\left\langle\varphi_\varepsilon'(\|X(s)\|_{-1}^2X(s)), X(s)dW(s)\right\rangle_{-1},\ 0\le r\le t<\infty,
\end{aligned}
$$

because, by (2.7), $\|x\|_{-1}\le\gamma^{-1}|x|_{m+1},\ \forall x\in L^{m+1}(\mathbb{R}^d)$. Letting $\varepsilon\to 0$, we get (5.5), as claimed.

Now, we conclude the proof as in [6]. Namely, by (5.5), it follows that

$$
\begin{aligned}
& e^{-C^*(1-m)t}\|X(t)\|_{-1}^{1-m}+\rho(1-m)\gamma^{m+1}\int_r^t e^{-C^*(1-m)s}\mathbb{1}_{[\|X_s\|_{-1}>0]}ds \\
& \quad\le e^{-C^*(1-m)r}\|X(r)\|_{-1}^{1-m} \\
& \quad+(1-m)\int_r^t e^{C^*(1-m)s}\left\langle\|X(s)\|_{-1}^{-(m+1)}X(s), X(s)dW(s)\right\rangle_{-1}
\end{aligned}
$$

and, therefore, $t \to e^{-C^*(1-m)t}\|X(t)\|_{-1}^{1-m}$ is an $\{\mathcal{F}_t\}$ supermartingale. Hence, $\|X(t)\|_{-1} = 0$ for $t \geq \tau$, because of Proposition 3.4, Chap. 2 of [22]. Moreover, taking expectation for $r = 0$, we get

$$e^{-C^*(1-m)t}\mathbb{E}\|X(t)\|_{-1}^{1-m} + \rho(1-m)\gamma^{m+1}\int_0^t e^{-C^*(1-m)s}\mathbb{P}(\tau > s)ds \leq \|x\|_{-1}^{1-m}.$$

This implies that

$$\mathbb{P}(\tau > t)\frac{1-e^{-C^*(1-m)t}}{C^*(1-m)} \leq \int_0^t e^{-C^*(1-m)s}\mathbb{P}(\tau > s)ds \leq \frac{\|x\|_{-1}^{1-m}}{\rho(1-m)\gamma^{m+1}},$$

and so (5.4) follows. This completes the proof. □

Corollary 5.2. Let $x \in \mathcal{H}^{-1} \cap L^{m+1}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ be such that $\|x\|_{-1} < \frac{\rho\gamma^{m+1}}{C^*}$. Let τ be the stopping time defined in (5.2). Then $\mathbb{P}(\tau < \infty) > 0$. In other words, there is extinction in finite time with positive probability.

Remark 5.3. In the case of bounded domain, Theorem 5.1 remains true for $m \in \left[\frac{d-2}{d+2}, 1\right)$ (see [6]). One might suspect that also in this case the extinction property (5.4) holds for a larger class of exponents m. However, the analysis carried out in [24] for deterministic fast diffusion equations in $\mathbb{R}^d$ shows that the extinction property is dependent not only on the exponent m, but also on the space $L^p(\mathbb{R}^d)$, where the solution exists (the so called extinction space).

Remark 5.4.

- The analysis in this section holds, in particular, if all the coefficients μ_k do vanish, i.e., in the deterministic framework. In that case, Theorem 5.1 implies the existence of a deterministic time $\tau > 0$ so that

$$t \geq \tau \Rightarrow \|X(t)\|_{-1} = 0,$$

  and so $X(t) = 0$, for all $t \geq \tau$.

- Let us set, for instance, $\psi(u) = u^m,\ d \geq 3,\ m = \frac{d-2}{d+2}\cdot$. Observe that $(L^1 \cap L^\infty)(\mathbb{R}^d) \subset (L^{m+1} \cap L^2)(\mathbb{R}^d) \cap \mathcal{H}^{-1}$. Consider, for instance, as initial condition $x \in (L^1 \cap L^\infty)(\mathbb{R}^d)$.

- By the Benilan-Crandall approach, see, e.g., Theorem 1 of [13], there is a solution $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, of

$$\begin{aligned} &dX - \Delta\psi(X)dt = 0 \ \text{ in } (0,T)\times\mathbb{R}^d,\\ &X(0) = x \ \text{ on } \mathbb{R}^d, \end{aligned} \tag{5.7}$$

  in the sense of distributions. u belongs to $(L^1 \cap L^\infty)((0,T)\times\mathbb{R}^d)$ and also the $\eta_u = \psi(u)$. In this case, u fulfills mass conservation.

- By use of Theorem 4.4, there is another solution $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ in the sense of distributions, such that $v \in L^p((0,T) \to \mathbb{R}^d)$, with $p = \max(1, 2m)$. Also, $\eta_v = \psi(v) \in L^{\frac{p}{m}}((0,T)\times\mathbb{R}^d)$. By Theorem 5.1, if x is small enough, there will be extinction, and so, v does not fulfill any mass conservation.

- In particular, there is no uniqueness for (5.7) in the sense of distributions. Remark that according to [16], uniqueness is guaranteed in the class $(L^1 \cap L^\infty)((0,T)\times\mathbb{R}^d)$.

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