A New Algorithm to Compute Inverse Dynamic of a Class of Nonlinear Systems

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Abstract

This paper supplies a new algorithm to compute the internal dynamics (or inversion dynamics) of affine MIMO control nonlinear systems. This algorithm consider both cases regular or singular characteristic matrix. Furthermore, it provides an extension to a class of nonlinear descriptor systems.

1. INTRODUCTION

Broadly speaking, the inverse dynamic of an input-output dynamical system involves its decomposition into an external part, that enables an explicit relationship between inputs and outputs, and an internal part that is governed by its dynamics without input. This last dynamics provides naturally the so-called zero dynamics when the external variables are maintained at zero. For, the single input single output (SISO) linear dynamical system the inverse dynamic has been completely characterized by its transfer function [17]. This problem was also solved in [11] within the SISO nonlinear case, where a full-order realization was given.
However, it is not an easy task when you dealing with a MIMO nonlinear dynamical system. Several research have dealt this problem, for example in [8] the concept of zero dynamics was connected to the inverse dynamic. Then, [13] provided a nice interpretation of input- output linearization via a feedback removing the dynamics of zero.
Despite intensive research in this subject, there are only few of them that supplies computational algorithms of inverse dynamic: [9] gave an algorithm to calculate the inverse and the zero dynamics see also [12].
Another solution proposed for such problem, is to determine the class of nonlinear dynamical systems which are input-output linearizable. Necessary and sufficient geometrical conditions have been stated [15] and [16].
Therefore the main priority objective of this paper begins to supply a new algorithm to compute the dynamic inversion for an affine MIMO nonlinear control systems with regular characteristic matrix as well singular matrix. In this last case, it provides an other algorithm to render the singular characteristic matrix regular.
This paper is organized as follows: Next section recalls, necessary and sufficient geometrical condition for solvability of the inversion dynamic problem for a MIMO dynamical system with regular characteristic matrix. It recalls also the involutivity concept by means of differential forms. Section three, for a MIMO dynamical system with singular characteristic matrix, in a first stage provides un algorithm to increase the rank of the singular characteristic matrix and then states sufficient condition for solvability of the inversion dynamic problem. The fourth section gives the geometrical algorithm by mean of a projector to compute the inverse dynamic. Section five extended this result to a class of descriptor dynamical systems. The last section is devoted to examples to highlight the proposed algorithm.

2. Notations and Dynamic Inversion for regular characteristic matrix

This section takes care to recall necessary and sufficient condition of the exitance of the inverse dynamic for a dynamical system with regular decoupling matrix (or regular characteristic matrix). To do so, let us consider the following MIMO dynamical system described as follows

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (1)$$

$$y = h(x) \qquad (2)$$

where $x \in U \subseteq \Re^n$ represents the state, $y = (y_1, ..., y_m)^T$ represent the outputs and $u = (u_1, ..., u_m)^T$ can be a vector of unknown inputs, perturbations or faults.
First, let us start with a definition of the so-called invert-

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ibility of a dynamical system [14].

Definition 1 *Dynamical system (1-2) is said to be invertible at x_0, if two distinct inputs $u^1 \neq u^2$ provide tow distinct inputs $y(t;u^1,x_0) \neq y(t;u^2,x_0)$.*

As the dynamical system (1-2) is supposed square i.e. it has as many inputs as outputs, then the realization of its inverse dynamic can be expressed as follows [14]:

$$\dot{\eta} = \varphi(\eta, y, \dot{y}, ...) \tag{3}$$

$$u = \omega(\eta, y, \dot{y}, ...) \tag{4}$$

where η is a function of sub-state of the state x to be determined. It represents also the internal state that doesn't have a relationship with inputs. It determination is a crucial issue on the inverse dynamic. To our knowledge there is no computational algorithm to determine η.
Before stating the conditions of existence of the inverse dynamic for (1-2), let us assume the following

1. the distribution $\Delta = Span\{g_1, ..., g_m\}$ has dimension m and is involutive i.e. is stable by Lie bracket

2. There exist integers $r_1, \dots, r_m$, such that for $1 \leq i \leq m$, $\exists k_i \in \{1, \dots, m\}$ such that

$$\begin{aligned} dL_f^{j-1}h_i(g_s) &= 0, \text{for all}\, s \neq k_i, 1 \leq j < r_i \\ dL_f^{r_i-1}h_i(g_{k_i}) &\neq 0 \end{aligned}$$

   where for $i = 1:m$ the function $L_f^{r_i-1}h_i$ is the $(r_i - 1)^{th}$ Lie derivative of h_i in the direction of the vector field f. The numbers $r_1, \dots, r_m$ are the so-called the relative degrees [12] and $r = r_1 + ... + r_m \leq n$ is the total relative degree.

3. the decoupling or characteristic matrix

$$\Gamma(x) = \begin{pmatrix} L_{g_1}L_f^{(r_1-1)}h_1 & ... & L_{g_m}L_f^{(r_1-1)}h_1 \\ ... & ... & ... \\ L_{g_1}L_f^{(r_m-1)}h_m & ... & L_{g_m}L_f^{(r_m-1)}h_m \end{pmatrix}$$

   is regular i.e. it has the full rank m

Now, consider the following partial change of coordinates

$$\xi_i = (\xi_{i,1}, \xi_{i,2}, ..., \xi_{i,r_i})^T = \left(h_i, L_f h_i, ..., L_f^{(r_i-1)}h_i\right)^T \tag{5}$$

For more details, the following result can be viewed in [12].

Theorem 2 *Under the second point of assumption 2 there exist locally $(n-r)$ variables $\eta = (\eta_1, ..., \eta_{n-r})$ independent from ξ such that in $(\xi, \eta =$ coordinates (1-2)can be rewritten as follows*

$$\begin{aligned} \dot{\xi}_{i,j} &= \xi_{i,j+1} \\ &\quad for \quad 1 \leq i \leq m \;\; and \;\; 1 \leq j \leq r_i - 1 \end{aligned} \tag{6}$$

$$\dot{\xi}_{i,r_i} = b_i(\xi, \eta) + \sum_{j=1}^{m} a_{i,j}(\xi, \eta) u_j \; for \; 1 \leq i \leq m \tag{7}$$

$$\dot{\eta} = \overline{f}(\xi, \eta) \tag{8}$$

where $b_i(\xi, \eta) = L_f^{r_i}h$ for $i = 1:m$ and $a_{i,j} = L_{g_j}L_f^{r_i-1}h_i$ for $j = 1:m$ are the $\Gamma(x)$ coefficients.

Therefore, from equation (6-8), we can deduce the inverse dynamic as follows

$$\left\{ \begin{array}{c} \dot{\eta} = \overline{f}(\xi, \eta) \\ \nu = \Gamma^{-1}(\xi, \eta) \left(\begin{pmatrix} \dot{\xi}_{1,r_1} \\ \dot{\xi}_{2,r_2} \\ ... \\ \dot{\xi}_{m,r_m} \end{pmatrix} - \begin{pmatrix} b_1(\xi, \eta) \\ b_2(\xi, \eta) \\ ... \\ b_m(\xi, \eta) \end{pmatrix} \right) \end{array} \right. \tag{9}$$

From the definition (5) of ξ and equations (6-8) it can be see that $\xi_{i,1} = y_i$ for $i = 1:m$ and for $j = 2:r_j - 1$ $\xi_{i,j}$ is equal to $= y_i^{(j-1)}$ the $(j-1)^{th}$ derivative of the output y_i, therefore, (9) is in the form (3)-(4).

Remark 3 *Let Δ^T to be the co-distribution annihilator of Δ. As Δ is of rank m and involutive then Δ^T is of rank $(n-m)$. Therefore, by Frobenius' theorem Δ^T is locally spanned by*

- *the $(r-m)$ independent exact 1-forms $\{d\xi_{i,j}, \;\; i = 1:m, \;\; r_i \geq 2\}$ together with*

- *$(n-r)$ independent 1-forms says $d\eta_i, \;\; i = 1:(n-r)$ to be determined.*

This two kind of 1-forms give us the coordinates to obtain the normal form (6-8). However, as can be seen, the above Theorem 2 supplies a characterization of the existence of an inverse dynamic rather than an algorithm to compute it. Specifically, the coordinates η is not readily determinable (see for example [12] page 225). In our knowledge there is no simple algorithm for determining these coordinates. Therefore, one of the main objectives being to supply an algorithm which enables us compute the internal state variable η.

We end this section by some background from differential geometry needed to understand within the rest of this paper, for more details see for example [4], [5].

Consider a distribution $\Delta = span\{g_1, ..., g_m\}$ of rank m and denote its annihilator by $\Delta^T = span\{\theta_1, ..., \theta_{n-m}\}$ to be the co-distribution spanned by differential 1-forms θ_i such that $\theta_i(g_j) = 0$ for $i = 1 : n-m \;\; j = 1 : m$.
The well-known following result expresses the integrability in the foliation theory [4], [5].

Lemma 4 *The distribution Δ (or co-distribution Δ^T) is involutive if and only if one of the following equivalent properties holds:*

- *The Lie bracket $[g_i, g_j] \in \Delta$ for all $1 \leq i, j \leq m$. Thus Δ is closed by Lie Bracket.*
- *The differential $d\theta_i = \sum_{j=1}^{m-n} \omega_{i,j} \wedge \theta_j$ for all $i = 1 : n-m$. Where $\wedge$ is the wedge product and $\omega_{i,j}$ are 1-forms.*

To highlight the equivalence between both properties stated in the previous lemma, the two following facts are recalled:

- the evaluation of the differential of the 1 forme θ of two vector fields X, Y is given by this formula:

$$d\theta(X,Y) = L_Y\theta(X) - L_X\theta(Y) + \theta([X,Y]).$$

- If ω is an other 1-form then

$$\omega \wedge \theta(X,Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X).$$

From this, Frobenius' theorem can be expressed as follows:
If Δ^T is involutive then there exist (locally) $(n-m)$ exact 1-forms $d\nu_1, ..., d\nu_{n-m}$ such that for all $i = 1 : n-m$ we have:

$$\theta_i = \sum_{j=1}^{m-n} \kappa_{i,j} d\nu_j \tag{10}$$

thus $\Delta^T = span\{d\nu_1, ..., d\nu_{n-m}\}$. In this coordinates, it claimed that Δ^T is integrable. This way to compute the coordinates ν will be enables us to compute the coordinates η.

The main purpose of this paper is to supply an algorithm for computing the internal state η. Nevertheless it is interesting to investigate before the case of singular decoupling matrix Γ and to generalize the theorem 2.

3. Dynamical inversion: singular characteristic matrix

This section is based on the algorithm proposed in [19], but in the context of dynamical inversion instead of left inversion [20, 10]. Considering again the system (1)-(2) but with singular matrix Γ (for the sake of algorithm's notation denoted Γ_0).

$$\Gamma_0 = \begin{pmatrix} L_{g_1} L_f^{(r_1-1)} h_1 & ... & L_{g_m} L_f^{(r_1-1)} h_1 \\ ... & ... & ... \\ ... & ... & ... \\ L_{g_1} L_f^{(r_m-1)} h_m & ... & L_{g_m} L_f^{(r_m-1)} h_m \end{pmatrix}$$

With $rank\{\Gamma_0\} < p$.
The main idea of the algorithm that we suggest here is to seek new outputs that enable to increase the rank of matrix Γ_0.

Algorithm I: Let $\pounds$ be the commutative algebra of the measured outputs and their successive Lie derivatives up to order $r_i - 1$: $\pounds = \text{span}\{h_1, ..., L_f^{r_1-1} h_1, ..., h_m, ..., L_f^{r_m-1} h_m\}$ and let $d\pounds$ be the co-distribution:

$$d\pounds = \text{span}\{dh_1, ..., dL_f^{r_1-1} h_1, ..., dh_m, ..., dL_f^{r_m-1} h_m\}.$$

As the rank of Γ_0 is strictly smaller than m, then there exists a $1 \times m$ row vector $K(x) = (k_1(x), ..., k_m(x)) \neq 0$ such that:

$$K(x)\Gamma_0(x) = 0 \text{ for all } x \in U \tag{11}$$

By the definition of relative degree, we have

$$\begin{pmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \\ ... \\ y_m^{(r_m)} \end{pmatrix} = \begin{pmatrix} L_f^{r_1} h_1 \\ L_f^{r_2} h_2 \\ ... \\ L_f^{r_m} h_m \end{pmatrix} + \Gamma_0(x)u \tag{12}$$

where $y_i^{(r_i)}$ the r_ith derivative of y_i. therefore by multiplying both members of the above equation by $K(x)$ we obtain

$$\sum_{i=1}^{m} k_i(x) L_f^{r_i} h_i(x) = \sum_{i=1}^{m} k_i(x) y_i^{(r_i)}$$

At each stage of the algorithm we will assume that $k_i(x) \in \pounds \; \forall i \in \{1, ..., m\}$. If not we stop the algorithm. Under this assumption we set

$$\bar{y} = \bar{h} = \sum_{i=1}^{m} k_i(x) L_f^{r_i} h_i(x) \tag{13}$$

If $\dot{\bar{y}}$ doesn't belong to $\pounds$[1], then it will be considered as a new output.

[1] In the reference [19] $\dot{\bar{y}} \notin \pounds$ is computed as follow $d\dot{\bar{y}} \notin d\pounds$

At this stage we set $y^{\bar{1}} := (y^{\bar{0}T}, \bar{y}^T)^T$ and we compute the corresponding Γ (i.e. Γ_1).

$$\Gamma_1 = \begin{pmatrix} L_{g_1} L_f^{(r_1-1)} h_1 & \dots & L_{g_m} L_f^{(r_1-1)} h_1 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ L_{g_1} L_f^{(r_m-1)} h_m & \dots & L_{g_m} L_f^{(r_m-1)} h_p \\ L_{g_1} L_f^{(r_{m+1}-1)} h_{m+1} & \dots & L_{g_m} L_f^{(r_{m+1}-1)} h_{p+1} \\ \dots & \dots & \dots \\ L_{g_1} L_f^{\left(r_{m+l_1}-1\right)} h_{m+l_1} & \dots & L_{g_m} L_f^{\left(r_{m+l_1}-1\right)} h_{m+l_1} \end{pmatrix}$$

If the rank of Γ_1 is equal to m the procedure stop. If not we began the same procedure

Remark 5 *1. Even if at some step j the dynamical together with output $y^{\bar{j}}$ is not square. which may be due to the fact that the new output $y^{\bar{j}}$ is a vector of dimension $p+l_1$ with l_1 the number of algebraically independents new selected outputs. So the relative degree for this new set of outputs are r_i for $i \in \{1,...,m+l_1\}$.*

2. *If one of the characteristic index of a new output is infinite, then it will be removed from the list, because here the purpose is not to deal with observability but only with dynamical inversion and to construct a matrix Γ_j with $j \in \{0,1,..\}$ of rank equal to m the input number.*

3. *We stop the algorithm in the case where Γ_γ has achieved the rank m or if at one step of the procedure we can not find a new $\bar{y}$ with finite relative degree and $\dot{\bar{y}} \notin \pounds$.*

If the algorithm supplies enough outputs at a step γ to obtain the rank of Γ_γ equal to m, then similarly to the previous section, we have the following sufficient conditions for dynamical inversion.

Theorem 6
Assume that the above algorithm supplies a matrix Γ_γ of rank m then there exist $\rho = n + l_\gamma - r_\gamma$ variables $\eta = (\eta_1, ..., \eta_\rho)$ independent from ξ, with l_γ the number of new outputs added by the algorithm, such that in this coordinates (1-2) became:

$$\begin{aligned} \dot{\xi}_{i,j} &= \xi_{i,j+1} \\ & \quad for \quad 1 \in \{1,..,m+l_\gamma\} \text{ and } 1 \le j \le r_i - 1 \quad (14) \\ \dot{\xi}_{i,r_i} &= b_i(\xi,\eta) + \sum_{j=1}^{m} a_{i,j}(\xi,\eta) u_j \\ & \quad for \quad 1 \le i \le m + l_\gamma \quad (15) \\ \dot{\eta} &= \overline{f}(\xi,\eta) \quad (16) \end{aligned}$$

where $b_i(\xi,\eta) = L_f^{r_i} h_i$ for $i = 1 : m + l_\gamma$ and $a_{i,j} = L_{g_j} L_f^{r_i - 1} h_i$ for $j = 1 : m + l_\gamma$ are the $\Gamma_\gamma(x)$ coefficients. Moreover h_i for $i \in \{m+1, ..m+l_\gamma\}$ is the new output given by the algorithm.

Again similarly to the previous section, from (14-16) it is possible to deduce the inverse dynamics as follow:

$$\left\{ \begin{array}{c} \dot{\eta} = \overline{f}(\xi,\eta) \\ u = (P\Gamma_\gamma)^{-1} P \left(\begin{pmatrix} \dot{\xi}_{1,r_1} \\ \dots \\ \dot{\xi}_{m,r_m} \\ \dots \\ \dot{\xi}_{m+l_\gamma, r_{m+l_\gamma}} \end{pmatrix} - \begin{pmatrix} b_1(\xi,\eta) \\ \dots \\ b_m(\xi,\eta) \\ \dots \\ b_{m+l_\gamma}(\xi,\eta) \end{pmatrix} \right) \end{array} \right. \quad (17)$$

Where $P \in \Re^{m \times m + l_\gamma}$ is a linear projection matrix such that $Rank\{P\Gamma_\gamma\} = Rank\{\Gamma_\gamma\} = m$. From the definition (5) of ξ and the normal form (14-16), it can be see that $\xi_{i,1} = y_i$ for $i = 1 : m + l_\gamma$ and for $j = 2 : r_j - 1$ $\xi_{i,j}$ is equal to $= y_i^{(j-1)}$ the $(j-1)^{th}$ derivative of the output y_i, therefore, (17) is in the form (3)-(4).

Remark 7 *The change of coordinates $(\xi;\eta) = \phi(x)$ is an immersion (see [21] for an interesting application of such concept)*

4. An algorithm to compute the inverse dynamic

This section presents an algorithm to determine the variables η in (3-4). For this, considering again the dynamical system (1)-(2):

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{aligned}$$

Let us set:

- $G = [g_1, ..., g_m]$ and
- $\sigma = P \begin{bmatrix} dL_f^{(r_1-1)} h_1 \\ \dots \\ dL_f^{(r_m-1)} h_m \\ \dots \\ dL_f^{\left(r_{m+l_\gamma}-1\right)} h_{m+l_\gamma} \end{bmatrix}$

  It easy to see that we have [2] $P\Gamma_\gamma = \sigma G$ is square and of rank equal to m. Where σG the evaluation of σ on G.

The following result define and characterize a projector which is the key of the computation of inverse dynamic

Proposition 8 *The map $\Pi : T\mathscr{X} \to T\mathscr{X}$ on the fibre bundle $T\mathscr{X}$ of $\mathscr{X}$ defined by*

$$\Pi = I_n - G(P\Gamma_\gamma)^{-1}\sigma.$$

[2] In the regular case $P\Gamma_\gamma = \Gamma_0$

Π *is a projector and its kernel is such that:*

$$ker\Pi = \Delta := span\{g_1,...,g_m\}.$$

Moreover, Π *verifies:*

i) $\Pi^2 = \Pi$,

ii) $\Pi G = 0$,

ii) $rank\Pi = m$.

Proof. Let us check this two properties:

i) $\Pi^2 = \Pi \circ \Pi = \Pi = I_n - 2G(P\Gamma_\gamma)^{-1}\sigma + G(P\Gamma_\gamma)^{-1}\sigma G(P\Gamma_\gamma)^{-1}\sigma = \Pi$

ii) $\Pi G = G - 2G(P\Gamma_\gamma)^{-1}\sigma G = G - G = 0$,

The fact that Π is of rank m is due to $Rank\sigma = m$ and $P\Gamma_\gamma = \sigma G$ invertible. ■

Before to state the main result of this work, it is necessary to recall some notations:

1. Let $\Pi := (\pi_{i,j})_{1\leq i,j\leq n}$. As Π acts on vector fields, it can be considered as multi valued 1-forms. Thus can view it in the following form:

$$\Pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix}$$

   where $\pi_i = (\pi_{i,1},...,\pi_{i,n}) = \sum_{j=1}^n \pi_{i,j}dx_j$ for $1 \leq i \leq n$.

2. As Δ is of dimension m, then rank$(\Pi) = n-m$ and the co-distribution Δ^T is spanned by $n-m$ independent 1-forms $\{\pi_{i_1},...,\pi_{i_{n-m}}\}$.

3. Among the 1-forms π_{i_j}, $r-m$ of them are linear combination of $\{dL_f^j h_i,\ 1\leq i\leq m\ 0\leq j\leq r_i-1\}$.

Now, it is possible to establish the following result which is the key point of the algorithm in order to determine η.

Corollary 9 *Dynamical system (1-2) with* $Rank\{\Gamma_\gamma\} = m$ *admits an inverse dynamic if one of the following equivalent conditions is fulfilled:*

1. *the distribution* Δ *is involutive*

2. *the co-distribution* Δ^T *is involutive.*

Moreover, the coordinates of the internal state η *are given by a subset of* π_{ik}*.*

This corollary gave the dual condition of theorem 2. The fact that conditions 1)-2) in corollary are equivalent is obviously true. In practices, before to construct Π, we check if condition 1) is true and then condition 2) allows us to determine the coordinate for the inverse system.
So, if condition 2) is fulfilled, then for $k = 1 : n+l_\gamma-r_\gamma$:

$$\pi_{i_k} = \sum_{j=1}^{n+l_\gamma-r_\gamma} \kappa_{i_k,j}(x)d\eta_j.$$

where $\kappa_{i_k,j}$ are functions and $\eta_1,...,\eta_{n+l_\gamma-r_\gamma}$ are $n+l_\gamma-r_\gamma$ independent functions. Moreover, Δ^T is

$$\Delta^T = Span\{d\eta_j, dL_f^i h_l\ \ 1\leq l\leq m+l_\gamma,\ \ 0\leq i\leq r_l-1\}$$

Before to present the algorithm, it is important to consider the following example in order to highlight some difficulties.

Example 10 *Let us consider the following dynamical system:*

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= u \\
\dot{\chi} &= \alpha(z,\chi)+\beta(z,\chi)u \\
y &= z_1
\end{aligned}$$

Set $g(x) = (0,1,\beta(z,\chi))^T$ *then it is clear that the relative degree is* 2*. The projector is as follows:*

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\beta & 1 \end{pmatrix}$$

thus the 1*-forms are* $\pi_1 = dz_1$*,* $\pi_2 = 0$ *and* $\pi_3 = -\beta dz_2 + d\chi$*. As* $\Delta = span\{g\}$ *is of dimension* 1 *then it is involutive the same holds true for* $\Delta^T = span\{\pi_1,\pi_3\}$*. Let us give some particular cases to point out some difficulties.*

- *if* $\beta = z_2$ *then* $\pi_3 = d(-\frac{1}{2}z_2^2+\chi) = d\eta$ *and the internal dynamic is*

$$\dot{\eta} = \alpha(z,\eta+\frac{1}{2}z_2^2)$$

- *if* $\beta = z_1$ *then* $\pi_3 = d(-z_1z_2+\chi)+z_2dz_1 = d\eta + z_2\pi_1$ *and the internal dynamic is*

$$\dot{\eta} = \alpha(z,\eta+z_1z_2)-z_2^2$$

- *if* $\beta = \chi$ *then* $\pi_3 = -\chi dz_2 + d\chi = l d\eta$*, where* $\eta = e^{\chi e^{-z_2}}$ *and* $l = e^{-z_2}e^{\chi e^{-z_2}}$ *and the internal dynamic is*

$$\dot{\eta} = -\chi e^{-z_2}e^{\chi e^{-z_2}}\alpha(z,ln(\eta e^{z_2}))$$

Algorithm II: Under the condition $\exists \Gamma_\gamma$ of rank m and assuming that Δ is involutive, it is easy to see that by successive derivative of output and a feedback, we can rewrite the dynamical system in the following form:

$$\begin{aligned}
\dot{\xi}_{i,j} &= \xi_{i,j+1} \\
&\text{for} \quad i \in \{1, ...m + l_\gamma\} \text{ and } 1 \leq j \leq r_i - 1 \quad (18)\\
\dot{\xi}_{i,r_i} &= b_i(\xi, \eta) + \sum_{j=1}^{m} a_{i,j}(\xi, \eta) u_j \\
&\text{for} \quad 1 \leq i \leq m + l_\gamma \quad (19)\\
\dot{\chi}_i &= B(\xi, \chi) + \sum_{j=1}^{m} \mu_{i,j}(\xi, \chi) u_j \\
&\text{for} \quad 1 \leq i \leq n + l - \gamma - r_\gamma \quad (20)\\
& \quad (21)
\end{aligned}$$

In fact, the projector has the following form: $\Pi = [M, R]$ where the matrix $M = diag[M_1, ..., M_{m+l_\gamma}]$ with M_i is a $r_i \times r_i$ matrix given by:

$$M_i = \begin{pmatrix} 1 & 0 & ... & 0 & 0 \\ ... & 1 & ... & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

and the matrix $R = [R_{n+l_\gamma - r_\gamma+1}, ..., R_n]$ where R_k is matrix $1 \times n$ such that R_k has $R_{k,r_j} = -\mu_{i,j}$, $R_{k,k} = 1$ for $k \geq n + l_\gamma - r_\gamma + 1$ in term of differential form we have: $R_k = d\chi_i - \sum_{j=1}^{m} \mu_{i,j} d\xi_{i,r_j}$

Therefore, if Δ is involutive then by using equation (10), there exist a function η_i such that:

$$d\chi_i - \sum_{j=1}^{m} \mu_{i,j} d\xi_{i,r_j} = \sum_{j=1}^{n+l_\gamma - r_\gamma} \kappa_{i,j} d\eta_j$$

where η_i are the sought coordinates.

Remark 11 *There is an interesting case in which $\pi : \chi \to \chi$ defined (locally) χ such that:*

$$d\pi := \pi_* = \Pi$$

the differential of π is equal (locally) to Π. Therefore by Poncaré's lemma the projector Π is integrable if and only if is closed i.e. $d\Pi = 0$.

5. The case of descriptor system

The previous algorithm below can be applied to a class of descriptor systems. Considering the following class of descriptor systems:

$$\begin{aligned}
\dot{x} &= f_1(x) + p^1(x)z + g^1(x)u \quad (22)\\
0 &= f_2(x) + p^2(x)z + g^2(x)u \quad (23)\\
y &= h(x) \quad (24)
\end{aligned}$$

where $x \in \mathbb{R}^n$ is the dynamic variable, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output and $z \in \mathbb{R}^s$ is the algebraic variable.

Moreover, f_1 is a vector field $n \times 1$, $p^1(x)$ is a matrix $n \times s$, h is the vector $m \times 1$ and $g^1(x)$ is a matrix $n \times k$ and $p^2(x)$ is a $s \times s$ matrix. Moreover, we assume: The algebraic variable of descriptor system (22-23) is u-solvable.

Definition 12 *System (22-23) is u-solvable, if there exists a feedback*

$$u = \alpha(x) + \gamma(x)z + \beta(x)v \quad (25)$$

with β invertible such that: the matrix $M(x) := p^2(x) + g^2(x)\gamma(x)$ is regular.

Remark 13

- *The above assumption is equivalent to $rank\{p_2, g_2\}$ is equal to s the dimension of the algebraic variables z.*
- *It is shown in [2] that under this assumption that (22-23) is strongly regular i.e. the system (22-23) has a unique solution for any initial condition x_0 and continuous u.*

It is also proved that this kind of feedback (25) doesn't change the relative degree. Therefore, the system (22-23) controlled by (25) can be rewritten on the follows z–form:

$$\left\{ \begin{array}{c} \dot{x} = \tilde{f}(x) + \sum_{i=1}^{m} \tilde{g}_i v_i \\ z = -M^{-1}\left(f_2 + g_2\alpha + g_2\beta v\right), \\ y = h(x) \end{array} \right. \quad (26)$$

where

- $\tilde{f} = f_1 + g_1\alpha - NM^{-1}\left(f_2 + g_2\alpha\right)$
- $\tilde{g} = g_1\beta - NM^{-1}g_2\beta$

with $M = p^2(x) + g^2(x)\gamma(x)$ and $N = p^1 + g^1\gamma(x)$

Definition 14 *The system (22-24) has an inverse dynamics if for the system (26) there exists an immersion (or a diffemorphism in a regular case i.e. $\Gamma_0 = \Gamma_\gamma$) $(\xi, \eta) = \phi(x)$ such that :*

$$\begin{aligned}
\dot{\eta} &= q(\eta, \xi) \quad (27)\\
u &= \psi(\eta, \xi) \quad (28)\\
z &= \varphi(\eta, \xi, u) \quad (29)
\end{aligned}$$

From this definition, it is possible to set the following corollary:

Corollary 15 *The system (22-24) admits an inverse dynamics, if it is $u-$solvable and its $z-$form (26) verifies the following conditions:*

- $\exists\Gamma_\gamma$ *of rank m*
- $\Delta = span\{\tilde{g}_1, ..., \tilde{g}_m\}$ *is involutive*

In this case $\xi = (\xi_{1,1}, ..., \xi_{1,r_1}, ..., \xi_{m+l_\gamma,1}, ..., \xi_{m+l_\gamma,r_{m+l_\gamma}})^T$ are the external variables given by $\xi_{i,j} = L^j_{\tilde{f}} h_i$ for $i = 1 : m + l_\gamma$ and $j = 1 : r_i - 1$ defined as follows:

$$\dot{\xi}_{i,j} = \xi_{i,j+1} \text{ for } i = 1 : m+l_\gamma \text{ and } j = 1 : r_i - 1 \quad (30)$$

$$\dot{\xi}_{i,r_i} = b_i(\xi,\eta) + \sum_{j=0}^{m} a_{i,j}(\xi,\eta) v_j \text{ for } i = 1 : m + l_\gamma \quad (31)$$

After that $\eta(x)$ is determined by using the algorithm II for the dynamic of (26). Moreover, the control u is obtained from v and (25). And finally v is replaced by u in (31)from the equation (25) and the fact that β is regular.

6. Examples

Example 16 *Consider the dynamical system given by*

$$\begin{aligned}
\dot{x}_1 &= x_2 - x_1^3 + u_2 \\
\dot{x}_2 &= x_3 + x_2^2 - x_2^3 - 3u_1 \\
\dot{x}_3 &= x_5 \\
\dot{x}_4 &= -x_4 + x_2^2 + 2u_1 \\
\dot{x}_5 &= -x_3 + x_6 u_2 \\
\dot{x}_6 &= -x_6 + u_2
\end{aligned}$$

The outputs are $y_1 = x_1$ and $y_2 = x_6$. The relative degrees r_1, r_2 are respectively $1, 1$. The unknown directions are

$$G = [g_1, g_2] = G = \begin{pmatrix} 0 & 1 \\ -3 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & x_6 \\ 0 & 1 \end{pmatrix}$$

It is easy to see that the involutivity condition is fulfilled. Now, as the matrix Γ_0 is singular because

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

So, it is necessary to use algorithm I. The first step using $K(x) = (1, -1)$ and by multiplying equation (12) by $K(x)$ we obtain

$$\bar{y} = \dot{y}_1 - \dot{y}_2 = x_2 - x_1^3 + x_6.$$

As this new output contains the term x_2, then it doesn't belong to £. Moreover, its relative degree is equal to 1. *Therefore the matrix Γ_1 is as follows*

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ -3 & 1 - 3x_1^2 \end{pmatrix}$$

which is of full rank 2*. Now, we choose the projection matrix P as follow:*

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A straightforward calculation provides

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -3x_1^2 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and matrix $P\Gamma_1$

$$P\Gamma_1 = \begin{pmatrix} 0 & 1 \\ -3 & 1 - 3x_1^2 \end{pmatrix}$$

And, the projector is given by:

$$\Pi(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 3x_1^2 & 0 & 0 & 0 & 0 & -3x_1^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2x_1^2 & \frac{2}{3} & 0 & 1 & 0 & 2x_1^2 \\ 0 & 0 & 0 & 0 & 1 & -x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This projector provides the following 1*-forms: $\pi_1 = d(x_1 - x_6) = d\xi_{1;2}$, $\pi_2 = 3x_1^2 d\xi_{1;2} = 3x_1^2\pi_1$, $\pi_3 = dx_3 = d\eta_1$, $\pi_4 = -2x_1^2 d\xi_{1;2} + \frac{2}{3}d(x_2 + x_4) = -2x_1^2\pi_1 + d\eta_2$ $\pi_5 = d(x_5 - \frac{1}{2}x_6) = d\eta_3$, $\pi_4 = -2x_1^2\pi_1 + d\eta_2$ and $\pi_6 = 0$.*
It is clear that $(4 = n - m)$ of them are independent: π_1, π_3, π_4 and π_5.
Because of π_4 this projector is not integrable following Remark 11. Nerveless, form (10) Lemma 4 we can conclude that Δ^T is spanned by:
$d\xi_{1;2}$, $d\eta_1$, $d\eta_2$ and $d\eta_3$.
Now, state ξ is defined as follow $\xi_{1,1}$, $\xi_{1,2} = \bar{y}$ and $\xi_{2,1} = x_6$.
Finally by the derivative of η coordinates we obtain the inverse dynamic:

$$\begin{aligned}
\dot{\eta}_1 &= \eta_3 + \frac{1}{2}\xi_{2,1}^2 \\
\dot{\eta}_2 &= \frac{2}{3}\eta_1 - \eta_2 + \frac{5}{3}\left(\bar{y} + y_1^3 - y_2\right)^2 \\
&+ \frac{2}{3}\left(\bar{y} + y_1^3 - y_2\right) + \left(\bar{y} + y_1^3 - y_2\right)^2 \\
\dot{\eta}_3 &= -\eta_1 - \xi_{2,1}^2
\end{aligned}$$

And finally, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -x_1^2+\frac{1}{3} \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} \dot{\xi}_{1,2} \\ \dot{\xi}_{2,1} \end{pmatrix} - \begin{pmatrix} \eta_1 + \left(\bar{y}+y_1^3-y_2\right)^2 - \left(\bar{y}+y_1^3-y_2\right)^3 - 3y_1^2\left(\bar{y}-y_2\right)+\xi_{2,1} \\ -\xi_{2,1} \end{pmatrix} \right)$

Example 17 *Consider the following descriptor dynamical system described as follows*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -cos(x_1) - z \\ \dot{x}_3 &= -x_3 + x_1^2 + \frac{1}{1+x_2^2} z \\ 0 &= cos(x_1) + x_2^2 + u + z \end{aligned}$$

The output is $y_1 = x_1$ *and setting* $u = v - x_2^2$*, the algebraic variable is:*

$$z = -cos(x_1) - v$$

so the system became:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v \\ \dot{x}_3 &= -x_3 + x_1^2 - \frac{1}{1+x_2^2}(cos(x_1) + v) \end{aligned}$$

The control direction is

$$g = \begin{pmatrix} 0 \\ 1 \\ \frac{-1}{1+x_2^2} \end{pmatrix}$$

and $\sigma = (0,1,0)$*, so* Π *is equal to*

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{1+x_2^2} & 1 \end{pmatrix}$$

So, $\pi_1 = dx_1$*,* $\pi_2 = 0$ *and* $\pi_3 = \frac{dx_2}{1+x_2^2} + dx_3$ *and consequently* $\eta = arctg(x_2) + x_3$ *and the dynamic of* η *is:*

$$\dot{\eta} = -\eta + arctg(\dot{y}) + y^2 + \frac{cos(y)}{1+\dot{y}^2}$$

Finally, as $v = \ddot{y}$*, u is equal to:*

$$u = \ddot{y} - \dot{y}^2$$

Conclusion 18 *In this paper it was given a dual formulation of the geometrical condition for the existence of an inverse dynamic of a nonlinear dynamical system. This was done for system with regular and singular 'decoupling' matrix. This algorithm is based on a projector on the tangent fibre bundle and for singular characteristic matrix some algebraic manipulations and an iterative algorithm was proposed. At the end of paper an extension of the method, to descriptor system with the concept of* $u-$*solvable was also proposed. In future works, the problem of system with known and unknown input will be investigated and the stability problem of* η *dynamics will be treated with a particular choice of immersion.*

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