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Abstract: We propose the first sound and complete bisimilarities for the call-by-name and call-by-value untyped $\lambda\mu$-calculus. We define applicative bisimilarities for both semantics and environmental bisimilarity for call-by-name. We give equivalence examples to illustrate how our relations can be used; in particular, we prove David and Py's counter-example, which cannot be proved with Lassen's preexisting normal form bisimilarities for the $\lambda\mu$-calculus.

Key-words: Applicative bisimilarity, $\lambda\mu$-calculus, call-by-name, call-by-value

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Bisimilarités correctes et complètes pour le $\lambda\mu$-calcul en appel par nom et par valeur

Résumé : Nous proposons les premières définitions de bisimilarités correctes et complètes pour le $\lambda\mu$-calcul non typé en appel par nom et en appel par valeur. Nous définissons une bisimilarité applicative pour chacune des sémantiques, et une bisimilarité environnementale en appel par nom. Nous donnons des examples d'équivalences pour montrer comment ces relations peuvent être utilisées ; en particulier, nous prouvons le contre-exemple de David et Py, qui ne peut être démontré avec la bisimilarité de forme normale définie auparavant par Lassen.

Mots-clés : Bisimilarité applicative, $\lambda\mu$-calcul, appel par nom, appel par valeur

1 Introduction

Contextual equivalence [13] is considered as the most natural behavioral equivalence in languages based on the λ-calculus. Two terms are contextually equivalent if an outside observer cannot tell them apart when they are evaluated within any *context* (a term with a hole). However, the quantification over contexts makes proving the equivalence of two given programs difficult. Consequently, characterizations of contextual equivalence are sought for, usually in the form of coinductively defined *bisimilarities*.

Several kinds of bisimilarity have been proposed, such as, e.g., *applicative* bisimilarity [1], which relates terms by reducing them to values (if possible), and then compares these values by applying them to an arbitrary argument. The idea is the same for *environmental* bisimilarity [18], except the values are tested with arguments built from an environment, which represents the knowledge of an observer about the tested terms. Finally, *normal form* bisimilarity [11] (initially called *open* bisimilarity [17]) reduces open terms to normal forms and then compares their subterms. Applicative and environmental bisimilarities still contain some quantification over arguments, and usually coincide with contextual equivalence. In contrast, normal form bisimilarity is easier to use as its definition does not contain any quantification over arguments, but it is generally not *complete*, i.e., there exist equivalent terms that are not normal form bisimilar.

This article treats the behavioral theory of the untyped $\lambda\mu$-calculus [15]. The $\lambda\mu$-calculus provides a computational interpretation of the classical natural deduction and thus extends the Curry-Howard correspondence [7] from intuitionistic to classical logic. Operationally, the reduction rules of the calculus express not only function applications but also captures of the current context of evaluation. Therefore, when considered in the untyped setting, the calculus offers an approach to the semantics of abortive control operators such as *callcc* known from the Scheme programming language and it may be viewed as a closely related alternative to Felleisen and Hieb's syntactic theory of control [5].

So far no characterization of contextual equivalence have been proposed in either call-by-value or call-by-name $\lambda\mu$-calculus. Lassen defines normal bisimilarities for call-by-name [10], for head reduction semantics [12], and, with Støvring, for call-by-value [20] that are not complete. However, normal form bisimilarity is complete for Saurin's $\Lambda\mu$-calculus [19] with head reduction semantics [12], and also for call-by-value $\lambda\mu$-calculus with store [20]. Lassen also defines a not complete applicative bisimilarity for call-by-name in [10]. A definition of applicative bisimilarity has also been proposed for a call-by-value typed μPCF [14], but the resulting relation is neither sound nor complete.

In this work we propose the first characterizations of contextual equivalence for call-by-name and call-by-value $\lambda\mu$-calculus. We define sound and complete applicative bisimilarities for both semantics, and we also discuss environmental bisimilarity for call-by-name. The relations we obtain are harder to use than Lassen's normal form bisimilarity to prove the equivalence of two given terms, but because they are complete, we can equate terms that cannot be related with normal form bisimilarity, such as David and Py's counter-example [3]. Even though the applicative bisimilarities we define for call-by-name and call-by-value are built along the same principles, the relation we obtain in call-by-value is much more difficult to use than the one for call-by-name. However, we provide counter-examples showing that it cannot be simplified.

The rest of the paper is organized as follows. We first discuss the behavioral theory of the call-by-name $\lambda\mu$-calculus in Section 2. We propose a notion of contextual equivalence (in Section 2.2) which observes top-level names, and we then characterize it with, respectively, applicative (Section 2.3) and environmental (Section 2.4) bisimilarities. In particular, we compare our definition of applicative bisimilarity with Lassen's one and we prove David and Py's counter-example using applicative bisimilarity. We then discuss call by value in Section 3. We propose a definition of applicative bisimilarity (Section 3.2) which coincides with contextual equivalence. We then show that the definition cannot be simplified to match the call-by-value one by providing counter-examples. Although the relation we obtain is harder to use than the one for call-by-name, we can still prove some equivalences of terms, as we demonstrate in Section 3.3. We conclude in Section 4, and the appendices contain some of the proofs missing from the main text (Appendix A for call-by-name and Appendix B for call-by-value).

2 Call-by-Name $\lambda\mu$-calculus

In this section, we define applicative and environmental bisimilarities for the call-by-name (CBN) $\lambda\mu$-calculus. We first recall its syntax and semantics, and we then discuss the definition of contextual equivalence for the calculus.

2.1 Syntax and Semantics

The $\lambda\mu$-calculus [15] extends the λ-calculus with named terms and a μ constructor that binds names in terms. We assume a set X of *variables*, ranged over by x, y, etc., and a distinct set A of *names*, ranged over by a, b, etc. Terms (T) and named terms (U) are defined by the following grammar:

$$\begin{array}{rl} \text{Terms:} & t ::= x \mid \lambda x.t \mid t\,t \mid \mu a.u \\ \text{Named terms:} & u ::= [a]t \end{array}$$

Values (V), ranged over by v, are terms of the form $\lambda x.t$. A λ-abstraction $\lambda x.t$ binds x in t and a μ-abstraction $\mu a.t$ binds a in t. We equate terms up to α-conversion of their bound variables and names, and we assume bound names to be pairwise distinct, as well as distinct from free names. We write $\mathsf{fv}(t)$ and $\mathsf{fv}(u)$ for the set of free variables of, respectively, t and u, and we write $\mathsf{fn}(t)$ and $\mathsf{fn}(u)$ for their set of free names. A term t or named term u is said *closed* if, respectively, $\mathsf{fv}(t) = \emptyset$ or $\mathsf{fv}(u) = \emptyset$. Note that a closed (named) term may contain free names. The sets of closed terms, closed values, and named terms are T^0, V^0, and U^0, respectively. In any discussion or proof, we say a variable or a name is *fresh* if it does not occur in any term under consideration.

We distinguish several kinds of contexts, represented outside-in, as follows:

$$\begin{array}{rl}
\text{Contexts:} & C ::= \Box \mid C\ t \mid t\ C \mid \lambda x.C \mid \mu a.\mathbb{C} \\
\text{Named contexts:} & \mathbb{C} ::= [a]C \\
\text{CBN evaluation contexts:} & E ::= \Box \mid E\ t \\
\text{Named evaluation contexts:} & \mathbb{E} ::= [a]E
\end{array}$$

The syntax of (named) evaluation contexts reflects the chosen reduction strategy, here call-by-name. Contexts can be filled only with a term t, to produce either regular terms $C[t]$, $E[t]$, or named terms $\mathbb{C}[t]$, $\mathbb{E}[t]$; the free names and free variables of t may be captured in the process.

We write $t_0\{t_1/x\}$ and $u_0\{t_1/x\}$ for the usual capture-avoiding substitution of terms for variables. We define the capture-avoiding substitution of named contexts for names, written $t\langle{}^{\mathbb{E}}/{}_a\rangle$ and $u\langle{}^{\mathbb{E}}/{}_a\rangle$, as follows. Note that the side-condition in the μ-binding case can always be fulfilled using α-conversion.

$$\begin{array}{rlrl}
x\langle{}^{\mathbb{E}}/{}_a\rangle & \stackrel{\text{def}}{=} x & (\mu b.u)\langle{}^{\mathbb{E}}/{}_a\rangle & \stackrel{\text{def}}{=} \mu b.u\langle{}^{\mathbb{E}}/{}_a\rangle \text{ if } b \notin \mathsf{fn}(\mathbb{E}) \cup \{a\} \\
(\lambda x.t)\langle{}^{\mathbb{E}}/{}_a\rangle & \stackrel{\text{def}}{=} \lambda x.t\langle{}^{\mathbb{E}}/{}_a\rangle & ([b]t)\langle{}^{\mathbb{E}}/{}_a\rangle & \stackrel{\text{def}}{=} \begin{cases} [b]t\langle{}^{\mathbb{E}}/{}_a\rangle & \text{if } a \neq b \\ \mathbb{E}[t\langle{}^{\mathbb{E}}/{}_a\rangle] & \text{if } a = b \end{cases} \\
(t_0\ t_1)\langle{}^{\mathbb{E}}/{}_a\rangle & \stackrel{\text{def}}{=} t_0\langle{}^{\mathbb{E}}/{}_a\rangle\ t_1\langle{}^{\mathbb{E}}/{}_a\rangle & &
\end{array}$$

We define the CBN reduction relation $\rightarrow_{\mathrm{n}}$ inductively by the following rules:

$$\begin{array}{lr}
(\beta_n) & [a](\lambda x.t_0)\ t_1 \rightarrow_{\mathrm{n}} [a]t_0\{t_1/x\} \\
(\mu) & [a]\mu b.u \rightarrow_{\mathrm{n}} u\langle{}^{[a]\Box}/{}_b\rangle \\
(\mathit{app}) & [a]t_0\ t_1 \rightarrow_{\mathrm{n}} u\langle{}^{[a]\Box\ t_1}/{}_b\rangle \text{ if } [b]t_0 \rightarrow_{\mathrm{n}} u \text{ and } b \notin \mathsf{fn}([a]t_0\ t_1)
\end{array}$$

Reduction is defined on named terms only. The rule (β_n) is the usual call-by-name β-reduction. In rule (μ), the current continuation, represented by a, is captured and substituted for b in u. In an application (cf. rule (*app*)), we reduce the term t_0 in function position by introducing a fresh name b which represents the top level. We then replace b with $[a]\Box\ t_1$ in the result u of the reduction of $[b]t_0$. We can also express reduction with top-level evaluation contexts as follows.

Lemma 1. *$u \rightarrow_{\mathrm{n}} u'$ iff $u = \mathbb{E}[(\lambda x.t_0)\ t_1]$ and $u' = \mathbb{E}[t_0\{t_1/x\}]$, or $u = \mathbb{E}[\mu a.u'']$ and $u' = u''\langle \mathbb{E}/a\rangle$.*

Reduction is also compatible with evaluation contexts in the following sense.

Lemma 2. *If $u \rightarrow_{\mathrm{n}} u'$, then $u\langle{}^{\mathbb{E}}/{}_a\rangle \rightarrow_{\mathrm{n}} u'\langle{}^{\mathbb{E}}/{}_a\rangle$.*

We write $\rightarrow^*_{\mathrm{n}}$ for the transitive and reflexive closure of $\rightarrow_{\mathrm{n}}$, and we define the evaluation relation of the calculus as follows.

Definition 1. *We write $u \Downarrow_{\mathrm{n}} u'$ if $u \rightarrow^*_{\mathrm{n}} u'$ and u' cannot reduce further.*

If $u \Downarrow_{\mathrm{n}} u'$, then u' is a named value. If u admits an infinite reduction sequence, we say it *diverges*, written $u \Uparrow_{\mathrm{n}}$. For example, let $\Omega \stackrel{\text{def}}{=} (\lambda x.x\ x)\ (\lambda x.x\ x)$; then $[a]\Omega \Uparrow_{\mathrm{n}}$ for all a.

2.2 Contextual Equivalence

As in the λ-calculus, contextual equivalence in the $\lambda\mu$-calculus is defined in terms of convergence. However, unlike previous definitions [10,12], we define contextual equivalence on named terms first, before extending it to any terms.

Definition 2. *Two closed terms u_0, u_1 are contextually equivalent, written $u_0 \approx_c u_1$, if for all closed contexts $\mathbb{C}$ and names a, there exist b, v_0, and v_1 such that $\mathbb{C}[\mu a.u_0] \Downarrow_{\mathrm{n}} [b]v_0$ iff $\mathbb{C}[\mu a.u_1] \Downarrow_{\mathrm{n}} [b]v_1$.*

Note that we can plug only terms in a context, therefore we prefix u_0 and u_1 with a μ-abstraction. Definition 2 is not as generic as it could be, because we require the resulting named values to have the same top-level name b; a more general definition would simply say "$\mathbb{C}[\mu a.u_0] \Downarrow_{\mathrm{n}}$ iff $\mathbb{C}[\mu a.u_1] \Downarrow_{\mathrm{n}}$." Our definition is strictly finer than the general one, because contexts cannot discriminate upon top-level names in some cases, as we can see with the next example.

Example 1. Let $\Theta \stackrel{\text{def}}{=} (\lambda x.\lambda y.y\,(x\,x\,y))\,(\lambda x.\lambda y.y\,(x\,x\,y))$ be Turing's CBN fixed-point combinator, and let $v \stackrel{\text{def}}{=} \lambda x.\lambda y.x$. The terms $u_0 \stackrel{\text{def}}{=} [a]\lambda x.\mu c.[b]\lambda y.\Theta\,v$ and $u_1 \stackrel{\text{def}}{=} [b]\lambda y.\Theta\,v$ are distinguished by Definition 2 if $a \neq b$, but we show they are related by the general contextual equivalence. To do so, we verify that $\mathbb{E}[\mu c.u_0] \Downarrow_{\mathrm{n}}$ iff $\mathbb{E}[\mu c.u_1] \Downarrow_{\mathrm{n}}$ holds for all $\mathbb{E}$ and c, and we can then conclude that u_0 and u_1 are in the general equivalence with David and Py's context lemma [3]. Let $\mathbb{E}$ be of the form $[d]E\;t$ for some d, E, t. Then $\mathbb{E}[\mu a.u_0] \Downarrow_{\mathrm{n}} [b]\lambda y.\Theta\,v$ and $\mathbb{E}[\mu a.u_1] \Downarrow_{\mathrm{n}} [b]\lambda y.\Theta\,v$, $\mathbb{E}[\mu b.u_0] \Downarrow_{\mathrm{n}} [a]\lambda x.\mu c.\mathbb{E}[\lambda y.\Theta\,v]$ and $\mathbb{E}[\mu b.u_1] \Downarrow_{\mathrm{n}} [d]\lambda y.\Theta\,v$, and finally $\mathbb{E}[\mu c.u_0] \Downarrow_{\mathrm{n}} u_0$ and $\mathbb{E}[\mu c.u_1] \Downarrow_{\mathrm{n}} u_1$ for $c \notin \{a, b\}$. The case $\mathbb{E} = [d]\Box$ is easy to check as well.

We choose Definition 2 because it gives more information on the behaviors of terms than the general equivalence. Besides, only very peculiar terms u_0 and u_1 are related by the general equivalence but not by Definition 2. These terms are like black holes: they reduce (in some context $\mathbb{C}$) to values $[a]v_0$ and $[b]v_1$ with $a \neq b$ that never evaluate their arguments. Indeed, if $\mathbb{E} = [c]\Box\;t_0 \ldots t_n$, then $\mathbb{E}[\mu a.[a]v_0] \rightarrow_{\mathrm{n}} \mathbb{E}[v_0\langle{}^{\mathbb{E}}\!/\!{}_a\rangle]$, and $\mathbb{E}[\mu a.[b]v_1] \Downarrow_{\mathrm{n}} [b]v_1\langle{}^{\mathbb{E}}\!/\!{}_a\rangle$. Suppose that when evaluating $\mathbb{E}[v_0\langle{}^{\mathbb{E}}\!/\!{}_a\rangle]$, we evaluate one of the t_i's. Then by replacing t_i with Ω, we obtain a context $\mathbb{E}'$ such that $\mathbb{E}'[\mu a.[a]v_0] \Uparrow_{\mathrm{n}}$ (because Ω will be evaluated), and $\mathbb{E}'[\mu a.[b]v_1] \Downarrow_{\mathrm{n}}$, which is in contradiction with the fact that u_0 and u_1 are in the general equivalence (they are distinguished by $\mathbb{E}'[\mu a.\mathbb{C}]$).

We extend Definition 2 to any closed terms t_0, t_1, by saying that $t_0 \approx_c t_1$ if $[a]t_0 \approx_c [a]t_1$ for any fresh a. Other versions of the extension are possible, for example by replacing "for any a" by "for some a", or by dropping the freshness requirement; as can be shown using the results of Section 2.3, all these definitions are equivalent. We can also define contextual equivalence on open terms, using the notion of *open extension*, which extends any relation on closed (named) terms to open (named) terms. We say a substitution σ closes t (or u) if σ replaces the variables in $\mathsf{fv}(t)$ (or $\mathsf{fv}(u)$) with closed terms.

Definition 3. *Let $\mathcal{R}$ be a relation on closed (named) terms. Two terms t_0 and t_1 are in the open extension of $\mathcal{R}$, written $t_0\ \mathcal{R}^\circ\ t_1$, if for all substitutions σ closing t_0 and t_1, we have $t_0\sigma\ \mathcal{R}\ t_1\sigma$ (and similarly for $u_0\ \mathcal{R}^\circ\ u_1$).*

2.3 Applicative Bisimilarity

We propose a notion of applicative bisimulation, which tests values by applying them to a random closed argument. As with contextual equivalence, we give the definitions for named terms, before extending it to regular terms.

Definition 4. *A relation $\mathcal{R}$ on closed named terms is an applicative bisimulation if $u_0\ \mathcal{R}\ u_1$ implies*

- *if $u_0 \to_{\mathrm{n}} u_0'$, then there exists u_1' such that $u_1 \to_{\mathrm{n}}^* u_1'$ and $u_0'\ \mathcal{R}\ u_1'$;*
- *if $u_0 = [a]\lambda x.t_0$, then there exists t_1 such that $u_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1$ and for all t, we have $[a]t_0\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}\ \mathcal{R}\ [a]t_1\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}$;*
- *the symmetric conditions on u_1.*

Applicative bisimilarity, written $\approx$, is the largest applicative bisimulation.

For regular terms, we write $t_0\ \mathcal{R}\ t_1$ if $[a]t_0\ \mathcal{R}\ [a]t_1$ for any $a \notin \mathsf{fn}(t_0, t_1)$. The first item of Definition 4 plays the bisimulation game for named terms which are not named values. If u_0 is a named value $[a]\lambda x.t_0$, then u_1 has to reduce to a named value $[a]\lambda x.t_1$, and we compare the values by applying them to an argument t. However, a context cannot interact with $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ by simply applying them to t, because $([a]\lambda x.t_0)\ t$ is not allowed by the syntax. Consequently, we have to prefix them first with μa. As a result, we consider the named terms $[a](\mu a.[a]\lambda x.t_0)\,t$ and $[a](\mu a.[a]\lambda x.t_1)\,t$, which reduce to, respectively, $[a](\lambda x.t_0\langle {}^{[a]\Box\ t}/_a\rangle)t$ and $[a](\lambda x.t_1\langle {}^{[a]\Box\ t}/_a\rangle)t$, and then to $[a]t_0\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}$ and $[a]t_1\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}$; we obtain the terms in the clause for values of Definition 4.

Remark 1. When considering $[a](\mu a.[a]\lambda x.t_0)\,t$ and $[a](\mu a.[a]\lambda x.t_1)\,t$, we use the same top-level name a as the one of the named values $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$. We could use a fresh name b instead; reusing the same name makes the bisimulation proofs easier (we do not have to introduce unnecessary fresh names).

We can also define a big-step version of the bisimulation, where we consider only evaluation to a value.

Definition 5. *A relation $\mathcal{R}$ on closed named terms is a big-step applicative bisimulation if $u_0\ \mathcal{R}\ u_1$ implies*

- *if $u_0 \to_{\mathrm{n}}^* [a]\lambda x.t_0$, then there exists t_1 such that $u_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1$ and for all t, we have $[a]t_0\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}\ \mathcal{R}\ [a]t_1\langle {}^{[a]\Box\ t}/_a\rangle\{t/x\}$;*
- *the symmetric condition on u_1.*

Lemma 3. *If $\mathcal{R}$ is a big-step applicative bisimulation, then $\mathcal{R} \subseteq\ \approx$.*

As a first property, we prove that reduction (and therefore, evaluation) is included in bisimilarity.

Lemma 4. *We have* $\rightarrow_{\mathrm{n}}^{*} \subseteq \approx$.

Proof. By showing that $\{(u, u') \mid u \rightarrow_{\mathrm{n}}^{*} u'\} \cup \{(u, u)\}$ is a big-step bisimulation.

We give a basic example to show how applicative bisimulation can be used.

Example 2. For all closed v and $a, b \notin \mathsf{fn}(v)$[1], we prove that $[a]v \approx [a]\lambda x.\mu b.[a]v$ by showing that $\{([a]v, [a]\lambda x.\mu b.[a]v) \mid b \notin \mathsf{fn}(v)\} \cup \approx$ is an applicative bisimulation. Indeed, if $v = \lambda x.t$, then for all t', we have $[a]t\{t'/x\} \approx [a]\mu b.[a]v\, t'$, because $[a]\mu b.[a]v\, t' \rightarrow_{\mathrm{n}}^{*} [a]t\{t'/x\}$ (and by Lemma 4).

Soundness and completeness. We now prove that $\approx$ coincides with $\approx_c$. We first show that $\approx$ is a congruence using Howe's method [8,6], which is a classic proof method to show that an applicative bisimilarity is a congruence. As in [10], we need to slightly adapt the proof to the $\lambda\mu$-calculus. Here we only sketch the application of the method, all the details can be found in Appendix A.1.

The principle of the method is to prove that a relation called the *Howe's closure* of $\approx$, which is a congruence by construction, is also a bisimulation. The definition of Howe's closure relies on an auxiliary relation, called the *compatible refinement* $\widetilde{\mathcal{R}}$ of a relation $\mathcal{R}$, and inductively defined by the following rules:

$$\frac{}{x \mathrel{\widetilde{\mathcal{R}}} x} \qquad \frac{t_0 \mathrel{\mathcal{R}} t_1}{\lambda x.t_0 \mathrel{\widetilde{\mathcal{R}}} \lambda x.t_1} \qquad \frac{t_0 \mathrel{\mathcal{R}} t_1 \qquad t_0' \mathrel{\mathcal{R}} t_1'}{t_0\, t_0' \mathrel{\widetilde{\mathcal{R}}} t_1\, t_1'} \qquad \frac{u_0 \mathrel{\mathcal{R}} u_1}{\mu a.u_0 \mathrel{\widetilde{\mathcal{R}}} \mu a.u_1}$$

$$\frac{t_0 \mathrel{\mathcal{R}} t_1}{[a]t_0 \mathrel{\widetilde{\mathcal{R}}} [a]t_1} \qquad \frac{t_0 \mathrel{\mathcal{R}} t_1 \qquad \mathbb{E}_0 \mathrel{\widetilde{\mathcal{R}}} \mathbb{E}_1}{t_0\langle \mathbb{E}_0/a\rangle \mathrel{\widetilde{\mathcal{R}}} t_1\langle \mathbb{E}_1/a\rangle} \qquad \frac{u_0 \mathrel{\mathcal{R}} u_1 \qquad \mathbb{E}_0 \mathrel{\widetilde{\mathcal{R}}} \mathbb{E}_1}{u_0\langle \mathbb{E}_0/a\rangle \mathrel{\widetilde{\mathcal{R}}} u_1\langle \mathbb{E}_1/a\rangle}$$

$$\frac{}{\Box \mathrel{\widetilde{\mathcal{R}}} \Box} \qquad \frac{E_0 \mathrel{\widetilde{\mathcal{R}}} E_1 \qquad t_0 \mathrel{\mathcal{R}} t_1}{E_0\ t_0 \mathrel{\widetilde{\mathcal{R}}} E_1\ t_1} \qquad \frac{E_0 \mathrel{\widetilde{\mathcal{R}}} E_1}{[a]E_0 \mathrel{\widetilde{\mathcal{R}}} [a]E_1}$$

In the original definition of compatible refinement [6], two terms are related by $\widetilde{\mathcal{R}}$ if they have the same outer language constructor, and their subterms are related by $\mathcal{R}$. In the $\lambda\mu$-calculus, compatible refinement is extended to (named) evaluation contexts, and we allow for the substitution of names with related named contexts.

Given two relations $\mathcal{R}_1$ and $\mathcal{R}_2$, we write $\mathcal{R}_1\mathcal{R}_2$ for their composition, e.g., $t_0\ \mathcal{R}_1\mathcal{R}_2\ t_2$ holds if there exists t_1 such that $t_0\ \mathcal{R}_1\ t_1$ and $t_1\ \mathcal{R}_2\ t_2$. We can now define Howe's closure of $\approx$, written $\approx^\bullet$, as follows.

Definition 6. *The Howe's closure* $\approx^\bullet$ *is the smallest relation verifying:*

$$\approx^\circ \subseteq \approx^\bullet \qquad \approx^\bullet\approx^\circ \subseteq \approx^\bullet \qquad \widetilde{\approx^\bullet} \subseteq \approx^\bullet$$

[1] Note that the result still holds if $a \in \mathsf{fn}(v)$

Howe's closure is defined on open (named) terms as well as on (named) evaluation contexts. Because it contains its compatible refinement, $\approx^\bullet$ is a congruence. To prove it is a bisimulation, we need a stronger result, called a pseudo-simulation lemma, where we test named values not with the same argument, but with arguments t'_0, t'_1 related by $\approx^\bullet$.

Lemma 5. *Let $(\approx^\bullet)^c$ be $\approx^\bullet$ restricted to closed terms, and let $u_0 \ (\approx^\bullet)^c \ u_1$.*

- *If $u_0 \rightarrow_{\mathrm{n}} u'_0$, then $u_1 \rightarrow^*_{\mathrm{n}} u'_1$ and $u'_0 \ (\approx^\bullet)^c \ u'_1$.*
- *If $u_0 = [a]\lambda x.t_0$, then $u_1 \rightarrow^*_{\mathrm{n}} [a]\lambda x.t_1$ and for all $t'_0 \ (\approx^\bullet)^c \ t'_1$, we have $[a]t_0\langle{}^{[a]\Box\, t'_0}/a\rangle\{t'_0/x\} \ (\approx^\bullet)^c \ [a]t_1\langle{}^{[a]\Box\, t'_1}/a\rangle\{t'_1/x\}$.*

With this result, we can prove that $(\approx^\bullet)^c$ is a bisimulation, and therefore included in $\approx$. Because it also contains $\approx$ by definition, we have $\approx = (\approx^\bullet)^c$, and this implies that $\approx$ is a congruence. As a result, $\approx$ is sound w.r.t. to $\approx_c$.

Theorem 1. $\approx \subseteq \approx_c$.

To simplify the proof of completeness (the reverse inclusion), we consider an alternate definition of contextual equivalence, where we test terms with named evaluation contexts only. By doing so, we prove a context lemma in the process.[2]

Definition 7. *Let u_0, u_1 be closed terms. We write $u_0 \ \dot{\approx}_c \ u_1$ if for all closed contexts $\mathbb{E}$ and names a, there exist b, v_0, v_1 such that $\mathbb{E}[\mu a.u_0] \Downarrow_{\mathrm{n}} [b]v_0$ iff $\mathbb{E}[\mu a.u_1] \Downarrow_{\mathrm{n}} [b]v_1$.*

Theorem 2. $\approx_c \subseteq \dot{\approx}_c \subseteq \approx$.

The first inclusion is by definition, and the second one is by showing that $\dot{\approx}_c$ is a big-step applicative bisimulation.

Comparison with Lassen's work. In [10], Lassen also proposes a definition of applicative bisimilarity that he proves sound, but he conjectures that it is not complete. We discuss here the differences between the two approaches.

Lassen defines a notion of bisimulation for regular terms only, and not for named terms. The definition is as follows.

Definition 8. *A relation $\mathcal{R}$ on closed terms is a Lassen applicative bisimulation if $t_0 \ \mathcal{R} \ t_1$ implies:*

- *for all a, if $[a]t_0 \rightarrow^*_{\mathrm{n}} [b]\lambda x.t'_0$, then there exists t'_1 such that $[a]t_1 \rightarrow^*_{\mathrm{n}} [b]\lambda x.t'_1$, and for all t, we have $t'_0\langle{}^{[b]\Box\, t}/b\rangle\{t/x\} \ \mathcal{R} \ t'_1\langle{}^{[b]\Box\, t}/b\rangle\{t/x\}$;*
- *the symmetric condition on t_1.*

[2] We cannot directly use David and Py's context lemma [3], because we use a different notion of contextual equivalence.

Lassen's definition is quite similar to our definition of big-step applicative bisimulation (Definition 5), except it requires $t_0\langle{}^{[b]\Box\, t}/_{b}\rangle\{t/x\}\ \mathcal{R}\ t_1\langle{}^{[b]\Box\, t}/_{b}\rangle\{t/x\}$, which implies that these terms must be related when reduced with any top-level name a. This is more restrictive than our definition, where we compare these terms only with the top-level name b (or, as discussed in Remark 1, we could instead compare $[c]t_0\langle{}^{[c]\Box\, t}/_{b}\rangle\{t/x\}$ and $[c]t_1\langle{}^{[c]\Box\, t}/_{b}\rangle\{t/x\}$ for some fresh name c). To illustrate the difference, we consider Lassen's counter-example from [10].

Example 3. Let $t_0 \stackrel{\text{def}}{=} (\lambda x.\lambda y.x\ x)\ (\lambda x.\lambda y.x\ x)$, and $t_1 \stackrel{\text{def}}{=} \mu a.[a]\lambda y.\mu c.[a]t_0$ (with $c \neq a$). These terms are not bisimilar according to Lassen's definition. For all b, we have $[b]t_0 \rightarrow_{\text{n}}^{*} [b]\lambda y.t_0$ and $[b]t_1 \rightarrow_{\text{n}}^{*} [b]\lambda y.\mu c.[b]t_0$. With Lassen's definition, one has to relate t_0 and $\mu c.[b]t_0\ t$ for any t, which means comparing $[d]t_0$ and $[d]\mu c.[b]t_0\ t$ for all d. But these two terms are not equivalent if $d \neq b$.

Lassen conjectures in [10] that these terms are contextually equivalent, and we can indeed prove that they are (big-step) bisimilar with our definition: we just have to compare $[b]t_0$ and $[b]\mu a'.[b]t_0\ t$ (or $[c]t_0$ and $[c]\mu a'.[c]t_0\ t$ for some fresh c) for any t, and both terms evaluate to $[b]\lambda x.t_0$ (or $[c]\lambda x.t_0$) and are therefore equivalent.

By comparing primarily named terms, as we do in our definition, we can keep track of what happens to the top level, and especially of any connection between the top level and a subterm. In Example 3, we can see that it is essential to remember that b represents the top level in $\mu c.[b]t_0\ t$, and therefore it does not make sense to compare $[d]t_0$ and $[d]\mu c.[b]t_0\ t$ for any $d \neq b$, as we have to do with Lassen's definition. We believe that comparing named terms is essential to obtain completeness w.r.t. contextual equivalence; note that the sound and complete normal form bisimilarity for the $\lambda\mu\rho$-calculus [20] is also defined on named terms.

David and Py's counter-example. In [3], David and Py give a counter-example showing that Böhm's theorem fails in CBN $\lambda\mu$-calculus. They prove that their terms are contextually equivalent using a context lemma. Here we slightly simplify their counter-example, and prove equivalence using applicative bisimilarity. Note that these terms cannot be proved equivalent with (a CBN variant of) eager normal form bisimilarity [10,20].

Example 4. Let $0 \stackrel{\text{def}}{=} \lambda x.\lambda y.y$, $1 \stackrel{\text{def}}{=} \lambda x.\lambda y.x$, and $t_a \stackrel{\text{def}}{=} \mu c.[a]0$. Then we have $\lambda x.\mu a.[a]x\ \mu b.[a]x\ t_a\ 0 \approx \lambda x.\mu a.[a]x\ \mu b.[a]x\ t_a\ 1$[3].

Proof (Sketch). We only give the main ideas here, the complete equivalence proof can be found in Appendix A.2. First, $\lambda x.\mu a.[a]x\ \mu b.[a]x\ t_a\ 0$ is not normal form bisimilar to $\lambda x.\mu a.[a]x\ \mu b.[a]x\ t_a\ 1$, because the subterms of these two terms are not normal form bisimilar (0 is not equivalent to 1).

[3] The terms David and Py consider in their work are $\lambda x.\mu a.[a]x\ \mu b.[a](x\ t_a\ 0)\ t_a$ and $\lambda x.\mu a.[a]x\ \mu b.[a](x\ t_a\ 1)\ t_a$. However, the additional argument t_a would not come into play in the proof we present, so we have elided it.

To prove applicative bisimilarity, let c be a fresh name and t be a closed term. We want to relate $[c]\mu a.[a]t\ \mu b.[a]t\ t_a\ 0$ and $[c]\mu a.[a]t\ \mu b.[a]t\ t_a\ 1$, which reduce respectively to $[c]t\,\mu b.[c]t\,t_c\,0$ (1) and $[c]t\,\mu b.[c]t\,t_c\,1$ (2). Let $d \notin \mathsf{fn}(t)$; we distinguish several cases depending on the behavior of $[d]t$. The interesting case is when $[d]t \Downarrow_{\mathrm{n}} [d]\lambda y.t'$; then $\mu b.[c]t\,t_c\,0$ or $\mu b.[c]t\,t_c\,1$ is passed as an argument to $\lambda y.t'$ in respectively (1) and (2). If t' executes its argument (that is, if t' reduces to $E[y]$ for some E), then (1) reduces to $[c]t\,t_c\,0$ (3), and (2) to $[c]t\,t_c\,1$ (4). But we know that $[d]t \Downarrow_{\mathrm{n}} [d]\lambda y.t'$, and t' executes its argument, so when evaluating (3) and (4), t_c will be reduced, and therefore (3) and (4) will evaluate to $[c]0$.

In the other cases (e.g., $[d]t \Downarrow_{\mathrm{n}} [e]\lambda y.t'$ with $e \neq d$), either (1) and (2) eventually get to a point similar to the situation above where t_c is executed, or they diverge. In all cases, they are applicative bisimilar.

2.4 Environmental Bisimilarity

Definition and basic properties. Environmental bisimilarity [18] uses environments (denoted by $\mathcal{E}$) to accumulate knowledge when comparing terms. Like applicative bisimilarity, it challenges values by passing them arguments, except the arguments are built from the environments instead of being completely arbitrary. In $\lambda\mu$-calculus, we define an environment $\mathcal{E}$ as a relation on named closed values; for example, the identity environment $\mathcal{I}$ is defined as $\{([a]v, [a]v) \mid a \in A, v \in V^0\}$. To build arguments from $\mathcal{E}$, we use the notion of *context closure*, defined as follows: given a relation $\mathcal{R}$ on (named) terms, its context closure $\widehat{\mathcal{R}}$ is the smallest relation verifying $\mathcal{R} \subseteq \widehat{\mathcal{R}}$ and $\widehat{\widehat{\mathcal{R}}} \subseteq \widehat{\mathcal{R}}$. Like compatible refinement, context closure is defined on open (named) terms and evaluation contexts, but we restrict it to closed (named) terms and contexts, unless specified otherwise. Note that, unlike usual definitions of context closure, our definition requires context closure to be stable by substitution of a name by a named evaluation context.

An environmental relation $\mathcal{X}$ is a set of environments $\mathcal{E}$ and triples $(\mathcal{E}, u_0, u_1)$, where u_0 and u_1 are closed. We write $u_0\ \mathcal{X}_{\mathcal{E}}\ u_1$ for $(\mathcal{E}, u_0, u_1) \in \mathcal{X}$; it means that u_0 and u_1 are compared within the current knowledge $\mathcal{E}$. We define environmental bisimulation as follows.

Definition 9. *An environmental relation $\mathcal{X}$ is an environmental bisimulation if $u_0\ \mathcal{X}_{\mathcal{E}}\ u_1$ implies*

- *if $u_0 \rightarrow_{\mathrm{n}} u_0'$, then there exists u_1' such that $u_1 \rightarrow_{\mathrm{n}}^* u_1'$ and $u_0'\ \mathcal{X}_{\mathcal{E}}\ u_1'$;*
- *if $u_0 = [a]v_0$, then there exists v_1 such that $u_1 \rightarrow_{\mathrm{n}}^* [a]v_1$ and $\{([a]v_0, [a]v_1)\} \cup \mathcal{E} \in \mathcal{X}$;*
- *the converse of the above condition on u_1.*

Furthermore, if $\mathcal{E} \in \mathcal{X}$ and $[a]\lambda x.t_0\ \mathcal{E}\ [a]\lambda x.t_1$, then for all $t_0'\ \widehat{\mathcal{E}}\ t_1'$, we have $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\}\ \mathcal{X}_{\mathcal{E}}\ [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$.

Environmental bisimilarity $\approxeq$ is the largest environmental bisimulation. To prove the equivalence between u_0 and u_1, we want to relate them without any predefined knowledge, i.e., we want to prove that $u_0 \approxeq_\emptyset u_1$ holds; we also write $\simeq$ for $\approxeq_\emptyset$. We extend these definitions to regular terms by saying that $t_0 \mathcal{X}_\mathcal{E} t_1$ holds if we have $[a]t_0 \mathcal{X}_\mathcal{E} [a]t_1$ for any $a \notin \mathsf{fn}(t_0, t_1)$.

Definition 9 is divided into two parts, the first one making explicit the bisimulation game for named terms, and the second one focusing on environments. If u_0 is a named value $[a]v_0$, then u_1 has to evaluate to named value with the same top-level name $[a]v_1$, and $\mathcal{E}$ extended with the newly acquired knowledge $([a]v_0, [a]v_1)$ must belong to $\mathcal{X}$. We then compare two named values $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ in $\mathcal{E}$ by passing them two terms t'_0 and t'_1 built from $\mathcal{E}$. As in the applicative bisimulation definition, we have to prefix the named values with μa first, and we therefore have to relate $[a]t_0\langle{}^{[a]\Box\, t'_0}/{}_a\rangle\{t'_0/x\}$ and $[a]t_1\langle{}^{[a]\Box\, t'_1}/{}_a\rangle\{t'_1/x\}$.

It is possible to define a big-step variant of environmental bisimulation by removing the first item of Definition 9 and by changing equality in the value case by $\to_\mathrm{n}^*$. We also use the following properties.

Lemma 6 (Weakening). *If $u_0 \approxeq_\mathcal{E} u_1$ and $\mathcal{E}' \subseteq \mathcal{E}$ then $u_0 \approxeq_{\mathcal{E}'} u_1$.*

A smaller environment is a weaker constraint, because we can build less arguments to test the named values in $\mathcal{E}$. The proof is as in [18]. Like with applicative bisimilarity, reduction and evaluation are included in $\simeq$.

Lemma 7. *If $u \to_\mathrm{n}^* u'$, then $u \simeq u'$.*

Bisimulation proofs with environmental bisimulation are usually harder than with applicative bisimulation, as we can see with the next example.

Example 5. Following Example 2, we want to prove $[a]v \simeq [a]\lambda x.\mu b.[a]v$ for all closed v and $b \notin \mathsf{fn}(v)$. To this end, we start with $\mathcal{X} \stackrel{\mathrm{def}}{=} \{(\emptyset, [a]v, [a]\lambda x.\mu b.[a]v) \mid a \in A, v \in V^0, b \notin \mathsf{fn}(v)\} \cup \{\mathcal{E} \mid \mathcal{E} \subseteq \overline{\mathcal{E}}\}$, where $\overline{\mathcal{E}} = \{([a]v, [a]\lambda x.\mu b.[a]v) \mid a \in A, v \in V^0, b \notin \mathsf{fn}(v)\}$. Let $[a]v \;\mathcal{E}\; [a]\lambda x.\mu b.[a]v$, where $v = \lambda x.t$. For all $t_0 \;\widehat{\mathcal{E}}\; t_1$, we want $[a]t\{t_0/x\} \;\mathcal{X}_\mathcal{E}\; [a]\mu b.[a]v\, t_1$ to hold. In contrast with Example 2, $[a]\mu b.[a]v\, t_1$ does not reduce to $[a]t\{t_0/x\}$; to conclude the proof easily, we need some up-to techniques, introduced later (see Example 6).

Soundness and completeness. Here we sketch the soundness and completeness proofs for $\simeq$, which are adaptations of the proofs for the λ-calculus [18]. More details can be found in Appendix A.3. The first step is to define some basic up-to techniques, specifically bisimulation up to environment (which allows for bigger environments in the bisimulation clauses) and up to bisimilarity (where $\simeq$ is used at some specific points in the bisimulation clauses); the definitions and proofs of soundness mimic the ones for the λ-calculus [18].

We then prove congruence w.r.t. evaluation contexts. Given a relation $\mathcal{R}$ on named terms, we write $\mathcal{R}^{\mathrm{nv}}$ for its restriction to named values. Let $\mathcal{Y}$ be an environmental bisimulation. Then the relation

$$\begin{aligned}\{(\widehat{\mathcal{E}}^{\mathrm{nv}}, u_0\langle{}^{\mathbb{E}_0^0}/{}_{a_0}\rangle \dots \langle{}^{\mathbb{E}_0^n}/{}_{a_n}\rangle, u_1\langle{}^{\mathbb{E}_1^0}/{}_{a_0}\rangle \dots \langle{}^{\mathbb{E}_1^n}/{}_{a_n}\rangle) \mid u_0 \;\mathcal{Y}_\mathcal{E}\; u_1, \mathbb{E}_0^i \;\widehat{\mathcal{E}}\; \mathbb{E}_1^i\} \\ \cup \{(\widehat{\mathcal{E}}^{\mathrm{nv}}, u_0, u_1) \mid \mathcal{E} \in \mathcal{Y}, u_0 \;\widehat{\mathcal{E}}\; u_1\} \cup \{\widehat{\mathcal{E}}^{\mathrm{nv}} \mid \mathcal{E} \in \mathcal{Y}\}\end{aligned}$$

is an environmental bisimulation up to environment. From that, we can deduce the following lemma.

Lemma 8. *If $u_0 \approxeq_{\mathcal{E}} u_1$, then $u_0\langle\mathbb{E}/a\rangle \approxeq_{\mathcal{E}} u_1\langle\mathbb{E}/a\rangle$.*

Next, we prove that $\simeq$ is a congruence by showing that $\{(\widehat{\simeq}^{\mathrm{nv}}, u_0, u_1) \mid u_0 \mathrel{\widehat{\simeq}} u_1\} \cup \{\widehat{\simeq}^{\mathrm{nv}}\}$ is an environmental bisimulation up to bisimilarity.

Lemma 9. *If $u_0 \simeq u_1$, then $C[u_0] \approxeq_{\widehat{\simeq}^{\mathrm{nv}}} C[u_1]$ and $\mathbb{C}[u_0] \approxeq_{\widehat{\simeq}^{\mathrm{nv}}} \mathbb{C}[u_1]$.*

Using congruence, we can prove that $\simeq$ is sound w.r.t. $\approx_c$, and completeness is proved using Definition 7, as in the case for applicative bisimulations.

Theorem 3. *We have $\simeq = \approx_c$.*

Up-to techniques. As witnessed by Example 5, proving the equivalence of two terms with Definition 9 alone is usually too difficult. To ease the proofs, it is common to rely on up-to techniques. We already mentioned bisimulation up to bisimilarity and up to environment; other techniques include bisimulation up to reduction, up to expansion, and up to context. We only discuss the last for space reasons, the others are defined and proved sound as in [18].

Bisimulation up to context allows to factor out a context common to both terms. Formally, we define the context closure of $\mathcal{X}$, written $\overline{\mathcal{X}}$, as follows: we have $u_0 \mathrel{\overline{\mathcal{X}_{\mathcal{E}}}} u_1$ if

- either $u_0 = u_0'\langle\mathbb{E}_0/a\rangle$, $u_1 = u_1'\langle\mathbb{E}_1/a\rangle$ with $u_0' \mathrel{\mathcal{X}_{\mathcal{E}}} u_1'$ and $\mathbb{E}_0 \mathrel{\widehat{\mathcal{E}}} \mathbb{E}_1$
- or $u_0 \mathrel{\widehat{\mathcal{E}}} u_1$.

Terms related by $\overline{\mathcal{X}_{\mathcal{E}}}$ are either built from $\mathcal{E}$ (second case), or we can decompose them into terms related by $\mathcal{X}_{\mathcal{E}}$ and evaluation contexts built from $\mathcal{E}$. We then define bisimulation up to context as follows.

Definition 10. *A relation $\mathcal{X}$ is an environmental bisimulation up to context if $u_0 \mathrel{\mathcal{X}_{\mathcal{E}}} u_1$ implies*

- *if $u_0 \rightarrow_{\mathrm{n}} u_0'$, then there exists u_1' such that $u_1 \rightarrow_{\mathrm{n}}^* u_1'$ and $u_0' \mathrel{\overline{\mathcal{X}_{\mathcal{E}}}} u_1'$;*
- *if $u_0 = [a]v_0$, then there exist v_1 and $\mathcal{E}' \subseteq \mathcal{X}$ such that $u_1 \rightarrow_{\mathrm{n}}^* [a]v_1$ and $\{([a]v_0, [a]v_1)\} \cup \mathcal{E} \subseteq \widehat{\mathcal{E}'}$;*
- *the converse of the above condition on u_1.*

Furthermore, if $\mathcal{E} \in \mathcal{X}$ and $[a]\lambda x.t_0 \mathrel{\mathcal{E}} [a]\lambda x.t_1$, then for all $t_0' \mathrel{\widehat{\mathcal{E}}} t_1'$, we have $[a]t_0\langle[a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\overline{\mathcal{X}_{\mathcal{E}}}} [a]t_1\langle[a]\Box\, t_1'/a\rangle\{t_1'/x\}$.

Lemma 10. *If $\mathcal{X}$ is a bisimulation up to context, then $\mathcal{X} \subseteq \approxeq$.*

The proof of soundness is as in [18]. We now give some examples showing how bisimulation up to context (combined with other up-to techniques) can be used to simplify proofs.

Example 6. Continuing Example 5, we can prove that the candidate relation $\mathcal{X}$ defined in this example is a bisimulation up to context up to reduction (meaning that we can do some reduction steps to get terms related by $\overline{\mathcal{X}}$). Indeed, we remind that we want to relate $[a]t\{t_0/x\}$ and $[a]\mu b.[a]v\,t_1$ for all $t_0\ \widehat{\mathcal{E}}\ t_1$, $v = \lambda x.t$, and $b \notin \mathsf{fn}(v)$. But $[a]\mu b.\underline{[a]v\,t_1} \rightarrow^2_{\mathrm{n}} [a]t\{t_1/x\}$, and $[a]t\{t_0/x\}\ \widehat{\mathcal{E}}\ [a]t\{t_1/x\}$ holds, which implies $[a]t\{t_0/x\}\ \overline{\mathcal{X}_{\mathcal{E}}}\ [a]t\{t_1/x\}$, as wished.

3 Call-by-Value $\lambda\mu$-calculus

We now discuss the behavioral theory of the call-by-value (CBV) $\lambda\mu$-calculus and point out the differences with call-by-name.

3.1 Semantics and Contextual Equivalence

Semantics. In this section, we use CBV left-to-right evaluation, which is encoded in the syntax of the CBV evaluation contexts:

$$E ::= \Box \mid E\ t \mid v\ E$$

The CBV reduction relation $\rightarrow_{\mathrm{v}}$ is defined by the following rules.

$$\begin{array}{rrl}
(\beta_v) & [a](\lambda x.t)\ v \rightarrow_{\mathrm{v}} [a]t\{v/x\} & \\
(\mu) & [a]\mu b.u \rightarrow_{\mathrm{v}} u\langle {}^{[a]\Box}/_{b}\rangle & \\
(app) & [a]t_0\ t_1 \rightarrow_{\mathrm{v}} u\langle {}^{[a]\Box\ t_1}/_{b}\rangle & \text{if } [b]t_0 \rightarrow_{\mathrm{v}} u \text{ and } b \notin \mathsf{fn}([a]t_0\ t_1) \\
(app_v) & [a]v\ t \rightarrow_{\mathrm{v}} u\langle {}^{[a]v\ \Box}/_{b}\rangle & \text{if } [b]t \rightarrow_{\mathrm{v}} u \text{ and } b \notin \mathsf{fn}([a]v\ t)
\end{array}$$

With rule (app_v), we reduce arguments to values, to be able to apply CBV β-reduction (rule (β_v)). The rules (μ) and (app) are the same as in CBN. We could also express reduction with top-level named evaluation contexts, as in Lemma 1. Furthermore, CBV reduction is compatible with CBV contexts, as in Lemma 2. We write $\rightarrow^*_{\mathrm{v}}$ for the reflexive and transitive closure of $\rightarrow_{\mathrm{v}}$, $\Downarrow_{\mathrm{v}}$ for CBV evaluation, and $\Uparrow_{\mathrm{v}}$ for CBV divergence.

Contextual equivalence. We use the same definition as in CBN.

Definition 11. *Two closed named terms u_0, u_1 are contextually equivalent, written $u_0 \approx_c u_1$, if for all closed $\mathbb{C}$, $\mathbb{C}[\mu a.u_0] \Downarrow_{\mathrm{v}} [b]v_0$ iff $\mathbb{C}[\mu a.u_1] \Downarrow_{\mathrm{v}} [b]v_1$.*

However, unlike in CBN, this definition (where we require the resulting values to have the same top-level names) coincides with the general definition where we simply say "$\mathbb{C}[\mu a.u_0] \Downarrow_{\mathrm{v}}$ iff $\mathbb{C}[\mu a.u_1] \Downarrow_{\mathrm{v}}$." Indeed, if $\mathbb{C}[\mu a.u_0] \Downarrow_{\mathrm{v}} [b]v_0$ and $\mathbb{C}[\mu a.u_1] \Downarrow_{\mathrm{v}} [c]v_1$ with $c \neq b$, then we can easily distinguish them, because $[b]\mu b.\mathbb{C}[\mu a.u_0]\ \Omega \rightarrow^*_{\mathrm{v}} [b]v_0\langle {}^{[b]\Box\ \Omega}/_{b}\rangle\ \Omega \Uparrow_{\mathrm{v}}$, and $[b]\mu b.\mathbb{C}[\mu a.u_1]\ \Omega \Downarrow_{\mathrm{v}} [c]v_1\langle {}^{[b]\Box\ \Omega}/_{b}\rangle$.

3.2 Applicative Bisimilarity

Before giving its complete definition, we explain how applicative bisimilarity $\approx$ should compare two named values $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$. The following reasoning explains and justifies the clauses in Definition 12. In particular, we provide counter-examples to show that we cannot simplify this definition.

In CBV λ-calculus (and also with delimited control [2]), values are tested by applying them to an arbitrary value argument. Following this principle, it is natural to propose the following clause for CBV $\lambda\mu$-calculus.

(1) For all v, we have $[a]t_0\langle {}^{[a]\square\ v}/_{a}\rangle\{v/x\} \approx [a]t_1\langle {}^{[a]\square\ v}/_{a}\rangle\{v/x\}$.

As with Definition 4, we in fact compare $[a](\mu a.[a]\lambda x.t_0)\,v$ with $[a](\mu a.[a]\lambda x.t_1)\,v$, which reduce to the terms in clause (1). However, such a clause would produce an unsound applicative bisimilarity; it would relate terms that are not contextually equivalent, like the ones in the next example.

Example 7. Let $v_0 \stackrel{\text{def}}{=} \lambda x.\mu b.[a]w\ x$, $v_1 \stackrel{\text{def}}{=} \lambda x.w\ x\ x$, with $w \stackrel{\text{def}}{=} \lambda y.\lambda z.z\ y$. Then we have $([a]v_0)\langle {}^{[a]\square\ v}/_{a}\rangle = [a](\lambda x.\mu b.[a]w\,x\,v)\,v \to_{\text{v}}^{*} [a]w\,v\,v$ and $([a]v_1)\langle {}^{[a]\square\ v}/_{a}\rangle = [a](\lambda x.w x x)v \to_{\text{v}}^{*} [a]wvv$. Because they reduce to the same term, $([a]v_0)\langle {}^{[a]\square\ v}/_{a}\rangle$ is contextually equivalent to $([a]v_1)\langle {}^{[a]\square\ v}/_{a}\rangle$, and using clause (1) would lead us to conclude that $[a]v_0$ and $[a]v_1$ are equivalent as well.

However, $[a]v_0$ and $[a]v_1$ can be distinguished with $t \stackrel{\text{def}}{=} \mu d.[d]\lambda y.\mu c.[d]w'$, where $w' \stackrel{\text{def}}{=} \lambda x.\mu c.[d]x\ w''$ and $w'' \stackrel{\text{def}}{=} \lambda x.\lambda y.\lambda z.\Omega$. Indeed, we can check that $([a]v_0)\langle {}^{[a]\square\ t}/_{a}\rangle \to_{\text{v}}^{*} \lambda z.\Omega$ and $([a]v_1)\langle {}^{[a]\square\ t}/_{a}\rangle \to_{\text{v}}^{*} \Omega$. This discrepancy comes from the fact that, in $([a]v_1)\langle {}^{[a]\square\ t}/_{a}\rangle$, t is reduced to a value once, capturing $[a]v_1\ \square$ in the process, while t is reduced twice to a value in $([a]v_0)\langle {}^{[a]\square\ t}/_{a}\rangle$, and each time it captures a different context. Therefore, $[a]v_0$ and $[a]v_1$ are distinguished by the context $[a](\mu a.\square)\ t$, and they are consequently not contextually equivalent.

Example 7 suggests that we should compare $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ with contexts of the form $[a]\square\ t$, instead of $[a]\square\ v$. Therefore, we should compare $u_0 \stackrel{\text{def}}{=} [a](\lambda x.t_0\langle {}^{[a]\square\ t}/_{a}\rangle)\ t$ with $u_1 \stackrel{\text{def}}{=} [a](\lambda x.t_1\langle {}^{[a]\square\ t}/_{a}\rangle)\ t$. However, we can restrict a bit the choice of the testing term t, based on its behavior. Let $b \notin \mathsf{fn}(t)$; if $[b]t$ diverges, then u_0 and u_1 diverge as well, and we gain no information on $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ themselves. If $[b]t \to_{\text{v}}^{*} [c]v$ with $b \neq c$, then $u_0 \to_{\text{v}}^{*} [c]v\langle {}^{[a]\lambda x.t_0\langle {}^{[a]\square\ t}/_{a}\rangle\ \square}/_{b}\rangle$, and similarly with u_1. The values $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ are captured by $[b]t$, and no interaction between t and the two named values takes place in the process ($[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ are not applied to any value); again, we do not gain any new knowledge on the behavior of $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$. Finally, if $[b]t \to_{\text{v}}^{*} [b]v$, then $u_0 \to_{\text{v}}^{*} [b](\lambda x.t_0\langle {}^{[a]\square\ t}/_{a}\rangle)\ v\langle {}^{[a]\lambda x.t_0\langle {}^{[a]\square\ t}/_{a}\rangle\ \square}/_{b}\rangle$, and similarly with u_1; in this case, a value is indeed passed to $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$, and we can compare their respective behaviors. Therefore, an interaction happens between t and the tested values iff $[b]t \to_{\text{v}}^{*} [b]v$, and the results of the interaction (after β-reduction) are the two terms in the clause below.

(2) For all t, b, v such that $[b]t \to_{\mathrm{v}}^* [b]v$ and $b \notin \mathsf{fn}(t)$, we have

$$[a]t_0\langle{}^{[a]\Box\ t}/_{a}\rangle\{v\langle{}^{[a]\lambda x.t_0\langle{}^{[a]\Box\ t}/_{a}\rangle\ \Box}/_{b}\rangle/x\} \\ \approx [a]t_1\langle{}^{[a]\Box\ t}/_{a}\rangle\{v\langle{}^{[a]\lambda x.t_1\langle{}^{[a]\Box\ t}/_{a}\rangle\ \Box}/_{b}\rangle/x\}.$$

Unfortunately, clause (2) is not enough to obtain a sound bisimilarity. The next example shows that an extra clause is needed.

Example 8. Let $v_0 \stackrel{\text{def}}{=} \lambda x.\mu b.[a](\lambda y.\lambda z.wy)x$ and $v_1 \stackrel{\text{def}}{=} w$ with $w \stackrel{\text{def}}{=} \lambda x.w'(x\lambda y.y)$, and $w' \stackrel{\text{def}}{=} \lambda y.y\,\lambda z.\Omega$. We first show that $[a]v_0$ and $[a]v_1$ are related by clause (2). Let t such that $[b]t \Downarrow_{\mathrm{v}} [b]v$ for $b \notin \mathsf{fn}(t)$. Then we have $([a]v_0)\langle{}^{[a]\Box\ t}/_{a}\rangle \to_{\mathrm{v}}^* [a]w'(v\langle{}^{[a]v_0\langle{}^{[a]\Box\ t}/_{a}\rangle\ \Box}/_{b}\rangle\,\lambda x.x)$ and $([a]v_1)\langle{}^{[a]\Box\ t}/_{a}\rangle \to_{\mathrm{v}}^* [a]w'(v\langle{}^{[a]v_1\ \Box}/_{b}\rangle\,\lambda x.x)$. We can prove that the two resulting terms are contextually equivalent by showing that the relation $\{(u\langle{}^{[a]E[v_0\langle{}^{[a]E[\Box\ t]}/_{a}\rangle\ \Box]}/_{b}\rangle, u\langle{}^{[a]E[v_1\ \Box]}/_{b}\rangle) \mid [b]t \Downarrow_{\mathrm{v}} [b]v, b \notin \mathsf{fn}(t)\}$ is an applicative bisimulation according to Definition 12, and by using Theorem 4. The proof can be found in Appendix B.1. Because $([a]v_0)\langle{}^{[a]\Box\ t}/_{a}\rangle$ and $([a]v_1)\langle{}^{[a]\Box\ t}/_{a}\rangle$ are contextually equivalent, using only clause (2) would lead us to conclude that $[a]v_0$ and $[a]v_1$ are also equivalent.

However, these two named values can be distinguished with the context $[a](\lambda x.x\ x)\ \mu a.\Box$, because in one case we have $([a]v_0)\langle{}^{[a](\lambda x.x\ x)\ \Box}/_{a}\rangle \to_{\mathrm{v}}^* \lambda z.\Omega$, and in the other $([a]v_1)\langle{}^{[a](\lambda x.x\ x)\ \Box}/_{a}\rangle \to_{\mathrm{v}}^* \Omega$. As in Example 7, when evaluating $([a]v_0)\langle{}^{[a](\lambda x.x\ x)\ \Box}/_{a}\rangle$, the body of v_0 is evaluated twice, and two different contexts are captured each time. In contrast, v_1 does not contain any control effect, so when its body is evaluated twice, we get the same result.

Example 8 shows that we have to compare two values $[a]\lambda x.t_0$ and $[a]\lambda x.t_1$ by also testing them with contexts of the form $[a]v\ \Box$, i.e., by considering $[a]v\ \lambda x.t_0\langle{}^{[a]v\ \Box}/_{a}\rangle$ and $[a]v\ \lambda x.t_1\langle{}^{[a]v\ \Box}/_{a}\rangle$. If $v = \lambda x.t$, then these terms reduce in one β-reduction step into $[a]t\{\lambda x.t_0\langle{}^{[a]v\ \Box}/_{a}\rangle/x\}$, and $[a]t\{\lambda x.t_1\langle{}^{[a]v\ \Box}/_{a}\rangle/x\}$. Taking this and clause (2) into account, we obtain the following definition of applicative bisimulation.

Definition 12. *A relation $\mathcal{R}$ on closed named terms is an applicative bisimulation if $u_0\ \mathcal{R}\ u_1$ implies*

- *if $u_0 \to_{\mathrm{v}} u_0'$, then there exists u_1' such that $u_1 \to_{\mathrm{v}}^* u_1'$ and $u_0'\ \mathcal{R}\ u_1'$;*
- *if $u_0 = [a]\lambda x.t_0$, then there exists t_1 such that $u_1 \to_{\mathrm{v}}^* [a]\lambda x.t_1$, and:*
  1. *for all t, b, v such that $[b]t \to_{\mathrm{v}}^* [b]v$ and $b \notin \mathsf{fn}(t)$, we have*

  $$[a]t_0\langle{}^{[a]\Box\ t}/_{a}\rangle\{v\langle{}^{[a]\lambda x.t_0\langle{}^{[a]\Box\ t}/_{a}\rangle\ \Box}/_{b}\rangle/x\} \\ \mathcal{R}\ [a]t_1\langle{}^{[a]\Box\ t}/_{a}\rangle\{v\langle{}^{[a]\lambda x.t_1\langle{}^{[a]\Box\ t}/_{a}\rangle\ \Box}/_{b}\rangle/x\};$$

  2. *for all $v = \lambda x.t$, we have*

  $$[a]t\{\lambda x.t_0\langle{}^{[a]v\ \Box}/_{a}\rangle/x\}\ \mathcal{R}\ [a]t\{\lambda x.t_1\langle{}^{[a]v\ \Box}/_{a}\rangle/x\};$$

- *the symmetric conditions on u_1.*

Applicative bisimilarity, written $\approx$*, is the largest applicative bisimulation.*

The definition is extended to regular terms t_0, t_1 as in CBN, by using a fresh top-level name a. Note that clause (2) implies that a bisimulation $\mathcal{R}$ is a congruence w.r.t. (regular) values; indeed, if $v_0 \mathrel{\mathcal{R}} v_1$, then $[a]v_0 \mathrel{\mathcal{R}} [a]v_1$ for a fresh a, and so we have $[a]t\{v_0/x\} \mathrel{\mathcal{R}} [a]t\{v_1/x\}$ for all t (by clause (2)). This property simplifies the congruence proof of $\approx$ with Howe's method.

As in CBN, we can define a big-step version of the bisimulation (where we use evaluation instead of reduction), and bisimilarity contains reduction.

Lemma 11. *We have* $\rightarrow_{\mathrm{v}}^{*} \subseteq \approx$.

The applicative bisimulation for CBV is more difficult to use than the one for CBN, as we can see by considering again the terms of Example 2.

Example 9. Let $v = \lambda x.t$ and $a, b \notin \mathsf{fn}(v)$; then $[a]v \approx [a]\lambda x.\mu b.[a]v$. To prove clause (1), we consider t' be such that $[b]t' \rightarrow_{\mathrm{v}}^{*} [b]v'$ for $b \notin \mathsf{fn}(t')$; we have to compare $[a]t\{v'\langle {}^{[a]v\,\square}/_{b}\rangle/x\}$ with $[a]\mu b.[a]v\; t'$. But $[a]\mu b.[a]v\; t' \rightarrow_{\mathrm{v}} [a]v\; t' \rightarrow_{\mathrm{v}}^{*} [a]t\{v'\langle {}^{[a]v\,\square}/_{b}\rangle/x\}$, therefore we can conclude with Lemma 11.

For clause (2), we have to relate $[a]t'\{v/y\}$ and $[a]t'\{\lambda x.\mu b.[a]v'\; v/y\}$ for all $v' = \lambda y.t'$. We proceed by case analysis on t'; the most interesting case is $t' = E[y\, v'']$. In this case, we have $[a]t'\{\lambda x.\mu b.[a]v'\, v/y\} \rightarrow_{\mathrm{v}}^{*} [a]v'\, v \rightarrow_{\mathrm{v}} [a]t'\{v/y\}$, therefore we can conclude with Lemma 11. To handle all the possible cases, we prove in Appendix B.1 that $\{(u\{v/y\}, u\{\lambda x.\mu b.[a]t_0/y\}) \mid [a]t_0 \rightarrow_{\mathrm{v}}^{*} u\{v/y\}\} \cup \approx$ is an applicative bisimulation.

In the next example, we give two terms that can be proved equivalent with applicative bisimilarity but not with eager normal form bisimilarity [20].

Example 10. Let $u_0 \stackrel{\text{def}}{=} [b]\lambda xy.\Omega$, $v \stackrel{\text{def}}{=} \lambda y.\mu a.[b]\lambda x.y$, and $u_1 \stackrel{\text{def}}{=} [b]\lambda xy.\Theta_{\mathsf{v}}\; v\; y$, where $\Theta_{\mathsf{v}} \stackrel{\text{def}}{=} (\lambda xy.y\,(\lambda z.xx\,y\,z))\,(\lambda xy.y\,(\lambda z.xx\,y\,z))$ is Turing's call-by-value fixed-point combinator. For u_0 and u_1 to be normal form bisimilar, we need $[c]\Omega$ to be related to $[c]\Theta_{\mathsf{v}}\, v\, y$ for a fresh c, but $[c]\Theta_{\mathsf{v}}\, v\, y \Downarrow_{\mathrm{v}} [b]\lambda y.\Theta_{\mathsf{v}} v\, y$ and $[c]\Omega \Uparrow_{\mathrm{v}}$. In contrast, we can prove that $u_0 \approx u_1$ (see Appendix B.1).

We now briefly sketch the proofs of soundness and completeness; more details can be found in Appendix B.2. The application of Howe's method is easier than in CBN because, as already pointed out, an applicative bisimulation (and, therefore, the applicative bisimilarity) is already a congruence for regular values by definition. What is left to prove is congruence for (named) terms. We use the same definitions of compatible refinement and Howe's closure $\approx^{\bullet}$ as in CBN. However, because $\approx$ is a congruence for values, we can prove directly that the restriction of $\approx^{\bullet}$ to closed terms (written $(\approx^{\bullet})^c$) is an applicative bisimulation, without having to prove a pseudo-simulation lemma (similar to Lemma 5) beforehand.

Lemma 12. *The relation* $(\approx^{\bullet})^c$ *is an applicative bisimulation.*

As in CBN, we can conclude that $(\approx^\bullet)^c = \approx$, and therefore $\approx$ is a congruence. We can then deduce that $\approx$ is sound w.r.t. $\approx_c$. For the reverse inclusion, we use an alternate definition of contextual equivalence where we test terms with evaluation contexts (see Definition 7), and we prove it is an applicative bisimulation. As a result, $\approx$ coincides with $\approx_c$.

Theorem 4. $\approx = \approx_c$.

Remark 2. In [9], Koutavas *et al.* show that applicative bisimilarity cannot be sound in a CBV λ-calculus with exceptions, a mechanism that can be seen as a form of control. Our work agrees with their conclusions, as their definition of applicative bisimilarity compares λ-abstractions by applying them to values only, and Example 7 shows that it is indeed not sufficient.

3.3 Examples

Even if applicative bisimulation for CBV is difficult to use, we can still prove some equivalences with it. Here we give some examples inspired from Sabry and Felleisen's axiomatization of call/cc [16]. Given a name a, we write $a^\dagger$ for the term $\lambda x.\mu b.[a]x$, and we encode call/cc into $\lambda x.\mu a.[a]x\, a^\dagger$. Given a named context $\mathbb{E}$, we also write $\mathbb{E}^\dagger$ for $\lambda x.\mu b.\mathbb{E}[x]$, where $b \notin \mathsf{fn}(\mathbb{E})$. The first example is the axiom C_{tail} of [16], where call/cc is exchanged with a λ-abstraction.

Example 11. If $y \notin \mathsf{fv}(t_1)$ and b is fresh, then $[b](\lambda x.\mu a.[a]x\, a^\dagger)\, (\lambda y.(\lambda z.t_0)\, t_1) \approx [b](\lambda z.(\lambda x.\mu a.[a]x\, a^\dagger)\, (\lambda y.t_0))\, t_1$.

Proof. Let $v_0 \stackrel{\text{def}}{=} \lambda z.t_0\{b^\dagger/y\}$ and $v_1 \stackrel{\text{def}}{=} \lambda z.(\lambda x.\mu a.[a]x\, a^\dagger)\, (\lambda y.t_0)$. The term on the left reduces to $[b]v_0\, t_1$, so we relate this term to the one on the right, i.e., $[b]v_1\, t_1$. We distinguish several cases depending on t_1. Let c be a fresh name. If $[c]t_1 \Downarrow_{\text{v}} [c]v$, then $[b]v_0\, t_1 \rightarrow_{\text{v}}^* [b]t_0\{b^\dagger/y\}\{v\langle{}^{[b]v_0\, \Box}/_c\rangle/z\}$ and $[b]v_1\, t_1 \rightarrow_{\text{v}}^* [b]t_0\{b^\dagger/y\}\{v\langle{}^{[b]v_1\, \Box}/_c\rangle/z\}$; because c is fresh, it does not occur in t_0, and the previous terms can be written $u\langle{}^{[b]v_0\, \Box}/_c\rangle$ and $u\langle{}^{[b]v_1\, \Box}/_c\rangle$ with $u \stackrel{\text{def}}{=} [b]t_0\{b^\dagger/y\}\{v/z\}$.

Similarly, if $[c]t_1 \Downarrow_{\text{v}} [d]v$ with $c \neq d$, then $[b]v_0\, t_1 \Downarrow_{\text{v}} [d]v\langle{}^{[b]v_0\, \Box}/_c\rangle$ and $[b]v_1\, t_1 \Downarrow_{\text{v}} [d]v\langle{}^{[b]v_1\, \Box}/_c\rangle$. When testing these two values with clauses 1 and 2, we obtain each time terms of the form $u\langle{}^{[b]v_0\, \Box}/_c\rangle$ and $u\langle{}^{[b]v_1\, \Box}/_c\rangle$ for some u. With this reasoning, we can prove that $\{(u\langle{}^{[b]v_0\, \Box}/_c\rangle, u\langle{}^{[b]v_1\, \Box}/_c\rangle) \mid u \in U^0\}$ is an applicative bisimulation, by case analysis on u.

With the next example and congruence of $\approx$, we can prove the axiom C_{abort}.

Example 12. Let $a \neq b$; we have $[b]E[a^\dagger\, t] \approx [b]a^\dagger\, t$.

Proof. We prove that the relation $\mathcal{R} \stackrel{\text{def}}{=} \{(u\langle{}^{[b]E[\mathbb{E}^\dagger\, \Box]}/_c\rangle, u\langle{}^{[b]\mathbb{E}^\dagger\, \Box}/_c\rangle) \mid u \in U^0\}$ is an applicative bisimulation by case analysis on u. For example, if $u = [c]t$ and $u \Downarrow_{\text{v}} [c]v$, then $u\langle{}^{[b]E[\mathbb{E}^\dagger\, \Box]}/_c\rangle \rightarrow_{\text{v}}^* [b]E[\mathbb{E}^\dagger\, v\langle{}^{[b]E[\mathbb{E}^\dagger\, \Box]}/_c\rangle] \rightarrow_{\text{v}} \mathbb{E}[v\langle{}^{[b]E[\mathbb{E}^\dagger\, \Box]}/_c\rangle]$ and $u\langle{}^{[b]\mathbb{E}^\dagger\, \Box}/_c\rangle \rightarrow_{\text{v}}^* [b]\mathbb{E}^\dagger\, v\langle{}^{[b]\mathbb{E}^\dagger\, \Box}/_c\rangle \rightarrow_{\text{v}} \mathbb{E}[v\langle{}^{[b]\mathbb{E}^\dagger\, \Box}/_c\rangle]$. If $\mathbb{E} \neq [d]\Box$, then the resulting terms are in $\mathcal{R}$, otherwise we get two named values; when checking clauses 1 and 2, we obtain terms of the form $u'\langle {}^{[b]E[\mathbb{E}'^{\dagger}\,\Box]}/_{c}\rangle$ and $u'\langle {}^{[b]\mathbb{E}'^{\dagger}\,\Box}/_{c}\rangle$ that are in $\mathcal{R}$. The remaining cases are similar.

Example 13 (axiom C_{lift}). We have $[b]E[(\lambda x.\mu a.[a]x\,a^{\dagger})\,t] \approx [b]E[t\,(\lambda x.b^{\dagger}\,E[x])]$.

Proof. In this proof, we use an intermediary result, proved in Appendix B.1: if $\mathbb{E} = \mathbb{E}_0[E_1]$, then $\mathbb{E}^{\dagger} \approx \lambda x.\mathbb{E}_0^{\dagger}\,(E_1[x])$. The proof of the axiom itself is by case analysis on t. An interesting case is when $[d]t \Downarrow_{\mathrm{v}} [d]\lambda y.t'$ where $d \notin \mathsf{fn}(t)$. Then $[b]E[(\lambda x.\mu a.[a]x\;a^{\dagger})\;t] \to_{\mathrm{v}}^{*} [b]E[(\lambda x.\mu a.[a]x\;a^{\dagger})\;\lambda y.t'\langle {}^{[b]E[(\lambda x.\mu a.[a]x\,a^{\dagger})\,\Box]}/_{d}\rangle] \to_{\mathrm{v}}^{*}$ $[b]E[t'\langle {}^{[b]E[(\lambda x.\mu a.[a]x\,a^{\dagger})\,\Box]}/_{d}\rangle\{\mathbb{E}^{\dagger}/y\}]$ (with $\mathbb{E} = [b]E$), and $[b]E[t\,(\lambda x.b^{\dagger}\,E[x])] \to_{\mathrm{v}}^{*}$ $[b]E[t'\langle {}^{[b]E[\Box\,(\lambda x.b^{\dagger}\,E[x])]}/_{d}\rangle\{\lambda x.b^{\dagger}\,E[x]/y\}]$. From the intermediary result, and because $\approx$ is a congruence, we know that $[b]E[t'\{\mathbb{E}^{\dagger}/y\}] \approx [b]E[t'\{\lambda x.b^{\dagger}\,E[x]/y\}]$. Hence, to conclude the proof, one can show that

$$\{(u_0\langle {}^{\mathbb{E}[(\lambda x.\mu a.[a]x\,a^{\dagger})\,\Box]}/_{d}\rangle, u_1\langle {}^{\mathbb{E}[\Box\,(\lambda x.\mathbb{E}_0^{\dagger}\,E_1[x])]}/_{d}\rangle) \mid u_0 \approx u_1, \mathbb{E} = \mathbb{E}_0[E_1]\}$$

is an applicative bisimulation.

4 Conclusion

In this work we propose a definition of applicative bisimilarity for CBN and CBV $\lambda\mu$-calculus. Even if the two definitions seem quite different, they follow the same principles. First, we believe it is essential for completeness to hold to relate primarily named terms, and then extend the definition to all terms, as explained when discussing Lassen's definition of applicative bisimilarity (Section 2.3). The top-level names allow to keep track of how the top level is captured and manipulated in the compared terms.

Then, the idea is to test named values with elementary contexts, $[a]\Box\;t$ for CBN, and $[a]\Box\;t$ and $[a]v\;\Box$ for CBV. In the CBV case, we slightly restrict the terms t tested when considering $[a]\Box\;t$, but the resulting definition remains complex to use compared to CBN, as we can see with Examples 2 and 9. However, we provide counter-examples showing that we cannot simplify it further (see Examples 7 and 8). In CBV as well as in CBN, applicative bisimilarity is harder to use than eager normal form bisimilarity [20], but our relations are complete characterizations of contextual equivalence, and we can therefore prove equivalences of terms that cannot be related with normal form bisimilarity, such as David and Py's example (see Example 4) and Example 10. To prove the equivalence between two given $\lambda\mu$-terms, one should start with the bisimulation of [20], and if it fails, try next our applicative or environmental bisimulations.

We believe the relations we define remain complete w.r.t. contextual equivalence in other variants of the $\lambda\mu$-calculus (perhaps with some slight variations), such as $\lambda\mu$ with different reduction semantics (like, e.g., in [3]), typed $\lambda\mu$-calculus [15], or de Groote's extended calculus ($\Lambda\mu$-calculus [4]). However, any direct implications of this work for other calculi for abortive continuations such as the syntactic theory of control [5] are unclear and remain to be investigated. The reason is that our approach hinges on the syntactic notion of names, unique to the $\lambda\mu$-calculus, that allows one to keep track of the whereabouts of the top level.

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A Call-by-Name $\lambda\mu$-calculus

A.1 Soundness and Completeness of Applicative Bisimilarity

Let $(\approx^\bullet)^c$ be the restriction of $\approx^\bullet$ to closed terms.

Lemma 13. *If $t_0 \approx^\bullet t_1$, then there exists a substitution σ which closes t_0 and t_1 such that $t_0\sigma$ $(\approx^\bullet)^c$ $t_1\sigma$, and the size of the derivation of $t_0\sigma$ $(\approx^\bullet)^c$ $t_1\sigma$ is equal to the size of the derivation of $t_0 \approx^\bullet t_1$. A similar result holds if $u_0 \approx^\bullet u_1$.*

Proof. As usual.

Lemma 14. *Let t_0 $(\approx^\bullet)^c$ t_1, and $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$.*

- *If $[a]t_0 \to_{\mathrm{n}} u_0$, then $[a]t_1 \to^*_{\mathrm{n}} u_1$ and u_0 $(\approx^\bullet)^c$ u_1.*
- *If $t_0 = \lambda x.t'_0$, then $[a]t_1 \to^*_{\mathrm{n}} [a]\lambda x.t'_1$ and for all t''_0 $(\approx^\bullet)^c$ t''_1, we have $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $[a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$.*

Proof. By induction on t_0 $(\approx^\bullet)^c$ t_1.

Suppose $t_0 \approx t_1$. Then $[a]t_0 \approx [a]t_1$. If $[a]t_0 \to_{\mathrm{n}} u_0$, then by bisimilarity, we have $[a]t_1 \to^*_{\mathrm{n}} u_1$ and $u_0 \approx u_1$, i.e., u_0 $(\approx^\bullet)^c$ u_1, as required.

Suppose $t_0 = \lambda x.t'_0$, and let t''_0 $(\approx^\bullet)^c$ t''_1. By the bisimilarity definition, we have $[a]t_1 \to^*_{\mathrm{n}} [a]\lambda x.t'_1$ and $[a]t'_0\{t''_1/x\} \approx [a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$ (using the fact that $a \notin \mathsf{fn}(t'_0)$). From t''_0 $(\approx^\bullet)^c$ t''_1, we deduce $t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $t'_0\{t''_1/x\}$, and then $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c\!\approx$ $[a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$, which implies $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $[a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$, as required.

Suppose $t_0 \approx^\bullet t \approx^\circ t_1$, so that t is closed (using Lemma 13 if necessary). In fact, we have t_0 $(\approx^\bullet)^c$ $t \approx t_1$. If $[a]t_0 \to_{\mathrm{n}} u_0$, then by the induction hypothesis, there exists u such that $[a]t \to^*_{\mathrm{n}} u$ and u_0 $(\approx^\bullet)^c$ u. By bisimilarity, there exists u_1 such that $[a]t_1 \to^*_{\mathrm{n}} u_1$ and $u \approx u_1$. From u_0 $(\approx^\bullet)^c$ $u \approx u_1$, we deduce u_0 $(\approx^\bullet)^c$ u_1, as wished.

Suppose $t_0 = \lambda x.t'_0$, and let t''_0 $(\approx^\bullet)^c$ t''_1. By induction, there exists t' such that $[a]t \to^*_{\mathrm{n}} [a]\lambda x.t'$ and $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $[a]t'\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$. By bisimilarity, there exists t'_1 such that $[a]t_1 \to^*_{\mathrm{n}} [a]\lambda x.t'_1$ and $[a]t'\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\} \approx [a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$. Hence, we have $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c\!\approx$ $[a]t'\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$, i.e., $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $[a]t'_1\langle [a]\Box\, t''_1/a\rangle\{t''_1/x\}$ as required.

If t_0 $\widetilde{(\approx^\bullet)^c}$ t_1, then we have several cases to consider.

Suppose $t_0 = \lambda x.t'_0$ and $t_1 = \lambda x.t'_1$ with $t'_0 \approx^\bullet t'_1$. Let t''_0 $(\approx^\bullet)^c$ t''_1. We have $[a]t'_0\{t''_0/x\}$ $(\approx^\bullet)^c$ $[a]t'_1\{t''_1/x\}$, hence the result holds (note that $a \notin \mathsf{fv}(t'_1)$ because $a \notin \mathsf{fv}(t_1)$).

Suppose $t_0 = \mu b.u_0$ and $t_1 = \mu b.u_1$ with u_0 $(\approx^\bullet)^c$ u_1. We have $[a]t_0 \to_{\mathrm{n}} u_0\langle [a]\Box/b\rangle$, $[a]t_1 \to_{\mathrm{n}} u_1\langle [a]\Box/b\rangle$, and $u_0\langle [a]\Box/b\rangle$ $(\approx^\bullet)^c$ $u_1\langle [a]\Box/b\rangle$, hence the result holds.

Suppose $t_0 = t^1_0\, t^2_0$, $t_1 = t^1_1\, t^2_1$ with t^1_0 $(\approx^\bullet)^c$ t^1_1 and t^2_0 $(\approx^\bullet)^c$ t^2_1. We distinguish two cases.

– If $[b]t_0^1 \to_{\mathrm{n}} u_0$ (for some fresh b), then $[a]t_0 \to_{\mathrm{n}} u_0\langle {}^{[a]\Box\ t_0^2}/b\rangle$. By the induction hypothesis, there exists u_1 such that $[b]t_1^1 \to_{\mathrm{n}}^* u_1$ and $u_0\ (\approx^\bullet)^c\ u_1$. Consequently, we have $[a]t_1 \to_{\mathrm{n}}^* u_1\langle {}^{[a]\Box\ t_1^2}/b\rangle$, and by definition of $\approx^\bullet$, we have $u_0\langle {}^{[a]\Box\ t_0^2}/b\rangle\ (\approx^\bullet)^c\ u_1\langle {}^{[a]\Box\ t_1^2}/b\rangle$, as required.
– If $t_0^1 = \lambda x.t_0'$, then we have $[a]t_0 \to_{\mathrm{n}} [a]t_0'\{t_0^2/x\}$. By the induction hypothesis, there exists t_1' such that $[a]t_1^1 \to_{\mathrm{n}}^* [a]\lambda x.t_1'$ and $[a]t_0'\{t_0^2/x\}\ (\approx^\bullet)^c\ [a]t_1'\langle {}^{[a]\Box\ t_1^2}/a\rangle\{t_1^2/x\}$. From $[a]t_1^1 \to_{\mathrm{n}}^* [a]\lambda x.t_1'$, we deduce

$$[a]t_1 \to_{\mathrm{n}}^* [a](\lambda x.t_1'\langle {}^{[a]\Box\ t_1^2}/a\rangle)t_1^2 \to_{\mathrm{n}} [a]t_1'\langle {}^{[a]\Box\ t_1^2}/a\rangle\{t_1^2/x\},$$

hence the result holds.

Suppose $t_0 = t_0'\langle {}^{\mathbb{E}_0}/b\rangle$, $t_1 = t_1'\langle {}^{\mathbb{E}_1}/b\rangle$ with $t_0'\ (\approx^\bullet)^c\ t_1'$, $\mathbb{E}_0\ (\approx^\bullet)^c\ \mathbb{E}_1$. If $[a]t_0 \to_{\mathrm{n}} u_0$, then in fact $[a]t_0' \to_{\mathrm{n}} u_0'$ with $u_0 = u_0'\langle {}^{\mathbb{E}_0}/b\rangle$. By the induction hypothesis, there exists u_1' such that $[a]t_1' \to_{\mathrm{n}}^* u_1'$ and $u_0'\ (\approx^\bullet)^c\ u_1'$. Consequently, we have $[a]t_1 \to_{\mathrm{n}}^* u_1'\langle {}^{\mathbb{E}_1}/b\rangle$, and $u_0'\langle {}^{\mathbb{E}_0}/b\rangle\ (\approx^\bullet)^c\ u_1'\langle {}^{\mathbb{E}_1}/b\rangle$ holds, as wished. If $[a]t_0$ is a named value, then in fact $t_0' = \lambda x.t_0'''$ and $t_0 = \lambda x.t_0'''\langle {}^{\mathbb{E}_0}/b\rangle$. Let $t_0''\ (\approx^\bullet)^c\ t_1''$. Then $t_0''\langle {}^{[c]\Box}/b\rangle\ (\approx^\bullet)^c\ t_1''\langle {}^{[c]\Box}/b\rangle$ for a fresh c. By the induction hypothesis, there exists t_1''' such that $[a]t_1' \to_{\mathrm{n}}^* [a]\lambda x.t_1'''$ and

$$[a]t_0'''\{t_0''\langle {}^{[c]\Box}/b\rangle/x\}\ (\approx^\bullet)^c\ [a]t_1'''\langle {}^{[a]\Box\ t_1''\langle {}^{[c]\Box}/b\rangle}/a\rangle\{t_1''\langle {}^{[c]\Box}/b\rangle/x\}.$$

Therefore, we have $[a]t_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1'''\langle {}^{\mathbb{E}_1}/b\rangle$, and

$$[a]t_0'''\{t_0''\langle {}^{[c]\Box}/b\rangle/x\}\langle {}^{\mathbb{E}_0}/b\rangle\ (\approx^\bullet)^c\ [a]t_1'''\langle {}^{[a]\Box\ t_1''\langle {}^{[c]\Box}/b\rangle}/a\rangle\{t_1''\langle {}^{[c]\Box}/b\rangle/x\}\langle {}^{\mathbb{E}_1}/b\rangle.$$

Because b does not occur in $t_0''\langle {}^{[c]\Box}/b\rangle$, $t_1''\langle {}^{[c]\Box}/b\rangle$ thanks to the renaming to a fresh c, we can switch the substitutions around, and in fact

$$[a]t_0'''\langle {}^{\mathbb{E}_0}/b\rangle\{t_0''\langle {}^{[c]\Box}/b\rangle/x\}\ (\approx^\bullet)^c\ [a]t_1'''\langle {}^{\mathbb{E}_1}/b\rangle\langle {}^{[a]\Box\ t_1''\langle {}^{[c]\Box}/b\rangle}/a\rangle\{t_1''\langle {}^{[c]\Box}/b\rangle/x\}$$

holds. Renaming c back into b, we obtain

$$[a]t_0'''\langle {}^{\mathbb{E}_0}/b\rangle\{t_0''/x\}\ (\approx^\bullet)^c\ [a]t_1'''\langle {}^{\mathbb{E}_1}/b\rangle\langle {}^{[a]\Box\ t_1''}/a\rangle\{t_1''/x\},$$

which gives us the required result.

Lemma 15. *Let $u_0\ (\approx^\bullet)^c\ u_1$.*

– *If $u_0 \to_{\mathrm{n}} u_0'$, then $u_1 \to_{\mathrm{n}}^* u_1'$ and $u_0'\ (\approx^\bullet)^c\ u_1'$.*
– *If $u_0 = [a]\lambda x.t_0$, then $u_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1$ and for all $t_0'\ (\approx^\bullet)^c\ t_1'$, we have $[a]t_0\langle {}^{[a]\Box\ t_0'}/a\rangle\{t_0'/x\}\ (\approx^\bullet)^c\ [a]t_1\langle {}^{[a]\Box\ t_1'}/a\rangle\{t_1'/x\}$.*

Proof. By induction on $u_0\ (\approx^\bullet)^c\ u_1$.

Suppose $u_0 \approx u_1$. The first item holds by bisimilarity. Suppose $u_0 = [a]\lambda x.t_0$, and let $t_0'\ (\approx^\bullet)^c\ t_1'$. By definition of the bisimilarity, we have $u_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1$ and $[a]t_0\langle {}^{[a]\Box\ t_1'}/a\rangle\{t_1'/x\} \approx [a]t_1\langle {}^{[a]\Box\ t_1'}/a\rangle\{t_1'/x\}$. From $t_0'\ (\approx^\bullet)^c\ t_1'$, we deduce $t_0\langle {}^{[a]\Box\ t_0'}/a\rangle\{t_0'/x\}\ (\approx^\bullet)^c\ t_0\langle {}^{[a]\Box\ t_1'}/a\rangle\{t_1'/x\}$, which implies

$$t_0\langle {}^{[a]\Box\ t_0'}/a\rangle\{t_0'/x\}\ (\approx^\bullet)^c \approx [a]t_1\langle {}^{[a]\Box\ t_1'}/a\rangle\{t_1'/x\},$$

which in turn implies

$$[a]t_0\langle {}^{[a]\Box\, t'_0}/_a\rangle\{t'_0/x\} \;(\approx^\bullet)^c\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\},$$

as required.

Suppose $u_0 \approx^\bullet u \approx^\circ u_1$, so that u is closed (using Lemma 13 if necessary). In fact, we have $u_0 \;(\approx^\bullet)^c\; u \approx u_1$. If $u_0 \rightarrow_{\mathrm{n}} u'_0$, then by the induction hypothesis, there exists u' such that $u \rightarrow^*_{\mathrm{n}} u'$ and $u'_0 \;(\approx^\bullet)^c\; u'$. By bisimilarity, there exists u'_1 such that $u_1 \rightarrow^*_{\mathrm{n}} u'_1$ and $u' \approx u'_1$. From $u'_0 \;(\approx^\bullet)^c\; u' \approx u'_1$, we deduce $u'_0 \;(\approx^\bullet)^c\; u'_1$, as wished.

Suppose $u_0 = [a]\lambda x.t_0$, and let $t'_0 \;(\approx^\bullet)^c\; t'_1$. By induction, there exists t such that $u \rightarrow^*_{\mathrm{n}} [a]\lambda x.t$ and $[a]t_0\langle {}^{[a]\Box\, t'_0}/_a\rangle\{t'_0/x\} \;(\approx^\bullet)^c\; [a]t\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\}$. By bisimilarity, there exists t_1 such that $u_1 \rightarrow^*_{\mathrm{n}} [a]\lambda x.t_1$ and $[a]t\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\} \approx [a]t_1\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\}$. Consequently, we have

$$[a]t_0\langle {}^{[a]\Box\, t'_0}/_a\rangle\{t'_0/x\} \;(\approx^\bullet)^c\!\approx [a]t\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\},$$

i.e., $[a]t_0\langle {}^{[a]\Box\, t'_0}/_a\rangle\{t'_0/x\} \;(\approx^\bullet)^c\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_a\rangle\{t'_1/x\}$ holds as required.

If $u_0 \;\widetilde{(\approx^\bullet)^c}\; u_1$, then we have two cases.

Suppose $u_0 = [a]t_0$ and $u_1 = [a]t_1$ with $t_0 \;(\approx^\bullet)^c\; t_1$. If $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$, then we can apply Lemma 14 directly to get the required result. Otherwise, let $b \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$. If $u_0 \rightarrow_{\mathrm{n}} u'_0$, then $[b]t_0 \rightarrow_{\mathrm{n}} u''_0$ and $u'_0 = u'_0\langle {}^{[a]\Box}/_b\rangle$. We can apply Lemma 14 to $[b]t_0$ and $[b]t_1$, and then rename b into a.

Suppose $u_0 = u'_0\langle {}^{\mathbb{E}_0}/_a\rangle$, $u_1 = u'_1\langle {}^{\mathbb{E}_1}/_a\rangle$ with $u'_0 \;(\approx^\bullet)^c\; u'_1$ and $\mathbb{E}_0 \;(\approx^\bullet)^c\; \mathbb{E}_1$.

If u_0 is a named value, then we distinguish two cases. First, we may have $u'_0 = [a]\lambda x.t_0$, $\mathbb{E}_0 = \mathbb{E}_1 = [b]\Box$, and $u_0 = [b]\lambda x.t_0\langle {}^{\mathbb{E}_0}/_a\rangle$. Let $t'_0 \;(\approx^\bullet)^c\; t'_1$. Let t''_0, t''_1 be t'_0 and t'_1 where a, b are renamed into fresh c, d to avoid some name clashes (we still have $t''_0 \;(\approx^\bullet)^c\; t''_1$). By the induction hypothesis, there exists t_1 such that $u'_1 \rightarrow^*_{\mathrm{n}} [a]\lambda x.t_1$ and $[a]t_0\langle {}^{[a]\Box\, t''_0}/_a\rangle\{t''_0/x\} \;(\approx^\bullet)^c\; [a]t_1\langle {}^{[a]\Box\, t''_1}/_a\rangle\{t''_1/x\}$. This implies

$$[a]t_0\langle {}^{[a]\Box\, t''_0}/_a\rangle\{t''_0/x\}\langle {}^{[a]\Box\, t''_0}/_b\rangle \;(\approx^\bullet)^c\; [a]t_1\langle {}^{[a]\Box\, t''_1}/_a\rangle\{t''_1/x\}\langle {}^{[a]\Box\, t''_1}/_b\rangle,$$

which is the same as

$$[a]t_0\langle {}^{[a]\Box\, t''_0}/_a\rangle\langle {}^{[a]\Box\, t''_0}/_b\rangle\{t''_0/x\} \;(\approx^\bullet)^c\; [a]t_1\langle {}^{[a]\Box\, t''_1}/_a\rangle\langle {}^{[a]\Box\, t''_1}/_b\rangle\{t''_1/x\}$$

because b does not occur in t''_0, t''_1. In turn, we have

$$[b]t_0\langle {}^{[a]\Box\, t''_0}/_a\rangle\langle {}^{[a]\Box\, t''_0}/_b\rangle\{t''_0/x\}\langle {}^{\mathbb{E}_0}/_a\rangle \qquad (\approx^\bullet)^c\; [b]t_1\langle {}^{[a]\Box\, t''_1}/_a\rangle\langle {}^{[a]\Box\, t''_1}/_b\rangle\{t''_1/x\}\langle {}^{\mathbb{E}_1}/_a\rangle,$$

which is equal to

$$[b]t_0\langle {}^{\mathbb{E}_0}/_a\rangle\langle {}^{[b]\Box\, t''_0}/_b\rangle\{t''_0/x\} \;(\approx^\bullet)^c\; [b]t_1\langle {}^{\mathbb{E}_1}/_a\rangle\langle {}^{[b]\Box\, t''_1}/_b\rangle\{t''_1/x\}$$

because a does not occur in t_0'', t_1''. Renaming c, d back into a, b, we obtain

$$[b]t_0\langle \mathbb{E}_0/a\rangle\langle [b]\Box\, t_0'/b\rangle\{t_0'/x\}\ (\approx^\bullet)^c\ [b]t_1\langle \mathbb{E}_1/a\rangle\langle [b]\Box\, t_1'/b\rangle\{t_1'/x\},$$

and because $u_1 \to_{\mathrm{n}}^* [b]\lambda x.t_1\langle \mathbb{E}_1/a\rangle$, the result holds.

In the second case, we have $u_0' = [b]\lambda x.t_0$ and $u_0 = [b]\lambda x.t_0\langle \mathbb{E}_0/a\rangle$ with $b \neq a$. Let $t_0'\ (\approx^\bullet)^c\ t_1'$. Let t_0'', t_1'' be t_0' and t_1' where a is renamed into a fresh c. By the induction hypothesis, there exists t_1 such that $u_1' \to_{\mathrm{n}}^* [b]\lambda x.t_1$ and

$$[b]t_0\langle [b]\Box\, t_0''/b\rangle\{t_0''/x\}\ (\approx^\bullet)^c\ [b]t_1\langle [b]\Box\, t_1''/b\rangle\{t_1''/x\}.$$

From $\mathbb{E}_0\langle [b]\Box\, t_0''/b\rangle\ (\approx^\bullet)^c\ \mathbb{E}_1\langle [b]\Box\, t_1''/b\rangle$ and the previous relation, we can deduce

$$[b]t_0\langle [b]\Box\, t_0''/b\rangle\{t_0''/x\}\langle \mathbb{E}_0\langle [b]\Box\, t_0''/b\rangle/a\rangle\ (\approx^\bullet)^c\ [b]t_1\langle [b]\Box\, t_1''/b\rangle\{t_1''/x\}\langle \mathbb{E}_1\langle [b]\Box\, t_1''/b\rangle/a\rangle.$$

Because a does not occur in t_0'', t_1'', this can be rewritten into

$$[b]t_0\langle \mathbb{E}_0/a\rangle\langle [b]\Box\, t_0''/b\rangle\{t_0''/x\}\ (\approx^\bullet)^c\ [b]t_1\langle \mathbb{E}_1/a\rangle\langle [b]\Box\, t_1''/b\rangle\{t_1''/x\}.$$

By renaming c back into a, we obtain

$$[b]t_0\langle \mathbb{E}_0/a\rangle\langle [b]\Box\, t_0'/b\rangle\{t_0'/x\}\ (\approx^\bullet)^c\ [b]t_1\langle \mathbb{E}_1/a\rangle\langle [b]\Box\, t_1'/b\rangle\{t_1'/x\},$$

and because $u_1 \to_{\mathrm{n}}^* [b]\lambda x.t_1\langle \mathbb{E}_1/a\rangle$, the result holds.

If $u_0 \to_{\mathrm{n}}$, then again we distinguish two cases. First, suppose $u_0' \to_{\mathrm{n}} u_0''$; then $u_0 \to_{\mathrm{n}} u_0''\langle E_0/a\rangle$. By the induction hypothesis, there exists u_1'' such that $u_1' \to_{\mathrm{n}}^* u_1''$ and $u_0''\ (\approx^\bullet)^c\ u_1''$. Then $u_1 \to_{\mathrm{n}}^* u_1''\langle \mathbb{E}_1/a\rangle$ and $u_0''\langle \mathbb{E}_0/a\rangle\ (\approx^\bullet)^c\ u_1''\langle \mathbb{E}_1/a\rangle$, hence the result holds.

Otherwise, $u_0' = [a]\lambda x.t_0$, $\mathbb{E}_0 = \mathbb{E}_0'[\Box\, t_0']$, and $u_0 \to_{\mathrm{n}} \mathbb{E}_0'[t_0\langle \mathbb{E}_0/a\rangle\{t_0'/x\}]$. We can prove by induction on $\mathbb{E}_0\ (\approx^\bullet)^c\ \mathbb{E}_1$ that $\mathbb{E}_1 = \mathbb{E}_1'[\Box\, t_1']$ with $\mathbb{E}_0'\ (\approx^\bullet)^c\ \mathbb{E}_1'$ and $t_0'\ (\approx^\bullet)^c\ t_1'$. Let $\mathbb{E}_i''$, t_i'' be $\mathbb{E}_i'$, t_i' with a renamed into a fresh b. By the induction hypothesis, there exists t_1 such that $u_1' \to_{\mathrm{n}}^* [a]\lambda x.t_1$, and

$$[a]t_0\langle [a]\Box\, t_0''/a\rangle\{t_0''/x\}\ (\approx^\bullet)^c\ [a]t_1\langle [a]\Box\, t_1''/a\rangle\{t_1''/x\}.$$

This implies

$$([a]t_0\langle [a]\Box\, t_0''/a\rangle\{t_0''/x\})\langle \mathbb{E}_0''/a\rangle\ (\approx^\bullet)^c\ ([a]t_1\langle [a]\Box\, t_1''/a\rangle\{t_1''/x\})\langle \mathbb{E}_1''/a\rangle,$$

i.e., $\mathbb{E}_0''[t_0\langle \mathbb{E}_0''[\Box\, t_0'']/a\rangle\{t_0''/x\}]\ (\approx^\bullet)^c\ \mathbb{E}_1''[t_1\langle \mathbb{E}_1''[\Box\, t_1'']/a\rangle\{t_1''/x\}]$ because a does not occurs in t_0'', t_1''. Renaming b into a, we obtain

$$\mathbb{E}_0'[t_0\langle \mathbb{E}_0/a\rangle\{t_0'/x\}]\ (\approx^\bullet)^c\ \mathbb{E}_1'[t_1\langle \mathbb{E}_1/a\rangle\{t_1'/x\}],$$

and because $u_1 \to_{\mathrm{n}}^* \mathbb{E}_1'[t_1\langle \mathbb{E}_1/a\rangle\{t_1'/x\}]$, we have the required result.

From there, we can prove that $(\approx^\bullet)^c{=}\approx$ using the usual techniques [6], and then we deduce soundness of $\approx$. To prove completeness, we show that $\dot{\approx}_c$ is an applicative bisimulation.

Lemma 16. *The relation $\dot{\approx}_c$ is a big step applicative bisimulation.*

Proof. Suppose $u_0\ \dot{\approx}_c\ u_1$. If $u_0 \Downarrow_{\mathrm{n}} [a]\lambda x.t_0$, then $u_1 \Downarrow_{\mathrm{n}} [a]\lambda x.t_1$. We have $[a](\mu a.u_0)\, t \to_{\mathrm{n}}^* [a]t_0\langle [a]\Box\, t/a\rangle\{t/x\}$, $[a](\mu a.u_1)\, t \to_{\mathrm{n}}^* [a]t_1\langle [a]\Box\, t/a\rangle\{t/x\}$, but also $[a](\mu a.u_0)\, t \mathrel{\dot{\approx}_c} [a](\mu a.u_1)\, t$. From $\to_{\mathrm{n}}^* \subseteq \approx \subseteq \dot{\approx}_c$, we have $[a]t_0\langle [a]\Box\, t/a\rangle\{t/x\} \mathrel{\dot{\approx}_c} [a]t_1\langle [a]\Box\, t/a\rangle\{t/x\}$ as wished.

A.2 David and Py's Counter-Example

Lemma 17. *Let* $0 \overset{def}{=} \lambda x.\lambda y.y$, $1 \overset{def}{=} \lambda x.\lambda y.x$, *and* $t_a \overset{def}{=} \mu c.[a]0$. *Then we have* $\lambda x.\mu a.[a]x\, \mu b.[a]x\, t_a\, 0 \approx \lambda x.\mu a.[a]x\, \mu b.[a]x\, t_a\, 1$.

Proof. We fix a name c, and for all t, we want to relate $[c]\mu a.[a]t\, \mu b.[a]t\, t_a\, 0$ and $[c]\mu a.[a]t\, \mu b.[a]t\, t_a\, 1$, which reduce respectively to $[c]t\, \mu b.[c]t\, t_c\, 0$ and $[c]t\, \mu b.[c]t\, t_c\, 1$. Let $s_0^t \overset{\text{def}}{=} \mu b.[c]t t_c 0$, $s_1^t \overset{\text{def}}{=} \mu b.[c]t t_c 1$, $\mathbb{E}_0^t \overset{\text{def}}{=} [c]\Box\ s_0^t$, and $\mathbb{E}_1^t \overset{\text{def}}{=} [c]\Box\ s_1^t$. We define a relation $[d]t \rightsquigarrow^k u$ as $[d]t \rightarrow_{\mathrm{n}}^* u$ if $k = 0$ and as $[d]t \Downarrow_{\mathrm{n}} [d]\lambda x_1.t_1$, $[c]t_1 \Downarrow_{\mathrm{n}} [d]\lambda x_2.t_2$, $\ldots [c]t_k \rightarrow_{\mathrm{n}}^* u$ for some $t_1 \ldots t_k$ if $k > 0$. The rationale behind this relation appears in the proof. Note that if $[d]t \rightsquigarrow^k u$, then $\mathsf{fv}(u) \subseteq \{x_1, \ldots x_k\}$, and $[d]t\langle \mathbb{E}/e\rangle \rightsquigarrow^k u\langle \mathbb{E}/e\rangle$ for all $\mathbb{E}$ and $e \neq d$. We define $\mathcal{R}$ as

$$
\begin{aligned}
&\{(u\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k, u\langle \mathbb{E}_1^t/d\rangle\{s_1^t/x_i\}_{i=1}^k) \mid u \in U, t \in T^0, d \notin \mathsf{fn}(t), [d]t \rightsquigarrow^k u\}\\
\cup\ &\{([c]t\, t_c\, 0, [c]t\, t_c\, 1) \mid t \in T^0, d \notin \mathsf{fn}(t), [d]t \rightsquigarrow^k \mathbb{E}[x_i], k \geq 1, 1 \leq i \leq k\}\\
\cup\ &\{(u, u) \mid u \in U^0\}
\end{aligned}
$$

where $t\{t_i/x_i\}_{i=1}^k$ stands for $t\{t_1/x_1\} \ldots \{t_k/x_k\}$, and we show that $\mathcal{R}$ is an applicative bisimulation.

Let $u\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k\ \mathcal{R}\ u\langle \mathbb{E}_1^t/d\rangle\{s_1^t/x_i\}_{i=1}^k$. If $u \rightarrow_{\mathrm{n}} u'$, then it is easy to conclude. Otherwise, we distinguish several cases.

If $u = [e]\lambda y.t_0$ with $e \neq d$, then $[e]t_0\langle [e]\Box\ t'/e\rangle\{t'/y\}\langle \mathbb{E}_0^{t''}/d\rangle\{s_0^{t''}/x_i\}_{i=1}^k\ \mathcal{R}$ $[e]t_0\langle [e]\Box\ t'/e\rangle\{t'/y\}\langle \mathbb{E}_1^{t''}/d\rangle\{s_1^{t''}/x_i\}_{i=1}^k$ (where $t'' = t\langle [e]\Box\ t'/e\rangle$) for all t', because the relation $[d]t \rightsquigarrow^k [e]\lambda y.t_0$ implies $[d]t'' \rightsquigarrow^k [e]t_0\langle [e]\Box\ t'/e\rangle\{t'/y\}$.

If $u = [d]\lambda x_{k+1}.t_{k+1}$, then we have the reduction $u\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k \rightarrow_{\mathrm{n}}$ $[c]t_{k+1}\{s_0^t/x_{k+1}\}\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k$ as well as the reduction $u\langle \mathbb{E}_1^t/d\rangle\{s_1^t/x_i\}_{i=1}^k \rightarrow_{\mathrm{n}}$ $[c]t_{k+1}\{s_1^t/x_{k+1}\}\langle \mathbb{E}_1^t/d\rangle\{s_1^t/x_i\}_{i=1}^k$. We obtain terms in $\mathcal{R}$ because we can rewrite them into, respectively, $[c]t_{k+1}\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^{k+1}$ and $[c]t_{k+1}\langle \mathbb{E}_1^t/d\rangle\{s_1^t/x_i\}_{i=1}^{k+1}$, and $[d]t \rightsquigarrow^{k+1} [c]t_{k+1}$ holds.

Finally, if $u = \mathbb{E}[x_i]$, for $1 \leq i \leq k$ (assuming $k \geq 1$), then we have $u\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k \rightarrow_{\mathrm{n}} [c]t\, t_c\, 0$ and $u\langle \mathbb{E}_0^t/d\rangle\{s_0^t/x_i\}_{i=1}^k \rightarrow_{\mathrm{n}} [c]t\, t_c\, 1$, and $[c]t\, t_c\, 0\ \mathcal{R}$ $[c]t\, t_c\, 1$ holds.

Let $[c]t\ t_c\ 0\ \mathcal{R}\ [c]t\ t_c\ 1$ with $[d]t \rightsquigarrow^k \mathbb{E}[x_i]$. There exist $t_1, \ldots, t_k$ such that $[d]t \Downarrow_{\mathrm{n}} [d]\lambda x_1.t_1$, $[c]t_1 \Downarrow_{\mathrm{n}} [d]\lambda x_2.t_2$, $\ldots [c]t_{k-1} \Downarrow_{\mathrm{n}} [d]\lambda x_k.t_k$, and $[c]t_k \rightarrow_{\mathrm{n}}^* \mathbb{E}[x_i]$. Then $[c]t\ t_c\ 0 \rightarrow_{\mathrm{n}}^* [c](\lambda x_1.t_1\langle [c]\Box\ t_c\ 0/d\rangle)\ t_c\ 0 \rightarrow_{\mathrm{n}} [c]t_1\langle [c]\Box\ t_c\ 0/d\rangle\{t_c/x_1\}\ 0 \rightarrow_{\mathrm{n}}^*$ $[c]t_2\langle [c]\Box\ t_c\ 0/d\rangle\langle [c]\Box\ 0/c\rangle\{t_c/x_i\}_{i=1}^2\ 0 \rightarrow_{\mathrm{n}}^* [c]t_k\langle [c]\Box\ t_c\ 0/d\rangle\langle [c]\Box\ 0/c\rangle\{t_c/x_i\}_{i=1}^k\ 0 \rightarrow_{\mathrm{n}}^*$ $\mathbb{E}'[t_c] \rightarrow_{\mathrm{n}} [c]0$ with $\mathbb{E}' \overset{\text{def}}{=} \mathbb{E}\langle [c]\Box\ t_c\ 0/d\rangle\langle [c]\Box\ 0/c\rangle\{t_c/x_i\}_{i=1}^k$. Similarly $[c]t\ t_c\ 1 \rightarrow_{\mathrm{n}}^*$ $\mathbb{E}''[t_c] \rightarrow_{\mathrm{n}} [c]0$ with $\mathbb{E}'' \overset{\text{def}}{=} \mathbb{E}\langle [c]\Box\ t_c\ 1/d\rangle\langle [c]\Box\ 1/c\rangle\{t_c/x_i\}_{i=1}^k$. Because the two terms evaluate to $[c]0$, it is easy to conclude.

A.3 Soundness and Completeness of Environmental Bisimilarity

Lemma 18. *Let* $\mathcal{R}$ *be a relation on closed terms and named terms. If* $t_0\ \widehat{\mathcal{R}}\ t_1$ *and* $t'_0\ \widehat{\mathcal{R}}\ t'_1$, *then* $t_0\{t'_0/x\}\ \widehat{\mathcal{R}}\ t_1\{t'_1/x\}$ *(and similarly if* $u_0\ \widehat{\mathcal{R}}\ u_1$*)*

Proof. By induction on $t_0\ \widehat{\mathcal{R}}\ t_1$.

Lemma 19. *If $u_0 \approx_{\mathcal{E}} u_1$, then $u_0\langle {}^{\mathbb{E}}/_{a}\rangle \approx_{\mathcal{E}} u_1\langle {}^{\mathbb{E}}/_{a}\rangle$.*

Let $\mathcal{Y}$ be an environmental bisimulation. We define

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_1 \cup \mathcal{X}_2 \cup \{\widehat{\mathcal{E}}^{\mathrm{nv}}, \mathcal{E} \in \mathcal{Y}\} \\ \mathcal{X}_1 &= \{(\widehat{\mathcal{E}}^{\mathrm{nv}}, u_0\langle {}^{\mathbb{E}_0^0}/_{a_0}\rangle \dots \langle {}^{\mathbb{E}_0^n}/_{a_n}\rangle, u_1\langle {}^{\mathbb{E}_1^0}/_{a_0}\rangle \dots \langle {}^{\mathbb{E}_1^n}/_{a_n}\rangle), u_0 \;\mathcal{Y}_{\mathcal{E}}\; u_1, \mathbb{E}_0^i \;\widehat{\mathcal{E}}\; \mathbb{E}_1^i\} \\ \mathcal{X}_2 &= \{(\widehat{\mathcal{E}}^{\mathrm{nv}}, u_0, u_1), \mathcal{E} \in \mathcal{Y}, u_0 \;\widehat{\mathcal{E}}\; u_1\} \end{aligned}$$

We first prove some preliminary lemmas about $\mathcal{X}$.

Lemma 20. *Let $\mathcal{E} \in \mathcal{Y}$.*

- *If $\lambda x.t_0 \;\widehat{\mathcal{E}}\; \lambda x.t_1$ and $t'_0 \;\widehat{\mathcal{E}}\; t'_1$, then $[a]t_0\{t'_0/x\} \;\widehat{\mathcal{E}}\; [a]t_1\{t'_1/x\}$.*
- *If $[a]\lambda x.t_0 \;\widehat{\mathcal{E}}\; [a]\lambda x.t_1$ and $t'_0 \;\widehat{\mathcal{E}}\; t'_1$, then we have $[a]t_0\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t'_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t'_1/x\}$.*

Proof. The first item is proved by induction on $\lambda x.t_0 \;\widehat{\mathcal{E}}\; \lambda x.t_1$. If $t_0 \;\widehat{\mathcal{E}}\; t_1$, then $[a]t_0\{t'_0/x\} \;\widehat{\mathcal{E}}\; [a]t_1\{t'_1/x\}$ holds using Lemma 18. Otherwise, we have $\lambda x.t_0 = \lambda x.t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle$, $\lambda x.t_1 = \lambda x.t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle$ with $\lambda x.t''_0 \;\widehat{\mathcal{E}}\; \lambda x.t''_1$ and $\mathbb{E}_0 \;\widehat{\mathcal{E}}\; \mathbb{E}_1$. Let t'''_0, t'''_1 be t'_0, t'_1 where b is renamed to a fresh c (we still have $t'''_0 \;\widehat{\mathcal{E}}\; t'''_1$). By the induction hypothesis, we have $t''_0\{t'''_0/x\} \;\widehat{\mathcal{E}}\; t''_1\{t'''_1/x\}$, which implies $t''_0\{t'''_0/x\}\langle {}^{\mathbb{E}_0}/_{b}\rangle \;\widehat{\mathcal{E}}\; t''_1\{t'''_1/x\}\langle {}^{\mathbb{E}_1}/_{b}\rangle$, i.e., $t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle\{t'''_0/x\} \;\widehat{\mathcal{E}}\; t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle\{t'''_1/x\}$ because $b \notin \mathsf{fn}(t'''_0) \cup \mathsf{fn}(t'''_1)$. Renaming c back into b, we obtain $t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle\{t'_0/x\} \;\widehat{\mathcal{E}}\; t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle\{t'_1/x\}$ as wished.

The second item is proved by induction on $[a]\lambda x.t_0 \;\widehat{\mathcal{E}}\; [a]\lambda x.t_1$. If $[a]\lambda x.t_0 \;\mathcal{E}\; [a]\lambda x.t_1$, then we have $[a]t_0\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t'_0/x\} \;\mathcal{Y}_{\mathcal{E}}\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t'_1/x\}$ because $\mathcal{Y}$ is a bisimulation, i.e., $[a]t_0\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t'_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t'_1/x\}$ holds as wished.

Suppose $\lambda x.t_0 \;\widehat{\mathcal{E}}\; \lambda x.t_1$, and let t''_0, t''_1 be t'_0, t'_1 where a is renamed into a fresh b. By the first item, we have $[b]t_0\{t''_0/x\} \;\widehat{\mathcal{E}}\; [b]t_1\{t''_1/x\}$, which implies $[b]t_0\{t''_0/x\}\langle {}^{[a]\Box\, t'_0}/_{a}\rangle \;\widehat{\mathcal{E}}\; [b]t_1\{t''_1/x\}\langle {}^{[a]\Box\, t'_1}/_{a}\rangle$, which gives $[b]t_0\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t''_0/x\} \;\widehat{\mathcal{E}}\; [b]t_1\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t''_1/x\}$ because a does not occur in t''_0, t''_1. Renaming b in a, we obtain $[a]t_0\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t'_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t_1\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t'_1/x\}$, as wished.

Suppose $[a]\lambda x.t_0 = [a]\lambda x.t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle$, $[a]\lambda x.t_1 = [a]\lambda x.t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle$ with $[a]\lambda x.t''_0 \;\widehat{\mathcal{E}}\; [a]\lambda x.t''_1$ and $\mathbb{E}_0 \;\widehat{\mathcal{E}}\; \mathbb{E}_1$. Let t'''_0, t'''_1 be t'_0, t'_1 where b is renamed to a fresh c. By the induction hypothesis, we have $[a]t''_0\langle {}^{[a]\Box\, t'''_0}/_{a}\rangle\{t'''_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t''_1\langle {}^{[a]\Box\, t'''_1}/_{a}\rangle\{t'''_1/x\}$, which implies $[a]t''_0\langle {}^{[a]\Box\, t'''_0}/_{a}\rangle\{t'''_0/x\}\langle {}^{\mathbb{E}_0}/_{b}\rangle \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t''_1\langle {}^{[a]\Box\, t'''_1}/_{a}\rangle\{t'''_1/x\}\langle {}^{\mathbb{E}_1}/_{b}\rangle$ (according to the definition of $\mathcal{X}$), which gives us $[a]t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle\langle {}^{[a]\Box\, t'''_0}/_{a}\rangle\{t'''_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle\langle {}^{[a]\Box\, t'''_1}/_{a}\rangle\{t'''_1/x\}$ (because a does not occur in t'''_0, t'''_1). Renaming b into a, we obtain $[a]t''_0\langle {}^{\mathbb{E}_0}/_{b}\rangle\langle {}^{[a]\Box\, t'_0}/_{a}\rangle\{t'_0/x\} \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; [a]t''_1\langle {}^{\mathbb{E}_1}/_{b}\rangle\langle {}^{[a]\Box\, t'_1}/_{a}\rangle\{t'_1/x\}$, as wished.

Lemma 21. *Let $\mathcal{E} \in \mathcal{Y}$, $t_0 \;\widehat{\mathcal{E}}\; t_1$, and $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$. If $[a]t_0 \to_{\mathrm{n}} u_0$, then $[a]t_1 \to_{\mathrm{n}}^{*} u_1$, and $u_0 \;\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\; u_1$.*

Proof. We proceed by induction on $t_0 \;\widehat{\mathcal{E}}\; t_1$.

Suppose $t_0 = \mu b.u_0$, $t_1 = \mu b.u_1$ with $u_0 \mathrel{\widehat{\mathcal{E}}} u_1$. Then $[a]t_0 \to_{\mathrm{n}} u_0\langle{}^{[a]\Box}\!/_{b}\rangle$, $[a]t_1 \to_{\mathrm{n}} u_1\langle{}^{[a]\Box}\!/_{b}\rangle$, and $u_0\langle{}^{[a]\Box}\!/_{b}\rangle \mathrel{\widehat{\mathcal{E}}} u_1\langle{}^{[a]\Box}\!/_{b}\rangle$ holds, so we have $u_0\langle{}^{[a]\Box}\!/_{b}\rangle \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1\langle{}^{[a]\Box}\!/_{b}\rangle$, as wished.

Suppose $t_0 = t_0^1\, t_0^2$ and $t_1 = t_1^1\, t_1^2$ with $t_0^1 \mathrel{\widehat{\mathcal{E}}} t_1^1$ and $t_0^2 \mathrel{\widehat{\mathcal{E}}} t_1^2$. We have two cases to consider.

- Assume $[a]t_0^1 \to_{\mathrm{v}} u_0$, so that $[a]t_0 \to_{\mathrm{v}} u_0\langle{}^{[a]\Box\, t_0^2}\!/_{a}\rangle$. By the induction hypothesis, there exists u_1 such that $[a]t_1^1 \to_{\mathrm{v}}^* u_1$ and $u_0 \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1$. From $t_0^2 \mathrel{\widehat{\mathcal{E}}} t_1^2$ and $u_0 \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1$, we can deduce $u_0\langle{}^{[a]\Box\, t_0^2}\!/_{a}\rangle \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1\langle{}^{[a]\Box\, t_1^2}\!/_{a}\rangle$ by definition of $\mathcal{X}$. We also have $[a]t_1 \to_{\mathrm{v}}^* u_1\langle{}^{[a]\Box\, t_1^2}\!/_{a}\rangle$, hence the result holds.
- Assume $t_0^1 = \lambda x.t_0'$ so that $[a]t_0 \to_{\mathrm{v}} [a]t_0'\{t_0^2/x\}$. Because $\mathcal{E}$ relates only named values, t_1^1 must be value $\lambda x.t_1'$ as well. We have $[a]t_1 \to_{\mathrm{v}} [a]t_1'\{t_1^2/x\}$, and by Lemma 20, we have $[a]t_0'\{t_0^2/x\} \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} [a]t_1'\{t_1^2/x\}$, hence the result holds.

Suppose $t_0 = t_0'\langle{}^{\mathbb{E}_0}\!/_{b}\rangle$, $t_1 = t_1'\langle{}^{\mathbb{E}_1}\!/_{b}\rangle$ with $t_0' \mathrel{\widehat{\mathcal{E}}} t_1'$ and $\mathbb{E}_0 \mathrel{\widehat{\mathcal{E}}} \mathbb{E}_1$. If $[a]t_0 \to_{\mathrm{n}}$, then $[a]t_0' \to_{\mathrm{n}} u_0$ and $[a]t_0 \to_{\mathrm{n}} u_0\langle{}^{\mathbb{E}_0}\!/_{a}\rangle$. By the induction hypothesis, there exists u_1 such that $[a]t_1' \to_{\mathrm{n}}^* u_1$ and $u_0 \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1$. By definition of $\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}$ u_1, we have $u_0\langle{}^{\mathbb{E}_0}\!/_{b}\rangle \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1\langle{}^{\mathbb{E}_1}\!/_{b}\rangle$, and we have $[a]t_1 \to_{\mathrm{n}}^* u_1\langle{}^{\mathbb{E}_1}\!/_{b}\rangle$, as wished.

We now prove Lemma 19 by showing that $\mathcal{X}$ is a bisimulation up to environment.

Proof. We first prove the bisimulation for the elements in $\mathcal{X}_2$ (for these, we do not need the "up to environment"). Let $t_0 \mathrel{\widehat{\mathcal{E}}} t_1$, with $\mathcal{E} \in \mathcal{Y}$. If u_0 is a named value, then one can check that so is t_1 (because $\mathcal{E}$ relates only named values), and we have $\widehat{\mathcal{E}}^{\mathrm{nv}} \cup \{(t_0, t_1)\} = \widehat{\mathcal{E}}^{\mathrm{nv}} \in \mathcal{X}$.

Let $u_0 \mathrel{\widehat{\mathcal{E}}} u_1$ such that $u_0 \to_{\mathrm{n}} u_0'$. We proceed by induction on $u_0 \mathrel{\widehat{\mathcal{E}}} u_1$. Suppose $u_0 = [a]t_0$, $u_1 = [a]t_1$, with $t_0 \mathrel{\widehat{\mathcal{E}}} t_1$. If $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$, then we can apply Lemma 21 directly to get the required result. Otherwise, let $b \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$. If $u_0 \to_{\mathrm{n}} u_0'$, then $[b]t_0 \to_{\mathrm{n}} u_0''$ and $u_0' = u_0'\langle{}^{[a]\Box}\!/_{b}\rangle$. We can apply Lemma 21 to $[b]t_0$ and $[b]t_1$, and then rename b into a.

Suppose $u_0 = u_0'\langle{}^{\mathbb{E}_0}\!/_{a}\rangle$, $u_1 = u_1'\langle{}^{\mathbb{E}_1}\!/_{a}\rangle$ with $u_0' \mathrel{\widehat{\mathcal{E}}} u_1'$ and $\mathbb{E}_0 \mathrel{\widehat{\mathcal{E}}} \mathbb{E}_1$. If u_0 is a named value, then $u_0' = [c]v_0$, and $u_0 = [b]v_0\langle{}^{\mathbb{E}_0}\!/_{a}\rangle$. Because $\mathcal{E}$ relates only named values, u_1' is also a named value $[b]v_1$, and we have $[b]v_0\langle{}^{\mathbb{E}_0}\!/_{a}\rangle \mathrel{\widehat{\mathcal{E}}} [b]v_1\langle{}^{\mathbb{E}_1}\!/_{a}\rangle$, as wished.

If $u_0 \to_{\mathrm{n}}$, then we distinguish two cases. First, suppose $u_0' \to_{\mathrm{n}} u_0''$; then $u_0 \to_{\mathrm{n}} u_0''\langle{}^{E_0}\!/_{a}\rangle$. By the induction hypothesis, there exists u_1'' such that $u_1' \to_{\mathrm{n}}^* u_1''$ and $u_0'' \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1''$. Then $u_1 \to_{\mathrm{n}}^* u_1''\langle{}^{\mathbb{E}_1}\!/_{a}\rangle$ and $u_0''\langle{}^{\mathbb{E}_0}\!/_{a}\rangle \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} u_1''\langle{}^{\mathbb{E}_1}\!/_{a}\rangle$ (by definition of $\mathcal{X}$), hence the result holds.

Otherwise, we have $u_0' = [a]\lambda x.t_0$, $\mathbb{E}_0 = \mathbb{E}_0'[\Box\ t_0']$, which implies $u_0 \to_{\mathrm{n}} \mathbb{E}_0'[t_0\langle{}^{\mathbb{E}_0}\!/_{a}\rangle\{t_0'/x\}]$. We can prove by induction on $\mathbb{E}_0 \mathrel{\widehat{\mathcal{E}}} \mathbb{E}_1$ that $\mathbb{E}_1 = \mathbb{E}_1'[\Box\ t_1']$ with $\mathbb{E}_0' \mathrel{\widehat{\mathcal{E}}} \mathbb{E}_1'$ and $t_0' \mathrel{\widehat{\mathcal{E}}} t_1'$. Let $\mathbb{E}_i''$, t_i'' be $\mathbb{E}_i'$, t_i' with a renamed into a fresh b. Because $\mathcal{E}$ relates only named values, we must have $u_1' = [a]\lambda x.t_1$. By Lemma 20, we have $[a]t_0\langle{}^{[a]\Box\, t_0''}\!/_{a}\rangle\{t_0''/x\} \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} [a]t_1\langle{}^{[a]\Box\, t_1''}\!/_{a}\rangle\{t_1''/x\}$, which in turn implies $([a]t_0\langle{}^{[a]\Box\, t_0''}\!/_{a}\rangle\{t_0''/x\})\langle{}^{\mathbb{E}_0''}\!/_{a}\rangle \mathrel{\mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}} ([a]t_1\langle{}^{[a]\Box\, t_1''}\!/_{a}\rangle\{t_1''/x\})\langle{}^{\mathbb{E}_1''}\!/_{a}\rangle$ by definition of

$\mathcal{X}$. Because a does not occurs in t_0'', t_1'', the previous relation is equivalent to $\mathbb{E}_0''[t_0\langle \mathbb{E}_0''[\Box\, t_0'']/a\rangle\{t_0''/x\}]\ \mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\ \mathbb{E}_1''[t_1\langle \mathbb{E}_1''[\Box\, t_1'']/a\rangle\{t_1''/x\}]$. Renaming b into a, we obtain $\mathbb{E}_0'[t_0\langle \mathbb{E}_0/a\rangle\{t_0'/x\}]\ \mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\ \mathbb{E}_1'[t_1\langle \mathbb{E}_1/a\rangle\{t_1'/x\}]$, and because we have $u_1 \rightarrow_{\mathrm{n}}^{*} \mathbb{E}_1'[t_1\langle \mathbb{E}_1/a\rangle\{t_1'/x\}]$, the result holds.

We now prove bisimulation (up to environment) for elements in $\mathcal{X}_1$. Let $u_0\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle\ \mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\ u_1\langle \mathbb{E}_1^0/a_0\rangle \dots \langle \mathbb{E}_1^n/a_n\rangle)$ so that $u_0\ \mathcal{Y}_{\mathcal{E}}\ u_1$ and $\mathbb{E}_0^i\ \widehat{\mathcal{E}}\ \mathbb{E}_1^i$. If u_0 is a named value $[a]v_0$, then because $\mathcal{Y}$ is a bisimulation, there exists v_1 such that $u_1 \rightarrow_{\mathrm{v}}^{*} [a]v_1$ and $\mathcal{E}' = \mathcal{E} \cup \{([a]v_0, [a]v_1)\} \in \mathcal{Y}$. We then have $u_1\langle \mathbb{E}_1^0/a_0\rangle \dots \langle \mathbb{E}_1^n/a_n\rangle \rightarrow_{\mathrm{v}}^{*} ([a]v_1)\langle \mathbb{E}_1^0/a_0\rangle \dots \langle \mathbb{E}_1^n/a_n\rangle$, and $([a]v_0)\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle$ and $([a]v_0)\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle$ are in $\mathcal{X}_2$. We can prove the bisimulation property with these two named terms the same way we did with the named terms in $\mathcal{X}_2$, except that we reason up to environment, because $\mathcal{E} \subseteq \mathcal{E}'$. Suppose u_0 is not a named value. There exists u_0' such that $u_0 \rightarrow_{\mathrm{v}} u_0'$, and so $u_0\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle \rightarrow_{\mathrm{v}} u_0'\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle$. Because $\mathcal{Y}$ is a bisimulation, there exists u_1' such that $u_1 \rightarrow_{\mathrm{v}}^{*} u_1'$ and $u_0'\ \mathcal{Y}_{\mathcal{E}}\ u_1'$. We therefore have $u_1\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle \rightarrow_{\mathrm{v}}^{*} u_1'\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle$ with $u_0'\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle\ \mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\ u_1'\langle \mathbb{E}_0^0/a_0\rangle \dots \langle \mathbb{E}_0^n/a_n\rangle$, as wished.

For the last bisimulation condition, let $[a]\lambda x.t_0\ \widehat{\mathcal{E}}^{\mathrm{nv}}\ [a]\lambda x.t_1$ and $t_0'\ \widehat{\mathcal{E}}\ t_1'$. By Lemma 20, we have $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\}\ \mathcal{X}_{\widehat{\mathcal{E}}^{\mathrm{nv}}}\ [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$, hence the result holds.

Lemma 22. *If* $[a]\lambda x.t_0 \simeq [a]\lambda x.t_1$ *then* $[a]t_0\langle [a]\Box\, t/a\rangle\{t/x\} \simeq [a]t_1\langle [a]\Box\, t/a\rangle\{t/x\}$

Proof. Follows almost directly from the bisimilarity definition.

Lemma 23. *If* $[a]\lambda x.t_0 \mathrel{\widehat{\simeq}} [a]\lambda x.t_1$ *and* $t_0' \mathrel{\widehat{\simeq}} t_1'$ *then* $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\widehat{\simeq}\simeq} [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$.

Proof. By case analysis on $[a]\lambda x.t_0 \mathrel{\widehat{\simeq}} [a]\lambda x.t_1$. Suppose $[a]\lambda x.t_0 \simeq [a]\lambda x.t_1$. By Lemma 22, we have $[a]t_0\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\} \simeq [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$. From $t_0' \mathrel{\widehat{\simeq}} t_1'$, we deduce $t_0\langle [a]\Box\, t_0'/a\rangle \mathrel{\widehat{\simeq}} t_0\langle [a]\Box\, t_1'/a\rangle$, which implies $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\widehat{\simeq}} [a]t_0\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$ (using Lemma 18). Finally, we get $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\widehat{\simeq}\simeq} [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$ as wished.

Suppose $\lambda x.t_0 \mathrel{\widehat{\simeq}} \lambda x.t_1$. We distinguish two cases. If $\lambda x.t_0 \simeq \lambda x.t_1$, then $[a]\lambda x.t_0 \simeq [a]\lambda x.t_1$, and we proceed as before. Otherwise, we have $t_0 \mathrel{\widehat{\simeq}} t_1$, which implies $t_0\langle [a]\Box\, t_0'/a\rangle \mathrel{\widehat{\simeq}} t_1\langle [a]\Box\, t_1'/a\rangle$, which in turn implies $t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\widehat{\simeq}} t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$ by Lemma 18, and finally we have $[a]t_0\langle [a]\Box\, t_0'/a\rangle\{t_0'/x\} \mathrel{\widehat{\simeq}} [a]t_1\langle [a]\Box\, t_1'/a\rangle\{t_1'/x\}$, as wished.

Lemma 24. *Let* $t_0 \mathrel{\widehat{\simeq}} t_1$ *and* $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$.

- *If* $[a]t_0 \rightarrow_{\mathrm{n}} u_0$, *then* $[a]t_1 \rightarrow_{\mathrm{n}}^{*} u_1$, *and* $u_0 \mathrel{\widehat{\simeq}\simeq} u_1$.
- *If* $t_0 = \lambda x.t_0'$, *then* $[a]t_1 \rightarrow_{\mathrm{n}}^{*} [a]\lambda x.t_1'$ *and* $[a]\lambda x.t_0' \mathrel{\widehat{\simeq}} [a]v \simeq [a]\lambda x.t_1'$ *for some* v.

Proof. By induction on $t_0 \mathrel{\widehat{\simeq}} t_1$.

If $t_0 \simeq t_1$, then $[a]t_0 \simeq [a]t_1$, and the result holds by bisimilarity.

If $t_0 = \lambda x.t'_0$ and $t_1 = \lambda x.t'_1$ with $t'_0 \mathrel{\widehat{\simeq}} t'_1$, then $[a]t_0 \mathrel{\widehat{\simeq}} [a]t_1$, and the result holds.

Suppose $t_0 = \mu b.u_0$ and $t_1 = \mu b.u_1$ with $u_0 \mathrel{\widehat{\simeq}} u_1$. We have $[a]t_0 \rightarrow_{\mathrm{n}} u_0\langle [a]\Box/b\rangle$, $[a]t_1 \rightarrow_{\mathrm{n}} u_1\langle [a]\Box/b\rangle$, and $u_0\langle [a]\Box/b\rangle \mathrel{\widehat{\simeq}} u_1\langle [a]\Box/b\rangle$, hence the result holds.

Suppose $t_0 = t_0^1\, t_0^2$, $t_1 = t_1^1\, t_1^2$ with $t_0^1 \mathrel{\widehat{\simeq}} t_1^1$ and $t_0^2 \mathrel{\widehat{\simeq}} t_1^2$. We distinguish two cases.

- If $[b]t_0^1 \rightarrow_{\mathrm{n}} u_0$ (for some fresh b), then $[a]t_0 \rightarrow_{\mathrm{n}} u_0\langle [a]\Box\; t_0^2/b\rangle$. By the induction hypothesis, there exist u_1, u such that $[b]t_1^1 \rightarrow_{\mathrm{n}}^* u_1$ and $u_0 \mathrel{\widehat{\simeq}} u \simeq u_1$. Consequently, we have $[a]t_1 \rightarrow_{\mathrm{n}}^* u_1\langle [a]\Box\; t_1^2/b\rangle$. By congruence, we have $u_0\langle [a]\Box\; t_0^2/b\rangle \mathrel{\widehat{\simeq}} u\langle [a]\Box\; t_1^2/b\rangle$, and by Lemma 19, we have $u\langle [a]\Box\; t_1^2/b\rangle \simeq u_1\langle [a]\Box\; t_1^2/b\rangle$, hence the result holds.
- If $t_0^1 = \lambda x.t'_0$, then $[a]t_0 \rightarrow_{\mathrm{n}} [a]t'_0\{t_0^2/x\}$. By the induction hypothesis, there exist t, t'_1 such that $[a]t_1^1 \rightarrow_{\mathrm{n}}^* [a]\lambda x.t'_1$ and $[a]\lambda x.t'_1 \mathrel{\widehat{\simeq}} [a]\lambda x.t \simeq [a]\lambda x.t'_1$. By Lemma 23, we have $[a]t'_0\{t_0^2/x\} \mathrel{\widehat{\approxeq}} [a]t\langle [a]\Box\; t_1^2/a\rangle\{t_1^2/x\}$, and by Lemma 22, we have $[a]t\langle [a]\Box\; t_1^2/a\rangle\{t_1^2/x\} \simeq [a]t'_1\langle [a]\Box\; t_1^2/a\rangle\{t_1^2/x\}$. From $[a]t_1^1 \rightarrow_{\mathrm{n}}^* [a]\lambda x.t'_1$, we deduce $[a]t_1 \rightarrow_{\mathrm{n}}^* [a](\lambda x.t'_1\langle [a]\Box\; t_1^2/a\rangle)t_1^2 \rightarrow_{\mathrm{n}} [a]t'_1\langle [a]\Box\; t_1^2/a\rangle\{t_1^2/x\}$, hence the result holds.

Suppose $t_0 = t'_0\langle \mathbb{E}_0/b\rangle$, $t_1 = t'_1\langle \mathbb{E}_1/b\rangle$ with $t'_0 \mathrel{\widehat{\simeq}} t'_1$, $\mathbb{E}_0 \mathrel{\widehat{\simeq}} \mathbb{E}_1$. If $[a]t_0 \rightarrow_{\mathrm{n}} u_0$, then in fact $[a]t'_0 \rightarrow_{\mathrm{n}} u'_0$ with $u_0 = u'_0\langle \mathbb{E}_0/b\rangle$. By the induction hypothesis, there exists u'_1 such that $[a]t'_1 \rightarrow_{\mathrm{n}}^* u'_1$ and $u'_0 \mathrel{\widehat{\simeq}\simeq} u'_1$. Consequently, we have $[a]t_1 \rightarrow_{\mathrm{n}}^* u'_1\langle \mathbb{E}_1/b\rangle$, and $u'_0\langle \mathbb{E}_0/b\rangle \mathrel{\widehat{\simeq}\simeq} u'_1\langle \mathbb{E}_1/b\rangle$ holds (using Lemma 19), as wished. If $[a]t_0$ is a named value, then in fact $t'_0 = \lambda x.t''_0$ and $t_0 = \lambda x.t''_0\langle \mathbb{E}_0/b\rangle$. By induction, there exist t''_1 such that $[a]t'_1 \rightarrow_{\mathrm{n}}^* [a]\lambda x.t''_1$ and $[a]\lambda x.t''_0 \mathrel{\widehat{\simeq}\simeq} [a]\lambda x.t''_1$. Hence, we have $[a]t_1 \rightarrow_{\mathrm{n}}^* [a]\lambda x.t''_1\langle \mathbb{E}_1/b\rangle$, and $[a]\lambda x.t''_0\langle \mathbb{E}_0/b\rangle \mathrel{\widehat{\simeq}\simeq} [a]\lambda x.t''_1\langle \mathbb{E}_1/b\rangle$ holds using Lemma 19.

Lemma 25. $u_0 \simeq u_1$ *implies* $C[u_0] \approx_{\widehat{\simeq}^{\mathrm{nv}}} C[u_1]$ *and* $\mathbb{C}[u_0] \approx_{\widehat{\simeq}^{\mathrm{nv}}} \mathbb{C}[u_1]$.

Proof. We prove that

$$\mathcal{X} = \{(\widehat{\simeq}^{\mathrm{nv}}, u_0, u_1), u_0 \mathrel{\widehat{\simeq}} u_1\} \cup \{\widehat{\simeq}^{\mathrm{nv}}\}$$

is a bisimulation up-to bisimilarity.

Let $u_0 \mathrel{\widehat{\simeq}} u_1$; we proceed by induction on $u_0 \mathrel{\widehat{\simeq}} u_1$.

If $u_0 \simeq u_1$, then the result holds by bisimilarity.

Suppose $u_0 = [a]t_0$, $u_1 = [a]t_1$, with $t_0 \mathrel{\widehat{\simeq}} t_1$. If $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$, then we can apply Lemma 24 directly to get the required result. Otherwise, let $b \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$. If $u_0 \rightarrow_{\mathrm{n}} u'_0$, then $[b]t_0 \rightarrow_{\mathrm{n}} u''_0$ and $u'_0 = u'_0\langle [a]\Box/b\rangle$. We can apply Lemma 24 to $[b]t_0$ and $[b]t_1$, and then rename b into a.

Suppose $u_0 = u'_0\langle \mathbb{E}_0/a\rangle$, $u_1 = u'_1\langle \mathbb{E}_1/a\rangle$ with $u'_0 \mathrel{\widehat{\simeq}} u'_1$ and $\mathbb{E}_0 \mathrel{\widehat{\simeq}} \mathbb{E}_1$.

If u_0 is a named value, then $u'_0 = [c]v_0$, and $u_0 = [b]v_0\langle \mathbb{E}_0/a\rangle$. By the induction hypothesis, there exists v_1 such that $u'_1 \rightarrow_{\mathrm{n}}^* [c]v_1$ and $[c]v_0 \mathrel{\widehat{\simeq}\simeq} [c]v_1$. We have $u_1 \rightarrow_{\mathrm{n}}^* [b]v_1\langle \mathbb{E}_1/a\rangle$, and also $[b]v_0\langle \mathbb{E}_0/a\rangle \mathrel{\widehat{\simeq}\simeq} [b]v_1\langle \mathbb{E}_1/a\rangle$ (using Lemma 19), hence the result holds.

If $u_0 \rightarrow_{\mathrm{n}}$, then we distinguish two cases. First, suppose $u'_0 \rightarrow_{\mathrm{n}} u''_0$; then $u_0 \rightarrow_{\mathrm{n}} u''_0\langle{}^{\mathbb{E}_0}/_a\rangle$. By the induction hypothesis, there exists u''_1 such that $u'_1 \rightarrow^*_{\mathrm{n}} u''_1$ and $u''_0 \,\widehat{\simeq}\!\simeq\, u''_1$. Then $u_1 \rightarrow^*_{\mathrm{n}} u''_1\langle{}^{\mathbb{E}_1}/_a\rangle$ and $u''_0\langle{}^{\mathbb{E}_0}/_a\rangle \,\widehat{\simeq}\!\simeq\, u''_1\langle{}^{\mathbb{E}_1}/_a\rangle$ (using Lemma 19), hence the result holds.

Otherwise, we have $u'_0 = [a]\lambda x.t_0$, $\mathbb{E}_0 = \mathbb{E}'_0[\Box\ t'_0]$, which implies $u_0 \rightarrow_{\mathrm{n}} \mathbb{E}'_0[t_0\langle{}^{\mathbb{E}_0}/_a\rangle\{t'_0/x\}]$. We can prove by induction on $\mathbb{E}_0 \,\widehat{\simeq}\, \mathbb{E}_1$ that $\mathbb{E}_1 = \mathbb{E}'_1[\Box\ t'_1]$ with $\mathbb{E}'_0 \,\widehat{\simeq}\, \mathbb{E}'_1$ and $t'_0 \,\widehat{\simeq}\, t'_1$. Let $\mathbb{E}''_i$, t''_i be $\mathbb{E}'_i$, t'_i with a renamed into a fresh b. By the induction hypothesis, there exist t, t_1 such that $u'_1 \rightarrow^*_{\mathrm{n}} [a]\lambda x.t_1$, and $[a]\lambda x.t_0 \,\widehat{\simeq}\, [a]\lambda x.t \simeq [a]\lambda x.t_1$. By Lemma 23 and Lemma 22, we have $[a]t_0\langle{}^{[a]\Box\ t''_0}/_a\rangle\{t''_0/x\} \,\widehat{\simeq}\!\simeq\, [a]t\langle{}^{[a]\Box\ t''_1}/_a\rangle\{t''_1/x\} \simeq [a]t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\{t''_1/x\}$, which in turn implies $([a]t_0\langle{}^{[a]\Box\ t''_0}/_a\rangle\{t''_0/x\})\langle{}^{\mathbb{E}''_0}/_a\rangle \,\widehat{\simeq}\!\simeq\, ([a]t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\{t''_1/x\})\langle{}^{\mathbb{E}''_1}/_a\rangle$ using Lemma 19. Because a does not occurs in t''_0, t''_1, the previous relation is equivalent to $\mathbb{E}''_0[t_0\langle{}^{\mathbb{E}''_0[\Box\ t''_0]}/_a\rangle\{t''_0/x\}] \,\widehat{\simeq}\!\simeq\, \mathbb{E}''_1[t_1\langle{}^{\mathbb{E}''_1[\Box\ t''_1]}/_a\rangle\{t''_1/x\}]$. Renaming b into a, we obtain $\mathbb{E}'_0[t_0\langle{}^{\mathbb{E}_0}/_a\rangle\{t'_0/x\}] \,\widehat{\simeq}\!\simeq\, \mathbb{E}'_1[t_1\langle{}^{\mathbb{E}_1}/_a\rangle\{t'_1/x\}]$, and because $u_1 \rightarrow^*_{\mathrm{n}} \mathbb{E}'_1[t_1\langle{}^{\mathbb{E}_1}/_a\rangle\{t'_1/x\}]$, we have the required result.

To prove the last item, we take $[a]\lambda x.t_0 \,\widehat{\simeq}\, [a]\lambda x.t_1$ and $t'_0 \,\widehat{\simeq}\, t'_1$. Then $[a]t_0\langle{}^{[a]\Box\ t'_0}/_a\rangle\{t'_0/x\} \,\widehat{\simeq}\!\simeq\, [a]t_1\langle{}^{[a]\Box\ t'_1}/_a\rangle\{t'_1/x\}$ holds by Lemma 23.

B Call-by-Value $\lambda\mu$-calculus

B.1 Equivalence Proofs

Lemma 26 (see Example 9). *The relation* $\mathcal{R} \overset{def}{=} \{(u\{v/y\}, u\{\lambda x.\mu b.[a]t_0/y\}) \mid [a]t_0 \rightarrow^*_{\mathrm{v}} u\{v/y\}\} \cup \approx$ *is an applicative bisimulation.*

Proof. We proceed by case analysis on u. If $u \rightarrow_{\mathrm{v}} u'$ or if $u = \mathbb{E}[v'\ y]$, then it is easy to conclude. If $u = \mathbb{E}[y\ v']$, then $u\{\lambda x.\mu b.[a]t_0/y\} \rightarrow_{\mathrm{v}} [a]t_0$ and $[a]t_0 \rightarrow^*_{\mathrm{v}} u\{v/y\}$ by definition, so it we can conclude with Lemma 11. Similarly, the transition from $u\{v/y\}$ is matched by $u\{\lambda x.\mu b.[a]t_0/y\}$.

If $u = [c]v'$, then we have to compare (the result of the reduction of) $u\langle{}^{\mathbb{E}}/_c\rangle\{v\langle{}^{\mathbb{E}}/_c\rangle/y\}$ and $u\langle{}^{\mathbb{E}}/_c\rangle\{\lambda x.\mu b.([a]t_0)\langle{}^{\mathbb{E}}/_c\rangle/y\}$ for some $\mathbb{E}$ (which depends on the clause we check). The resulting terms are in $\mathcal{R}$, because $[a]t_0 \rightarrow^*_{\mathrm{v}} u\{v/y\}$ implies $([a]t_0)\langle{}^{\mathbb{E}}/_c\rangle \rightarrow^*_{\mathrm{v}} u\langle{}^{\mathbb{E}}/_c\rangle\{v\langle{}^{\mathbb{E}}/_c\rangle/y\}$.

We decompose the equivalence proof of Example 10 into several lemmas to improve readability. We remind that $u_0 \overset{\mathrm{def}}{=} [b]\lambda xy.\Omega$, $v \overset{\mathrm{def}}{=} \lambda y.\mu a.[b]\lambda x.y$, $u_1 \overset{\mathrm{def}}{=} [b]\lambda xy.\Theta_{\mathsf{v}}\ v\ y$, and $\Theta_{\mathsf{v}} \overset{\mathrm{def}}{=} (\lambda xy.y\ (\lambda z.xx\ y\ z))\ (\lambda xy.y\ (\lambda z.xx\ y\ z))$

Lemma 27. *Let t_1, t_2 such that $[c]t_1 \Downarrow_{\mathrm{v}} [c]v_1$ ($c \notin \mathsf{fn}(t_1)$), and $[d]t_2 \Downarrow_{\mathrm{v}} [d]v_2$ ($d \notin \mathsf{fn}(t_2)$). For all v', we have $[b]\Omega \approx [b]\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_b\rangle v'$.*

Proof. We have

$$
\begin{aligned}
&[b]\Theta_{\mathsf{v}}\, v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle v'\\
\to_{\mathrm{v}}^{*}&[b]v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ (\lambda x.\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ x)\ v'\\
\to_{\mathrm{v}}^{*}&[b](\lambda zx.\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ x)\ t_1\ t_2\\
\to_{\mathrm{v}}^{*}&[b](\lambda zx.\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ x)\ v_1'\ t_2\\
\to_{\mathrm{v}}^{*}&[b](\lambda zx.\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ x)\ v_1'\ v_2'\\
\to_{\mathrm{v}}^{*}&[b]\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle\ v_2'
\end{aligned}
$$

for some v_1', v_2' (which depend on v_1, v_2). So for all v', there exists v'' such that $[b]\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle v' \ \to_{\mathrm{v}}^{*}\ [b]\Theta_{\mathsf{v}}\ v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle v''$; from that, we deduce that $[b]\Theta_{\mathsf{v}}\, v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle v'$ is diverging, and therefore $[b]\Omega \approx [b]\Theta_{\mathsf{v}}\, v\langle{}^{[b]\Box\ t_1\ t_2}/_{b}\rangle v'$ holds.

Lemma 28. *Let* $v' \stackrel{\mathit{def}}{=} \lambda z.t'$*, and* t_1 *such that there exists* v_1 *such that* $[c]t_1 \Downarrow_{\mathrm{v}} [c]v_1$ *for* $c \notin \mathsf{fn}(t_1)$*. We have*

$$[b]t'\{\lambda y.\Omega/z\} \approx [b]t'\{\lambda y.\Theta_{\mathsf{v}}\ v\langle{}^{[b]v'\ (\Box\ t_1)}/_{b}\rangle\ y/z\}.$$

Proof. Let $\mathcal{R} \stackrel{\text{def}}{=} \{(u\{\lambda y.\Omega/z\}, u\{\lambda y.\Theta_{\mathsf{v}} v\langle{}^{\mathbb{E}[v'\ (\Box\ t_1)]}/_{b}\rangle y/z\}) \mid \exists v_1.[c]t_1 \Downarrow_{\mathrm{v}} [c]v_1, c \notin \mathsf{fn}(t_1), v' = \lambda z.t', \mathbb{E}[t'] \to_{\mathrm{v}}^{*} u\} \cup \approx$. We prove $\mathcal{R}$ is an applicative bisimilarity, by case analysis on u.

The case $u \to_{\mathrm{v}} u'$ is easy. Suppose $u = [d]v_2$, with $v_2 \stackrel{\text{def}}{=} \lambda z_2.t_2$. Then we have two items to prove.

- Let $[e]t_3 \Downarrow_{\mathrm{v}} [e]v_3$ ($e \notin \mathsf{fn}(t_3)$). Then we have to relate

$$[d]t_2\{\lambda y.\Omega/z\}\langle{}^{[d]\Box\ t_3}/_{d}\rangle\{v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle/z_2\}$$

to

$$[d](t_2\{\lambda y.\Theta_{\mathsf{v}}\ v\langle{}^{\mathbb{E}[v'\ (\Box\ t_1)]}/_{b}\rangle\ y/z\})\langle{}^{[d]\Box\ t_3}/_{d}\rangle\{v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle/z_2\}.$$

These terms can be rewritten into

$$[d]t_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\{v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle/z_2\}\{\lambda y.\Omega/z\} \tag{1}$$

and

$$[d]t_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\{v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle/z_2\}\{\lambda y.\Theta_{\mathsf{v}}\ v\langle{}^{\mathbb{E}'[v''\ (\Box\ t_1')]}/_{b}\rangle\ y/z\} \tag{2}$$

where $\mathbb{E}' \stackrel{\text{def}}{=} \mathbb{E}\langle{}^{[d]\Box\ t_3}/_{d}\rangle$, $v'' \stackrel{\text{def}}{=} v'\langle{}^{[d]\Box\ t_3}/_{d}\rangle$, and $t_1' \stackrel{\text{def}}{=} t_1\langle{}^{[d]\Box\ t_3}/_{d}\rangle$. We have $v'' = \lambda z.t'\langle{}^{[d]\Box\ t_3}/_{d}\rangle$, and

$$
\begin{aligned}
&\mathbb{E}'[t'\langle{}^{[d]\Box\ t_3}/_{d}\rangle]\\
\to_{\mathrm{v}}^{*}&[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ t_3 \text{ (because } \mathbb{E}[t'] \to_{\mathrm{v}}^{*} u = [d]v_2)\\
\to_{\mathrm{v}}^{*}&[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle \text{ (because } [e]t_3 \Downarrow_{\mathrm{v}} [e]v_3)\\
\to_{\mathrm{v}}&[d]t_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\{v_3\langle{}^{[d]v_2\langle{}^{[d]\Box\ t_3}/_{d}\rangle\ \Box}/_{e}\rangle/z_2\}.
\end{aligned}
$$

The side-conditions are satisfied, therefore (1) and (2) are in $\mathcal{R}$.

– Let $v_3 = \lambda z_3.t_3$. We have to relate

$$[d]t_3\{{}^{v_2\{\lambda y.\Omega/z\}\langle [d]v_3\ \Box/d\rangle}/_{z_3}\}$$

to

$$[d]t_3\{{}^{(v_2\{\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y/z\})\langle [d]v_3\ \Box/d\rangle}/_{z_3}\}$$

These terms can be rewritten into

$$[d]t_3\{{}^{v_2\langle [d]v_3\ \Box/d\rangle}/_{z_3}\}\{\lambda y.\Omega/z\} \tag{3}$$

and

$$[d]t_3\{{}^{v_2\langle [d]v_3\ \Box/d\rangle}/_{z_3}\}\{\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}'[v''\ (\Box\ t_1')]/b\rangle\ y/z\} \tag{4}$$

where $\mathbb{E}' \stackrel{\text{def}}{=} \mathbb{E}\langle [d]v_3\ \Box/d\rangle$, $v'' \stackrel{\text{def}}{=} v'\langle [d]v_3\ \Box/d\rangle$, and $t_1' \stackrel{\text{def}}{=} t_1\langle [d]v_3\ \Box/d\rangle$. We have $v'' = \lambda z.t'\langle [d]v_3\ \Box/d\rangle$, and

$$\begin{array}{l}\mathbb{E}'[t'\langle [d]v_3\ \Box/d\rangle]\\ \to_{\mathsf{v}}^{*}[d]v_3\ v_2\langle [d]v_3\ \Box/d\rangle \text{ (because } \mathbb{E}[t'] \to_{\mathsf{v}}^{*} u = [d]v_2)\\ \to_{\mathsf{v}}[d]t_3\{v_2\langle [d]v_3\ \Box/d\rangle/z_3\}\end{array}$$

The side-conditions are satisfied, therefore (3) and (4) are in $\mathcal{R}$.

The last case is when $u = \mathbb{E}'[x\ v'']$. Then $u\{\lambda y.\Omega/z\} \to_{\mathsf{v}} \mathbb{E}''[\Omega]$ for some $\mathbb{E}''$, and

$$\begin{array}{l}u\{\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y/z\}\\ \to_{\mathsf{v}}\mathbb{E}^{(3)}[\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ v^{(3)}] \text{ for some } \mathbb{E}^{(3)}, v^{(3)}\\ \to_{\mathsf{v}}^{*}\mathbb{E}^{(3)}[v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ (\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y)v^{(3)}]\\ \to_{\mathsf{v}}^{*}\mathbb{E}[v'\ ((\lambda zy.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y)\ t_1)]\\ \to_{\mathsf{v}}^{*}\mathbb{E}[v'\ ((\lambda zy.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y)\ v_1')] \text{ for some } v_1',\ \text{ because } [c]t_1 \Downarrow_{\mathsf{v}} [c]v_1\\ \to_{\mathsf{v}}\mathbb{E}[v'\ (\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y)]\\ \to_{\mathsf{v}}\mathbb{E}[t'\{\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y/z\}]\\ \to_{\mathsf{v}}^{*}u\{\lambda y.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ (\Box\ t_1)]/b\rangle\ y/z\} \text{ because } \mathbb{E}[t'] \to_{\mathsf{v}}^{*} u\end{array}$$

We obtain two non-terminating terms that are therefore bisimilar.

Lemma 29. *Let $v' \stackrel{def}{=} \lambda z.t'$. We have*

$$[b]t'\{\lambda xy.\Omega/z\} \approx [b]t'\{\lambda xy.\Theta_{\mathsf{v}}\ v\langle [b]v'\ \Box/b\rangle\ y/z\}$$

Proof. Let $\mathcal{R} \stackrel{\text{def}}{=} \{(u\{\lambda xy.\Omega/z\}, u\{\lambda xy.\Theta_{\mathsf{v}}\ v\langle \mathbb{E}[v'\ \Box]/b\rangle\ y/z\})\ \mid\ v' = \lambda z.t', \mathbb{E}[t'] \to_{\mathsf{v}}^{*} u\} \cup \approx$. We prove $\mathcal{R}$ is an applicative bisimilarity, by case analysis on u. The proof is the same as for Lemma 28; we only detail the last case, where $u = \mathbb{E}'[x\,v_1\,v_2]$. Then $u\{\lambda xy.\Omega/z\} \rightarrow_{\mathsf{v}}^{2} \mathbb{E}''[\Omega]$ for some $\mathbb{E}''$, and

$$\begin{aligned}
&u\{\lambda xy.\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; y/z\}\\
\rightarrow_{\mathsf{v}}^{2}&\mathbb{E}^{(3)}[\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; v_2'] \text{ for some } \mathbb{E}^{(3)}, v_2'\\
\rightarrow_{\mathsf{v}}^{*}&\mathbb{E}^{(3)}[v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; (\lambda y.\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; y) v_2']\\
\rightarrow_{\mathsf{v}}^{*}&\mathbb{E}[v'\; (\lambda xy.\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; y)]\\
\rightarrow_{\mathsf{v}}&\mathbb{E}[t'\{\lambda xy.\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; y/z\}]\\
\rightarrow_{\mathsf{v}}^{*}&u\{\lambda xy.\Theta_{\mathsf{v}}\, v\langle{}^{\mathbb{E}[v'\,\square]}/_{b}\rangle\; y/z\} \text{ because } \mathbb{E}[t'] \rightarrow_{\mathsf{v}}^{*} u
\end{aligned}$$

We obtain two non-terminating terms that are therefore bisimilar.

Lemma 30 (Example 10). *The relation* $\{([b]\lambda xy.\Omega, [b]\lambda xy.\Theta_{\mathsf{v}}\, v\, y)\} \cup \approx$ *is an applicative bisimulation.*

Proof. We have to prove two items

- Let t_1 such that $[c]t_1 \Downarrow_{\mathsf{v}} [c]v_1$ ($c \notin \mathsf{fn}(t_1)$). We have to prove that $[b]\lambda y.\Omega$ is bisimilar to $[b]\lambda y.\Theta_{\mathsf{v}}\, v\langle{}^{[b]\square\; t_1}/_{b}\rangle y$, which, in turn, requires that
  - for all $[d]t_2 \Downarrow_{\mathsf{v}} [d]v_2$ ($d \notin \mathsf{fn}(t_2)$), we need $[b]\Omega \approx [b]\Theta_{\mathsf{v}}\, v\langle{}^{[b]\square\; t_1\; t_2}/_{b}\rangle v_2'$ for some v_2'. This holds by Lemma 27;
  - for all $v' \stackrel{\text{def}}{=} \lambda z.t'$, we need $[b]t'\{\lambda y.\Omega/z\} \approx [b]t'\{\lambda y.\Theta_{\mathsf{v}} v\langle{}^{[b]v'\;(\square\; t_1)}/_{b}\rangle y/z\}$. This holds by Lemma 28.
- Let $v' \stackrel{\text{def}}{=} \lambda z.t'$. We have to prove that $[b]t'\{\lambda xy.\Omega/z\}$ is related to the term $[b]t'\{\lambda xy.\Theta_{\mathsf{v}}\, v\langle{}^{[b]v'\;\square}/_{b}\rangle\; y/z\}$. This a consequence of Lemma 29.

For the next lemma and its proof, we use the same definitions of terms as in Example 8.

Lemma 31 (see Example 8). *The relation*

$$\mathcal{R} \stackrel{def}{=} \{(u\langle{}^{[a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\;\square]}/_{b}\rangle, u\langle{}^{[a]E[v_1\;\square]}/_{b}\rangle) \;\mid\; [b]t \Downarrow_{\mathsf{v}} [b]v, b \notin \mathsf{fn}(t)\}$$

is an applicative bisimulation.

Proof. We proceed by case analysis on u. If $u \rightarrow_{\mathsf{v}} u'$, then the result holds. If $u = [c]v'$ with $c \neq a$ and $c \neq b$, then when checking clauses 1 and 2, we obtain terms that can be written $u'\langle{}^{[a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\;\square]}/_{b}\rangle$ and $u'\langle{}^{[a]E[v_1\;\square]}/_{b}\rangle$ for some u', and are therefore in $\mathcal{R}$.

If $u = [a]v'$, then when checking clauses 1 and 2, we obtain terms that can be written $u'\langle{}^{[a]E'[E[v_0\langle{}^{[a]E'[E[\square\; t]]}/_{a}\rangle\;\square]]}/_{b}\rangle$ and $u'\langle{}^{[a]E'[E[v_1\;\square]]}/_{b}\rangle$ for some u' and E' (depending on which clause we check). We obtain terms in $\mathcal{R}$.

If $u = [b]v'$, then

$$\begin{aligned}
u\langle{}^{[a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\;\square]}/_{b}\rangle &= [a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\; v'\langle{}^{[a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\;\square]}/_{b}\rangle]\\
&\rightarrow_{\mathsf{v}}^{*} [a]E[w'\; (v'\langle{}^{[a]E[v_0\langle{}^{[a]E[\square\; t]}/_{a}\rangle\;\square]}/_{b}\rangle\; \lambda x.x)]
\end{aligned}$$

and $u\langle{}^{[a]E[v_1\;\square]}/_{b}\rangle = [a]E[v_1\; v'\langle{}^{[a]E[v_1\;\square]}/_{b}\rangle] \rightarrow_{\mathsf{v}}^{*} [a]E[w'\; (v'\langle{}^{[a]E[v_1\;\square]}/_{b}\rangle\; \lambda x.x)]$; we obtain terms in $\mathcal{R}$.

The next lemma uses the notations of Section 3.3

Lemma 32 (see Example 13). *Let* $\mathbb{E}$, $\mathbb{E}_0$, *and* E_1 *be such that* $\mathbb{E} = \mathbb{E}_0[E_1]$. *Then* $\mathbb{E}^\dagger \approx \lambda x.\mathbb{E}_0^\dagger\, E_1[x]$.

Proof. Let a be a fresh name. Let t such that $[d]t \Downarrow_{\mathrm{v}} [d]v$. To check clause 1, we have to relate $[a]\mu c.\mathbb{E}[v\langle {}^{[a]\mathbb{E}^\dagger\,\Box}/_d\rangle]$ and $[a]\mathbb{E}_0^\dagger\, E_1[v\langle {}^{[a]\lambda x.\mathbb{E}_0^\dagger\, E_1[x]\,\Box}/_d\rangle]$. These terms reduce respectively to $\mathbb{E}[v\langle {}^{[a]\mathbb{E}^\dagger\,\Box}/_d\rangle]$ and $\mathbb{E}_0[E_1[v\langle {}^{[a]\lambda x.\mathbb{E}_0^\dagger\, E_1[x]\,\Box}/_d\rangle]]$, which we can rewrite into respectively $u\langle {}^{[a]\mathbb{E}^\dagger\,\Box}/_d\rangle$ and $u\langle {}^{[a]\lambda x.\mathbb{E}_0^\dagger\, E_1[x]\,\Box}/_d\rangle$. To check clause 2, we have to relate $[a]t\{\mathbb{E}^\dagger/y\}$ and $[a]t\{\lambda x.\mathbb{E}_0^\dagger\, E_1[x]/y\}$ for all t. The most interesting case is when $t = E[y\,v]$; then these terms reduce to respectively $\mathbb{E}[v]\{\mathbb{E}^\dagger/y\}$ and $\mathbb{E}[v]\{\lambda x.\mathbb{E}_0^\dagger E_1[x]/y\}$. In fact, one can prove the result by showing that

$$\{(u\langle {}^{[a]\mathbb{E}^\dagger\,\Box}/_d\rangle, u\langle {}^{[a]\lambda x.\mathbb{E}_0^\dagger\, E_1[x]\,\Box}/_d\rangle), (u\{\mathbb{E}^\dagger/y\}, u\{\lambda x.\mathbb{E}_0^\dagger\, E_1[x]/y\}) \mid \mathbb{E} = \mathbb{E}_0[E_1]\}$$

is an applicative bisimulation.

B.2 Proof of Soundness

Lemma 33. *If* $[a]v_0 \approx [a]v_1$, *then for all* $[b]t \to_{\mathrm{v}}^* u$ *(*$b \notin \mathsf{fn}(t)$*), we have* $u\langle {}^{[a]v_0\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle \approx u\langle {}^{[a]v_1\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle$.

Proof. The relation $\{(u\langle {}^{[a]v_0\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle, u\langle {}^{[a]v_1\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle) \mid \forall u, [b]t \to_{\mathrm{v}}^* u\} \cup \approx$ is a bisimulation.

Lemma 34. *If* $\lambda x.t_0\ (\approx^\bullet)^c\ t_1$, *then there exists* t *such that* $t_0 \approx^\bullet t$, $\mathsf{fv}(t) \subseteq \{x\}$ *and* $\lambda x.t \approx t_1$.

Proof. By induction on $\lambda x.t_0\ (\approx^\bullet)^c\ t_1$.

Lemma 35. *If* $[a]t_0\ (\approx^\bullet)^c\ u_1$, *then there exists a closed* t *such that* $t_0\ (\approx^\bullet)^c\ t$ *and* $[a]t \approx u_1$.

Proof. By induction on $[a]t_0\ (\approx^\bullet)^c\ u_1$.

Lemma 36. *Let* $t_0\ (\approx^\bullet)^c\ t_1$, *and* $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$.

- *If* $[a]t_0 \to_{\mathrm{n}} u_0$, *then* $[a]t_1 \to_{\mathrm{n}}^* u_1$ *and* $u_0\ (\approx^\bullet)^c\ u_1$.
- *If* $t_0 = \lambda x.t_0'$, *then* $[a]t_1 \to_{\mathrm{n}}^* [a]\lambda x.t_1'$ *and for all* t, b, v *such that* $[b]t \to_{\mathrm{v}}^* [b]v$ *and* $b \notin \mathsf{fn}(t)$, *we have*

$$[a]t_0'\langle {}^{[a]\Box\, t}/_a\rangle\{v\langle {}^{[a]\lambda x.t_0\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle/x\}\ (\approx^\bullet)^c \\ [a]t_1'\langle {}^{[a]\Box\, t}/_a\rangle\{v\langle {}^{[a]\lambda x.t_1\langle {}^{[a]\Box\, t}/_a\rangle\,\Box}/_b\rangle/x\}$$

  and for all $v' = \lambda x.t'$, *we have*

$$[a]t'\{\lambda x.t_0'\langle {}^{[a]v\,\Box}/_a\rangle/x\}\ (\approx^\bullet)^c\ [a]t'\{\lambda x.t_1'\langle {}^{[a]v\,\Box}/_a\rangle/x\}.$$

Proof. By induction on $t_0\ (\approx^\bullet)^c\ t_1$. If $t_0 \approx t_1$, then the result holds by bisimilarity. If $t_0 \approx^\bullet t \approx^\circ t_1$, so that t is closed (using Lemma 13 if necessary), then we can conclude with the induction hypothesis and the bisimilarity definition.

If $t_0\ \widetilde{(\approx^\bullet)}^c\ t_1$, then we have several cases to consider.

If $t_0 = \lambda x.t'_0$ and $t_1 = \lambda x.t'_1$ with $t'_0 \approx^\bullet t'_1$, we can prove the required result because $\approx^\bullet$ is substitutive. Suppose $t_0 = \mu b.u_0$ and $t_1 = \mu b.u_1$ with $u_0\ (\approx^\bullet)^c\ u_1$. We have $[a]t_0 \rightarrow_{\mathrm{v}} u_0\langle[a]\Box/b\rangle$, $[a]t_1 \rightarrow_{\mathrm{v}} u_1\langle[a]\Box/b\rangle$, and $u_0\langle[a]\Box/b\rangle\ (\approx^\bullet)^c\ u_1\langle[a]\Box/b\rangle$, hence the result holds.

Suppose $t_0 = t_0^1\, t_0^2$, $t_1 = t_1^1\, t_1^2$ with $t_0^1\ (\approx^\bullet)^c\ t_1^1$ and $t_0^2\ (\approx^\bullet)^c\ t_1^2$. We distinguish three cases.

- If $[b]t_0^1 \rightarrow_{\mathrm{v}} u_0$ (for some fresh b), then $[a]t_0 \rightarrow_{\mathrm{v}} u_0\langle[a]\Box\ t_0^2/b\rangle$. By the induction hypothesis, there exists u_1 such that $[b]t_1^1 \rightarrow_{\mathrm{v}}^* u_1$ and $u_0\ (\approx^\bullet)^c\ u_1$. Consequently, we have $[a]t_1 \rightarrow_{\mathrm{v}}^* u_1\langle[a]\Box\ t_1^2/b\rangle$, and by definition of $\approx^\bullet$, we have $u_0\langle[a]\Box\ t_0^2/b\rangle\ (\approx^\bullet)^c\ u_1\langle[a]\Box\ t_1^2/b\rangle$, as required.
- Suppose $t_0^1 = \lambda x.t'_0$ and $[b]t_0^2 \rightarrow_{\mathrm{v}} u_0$ for some fresh b; then we have $[a]t_0 \rightarrow_{\mathrm{v}} u_0\langle[a]\lambda x.t'_0\ \Box/b\rangle$. By the induction hypothesis, there exists u_1 such that $[b]t_1^2 \rightarrow_{\mathrm{v}}^* u_1$ and $u_0\ (\approx^\bullet)^c\ u_1$. Because $\lambda x.t'_0\ (\approx^\bullet)^c\ t_1^1$, there exists a t such that $t'_0 \approx^\bullet t$ and $\lambda x.t \approx t_1^1$ by Lemma 34. From $t'_0 \approx^\bullet t$, we deduce $[a]\lambda x.t'_0\ \Box\ (\approx^\bullet)^c\ [a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box$, which implies $u_0\langle{}^{[a]\lambda x.t'_0\,\Box}/_{b}\rangle\ (\approx^\bullet)^c\ u_1\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{b}\rangle$. Because $\lambda x.t \approx t_1^1$, there exists v_1 such that $[a]t_1^1 \rightarrow_{\mathrm{v}}^* [a]v_1$, and we have $[a]\lambda x.t \approx [a]v_1$. By Lemma 33, $u_1\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{b}\rangle \approx u_1\langle{}^{[a]v_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{b}\rangle$ holds, so $u_0\langle{}^{[a]\lambda x.t'_0\,\Box}/_{b}\rangle\ (\approx^\bullet)^c\ u_1\langle{}^{[a]v_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{b}\rangle$ holds as well. Besides, we have $[a]t_1 \rightarrow_{\mathrm{v}}^* u_1\langle{}^{[a]v_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{b}\rangle$, hence we have the required result.
- If $t_0^1 = \lambda x.t'_0$ and $t_0^2 = \lambda y.t''_0$, then $[a]t_0 \rightarrow_{\mathrm{v}} [a]t'_0\{\lambda x.t''_0/x\}$. By Lemma 34, there exist closed t, t' such that $t'_0\ (\approx^\bullet)^c\ t$, $\lambda x.t \approx t_1^1$, $t''_0\ (\approx^\bullet)^c\ t'$, and $\lambda y.t' \approx t_1^2$. From $t'_0\ (\approx^\bullet)^c\ t$, we deduce $t'_0\ (\approx^\bullet)^c\ t\langle[a]\Box\ t_1^2/a\rangle$, and from $t''_0\ (\approx^\bullet)^c\ t'$, we deduce $\lambda y.t''_0\ (\approx^\bullet)^c\ \lambda y.t'\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle$. Consequently, we have $[a]t'_0\{\lambda y.t''_0/x\}\ (\approx^\bullet)^c\ [a]t\langle[a]\Box\ t_1^2/a\rangle\{\lambda y.t'\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}$. From $\lambda y.t' \approx t_1^2$, by bisimilarity, there exists v_1 such that $[a]t_1^2 \rightarrow_{\mathrm{v}}^* [a]v_1$, and

  $$\begin{aligned}&[a]t\langle[a]\Box\ t_1^2/a\rangle\{\lambda y.t'\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}\\ &\qquad\qquad\approx [a]t\langle[a]\Box\ t_1^2/a\rangle\{v_1\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}\end{aligned}$$

  holds (clause 2). From $\lambda x.t \approx t_1^1$, by bisimilarity, there exists t_1 such that $[a]t_1^1 \rightarrow_{\mathrm{v}}^* [a]\lambda x.t_1$, and

  $$\begin{aligned}&[a]t\langle[a]\Box\ t_1^2/a\rangle\{v_1\langle{}^{[a]\lambda x.t\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}\\ &\qquad\qquad\approx [a]t_1\langle[a]\Box\ t_1^2/a\rangle\{v_1\langle{}^{[a]\lambda x.t_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}\end{aligned}$$

  holds (clause 1). Finally, we have

  $$[a]t'_0\{\lambda y.t''_0/x\}\ (\approx^\bullet)^c\ [a]t_1\langle[a]\Box\ t_1^2/a\rangle\{v_1\langle{}^{[a]\lambda x.t_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\},$$

  and because $[a]t_1 \rightarrow_{\mathrm{v}}^* [a]t_1\langle[a]\Box\ t_1^2/a\rangle\{v_1\langle{}^{[a]\lambda x.t_1\langle[a]\Box\ t_1^2/a\rangle\ \Box}/_{a}\rangle/x\}$, we have the required result.

Suppose $t_0 = t'_0\langle\mathbb{E}_0/b\rangle$, $t_1 = t'_1\langle\mathbb{E}_1/b\rangle$ with $t'_0\ (\approx^\bullet)^c\ t'_1$, $\mathbb{E}_0\ (\approx^\bullet)^c\ \mathbb{E}_1$. If $[a]t_0 \to_{\mathrm{v}} u_0$, then in fact $[a]t'_0 \to_{\mathrm{v}} u'_0$ with $u_0 = u'_0\langle\mathbb{E}_0/b\rangle$. By the induction hypothesis, there exists u'_1 such that $[a]t'_1 \to_{\mathrm{v}}^* u'_1$ and $u'_0\ (\approx^\bullet)^c\ u'_1$. Consequently, we have $[a]t_1 \to_{\mathrm{v}}^* u'_1\langle\mathbb{E}_1/b\rangle$, and $u'_0\langle\mathbb{E}_0/b\rangle\ (\approx^\bullet)^c\ u'_1\langle\mathbb{E}_1/b\rangle$ holds, as wished. If $[a]t_0$ is a named value, then in fact $t'_0 = \lambda x.t'''_0$ and $t_0 = \lambda x.t'''_0\langle\mathbb{E}_0/b\rangle$. The result holds by using the induction hypothesis (and with some renaming of b into a fresh c to avoid name clashes).

Lemma 37. *The relation $(\approx^\bullet)^c$ is an applicative simulation.*

Proof. Let $u_0\ (\approx^\bullet)^c\ u_1$; we prove the simulation clause by induction on $u_0\ (\approx^\bullet)^c\ u_1$.

If $u_0 \approx u_1$, then the result holds by bisimilarity. If $u_0 \approx^\bullet u \approx^\circ u_1$, then we can conclude using the induction hypothesis and the bisimilarity definition.

If $u_0\ \widetilde{(\approx^\bullet)^c}\ u_1$, then we have two cases. Suppose $u_0 = [a]t_0$ and $u_1 = [a]t_1$ with $t_0\ (\approx^\bullet)^c\ t_1$. If $a \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$, then we can apply Lemma 36 directly to get the required result. Otherwise, let $b \notin \mathsf{fn}(t_0) \cup \mathsf{fn}(t_1)$. If $u_0 \to_{\mathrm{v}} u'_0$, then $[b]t_0 \to_{\mathrm{v}} u''_0$ and $u'_0 = u'_0\langle[a]\square/b\rangle$. We can apply Lemma 36 to $[b]t_0$ and $[b]t_1$, and then rename b into a.

Suppose $u_0 = u'_0\langle\mathbb{E}_0/a\rangle$, $u_1 = u'_1\langle\mathbb{E}_1/a\rangle$ with $u'_0\ (\approx^\bullet)^c\ u'_1$ and $\mathbb{E}_0\ (\approx^\bullet)^c\ \mathbb{E}_1$.

If u_0 is a named value, then we use the induction hypothesis, with some renaming (as in the call-by-name case) to avoid some name clashes.

If $u_0 \to_{\mathrm{v}}$, then again we distinguish several cases. First, suppose $u'_0 \to_{\mathrm{v}} u''_0$; then $u_0 \to_{\mathrm{v}} u''_0\langle E_0/a\rangle$. By the induction hypothesis, there exists u''_1 such that $u'_1 \to_{\mathrm{v}}^* u''_1$ and $u''_0\ (\approx^\bullet)^c\ u''_1$. Then $u_1 \to_{\mathrm{v}}^* u''_1\langle\mathbb{E}_1/a\rangle$ and $u''_0\langle\mathbb{E}_0/a\rangle\ (\approx^\bullet)^c\ u''_1\langle\mathbb{E}_1/a\rangle$, hence the result holds.

Second, assume $u'_0 = [a]\lambda x.t_0$, $\mathbb{E}_0 = \mathbb{E}'_0[\square\ v_0]$ and $u_0 \to_{\mathrm{v}} \mathbb{E}'_0[t_0\langle\mathbb{E}_0/a\rangle\{v_0/x\}]$. By Lemma 35, there exists t such that $\lambda x.t_0\ (\approx^\bullet)^c\ t$ and $[a]t \approx u'_1$. By Lemma 34, there exists t' such that $t_0 \approx^\bullet t'$ and $\lambda x.t' \approx t$. We can prove by induction on $\mathbb{E}_0\ (\approx^\bullet)^c\ \mathbb{E}_1$ that $\mathbb{E}_1 = \mathbb{E}'_1[\square\ t'_1]$ with $\mathbb{E}'_0\ (\approx^\bullet)^c\ \mathbb{E}'_1$ and $v_0\ (\approx^\bullet)^c\ t'_1$. Suppose $v_0 = \lambda x.t'_0$. By Lemma 34, there exists s such that $t'_0 \approx^\bullet s$ and $\lambda x.s \approx t'_1$. Let $\mathbb{E}''_i$, s', t''_i be $\mathbb{E}'_i$, s, and t'_i with a renamed into a fresh c. From $t_0\ (\approx^\bullet)^c\ t'$, $\lambda x.t''_0\ (\approx^\bullet)^c\ t''_1$, and $\lambda x.t''_0\ (\approx^\bullet)^c\ \lambda x.s'$, we deduce

$$[a]t_0\langle{\scriptstyle [a]\square\ \lambda x.t''_0/a}\rangle\{\lambda x.t''_0/x\}\ (\approx^\bullet)^c\ [a]t'\langle{\scriptstyle [a]\square\ t''_1/a}\rangle\{\lambda x.s'/x\}.$$

Because $\lambda x.s' \approx t''_1$, there exists v'_1 such that $[b]t''_1 \to_{\mathrm{v}}^* [b]v'_1$ (for a fresh b), and $[a]t'\langle{\scriptstyle [a]\square\ t''_1/a}\rangle\{\lambda x.s'/x\} \approx [a]t'\langle{\scriptstyle [a]\square\ t''_1/a}\rangle\{v'_1\langle{\scriptstyle [a]\lambda x.t'\langle{\scriptscriptstyle [a]\square\ t''_1/a}\rangle\ \square/b}\rangle/x\}$ (using the second item of the value case of the bisimulation definition). Because $\lambda x.t' \approx t$, there exists t'' such that $[a]t \to_{\mathrm{v}}^* [a]\lambda x.t''$ and

$$\begin{aligned}&[a]t'\langle{\scriptstyle [a]\square\ t''_1/a}\rangle\{v'_1\langle{\scriptstyle [a]\lambda x.t'\langle{\scriptscriptstyle [a]\square\ t''_1/a}\rangle\ \square/b}\rangle/x\}\\ &\qquad\qquad \approx [a]t''\langle{\scriptstyle [a]\square\ t''_1/a}\rangle\{v'_1\langle{\scriptstyle [a]\lambda x.t''\langle{\scriptscriptstyle [a]\square\ t''_1/a}\rangle\ \square/b}\rangle/x\}\end{aligned}$$

(by clause 1). Because $[a]\lambda x.t' \approx u'_1$, there exists t_1 such that $u'_1 \rightarrow^*_{\mathrm{v}} [a]\lambda x.t_1$ and we have also

$$[a]t''\langle{}^{[a]\Box\ t''_1}/_a\rangle\{v'_1\langle{}^{[a]\lambda x.t''\langle{}^{[a]\Box\ t''_1}/_a\rangle\ \Box}/_b\rangle/x\} \approx [a]t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\{v'_1\langle{}^{[a]\lambda x.t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\ \Box}/_b\rangle/x\}$$

by clause 1. Finally, we obtain

$$[a]t_0\langle{}^{[a]\Box\ \lambda x.t''_0}/_a\rangle\{\lambda x.t''_0/x\}\ (\approx^\bullet)^c\ [a]t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\{v'_1\langle{}^{[a]\lambda x.t_1\langle{}^{[a]\Box\ t''_1}/_a\rangle\ \Box}/_b\rangle/x\}$$

by transitivity of $\approx$ and definition of $\approx^\bullet$, from which we can deduce

$$\mathbb{E}''_0[t_0\langle{}^{\mathbb{E}''_0[\Box\ \lambda x.t''_0]}/_a\rangle\{\lambda x.t''_0/x\}]\ (\approx^\bullet)^c\ \mathbb{E}''_1[t_1\langle{}^{\mathbb{E}''_1[\Box\ t''_1]}/_a\rangle\{v'_1\langle{}^{\mathbb{E}''_1[\lambda x.t_1\langle{}^{\mathbb{E}''_1[\Box\ t''_1]}/_a\rangle\ \Box]}/_b\rangle/x\}].$$

Renaming c back into a, we get

$$\mathbb{E}'_0[t_0\langle{}^{\mathbb{E}'_0[\Box\ \lambda x.t'_0]}/_a\rangle\{\lambda x.t'_0/x\}]\ (\approx^\bullet)^c\ \mathbb{E}'_1[t_1\langle{}^{\mathbb{E}'_1[\Box\ t'_1]}/_a\rangle\{v_1\langle{}^{\mathbb{E}'_1[\lambda x.t_1\langle{}^{\mathbb{E}'_1[\Box\ t'_1]}/_a\rangle\ \Box]}/_b\rangle/x\}]$$

(assuming v_1 is the result of renaming c into a in v'_1), which is the same as

$$\mathbb{E}'_0[t_0\langle{}^{\mathbb{E}_0}/_a\rangle\{\lambda x.t'_0/x\}]\ (\approx^\bullet)^c\ \mathbb{E}'_1[t_1\langle{}^{\mathbb{E}_1}/_a\rangle\{v_1\langle{}^{\mathbb{E}'_1[\lambda x.t_1\langle{}^{\mathbb{E}_1}/_a\rangle\ \Box]}/_b\rangle/x\}].$$

One can check that $u_1 \rightarrow^*_{\mathrm{v}} \mathbb{E}'_1[t_1\langle{}^{\mathbb{E}_1}/_a\rangle\{v_1\langle{}^{\mathbb{E}'_1[\lambda x.t_1\langle{}^{\mathbb{E}_1}/_a\rangle\ \Box]}/_b\rangle/x\}]$, hence the result holds.

Finally, the last case is $u'_0 = [a]v_0$, $\mathbb{E}_0 = \mathbb{E}'_0[\lambda x.t_0\ \Box]$, which give $u_0 \rightarrow_{\mathrm{v}} \mathbb{E}'_0[t_0\{v_0\langle\mathbb{E}_0/a\rangle/x\}]$. This case is similar to the previous one and is left to the reader.

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