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► **To cite this version:**

| Antoine Gloria, Felix Otto. Quantitative theory in stochastic homogenization. 2014. <hal-00933251>

**HAL Id: hal-00933251**

**<https://hal.inria.fr/hal-00933251>**

Submitted on 20 Jan 2014

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# Quantitative theory in stochastic homogenization

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October 7, 2013

## 1 Introduction, setting, and main results

This article corresponds to a course given by F. Otto at the summer school CEMRACS 2013 in Luminy, France. It is based on [3], which is a continuum version of [2]. It slightly differs from [3] because the present analysis does not rely on Green's functions and treats the periodic case. As opposed to [3], we also treat non-symmetric coefficients. For related work on an emerging quantitative theory of stochastic homogenization, including many references, we refer to three preprints: [6] requires the least machinery, [4] gives an extensive introduction next to a couple of quantitative results, and [5] uses both to give a full error estimate.

We start by introducing the relevant deterministic notions: The corrector  $\phi(a; \cdot)$  and the homogenized coefficient  $a_{hom}(a)$  for an arbitrary coefficient field  $a$  on the torus of side length  $L$ , which we sometimes denote as  $[-\frac{L}{2}, \frac{L}{2}]^d$  to single out the point 0.

### Definition 1.

SPACE OF COEFFICIENT FIELDS. *For a given side-length  $L$  let  $\Omega$  be the space of all  $[-\frac{L}{2}, \frac{L}{2}]^d$ -periodic fields of  $d \times d$  matrices  $a$  that are uniformly elliptic*

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in the sense

$$\forall x \in \left[-\frac{L}{2}, \frac{L}{2}\right)^d, \xi \in \mathbb{R}^d \quad \lambda|\xi|^2 \leq \xi \cdot a(x)\xi, \quad |a(x)\xi| \leq |\xi|,$$

where  $\lambda > 0$  is a number fixed throughout the article.

**CORRECTOR.** For given  $a \in \Omega$ , the corrector  $\phi(a; \cdot)$  is an  $[-\frac{L}{2}, \frac{L}{2}]^d$ -periodic function defined through the elliptic equation

$$-\nabla \cdot a(\nabla \phi(a; \cdot) + \xi) = 0 \quad \text{and} \quad \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi(a; \cdot) = 0, \quad (1)$$

where  $\xi \in \mathbb{R}^d$  with  $|\xi| = 1$  is a direction which is fixed throughout the article. For further reference we note that  $\phi$  is “stationary” in the sense of

$$\phi(a(\cdot + z), x) = \phi(a, x + z) \quad (2)$$

for all points  $x \in \mathbb{R}^d$ , coefficient fields  $a \in \Omega$ , and shift vectors  $z \in \mathbb{R}^d$ .

**HOMOGENIZED COEFFICIENT.** The homogenized coefficient in directions  $\xi, \xi'$  is defined via

$$\xi' \cdot a_{\text{hom}}(a)\xi := L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \xi' \cdot a(\nabla \phi(a; \cdot) + \xi), \quad (3)$$

where  $\xi'$  with  $|\xi'| = 1$  is a direction which is fixed throughout the article.

We now introduce our example of an ensemble on the space of coefficient fields on the torus.

**Definition 2.** By the “Poisson ensemble” we understand the following probability measure on  $\Omega$ :

Let the configuration of points  $X := \{X_n\}_{n=1, \dots, N}$  on the torus be distributed according to the Poisson point process with density one. This means the following

- For any two disjoint (Lebesgue measurable) subsets  $D$  and  $D'$  of the torus we have that the configuration of points in  $D$  and the configuration of points in  $D'$  are independent. In other words, if  $\zeta$  is a function of  $X$  that depends on  $X$  only through  $X|_D$  and  $\zeta'$  is a function of  $X$  that depends on  $X$  only through  $X|_{D'}$  we have

$$\langle \zeta \zeta' \rangle_0 = \langle \zeta \rangle_0 \langle \zeta' \rangle_0, \quad (4)$$

where  $\langle \cdot \rangle_0$  denotes the expectation w. r. t. the Poisson point process.

- For any (Lebesgue measurable) subset  $D$  of the torus, the number of points in  $D$  is Poisson distributed; the expected number is given by the Lebesgue measure of  $D$ .

Note that  $N$  is random, too.

With any realization  $X = \{X_n\}_{n=1, \dots, N}$  of the Poisson point process, we associate the coefficient field  $a \in \Omega$  via

$$a(x) = \left\{ \begin{array}{ll} \lambda & \text{if } x \in \bigcup_{n=1}^N B_1(X_n) \\ 1 & \text{else} \end{array} \right\} \text{id.} \quad (5)$$

Here and throughout the article, balls like  $B_1(X_n)$  refer to the distance function of the torus. This defines a probability measure on  $\Omega$  by “push-forward” of  $\langle \cdot \rangle_0$ . We denote the expectation w. r. t. this ensemble with  $\langle \cdot \rangle$ .

For our result, we only need the following two properties of the Poisson ensemble.

**Lemma 1.**

**STATIONARITY.** *The Poisson ensemble is stationary which means that for any shift vector  $z \in \mathbb{Z}^d$  the random field  $a$  and its shifted version  $a(\cdot + z): x \mapsto a(x + z)$  have the same distribution. In other words, for any (integrable) function  $\zeta: \Omega \rightarrow \mathbb{R}$  (which we think of as a random variable) we have that  $a \mapsto \zeta(a(\cdot + z))$  and  $\zeta$  have the same expectation:*

$$\langle \zeta(a(\cdot + z)) \rangle = \langle \zeta \rangle. \quad (6)$$

**SPECTRAL GAP ESTIMATE.** *The Poisson ensemble satisfies a Spectral Gap Estimate by which we understand the following: There exists a radius  $R$  only depending on  $d$  such that for any function  $\zeta: \Omega \rightarrow \mathbb{R}$ , we have*

$$\langle (\zeta - \langle \zeta \rangle)^2 \rangle \leq \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{R}{2}, \frac{R}{2}]^d} (\text{osc}_{B_R(z)} \zeta)^2 \right\rangle. \quad (7)$$

Here, for a (Lebesgue measurable) subset  $D$  of the torus, the (essential) oscillation  $\text{osc}_D \zeta$  of  $\zeta$  with respect to  $D$  is a random variable defined through

$$\begin{aligned} (\text{osc}_D \zeta)(a) &= \sup\{\zeta(\tilde{a}) \mid \tilde{a} \in \Omega \text{ with } \tilde{a} = a \text{ outside } D\} \\ &\quad - \inf\{\zeta(\tilde{a}) \mid \tilde{a} \in \Omega \text{ with } \tilde{a} = a \text{ outside } D\}. \end{aligned} \quad (8)$$

It measures how sensitively  $\zeta(a)$  depends on  $a|_D$ . Note that  $(\text{osc}_D \zeta)(a)$  does not depend on  $a|_D$ .

The main result of the article is a Central Limit Theorem-type scaling of the variance of the homogenized coefficient in terms of the system volume  $L^d$ .

**Theorem 1.** *Suppose  $\langle \cdot \rangle$  is stationary and satisfies the Spectral Gap Estimate. Then we have the following estimate on the variance of the homogenized coefficient*

$$\langle (\xi' \cdot a_{\text{hom}} \xi - \langle \xi' \cdot a_{\text{hom}} \xi \rangle)^2 \rangle \leq C(d, \lambda) L^{-d}.$$

In this article, we prove Theorem 1 only for  $d > 2$ . We shall derive it from the following result of independent interest, which is only true for  $d > 2$ .

**Proposition 1.** *Let  $d > 2$  and suppose  $\langle \cdot \rangle$  is stationary and satisfies the Spectral Gap Estimate. Then all moments of the corrector are bounded independently of  $L$ , that is, for any  $1 \leq p < \infty$  we have*

$$\langle \phi^{2p} \rangle \leq C(d, \lambda, p).$$

Here and in the entire text, we write  $\phi^{2p}$  for  $(\phi^2)^p$ , so that expressions like above make sense also for a non-integer exponent  $p$ .

## 2 Auxiliary results

We need the following  $L^p(\Omega)$ -version of the Spectral Gap Estimate.

**Lemma 2.** *Let  $\langle \cdot \rangle$  satisfy the Spectral Gap Estimate. Then it satisfies an  $L^p(\Omega)$ -version of a Spectral Gap Estimate in the following sense: Let  $R$  be the radius from  $(\mathcal{Y})$ . Then we have for any ( $2p$ -integrable) function  $\zeta: \Omega \rightarrow \mathbb{R}$  and any  $1 \leq p < \infty$*

$$\langle (\zeta - \langle \zeta \rangle)^{2p} \rangle \lesssim \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \zeta)^2 \right)^p \right\rangle. \quad (9)$$

Here  $\lesssim$  means up to a generic constant that only depends on  $p$ .

Together with the previous lemma, the following lemma gives an estimate of  $\phi$  in terms of  $\nabla \phi + \xi$ .

**Lemma 3.** *Suppose  $d > 2$  and that  $\langle \cdot \rangle$  is stationary. Then we have for any  $\frac{d}{d-2} < p < \infty$  and any  $R \lesssim 1$*

$$\left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \right)^p \right\rangle \lesssim \left\langle \left( \int_{B_1} |\nabla \phi + \xi|^2 \right)^p \right\rangle, \quad (10)$$

where  $\lesssim$  means up to a generic constant only depending on  $d, \lambda, p,$  and  $R$ .

The last lemma in turn gives an estimate of  $\nabla \phi + \xi$  in terms of  $\phi$ .

**Lemma 4.** *Suppose that  $\langle \cdot \rangle$  is stationary. Then we have for any  $2 \leq p < \infty$*

$$\left\langle \left( \int_{B_1} |\nabla \phi + \xi|^2 \right)^p \right\rangle \lesssim \langle \phi^{2(p-1)} \rangle + 1,$$

where  $\lesssim$  means up to a generic constant only depending on  $d, \lambda,$  and  $p$ .

### 3 Proofs

PROOF OF LEMMA 1.

**Step 1.** Generalization and reduction. The most natural form of the result of the lemma is the following: For any measurable partition  $D_1, \dots, D_N$  of the torus we have

$$\langle (\zeta - \langle \zeta \rangle)^2 \rangle \leq \left\langle \sum_{n=1}^N (\text{osc}_{B_1(D_n)} \zeta)^2 \right\rangle, \quad (11)$$

where  $B_1(D)$  is the set of all points on the torus that have distance less than one to  $D$ . In this step, we will derive this from the following similar estimate on the Poisson point process itself:

$$\langle (\zeta_0 - \langle \zeta_0 \rangle_0)^2 \rangle_0 \leq \left\langle \sum_{n=1}^N (\text{osc}_{0, D_n} \zeta_0)^2 \right\rangle_0. \quad (12)$$

Here  $\langle \cdot \rangle_0$  denotes the expectation w. r. t. to the Poisson point process  $X := \{X_n\}_{n=1, \dots, N}$ ,  $\zeta$  is a (square integrable) function of the point configuration  $X$ , and the oscillation  $\text{osc}_{0, D}$  is defined in a similar way to (8):

$$\begin{aligned} (\text{osc}_{0, D} \zeta_0)(X) &= \sup\{\zeta_0(\tilde{X}) \mid \tilde{X} = X \text{ outside } D\} \\ &\quad - \inf\{\zeta_0(\tilde{X}) \mid \tilde{X} = X \text{ outside } D\}. \end{aligned} \quad (13)$$

Indeed, (11) is an immediate consequence of (12) because of the following two facts.

- We recall that (5) defines a mapping  $X \mapsto a$  from point configurations to coefficient fields. As such, it pulls back functions according to  $\zeta_0(X) = \zeta(a(X))$  and pushes forward the ensemble according to

$$\langle \zeta \rangle = \langle \zeta_0 \rangle_0. \quad (14)$$

In particular we have for the variance

$$\langle (\zeta - \langle \zeta \rangle)^2 \rangle = \langle (\zeta_0 - \langle \zeta_0 \rangle_0)^2 \rangle_0.$$

- By definition (5), if the point configurations  $X$  and  $\tilde{X}$  coincide outside  $D$ , then the corresponding coefficient fields  $a(X; \cdot)$  and  $a(\tilde{X}; \cdot)$  coincide outside  $B_1(D)$ . Hence for a given configuration  $X$ , the set  $\{a(\tilde{X}) | \tilde{X} = X \text{ outside } D\}$  is contained in the set  $\{\tilde{a} | \tilde{a} = a(X) \text{ outside } B_1(D)\}$  so that

$$\begin{aligned} & \sup\{\zeta(a(\tilde{X})) | \tilde{X} = X \text{ outside } D\} \\ & \leq \sup\{\zeta(\tilde{a}) | \tilde{a} = a(X) \text{ outside } B_1(D)\}, \end{aligned}$$

and the opposite inequality if we replace the supremum by the infimum. From the definitions (8) and (13) of the oscillation we thus see

$$(\text{osc}_{0,D}\zeta_0)(X) \leq (\text{osc}_{B_1(D)}\zeta)(a(X)).$$

By (14), this implies as desired

$$\langle (\text{osc}_{0,D}\zeta_0)^2 \rangle_0 \leq \langle (\text{osc}_{B_1(D)}\zeta)^2 \rangle.$$

**Step 2.** Conditional expectations and independence. From now on, we prove statement (12) for the Poisson point process. For brevity, we drop the subscript 0.

For a given (Lebesgue measurable) subset  $D$  of the torus, we denote by  $\langle \cdot | D \rangle$  the expectation conditioned on the restriction  $X|_D$  of the (random) point configuration  $X$  on  $D$ . We note that for a function  $\zeta: \Omega \rightarrow \mathbb{R}$  which is square integrable,  $\langle \zeta | D \rangle$  is the  $L^2(\Omega)$ -orthogonal projection of  $\zeta$  onto the

space of square integrable functions  $\tilde{\zeta}: \Omega \rightarrow \mathbb{R}$  that only depend on  $X$  via  $X|_D$ .

With help of these conditional expectations the independence assumption (4) can be rephrased as follows: For any two (Lebesgue measurable) subsets  $D, D'$  that are *disjoint* and any (square integrable) function  $\zeta: \Omega \rightarrow \mathbb{R}$  that does *not* depend on  $X|_D$  we have

$$\langle \zeta | D \cup D' \rangle = \langle \zeta | D' \rangle. \quad (15)$$

Here comes the argument: By definition of conditional expectation, (15) follows if for any pair of (bounded and measurable) test functions  $u$  and  $u'$  which only depend on  $X|_D$  and  $X|_{D'}$ , respectively, we have

$$\langle \zeta uu' \rangle = \langle \langle \zeta | D' \rangle uu' \rangle.$$

Indeed, on the one hand, since  $\zeta$  only depends on  $X|_{D^c}$  (where  $D^c$  denotes the complement of  $D$ ) and  $u'$  does only depend on  $X|_{D'}$  (and thus a fortiori only on  $X|_{D^c}$ ) while  $u$  only depends on  $X|_D$ , we have from (4):

$$\langle \zeta uu' \rangle = \langle \zeta u' \rangle \langle u \rangle.$$

On the other hand, since  $\langle \zeta | D' \rangle u'$  only depends on  $X|_{D'}$  (and in particular only on  $X|_{D^c}$ ) while  $u$  only depends on  $X|_D$ , we have from (4):

$$\langle \langle \zeta | D' \rangle uu' \rangle = \langle \langle \zeta | D' \rangle u' \rangle \langle u \rangle = \langle \langle \zeta u' | D' \rangle \rangle \langle u \rangle = \langle \zeta u' \rangle \langle u \rangle,$$

where the middle identity holds since  $u'$  only depends on  $X|_{D'}$ .

**Step 3.** Conditional expectation and oscillation. For any (Lebesgue measurable) disjoint subsets  $D$  and  $D'$  of the torus and any (square integrable) function, we have

$$| \langle \zeta | D \cup D' \rangle - \langle \zeta | D' \rangle | \leq \langle \text{osc}_D \zeta | D' \rangle. \quad (16)$$

By exchanging  $\zeta$  with  $-\zeta$ , we see that it is enough to show

$$\langle \zeta | D \cup D' \rangle \leq \langle \zeta | D' \rangle + \langle \text{osc}_D \zeta | D' \rangle, \quad (17)$$

We note that  $\sup_D \zeta \leq \zeta + \text{osc}_D \zeta$ , where we've set for abbreviation

$$(\sup_D \zeta)(X) := \sup\{\zeta(\tilde{X}) | \tilde{X} = X \text{ outside } D\}.$$



Hence (17) follows from

$$\langle \zeta | D \cup D' \rangle \leq \left\langle \sup_D \zeta | D' \right\rangle.$$

The latter inequality can be seen as follows

$$\begin{aligned} & \langle \zeta | D \cup D' \rangle \\ & \leq \left\langle \sup_D \zeta | D \cup D' \right\rangle \quad \text{since } \zeta \leq \sup_D \zeta \\ & \stackrel{(15)}{=} \left\langle \sup_D \zeta | D' \right\rangle \quad \text{since } \sup_D \zeta \text{ does not depend on } X|_D. \end{aligned}$$

**Step 4.** Martingale decomposition. For conciseness, we only prove (12) for  $N = 3$ . So let  $\{D_1, D_2, D_3\}$  be a partition of the torus, we claim

$$\begin{aligned} & \langle (\zeta - \langle \zeta \rangle)^2 \rangle \tag{18} \\ & = \langle (\zeta - \langle \zeta | D_1 \cup D_2 \rangle)^2 \rangle + \langle (\langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle)^2 \rangle + \langle (\langle \zeta | D_1 \rangle - \langle \zeta \rangle)^2 \rangle. \end{aligned}$$

Indeed, this follows from the fact that

$$\zeta - \langle \zeta | D_1 \cup D_2 \rangle, \langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle, \langle \zeta | D_1 \rangle - \langle \zeta \rangle \text{ are } L^2(\Omega)\text{-orthogonal.}$$

The latter can be seen as follows: By definition of  $\langle \cdot | D \rangle$  as  $L^2(\Omega)$ -orthogonal projection, the two last functions  $\langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle$  and  $\langle \zeta | D_1 \rangle - \langle \zeta \rangle$  do only depend on  $X|_{D_1 \cup D_2}$ , so that they are orthogonal to the first function  $\zeta - \langle \zeta | D_1 \cup D_2 \rangle$ . It remains to argue that the two last functions  $\langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle$  and  $\langle \zeta | D_1 \rangle - \langle \zeta \rangle$  are orthogonal. To that purpose, we rewrite the middle function as

$$\langle \zeta | D_1 \cup D_2 \rangle - \langle \zeta | D_1 \rangle = \zeta' - \langle \zeta' | D_1 \rangle \quad \text{where } \zeta' := \langle \zeta | D_1 \cup D_2 \rangle.$$

Since the last function only depends on  $X|_{D_1}$ , they are orthogonal.

**Step 5.** Conclusion, i. e. (11) for  $N = 3$ . By Step 4, it remains to estimate the three r. h. s. terms of (18). For the first term, we use (16) with  $D' = D_1 \cup D_2$  and  $D = D_3$  and obtain because of  $\zeta = \langle \zeta | D_1 \cup D_2 \cup D_3 \rangle$

$$\begin{aligned} \langle (\zeta - \langle \zeta | D_1 \cup D_2 \rangle)^2 \rangle & \leq \langle (\text{osc}_{D_3} \zeta | D_1 \cup D_2)^2 \rangle \\ & \stackrel{\text{Jensen}}{\leq} \langle \langle (\text{osc}_{D_3} \zeta)^2 | D_1 \cup D_2 \rangle \rangle = \langle (\text{osc}_{D_3} \zeta)^2 \rangle. \end{aligned}$$

The other two terms follow the same way.

PROOF OF LEMMA 2.

W. l. o. g. we may assume that  $\langle \zeta \rangle = 0$ .

**Step 1.** Application of the original Spectral Gap Estimate to  $\zeta^p$ . We claim that this yields

$$\langle \zeta^{2p} \rangle \lesssim \langle \zeta^p \rangle^2 + \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \zeta)^2 \right)^p \right\rangle. \quad (19)$$

Indeed, (7) applied to  $\zeta^p$  at first gives

$$\langle (\zeta^p - \langle \zeta^p \rangle)^2 \rangle \lesssim \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}(\zeta^p))^2 \right\rangle. \quad (20)$$

Using the triangle inequality in  $L^2(\Omega)$  on the l. h. s. of (20) in form of

$$\langle (\zeta^p)^2 \rangle^{\frac{1}{2}} \leq \langle (\zeta^p - \langle \zeta^p \rangle)^2 \rangle^{\frac{1}{2}} + |\langle \zeta^p \rangle|,$$

we see that (19) follows from (20) by Young's inequality provided we can show

$$\begin{aligned} \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}(\zeta^p))^2 \right\rangle &\lesssim \langle \zeta^{2p} \rangle^{1-\frac{1}{p}} \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \zeta)^2 \right)^p \right\rangle^{\frac{1}{p}} \\ &\quad + \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \zeta)^2 \right)^p \right\rangle. \end{aligned} \quad (21)$$

The latter can be seen as follows: From the elementary real-variable estimate

$$|\tilde{\zeta}^p - \zeta^p| \lesssim |\zeta|^{p-1} |\tilde{\zeta} - \zeta| + |\tilde{\zeta} - \zeta|^p,$$

we obtain by definition of osc that

$$\text{osc}_{B_R(z)}(\zeta^p) \lesssim |\zeta|^{p-1} \text{osc}_{B_R(z)} \zeta + (\text{osc}_{B_R(z)} \zeta)^p.$$

Using that the discrete  $\ell^{2p}(\mathbb{Z}^d)$ -norm is estimated by the discrete  $\ell^2(\mathbb{Z}^d)$ -norm, this implies

$$\begin{aligned} & \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}(\zeta^p))^2 \\ & \lesssim \zeta^{2(p-1)} \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}\zeta)^2 + \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}\zeta)^2 \right)^p. \end{aligned}$$

Hölder's inequality w. r. t. to  $\langle \cdot \rangle$  applied to the first r. h. s. term with exponents  $(\frac{p}{p-1}, p)$  yields (21).

**Step 2.** Conclusion in case of  $p \geq 2$  (the other case is easier and not needed later). It remains to treat the first r. h. s. term of (19). By Hölder's inequality w. r. t.  $\langle \cdot \rangle$  we have

$$\langle \zeta^p \rangle \leq \langle \zeta^{2p} \rangle^{\frac{p-2}{2p-2}} \langle \zeta^2 \rangle^{\frac{p}{2p-2}}. \quad (22)$$

Using  $\langle \zeta \rangle = 0$  we obtain from the original Spectral Gap Estimate applied to  $\zeta$  itself

$$\langle \zeta^2 \rangle \lesssim \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}\zeta)^2 \right\rangle \stackrel{\text{Jensen}}{\leq} \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}\zeta)^2 \right)^p \right\rangle^{\frac{1}{p}}. \quad (23)$$

Inserting (23) into (22) we obtain

$$\langle \zeta^p \rangle^2 \lesssim \langle \zeta^{2p} \rangle^{\frac{p-2}{p-1}} \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)}\zeta)^2 \right)^p \right\rangle^{\frac{1}{p-1}}.$$

Inserting this into (19) and using Young's inequality yields the claim of the lemma.

**PROOF OF LEMMA 3.**

**Step 1.** Regularity theory for  $a$ -harmonic functions. We will use the following two ingredients from De Giorgi's theory for uniformly elliptic equations: For any  $a \in \Omega$  and any  $a$ -harmonic function  $u$  in  $B_2$  we have

$$\sup_{B_1} |u| \lesssim \left( \int_{B_2} u^2 \right)^{\frac{1}{2}}. \quad (24)$$

Moreover, there exists a Hölder exponent  $\alpha > 0$  only depending on  $d$  and  $\lambda$  such that

$$\sup_{x_1, x_2 \in B_1} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \lesssim \left( \int_{B_2} |\nabla u|^2 \right)^{\frac{1}{2}}. \quad (25)$$

Both ingredients follow from De Giorgi's theorem (see for instance [1, Theorem 4.11]):

$$\sup_{B_1} |u| + \sup_{x_1, x_2 \in B_1} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \lesssim \left( \int_{B_2} u^2 \right)^{\frac{1}{2}}. \quad (26)$$

Indeed, for (25), one may assume w. l. o. g. that  $\int_{B_2} u = 0$  so that (25) follows from (26) and Poincaré's inequality on  $B_2$  for functions with mean value zero. Here and in the sequel, we write  $B_R = B_R(0)$  for brevity. The crucial element of these estimates is that the constants depend on the coefficient field  $a$  only through the ellipticity ratio  $\lambda$  (as indicated by the use of  $\lesssim$ ).

**Step 2.** In this step, we derive an auxiliary a priori estimate involving dyadic annuli. Let  $u$  be a function and  $g$  a vector field on the torus related by the elliptic equation  $-\nabla \cdot a \nabla u = \nabla \cdot g$  and normalized by  $\int_{[-\frac{L}{2}, \frac{L}{2}]^d} u = 0$ . We claim that if  $g$  vanishes in  $B_1$  we have

$$|u(0)| \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \int_{B_{2^n} \setminus B_{2^{n-1}}} |g|^2 \right)^{\frac{1}{2}}. \quad (27)$$

We note that this sum is actually finite since for  $2^n \gg L$ , the ball  $B_{2^{n-1}}$  invades the entire torus so that the "annulus"  $B_{2^n} \setminus B_{2^{n-1}}$  is actually void. Estimate (27) will be derived from (24) and an elementary scaling argument. Indeed, for  $n \in \mathbb{N}$ , we introduce

$$g_n := \left\{ \begin{array}{ll} g & \text{on } B_{2^n} \setminus B_{2^{n-1}} \\ 0 & \text{else} \end{array} \right\},$$

so that  $g = \sum_{n=1}^{\infty} g_n$ . Let  $u_n$  denote the solution of  $-\nabla \cdot a \nabla u_n = \nabla \cdot g_n$  on the torus normalized by  $\int_{[-\frac{L}{2}, \frac{L}{2}]^d} u_n = 0$ , so that  $u = \sum_{n=1}^{\infty} u_n$ . Hence by the triangle inequality, for (27) it is enough to show

$$|u_n(0)| \lesssim (2^n)^{1-\frac{d}{2}} \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |g_n|^2 \right)^{\frac{1}{2}}. \quad (28)$$

We now give the argument for (28). Testing  $-\nabla \cdot a \nabla u_n = \nabla \cdot g_n$  with  $u_n$ , using the uniform ellipticity, and Cauchy-Schwarz' inequality, we obtain

$$\left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |\nabla u_n|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |g_n|^2 \right)^{\frac{1}{2}}.$$

Since  $d > 2$ , Sobolev's embedding together with  $\int_{[-\frac{L}{2}, \frac{L}{2}]^d} u_n = 0$  yields

$$\left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |u_n|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \lesssim \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |g_n|^2 \right)^{\frac{1}{2}}.$$

By Hölder's inequality on  $B_{2^{n-1}}$  with exponents  $(\frac{d}{d-2}, \frac{d}{2})$  we obtain

$$\left( \int_{B_{2^{n-1}}} |u_n|^2 \right)^{\frac{1}{2}} \lesssim 2^n \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |g_n|^2 \right)^{\frac{1}{2}}.$$

We note that  $u_n$  is  $a$ -harmonic on  $B_{2^{n-1}}$  (since  $g_n$  vanishes there). Hence by (24), which we rescale from  $B_2$  to  $B_{2^{n-1}}$ , we have

$$|u_n(0)| \leq \sup_{B_{2^{n-2}}} |u_n| \lesssim \left( (2^n)^{-d} \int_{B_{2^{n-1}}} |u_n|^2 \right)^{\frac{1}{2}}.$$

The combination of the two last estimates yields (28).

**Step 3.** As a preliminary, we study the local dependence of  $\nabla \phi + \xi$  on  $a$ : Let the two coefficient fields  $a$  and  $\tilde{a}$  agree outside  $B_R$ . Then we have

$$\int_{B_R} |\nabla \phi(\tilde{a}; \cdot) + \xi|^2 \lesssim \int_{B_R} |\nabla \phi(a; \cdot) + \xi|^2. \quad (29)$$

Indeed, we note that the function  $\phi(\tilde{a}; \cdot) - \phi(a; \cdot)$  satisfies

$$-\nabla \cdot \tilde{a} \nabla (\phi(\tilde{a}; \cdot) - \phi(a; \cdot)) = \nabla \cdot (\tilde{a} - a) (\nabla \phi(a; \cdot) + \xi).$$

We test this equation with  $\phi(\tilde{a}; \cdot) - \phi(a; \cdot)$  and obtain from uniform ellipticity and Cauchy-Schwarz' inequality

$$\int_{[-\frac{L}{2}, \frac{L}{2}]^d} |\nabla (\phi(\tilde{a}; \cdot) - \phi(a; \cdot))|^2 \lesssim \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |(\tilde{a} - a) (\nabla \phi(a; \cdot) + \xi)|^2.$$

Since by assumption,  $\tilde{a} - a$  vanishes outside  $B_R$ , the above yields

$$\begin{aligned} & \int_{B_R} |\nabla(\phi(\tilde{a}; \cdot) - \phi(a; \cdot))|^2 \\ & \leq \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |\nabla(\phi(\tilde{a}; \cdot) - \phi(a; \cdot))|^2 \lesssim \int_{B_R} |\nabla\phi(a; \cdot) + \xi|^2. \end{aligned} \quad (30)$$

This implies (29) by the triangle inequality in  $L^2(B_R)$ .

**Step 4.** In this step, we derive the central deterministic estimate

$$\begin{aligned} & \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \right)^{\frac{1}{2}} \\ & \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \left( \int_{B_R(z)} |\nabla\phi + \xi|^2 \right)^p \right)^{\frac{1}{2p}}. \end{aligned} \quad (31)$$

Given a coefficient field  $a$  on the torus and a point on the integer lattice  $z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d$ , we denote by  $a_z$  an arbitrary coefficient field on the torus that agrees with  $a$  outside  $B_R(z)$ . We note that the function  $\phi(a_z; \cdot) - \phi(a; \cdot)$  satisfies

$$-\nabla \cdot a \nabla(\phi(a_z; \cdot) - \phi(a; \cdot)) = \nabla \cdot (a_z - a)(\nabla\phi(a_z; \cdot) + \xi). \quad (32)$$

Given a discrete field  $\{\omega_z\}_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d}$  we consider the function  $u$  and the vector field  $g$  on the torus defined through

$$\begin{aligned} u(x) & := \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z (\phi(a_z; x) - \phi(a; x)), \\ g(x) & := \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z (a_z(x) - a(x)) (\nabla\phi(a_z; x) + \xi) \end{aligned}$$

and note that (32) translates into  $-\nabla \cdot a \nabla u = \nabla \cdot g$ . Provided  $\omega_z = 0$  for  $z \in B_{R+1}$ , we have  $g(x) = 0$  for  $x \in B_1$ . Under this assumption, we may apply (27) from Step 2 and obtain

$$\begin{aligned} & \left| \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z (\phi(a_z; 0) - \phi(a; 0)) \right| \\ & \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \int_{B_{2^n} \setminus B_{2^{n-1}}} \left| \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z (a_z - a)(\nabla\phi(a_z; \cdot) + \xi) \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $|a_z - a| \leq 1$  is supported in  $B_R(z)$  and since  $\{B_R(z)\}_{z \in \mathbb{Z}^d}$  locally have a finite overlap, this turns into

$$\begin{aligned}
& \left| \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z (\phi(a_z; 0) - \phi(a; 0)) \right| \\
& \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z^2 \int_{B_R(z) \cap B_{2^n}} |\nabla \phi(a_z, \cdot) + \xi|^2 \right)^{\frac{1}{2}} \\
& \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \omega_z^2 \int_{B_R(z)} |\nabla \phi(a_z, \cdot) + \xi|^2 \right)^{\frac{1}{2}} \\
& \stackrel{(29)}{\lesssim} \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \omega_z^2 \int_{B_R(z)} |\nabla \phi(a, \cdot) + \xi|^2 \right)^{\frac{1}{2}} \\
& \stackrel{\text{H\"older in } z}{\lesssim} \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \omega_z^{2q} \right)^{\frac{1}{2q}} \\
& \quad \times \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \left( \int_{B_R(z)} |\nabla \phi(a, \cdot) + \xi|^2 \right)^p \right)^{\frac{1}{2p}},
\end{aligned}$$

where  $p$  and  $q$  are dual exponents, that is,  $\frac{1}{p} + \frac{1}{q} = 1$ . Since  $\{\omega_z\}_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d}$  was arbitrary under the constraint that  $\omega_z = 0$  for  $z \in B_{R+1}$  this implies by the duality of  $\ell^{2q}(\mathbb{Z}^d)$  and  $\ell^{\frac{2q}{2q-1}}(\mathbb{Z}^d)$

$$\begin{aligned}
& \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} |\phi(a_z; 0) - \phi(a; 0)|^{\frac{2q}{2q-1}} \right)^{\frac{2q-1}{2q}} \\
& \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \left( \int_{B_R(z)} |\nabla \phi(a, \cdot) + \xi|^2 \right)^p \right)^{\frac{1}{2p}}.
\end{aligned}$$

Since for any  $z \in \mathbb{Z}^d$ ,  $a_z$  was an arbitrary coefficient field that agrees with  $a$  outside  $B_R(z)$ , this implies by the definition of  $\text{osc}_{B_R(z)}$

$$\begin{aligned}
& \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^{\frac{2q}{2q-1}} \right)^{\frac{2q-1}{2q}} \\
& \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \left( \int_{B_R(z)} |\nabla \phi + \xi|^2 \right)^p \right)^{\frac{1}{2p}}.
\end{aligned}$$

On the l. h. s. we use that since  $\frac{2q}{2q-1} \leq 2$ , the discrete  $\ell^{\frac{2q}{2q-1}}(\mathbb{Z}^d)$ -norm dominates the discrete  $\ell^2(\mathbb{Z}^d)$ -norm to obtain (31).

**Step 5.** Using stationarity, we upgrade Step 4 to the stochastic estimate

$$\left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \right)^p \right\rangle^{\frac{1}{2p}} \lesssim \left\langle \left( \int_{B_R} |\nabla \phi + \xi|^2 \right)^p \right\rangle^{\frac{1}{2p}}. \quad (33)$$

Indeed, we start from (31) in Step 4 and apply the triangle inequality to the sum over  $n$  w. r. t. the norm  $L^{2p}(\Omega)$ :

$$\begin{aligned} & \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} \text{osc}_{B_R(z)} (\phi(\cdot; 0))^2 \right)^p \right\rangle^{\frac{1}{2p}} \\ & \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} \left( \sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} \left\langle \left( \int_{B_R(z)} |\nabla \phi(a, \cdot) + \xi|^2 \right)^p \right\rangle \right)^{\frac{1}{2p}}. \quad (34) \end{aligned}$$

We now note that the stationarity (2) of  $\phi$  also yields

$$\nabla \phi(a; x+z) = \nabla \phi(a(\cdot+z); x)$$

and thus

$$\int_{B_R(z)} |\nabla \phi(a, x') + \xi|^2 dx' = \int_{B_R} |\nabla \phi(a(\cdot+z), x) + \xi|^2 dx.$$

By stationarity of  $\langle \cdot \rangle$ , cf. (6) applied to  $\zeta(a) = \int_{B_R(z)} |\nabla \phi(a, x) + \xi|^2 dx$ , this implies

$$\left\langle \left( \int_{B_R(z)} |\nabla \phi(\cdot, x') + \xi|^2 dx' \right)^p \right\rangle = \left\langle \left( \int_{B_R} |\nabla \phi(\cdot, x) + \xi|^2 dx \right)^p \right\rangle. \quad (35)$$

Inserting this into (34) yields (because of  $\sum_{z \in \mathbb{Z}^d \cap B_{2^n+R}} 1 \lesssim (2^n + R)^d$ )

$$\begin{aligned} & \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \setminus B_{R+1}} \text{osc}_{B_R(z)} (\phi(\cdot; 0))^2 \right)^p \right\rangle^{\frac{1}{2p}} \\ & \lesssim \sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} (2^n + R)^{\frac{d}{2p}} \left\langle \left( \int_{B_R} |\nabla \phi(a, \cdot) + \xi|^2 \right)^p \right\rangle^{\frac{1}{2p}}. \quad (36) \end{aligned}$$



Since for  $p > \frac{d}{d-2}$  the exponent  $1 - \frac{d}{2} + \frac{d}{2p} < 0$  is negative we have

$$\sum_{n=1}^{\infty} (2^n)^{1-\frac{d}{2}} (2^n + R)^{\frac{d}{2p}} \lesssim 1.$$

Hence (36) turns into the desired (33).

**Step 6.** It remains to treat  $z \in \mathbb{Z}^d \cap B_{R+1}$  in (10). By stationarity, it will be enough to consider  $z = 0$ , cf. Step 7. In this step, we will derive from the Hölder continuity a priori estimate (25) the deterministic estimate

$$\text{osc}_{B_R} \phi(a; 0) \lesssim \left( \int_{B_{2R}} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}. \quad (37)$$

Let  $a \in \Omega$  be given and  $\tilde{a} \in \Omega$  agree with  $a$  outside  $B_R$  and otherwise be arbitrary. On the one hand, since  $d > 2$  and  $\int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi(\tilde{a}; \cdot) - \phi(a; \cdot)) \stackrel{(1)}{=} 0$ , we have by Sobolev's embedding

$$\begin{aligned} \left( \int_{B_R} |\phi(\tilde{a}; \cdot) - \phi(a; \cdot)|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} &\leq \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |\phi(\tilde{a}; \cdot) - \phi(a; \cdot)|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \\ &\lesssim \left( \int_{[-\frac{L}{2}, \frac{L}{2}]^d} |\nabla(\phi(\tilde{a}; \cdot) - \phi(a; \cdot))|^2 \right)^{\frac{1}{2}} \\ &\stackrel{(30)}{\lesssim} \left( \int_{B_R} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (38)$$

On the other hand, we obtain from (25) applied to the  $a$ -harmonic function  $u(x) = \phi(a; x) + \xi \cdot x$  (rescaled from  $B_1$  to  $B_R$ ):

$$\sup_{x_1, x_2 \in B_R} \frac{|\phi(a; x_1) - \phi(a; x_2) + \xi \cdot (x_1 - x_2)|}{|x_1 - x_2|^\alpha} \lesssim \left( \int_{B_{2R}} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}. \quad (39)$$

Replacing  $a$  by  $\tilde{a}$  in the above and using (29) from Step 3 (with  $B_R$  replaced by  $B_{2R}$ ) we likewise have

$$\sup_{x_1, x_2 \in B_R} \frac{|\phi(\tilde{a}; x_1) - \phi(\tilde{a}; x_2) + \xi \cdot (x_1 - x_2)|}{|x_1 - x_2|^\alpha} \lesssim \left( \int_{B_{2R}} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}. \quad (40)$$

Combining (39) and (40), we obtain

$$\sup_{x_1, x_2 \in B_R} \frac{|(\phi(\tilde{a}; x_1) - \phi(a; x_1)) - (\phi(\tilde{a}; x_2) - \phi(a; x_2))|}{|x_1 - x_2|^\alpha} \lesssim \left( \int_{B_{2R}} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}. \quad (41)$$

By the following elementary interpolation estimate, valid for an arbitrary function  $u$ ,

$$\sup_{B_R} |u| \lesssim \left( \int_{B_R} |u|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} + \sup_{x_1, x_2 \in B_R} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha},$$

we see that (38) and (41) combine to

$$|\phi(\tilde{a}; 0) - \phi(a; 0)| \lesssim \left( \int_{B_{2R}} |\nabla \phi(a; \cdot) + \xi|^2 \right)^{\frac{1}{2}}.$$

Since  $\tilde{a}$  was arbitrary besides agreeing with  $a$  outside  $B_R$ , we obtain (37) by definition of  $\text{osc}$ .

**Step 7.** We upgrade Step 6 to the stochastic estimate

$$\left\langle \left( \sum_{z \in \mathbb{Z}^d \cap B_{R+1}} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \right)^p \right\rangle \lesssim \left\langle \left( \int_{B_{3R+1}} |\nabla \phi + \xi|^2 \right)^p \right\rangle. \quad (42)$$

Indeed, (37) from Step 6, with the origin replaced by  $z$ , implies after summation

$$\sum_{z \in \mathbb{Z}^d \cap B_{R+1}} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \lesssim \int_{B_{3R+1}} |\nabla \phi + \xi|^2.$$

Taking the  $p$ -th power and the expectation yields (42).

**Step 8.** From Steps 5 and 7 we learn that (10) is satisfied with  $B_1$  replaced by  $B_{3R+1}$  on the r. h. s. . We appeal once more to stationarity to get for a generic  $R \lesssim 1$

$$\left\langle \left( \int_{B_R} |\nabla \phi + \xi|^2 \right)^p \right\rangle \lesssim \left\langle \left( \int_{B_1} |\nabla \phi + \xi|^2 \right)^p \right\rangle. \quad (43)$$

Indeed, there exist points  $z_1, \dots, z_N$  on the torus such that  $B_R \subset \bigcup_{n=1}^N B_1(z_n)$  and we can arrange for  $N \lesssim 1$  because of  $R \lesssim 1$ . Thus we have

$$\int_{B_R} |\nabla \phi + \xi|^2 \leq \sum_{n=1}^N \int_{B_1(z_n)} |\nabla \phi + \xi|^2.$$

Taking the  $p$ -th power gives

$$\left( \int_{B_R} |\nabla \phi + \xi|^2 \right)^p \lesssim \sum_{n=1}^N \left( \int_{B_1(z_n)} |\nabla \phi + \xi|^2 \right)^p;$$

taking the expectation yields

$$\left\langle \left( \int_{B_R} |\nabla\phi + \xi|^2 \right)^p \right\rangle \leq \max_{n=1, \dots, N} \left\langle \left( \int_{B_1(z_n)} |\nabla\phi + \xi|^2 \right)^p \right\rangle.$$

By stationarity, cf. (35), this yields (43).

**PROOF OF LEMMA 4**

**Step 1.** We start by establishing the deterministic estimate

$$\left( \int_{B_1} |\nabla\phi + \xi|^2 \right)^p \lesssim \int_{B_2} (\phi + \xi \cdot x)^{2(p-1)} |\nabla\phi + \xi|^2, \quad (44)$$

which we will use in form of

$$\left( \int_{B_1} |\nabla\phi + \xi|^2 \right)^p \lesssim \int_{B_2} (\phi^{2(p-1)} + 1)(|\nabla\phi|^2 + 1). \quad (45)$$

Estimate (44) relies on the fact that  $u(x) := \phi(x) + \xi \cdot x$  is  $a$ -harmonic, that is,

$$-\nabla \cdot a \nabla u = 0.$$

We test this equation with  $\eta^2 u$ , where  $\eta$  is a cut-off function for  $B_1$  in  $B_2$ . By uniform ellipticity we obtain

$$\lambda \int (\eta |\nabla u|)^2 \leq 2 \int |\nabla \eta| |u| \eta |\nabla u|.$$

We now use Young's inequality (and  $p \geq 2 > 1$ ) on the r. h. s. integrand in form of

$$\frac{2}{\lambda} |\nabla \eta| |u| \eta |\nabla u| \leq \frac{1}{2} (\eta |\nabla u|)^2 + C (|\nabla \eta| |u|)^{2 \frac{p-1}{p}} (\eta |\nabla u|)^{\frac{2}{p}},$$

which yields

$$\int (\eta |\nabla u|)^2 \lesssim \int (|\nabla \eta| |u|)^{2 \frac{p-1}{p}} (\eta |\nabla u|)^{\frac{2}{p}}.$$

By the choice of  $\eta$ , this implies

$$\int_{B_1} |\nabla u|^2 \lesssim \int_{B_2} u^{2 \frac{p-1}{p}} |\nabla u|^{\frac{2}{p}}.$$

It remains to apply Jensen's inequality on the r. h. s. to obtain as desired

$$\int_{B_1} |\nabla u|^2 \lesssim \left( \int_{B_2} u^{2(p-1)} |\nabla u|^2 \right)^{\frac{1}{p}}.$$

**Step 2.** We continue with the deterministic estimate

$$\int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi^{2(p-1)} |\nabla \phi|^2 \lesssim \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi^{2(p-1)}, \quad (46)$$

which we will use in form of

$$\int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) |\nabla \phi|^2 \lesssim \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) \quad (47)$$

that follows from the combination of (46) once with the generic exponent  $p$  and once with the exponent  $p = 2$ . Indeed, we test  $-\nabla \cdot a(\nabla \phi + \xi) = 0$  with the monotone-in- $\phi$  expression  $\frac{1}{2p-1} \phi |\phi|^{2(p-1)}$  over the entire torus. Because of  $\nabla \frac{1}{2p-1} \phi |\phi|^{2(p-1)} = \phi^{2(p-1)} \nabla \phi$  and by uniform ellipticity, we obtain

$$\lambda \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi^{2(p-1)} |\nabla \phi|^2 \leq \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi^{2(p-1)} |\nabla \phi|.$$

Using Cauchy-Schwarz' inequality on the r. h. s. of that inequality yields (46).

**Step 3.** Conclusion using stationarity. We take the expectation of (45):

$$\left\langle \left( \int_{B_1} |\nabla \phi + \xi|^2 \right)^p \right\rangle \lesssim \left\langle \int_{B_2} (\phi^{2(p-1)} + 1) (|\nabla \phi|^2 + 1) \right\rangle.$$

By stationarity, we have

$$\left\langle \int_{B_2} (\phi^{2(p-1)} + 1) (|\nabla \phi|^2 + 1) \right\rangle = |B_2| L^{-d} \left\langle \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) (|\nabla \phi|^2 + 1) \right\rangle.$$

We now use the expectation of (47):

$$|B_2| L^{-d} \left\langle \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) |\nabla \phi|^2 \right\rangle \lesssim L^{-d} \left\langle \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) \right\rangle.$$

We use once more stationarity in form of

$$L^{-d} \left\langle \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\phi^{2(p-1)} + 1) \right\rangle = \langle \phi^{2(p-1)} \rangle + 1.$$

**PROOF OF PROPOSITION 1.**

By Jensen's inequality, it is enough to prove the statement for  $p > \frac{d}{d-2}$ . We apply Lemma 2 to  $\zeta(a) = \phi(a; 0)$ . We note that by stationarity of  $\phi$  and  $\langle \cdot \rangle$  we have

$$\langle \phi \rangle = \left\langle L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \phi \right\rangle \stackrel{(1)}{=} 0.$$

Hence the statement of Lemma 2 assumes the form

$$\langle \phi^{2p} \rangle \lesssim \left\langle \left( \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \phi(\cdot; 0))^2 \right)^p \right\rangle.$$

Estimating the r. h. s. by Lemmas 3 and 4, this turns into

$$\langle \phi^{2p} \rangle \leq C \left( \langle \phi^{2(p-1)} \rangle + 1 \right).$$

We conclude by using Jensen's and Young's inequalities in form of  $C \langle \phi^{2(p-1)} \rangle \leq C \langle \phi^{2p} \rangle^{\frac{p-1}{p}} \leq \frac{1}{2} \langle \phi^{2p} \rangle + \tilde{C}$ .

**PROOF OF THEOREM 1.**

**Step 1.** Application of Lemma 1 to  $\zeta = \xi' \cdot a_{\text{hom}} \xi$  yields

$$\langle (\xi' \cdot a_{\text{hom}} \xi - \langle \xi' \cdot a_{\text{hom}} \xi \rangle)^2 \rangle \lesssim \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \xi' \cdot a_{\text{hom}} \xi)^2 \right\rangle. \quad (48)$$

**Step 2.** Deterministic estimate of the oscillation. We first rewrite  $\xi' \cdot a_{\text{hom}} \xi$  as

$$\xi' \cdot a_{\text{hom}} \xi = L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi' + \xi') \cdot a(\nabla \phi + \xi), \quad (49)$$

where  $\phi'(a; x)$  is the corrector associated with the pointwise transpose field  ${}^T a$  of  $a$  and direction  $\xi'$ . Indeed, (49) holds by (1) since  $\phi'$  is periodic. We claim

$$\text{osc}_{B_R(z)} \xi' \cdot a_{\text{hom}} \xi \lesssim L^{-d} \left( \int_{B_R(z)} |\nabla \phi' + \xi'|^2 \right)^{\frac{1}{2}} \left( \int_{B_R(z)} |\nabla \phi + \xi|^2 \right)^{\frac{1}{2}}. \quad (50)$$

Indeed, consider two arbitrary coefficient fields  $a_0, a_1 \in \Omega$  that agree outside  $B_R(z)$ . We write for abbreviation  $\phi_i(x) = \phi(a_i; x)$ ,  $\phi'_i(x) = \phi'(a_i, x)$ , and  $a_{hom,i} = a_{hom}(a_i)$  for  $i = 0, 1$ . By definition of  $\text{osc}$  it is enough to show

$$L^d |\xi' \cdot a_{hom,1} \xi - \xi' \cdot a_{hom,0} \xi| \lesssim \left( \int_{B_R(z)} |\nabla \phi'_0 + \xi'|^2 \right)^{\frac{1}{2}} \left( \int_{B_R(z)} |\nabla \phi_0 + \xi|^2 \right)^{\frac{1}{2}}. \quad (51)$$

Indeed, we have by definition of  $a_{hom}$  and of  $\phi, \phi'$

$$\begin{aligned} & L^d (\xi' \cdot a_{hom,1} \xi - \xi' \cdot a_{hom,0} \xi) \\ & \stackrel{(49)}{=} \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_1 + \xi') \cdot a_1 (\nabla \phi_1 + \xi) - \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_0 + \xi') \cdot a_0 (\nabla \phi_0 + \xi) \\ & = \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \nabla (\phi'_1 - \phi'_0) \cdot a_1 (\nabla \phi_1 + \xi) + \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_0 + \xi') \cdot a_0 \nabla (\phi_1 - \phi_0) \\ & \quad + \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_0 + \xi') \cdot (a_1 - a_0) (\nabla \phi_1 + \xi) \\ & = \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \nabla (\phi'_1 - \phi'_0) \cdot a_1 (\nabla \phi_1 + \xi) + \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \nabla (\phi_1 - \phi_0) \cdot {}^T a_0 (\nabla \phi'_0 + \xi') \\ & \quad + \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_0 + \xi') \cdot (a_1 - a_0) (\nabla \phi_1 + \xi). \end{aligned}$$

Using the equation (1) for  $\phi_1$  and for  $\phi'_0$ , the first two r. h. s. terms vanish and this identity turns into

$$L^d (\xi' \cdot a_{hom,1} \xi - \xi' \cdot a_{hom,0} \xi) = \int_{[-\frac{L}{2}, \frac{L}{2}]^d} (\nabla \phi'_0 + \xi') \cdot (a_1 - a_0) (\nabla \phi_1 + \xi),$$

so that we obtain

$$L^d |\xi' \cdot a_{hom,1} \xi - \xi' \cdot a_{hom,0} \xi| \leq \left( \int_{B_R(z)} |\nabla \phi'_0 + \xi'|^2 \int_{B_R(z)} |\nabla \phi_1 + \xi|^2 \right)^{\frac{1}{2}}.$$

Now (51) follows from this and Step 3 in the proof of Lemma 3 in form of  $\int_{B_R(z)} |\nabla \phi_1 + \xi|^2 \lesssim \int_{B_R(z)} |\nabla \phi_0 + \xi|^2$ .

**Step 3.** Stochastic estimate based on Proposition 1. We claim

$$\left\langle \left( \int_{B_R(z)} |\nabla \phi + \xi|^2 \right)^2 \right\rangle \lesssim 1. \quad (52)$$

Indeed, by Step 8 from the proof of Lemma 3 (and stationarity to replace  $z$  by 0) we have

$$\left\langle \left( \int_{B_R(z)} |\nabla\phi + \xi|^2 \right)^2 \right\rangle \lesssim \left\langle \left( \int_{B_1} |\nabla\phi + \xi|^2 \right)^2 \right\rangle.$$

An application of Lemma 4 with  $p = 2$  and of Proposition 1 with  $p = 1$  yields (52). Since the ensemble  $\langle \cdot \rangle'$  that is obtained from  $\langle \cdot \rangle$  as pushforward under  $a \mapsto {}^T a$  satisfies our assumptions, we have

$$\left\langle \left( \int_{B_R(z)} |\nabla\phi' + \xi'|^2 \right)^2 \right\rangle = \left\langle \left( \int_{B_R(z)} |\nabla\phi + \xi|^2 \right)^2 \right\rangle' \lesssim 1. \quad (53)$$

**Step 4. Conclusion:**

$$\begin{aligned} & \langle (\xi' \cdot a_{hom}\xi - \langle \xi' \cdot a_{hom}\xi \rangle)^2 \rangle \\ & \stackrel{(48)}{\lesssim} \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} (\text{osc}_{B_R(z)} \xi' \cdot a_{hom}\xi)^2 \right\rangle \\ & \stackrel{(50)}{\lesssim} L^{-2d} \left\langle \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \int_{B_R(z)} |\nabla\phi' + \xi'|^2 \int_{B_R(z)} |\nabla\phi + \xi|^2 \right\rangle \\ & \leq L^{-2d} \sum_{z \in \mathbb{Z}^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \left\langle \left( \int_{B_R(z)} |\nabla\phi' + \xi'|^2 \right)^2 \right\rangle^{\frac{1}{2}} \left\langle \left( \int_{B_R(z)} |\nabla\phi + \xi|^2 \right)^2 \right\rangle^{\frac{1}{2}} \\ & \stackrel{(52), (53)}{\lesssim} L^{-d}. \end{aligned}$$

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