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## To cite this version:

Emmanuel Bernuau, Denis Efimov, Wilfrid Perruquetti, Andrey Polyakov. On homogeneity and its application in sliding mode control. Journal of The Franklin Institute, Elsevier, 2014, <http://dx.doi.org/10.1016/j.jfranklin.2014.01.007>. <10.1016/j.jfranklin.2014.01.007>. <hal-00942326>

## HAL Id: hal-00942326

## https://hal.inria.fr/hal-00942326

Submitted on 5 Feb 2014

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# On Homogeneity and Its Application in Sliding Mode Control[1]

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## Abstract

The paper is reviewing the tools to handle high-order sliding mode design and robustness. The main ingredient is homogeneity which can be checked using an algebraic test and which helps us in obtaining one of the most desired property in sliding mode control that is finite-time stability. This paper stresses some recently obtained results about homogeneity for differential inclusions and robustness with respect to perturbations in the context of input-to-state stability. Lastly within this framework, most of the popular high-order sliding mode control schemas are analyzed.

## 1. Introduction and related works

Sliding mode control has a long standing history starting from the early 20th century with the paper of G. Nikolskii [1]: the notion of "sliding mode" was introduced in the context of relay control systems. Indeed, the state dependent relay control may switch at high frequency (theoretically infinite) inducing a constraint motion on a manifold (called the sliding mode [2]). Later in the fifties-sixties an intensive research on sliding mode control steps ahead with the bang-bang control problems (just to mention a few works : B. Hamel in France [3] or Y.Z. Tzypkin [4] (and references therein) and S.V. Emelyanov [5] in USSR).

The sliding mode control design is a procedure with two stages :

- Sliding surface synthesis, which will provide the desired performances;

[1]This work was partially supported by ANR Grant CHASLIM (ANR-11-BS03-0007).

- Control design (discontinuous or not) ensuring that all motions will slide along the designed sliding manifold.

The obtained controls make the motions to be composed of two parts: the hitting (or reaching) phase and the sliding phase.

Such a sliding mode control exhibits interesting features for engineers: a relative simplicity of design, equivalent control motion (as long as sliding conditions are maintained), invariance to process characteristics and external perturbations, wide variety of application domains such as regulation, trajectory control [6], model following [7], observation [8] etc. Although the subject has already been treated in many papers [9, 10], surveys [11], or books [12, 13, 2], it still remains the object of many studies (theoretical or related to various applications).

The first generation of sliding modes (1960-1990) are called classical sliding modes or first order sliding modes and they possess the following characteristics:

**Pros**
- system order reduction on sliding surface;
- finite-time convergence;
- robustness with respect to parameter variations and/or matching disturbances (if the control amplitude can be large enough).

**Cons**
- chattering;
- noise sensitivity;
- do not reject unmatched disturbances.

V.I. Utkin [2] and co-workers had a tremendous impact on its development.

Later, the evolution of the sliding mode control theory and its applications required a more detailed specification of the sliding motion, which may have different order of "smoothness" (chattering removal). The modern control theory calls this extended concept the high-order sliding mode that it was first introduced by A. Levant in 1985 (see references in [14]).

From the very beginning, one of the most important properties of the sliding mode is the finite-time convergence of motions toward the sliding manifold. Making a scalar variable converging to zero in a finite time is quite obvious (see for example [15, 16, 17, 18]) whereas extending this qualitative property to a higher order vector variable becomes very challenging. The later has been intensively studied in the nineties with the introduction of higher order sliding mode concepts ([14] and references therein). This finite-time convergence property was intensively used within ordinary differential framework [17]. Since then, many papers were devoted to the finite-time convergence [19, 20, 21, 22, 23, 24]. This property was exploited in [25, 26, 20, 27, 28], with application to controller design in [29, 30, 31, 28] or to observer design in [32, 33, 34, 35, 24]. The output feedback design is treated in [36, 37, 38].

Until now, the main framework used to obtain FTS (finite-time stability) property, relies upon the notion of homogeneity. Indeed local attractiveness of a homogeneous system with negative degree implies the global FTS (see [29, 30, 39]). Homogeneity is an intrinsic property of an object, which remains consistent with respect to some scaling: level sets (resp. solutions) are preserved for homogeneous functions (resp. vector fields). Homogeneity has a long standing history that can be broken into three steps:

- *Standard homogeneity* goes back to L. Euler. In the sixties it was used to investigate stability properties (see [40, 41, 16, 42, 18]). A particular attention was paid to polynomial systems [43].

- *Weighted homogeneity* was introduced by V.I. Zubov [44] in late 1950s and independently by H. Hermes [45, 46] in the nineties when looking at a local approximation of nonlinear systems: asymptotic controllability is shown to be inherited by the original nonlinear system if this property holds for the homogeneous approximation [46, 47, 48]. With this property, many results were obtained for stability/stabilization [49, 50, 51, 52, 53, 54, 55, 56, 57, 58], or output feedback [59]. Let us recall the result obtained by V.I. Zubov in [44] and L. Rosier in [60]: if a homogeneous system is globally asymptotically stable (with a continuous vector field), then there exists a homogeneous proper Lyapunov function. This notion was also used in different contexts: polynomial systems [61] and switched systems [22], self-triggered systems [62], control and analysis of oscillations [63, 64]. Extensions were provided to vector fields which are homogeneous with respect to a general linear Euler vector field with degrees of homogeneity that are functions of the state [65] and to homogeneity in the bi-limit [44, 59], which makes homogeneous approximation valid both at the origin and at infinity. Those tools were useful for nonlinear observer and output feedback design. Extensions to local homogeneity have been proposed recently [59, 63].

- *Geometric homogeneity:* The weighted homogeneity is based on the dilation definition, which depends on the coordinates. There exist vector fields, which are homogeneous in some coordinates, but not using other coordinates. Moreover, since homogeneity is a kind of symmetry, it should be invariant under a change of coordinates. This motivates for a geometric, coordinate free definition of homogeneity. The very first geometric definitions appear by V.V. Khomenuk [66] in 1960s, then some extensions were given independently by M. Kawski [55, 67, 68] and L. Rosier [69] in 1990s. The paper [20] gives a counterpart to [60] in this context, and [62] has used it for self-triggered systems.

In the context of sliding-mode control and differential inclusions, some few papers were devoted to treatment of FTS by homogeneity [70, 22].

This paper aims at reviewing the main tools to handle higher order sliding mode design and its robustness via homogeneity (the advantage of the latter is that it can be checked applying some algebraic operations only): a particular attention will be paid to the finite-time stability. After some notations (Section 2), Section 3 will give background on the class of systems under consideration within the sliding mode context (Ordinary differential equations and Differential inclusions). Then, we will introduce the main ingredients:

- Homogeneity concepts and results (Section 4),
- Finite-Time Stability concepts and results (Section 5): in particular this section stresses some recently obtained results on robustness of FTS for DI with respect to perturbations in the context of input-to-state stability.

Lastly their consequences for sliding-mode control analysis via the homogeneity will be given in Section 6.

## 2. Notations

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $\|.\|$ denotes the Euclidian norm.
- $B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq \varepsilon\}$ is the closed ball in $\mathbb{R}^n$ with the center at $x_0 \in \mathbb{R}^n$ and the radius $\varepsilon > 0$.
- $\mathrm{diag}(\mu_i)$ denotes the diagonal matrix with elements $\mu_1, ..., \mu_n \in \mathbb{R}$.
- For a Lebesgue measurable function $d : \mathbb{R}_+ \to \mathbb{R}^p$ define the norm $||d||_{(t_0,t_1)} = \mathrm{ess}\sup_{t\in(t_0,t_1)} \|d(t)\|$, then $||d||_\infty = ||d||_{(0,+\infty)}$ and the set of functions $d$ with the property $||d||_\infty < +\infty$ we will further denote as $\mathcal{L}_\infty$ (the set of essentially bounded measurable functions).

- Let $A, B$ be two compacts subsets of $\mathbb{R}^n$ using $\|x\|_A = \inf_{y\in A} \|x-y\|$ one can define the Hausdorff distance

$$\|A - B\|_H = \max\{\sup_{y\in B} \|y\|_A, \sup_{x\in A} \|x\|_B\}.$$

- For $P \subset \mathbb{R}^n$ the notation $\overline{\mathrm{conv}}(P)$ is used for the closed convex hull of $P$, i.e the minimal closed convex set containing $P$.

- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. The function $r \in \mathcal{CL}$ if it is continuous and locally Lipschitz in some neighbourhood of the origin excluding the origin. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{GKL}$ if $\beta(s,t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$, $\beta$ is a strictly decreasing function of its second argument $t \in \mathbb{R}_+$ for any fixed first argument $s \in \mathbb{R}_+$ and $\beta(s,T) = 0$ for each fixed $s \in \mathbb{R}_+$ for some $0 \le T < +\infty$.

- For any real number $\alpha \ge 0$ and for all real $x$ we set

$$\lceil x \rfloor^\alpha = \mathrm{sign}(x)|x|^\alpha. \tag{1}$$

- For any $x \in \mathbb{R}^n$ the upper directional derivative $DV(x)$ of a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is defined as follows

$$DV(x)v \quad = \quad \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{V(y+tv) - V(y)}{t},$$

  where $\limsup$ denotes the upper limit and $v \in \mathbb{R}^n$.

- The set $\mathcal{C}^k$ contains continuous functions, which have continuous derivatives at least up to the order $k$, where $k$ is a positive natural number .

- For a function $s : \mathbb{R}^n \to \mathbb{R}$ and the vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ the Lie derivatives of the order $k$ are denoted by $\mathcal{L}_f^k s$ and defined recursively by

$$\mathcal{L}_f^0 s(x) = s(x), \quad \mathcal{L}_f^i s(x) = \left(\frac{\partial}{\partial x}\mathcal{L}_f^{i-1} s(x)\right) f(x).$$

  For shortness we define $\mathcal{L}_f s(x) = \mathcal{L}_f^1 s(x)$.

## 3. Preliminaries

### *3.1. Class of systems under consideration*

In the following we will consider:

**ODE** Ordinary Differential Equation (ODE) of the form

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in \mathbb{R}_+, \tag{2}$$

where $x$ is the state which belongs to an open nonempty set $\mathcal{X} \subset \mathbb{R}^n$ containing the origin, $d(t) \in \mathbb{R}^p$ is the disturbance input, which can be a constant vector of uncertain parameters or a function of time $d \in \mathcal{L}_\infty$. In some cases, we will also assume that $d(t) \in \mathrm{D} \subset \mathbb{R}^p$. The assumptions related to the vector field $f$ will be given later.

**DI** Differential Inclusion (DI) of the form

$$\dot{x}(t) \in F(x(t), d(t)), \quad t \in \mathbb{R}_+, \tag{3}$$

where $x$, $d$ have the same meaning as above for ODE and $F : \mathbb{R}^{n+p} \rightrightarrows \mathbb{R}^n$ is a set-valued map, which satisfies some conditions given below.

In the rest of the paper, the solutions of these systems starting at time $t = 0$ from initial state $x_0 \in \mathcal{X}$ will be denoted by $\Phi^t(x_0)$: the considered system will be mentioned if necessary and omitted if it is clear from the context.

We consider only the autonomous systems (2) and (3), since the non-autonomous can be reduced to autonomous one by an artificial variable $x_{n+1} = t$ and an additional equation $\dot{x}_{n+1} = 1$ with some minor modifications for stability analysis.

### *3.2. Ordinary Differential Equations*

Let us briefly recall some results about existence of solution(s) for the case $d = 0$. When speaking about solutions, one has to specify the problem associated to solutions of ODE. An *initial condition problem or Cauchy Problem* occurs when for a given initial state $x_0$ and a given initial time $t_0 \in \mathbb{R}_+$ it is required to find a time function defined on a time interval containing $t_0$ which satisfies the ODE (or an integral form in the Lebesgue sense) such that its value at $t_0$ is $x_0$. When the data of initial condition is replaced by the data at given times, one speak about a *boundary problem*. For solutions one can simply look at functions which are at least absolutely continuous[2] with respect to time. The Cauchy problem may not have a solution, and sometimes may have many solutions. Indeed, the system $\frac{dx}{dt} = |x|^{\frac{1}{2}}$, $x \in \mathbb{R}$, has an infinite number of solutions starting from zero ($x(t_0) = 0$), defined by:

$$\begin{aligned} &\varepsilon \in \mathbb{R}_+, \quad \phi_\varepsilon : \mathbb{R} \to \mathbb{R}, \\ &t \mapsto \phi_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t_0 \le t \le t_0 + \varepsilon, \\ \frac{(t - t_0 - \varepsilon)^2}{4} & \text{if} \quad t_0 + \varepsilon \le t. \end{cases} \end{aligned} \tag{4}$$

Thus, concerning existence of solutions, one can distinguish the four cases A–D according to the smoothness of the function $f$ in (2). Some of these cases are also ensuring uniqueness of solutions.

**Case A (Peano, 1886)** If the function $f$ is continuous with respect to both arguments and $d$ is a continuous function of $t$, then there exists a continuously differentiable solution of (2) (this is the class of $\mathcal{C}^1$ solutions).

**Case B (Carathéodory, 1918)** If the function $f$ is continuous with respect to both arguments and $d$ is essentially bounded measurable function, then there exist absolutely continuous solutions of (2).

**Case C (Coddington & Levinson, 1955)** If the function $f(x, \mathrm{d})$ is locally Lipschitz with respect to $x$, continuous in $\mathrm{d}$, then there exists a unique maximal solution (see below for a definition) of (2), which is absolutely continuous.

**Case D (Winter, 1945)** If the continuous function $f$ has a norm which is bounded by an affine function, i.e $\forall (x, \mathrm{d}) \in (\mathcal{X} \times \mathrm{D})$ (possibly almost everywhere): $\|f(x, \mathrm{d})\| \le c_1 \|x\| + c_2 \|\mathrm{d}\|$ with $c_1$ and $c_2$ being nonnegative constants, then any solution to the Cauchy Problem exists on $\mathcal{X}$.

When dealing with ODE, it will be assumed implicitly in the rest of the paper that they do possess unique maximal solutions in forward time on $\mathcal{X} \subset \mathbb{R}^n$, that is to say: for all $x_0 \in \mathcal{X}$, $\Phi_0^t(x_0)$ defined on $[0, T_{\Phi_0}[$ is a maximal solution, if for any other solution $\Phi_1^t(x_0)$ defined on $[0, T_{\Phi_1}[$, $T_{\Phi_1} \le T_{\Phi_0}$ and $\Phi_0^t(x_0) = \Phi_1^t(x_0)$ for all $t \in [0, T_{\Phi_1}[$.

[2] $\phi : [\alpha, \beta] \mapsto \mathbb{R}^n$ is **absolutely continuous** if $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \forall \{]\alpha_i, \beta_i[\}_{i \in \{1..n\}}, ]\alpha_i, \beta_i[ \subset [\alpha, \beta], \sum_{i=1}^n (\beta_i - \alpha_i) \le \delta(\varepsilon) \Rightarrow \sum_{i=1}^n \|\phi(\beta_i) - \phi(\alpha_i)\| \le \varepsilon$.

*3.3. Differential Inclusion*

Let us consider the following Cauchy Problem:

$$\dot{x} = -\frac{x}{|x|} \text{ if } x \neq 0, \qquad x(0) = 0,$$

which does not have a solution in the classical sense since $\dot{x}(0)$ is not defined. However, replacing the right-hand side with the multi-valued function $F(x) = \left\{-\frac{x}{|x|}\right\}$ if $x \neq 0$, $F(0) = [-1, 1]$, one can show that $x(t) = 0$ satisfies the relation $\frac{dx}{dt} \in F(x)$ for all $t$. Similarly, the Cauchy Problem associated to the initial condition $x(0) = x_0$ admits a solution.

In a more general framework, in order to deal with an ODE having a discontinuous right-hand side, let us introduce the following model:

$$\frac{dx}{dt} = f(x), x \in \mathcal{X}\backslash\mathcal{N}, \tag{5}$$

where $x$ has the same meaning as above, $f(x)$ is defined and continuous on $\mathcal{X}\backslash\mathcal{N}$, and $\mathcal{N}$ is a set of zero Lebesgue measure. Such models appear in variable structure systems, systems with adaptive control, power electronic systems with switching devices, mechanical systems with friction, etc. It is worth to replace it with the following *differential inclusion*:

$$\frac{dx}{dt} \in F(x), \tag{6}$$

where $F$ is a set-valued map, which construction will be given later, defined for $x \in \mathcal{X}\backslash\mathcal{N}$ as $F(x) = \{f(x)\}$ and taking multi-values at each point in $\mathcal{N}$. This differential inclusion construction should capture the behaviors of (5).

**Remark 3.1.** *The control system $\frac{dx}{dt} = f(x, u), x \in \mathcal{X}, u \in \mathcal{U}$ sometimes can be reduced to the differential inclusion $\frac{dx}{dt} \in F(x, \mathcal{U})$. This property is often used when dealing with some time-optimal control analysis.*

**Remark 3.2.** *Later we will see how to handle ODE with discontinuous right-hand side in the presence of perturbation $d$ and/or control $u$. The main question is how to construct the corresponding right-hand side? An answer is given below.*

When addressing the Cauchy Problem for such a DI, we are looking for a function $\phi$ such that $\frac{d\phi}{dt} \in F(\phi(t), d(t))$ almost everywhere (a.e.) on some set and satisfying $\phi(t_0) = x_0$. Thus we will use the following

**Definition 3.3.** *[71] A solution of (6) passing through $x_0$ at $t_0$, is any absolutely continuous function $\phi$ defined on a non empty interval $\mathcal{I}_{(t_0,x_0)} \subset \mathbb{R}_+$ containing $t_0$:*

$$\begin{array}{rcl} \phi : \mathcal{I} \subset \mathbb{R}_+ & \to & \mathcal{X} \subset \mathbb{R}^n, \\ t & \mapsto & \phi(t; t_0, x_0), \end{array}$$

*simply denoted as $\phi(t)$, satisfying $\frac{d\phi}{dt} \in F(\phi(t))$ a.e. on $\mathcal{I}_{(t_0,x_0)}$ and such that $\phi(t_0) = x_0$.*

**Remark 3.4.** *If $\phi(t)$ is an* absolutely continuous *function, then there exists a Lebesgue integrable function $\chi$ which is the derivative of $\phi$ a.e. : $\phi(t) = \phi(t_0) + \int_{t_0}^{t} \chi(\nu) d\nu$ and if $\frac{d\phi}{dt} \underset{a.e.}{\in} F(\phi(t))$, then $\chi(\nu) \underset{a.e.}{\in} F(\phi(\nu))$. Reverse is true only under some specific regularity hypothesis on the multi-valued map (for example, if $F$ is compact, convex and upper semi-continuous[3]).*

**Theorem 3.5.** *[71] Assume that the multi-valued function $F$ is non-empty, compact, convex and upper semi-continuous on the ball $B(x_0, a) \subset \mathrm{dom}(F)$. If for some $K > 0$ and for all $x \in B(x_0, a)$ we have $\|F(x) - \{0\}\|_H \leq K$, then there exist at least one solution defined at least on the interval: $[t_0 - a/K, t_0 + a/K]$.*

**Theorem 3.6.** *[71] Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be non-empty, compact, convex and upper semi-continuous.*

- *Then for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$ there exists at least one solution to the Cauchy Problem defined on an open interval containing the initial time;*
- *If there exist two positives constants $c_1$ and $c_2$ such that for all $x \in \mathbb{R}^n$ we have $\|F(x) - \{0\}\|_H \leq c_1 \|x\| + c_2$, then there exists at least one solution to the Cauchy Problem defined on $\mathbb{R}$.*

**Remark 3.7.** *The second condition can be replaced by the existence of two locally integrable functions $c_1(t)$ and $c_2(t)$ satisfying a similar inequality.*

Let us stress, that in the two above mentioned existence results, a central condition is played by the convexity of $F$. However, the following relaxation theorem

[3]Let $E_1$ and $E_2$ be two topological Hausdorff spaces, a set-valued map $F : E_1 \rightrightarrows E_2$ is **upper semi-continuous** at $x \in \mathrm{dom}(F)$ if, for any neighbourhood $V \subset E_2$ of $F(x)$, there exists a neighbourhood $W$ of $x$ such that $F(W) \subset V$.

shows that if $F$ is *Lipschitz* , i.e $\exists L > 0 : \|F(x) - F(y)\|_H \leq L\|x - y\|, \forall x, y \in \mathcal{X}$, then the set of solutions of the DI (6) is dense in the set of solutions of the following convexified DI:

$$\frac{dx}{dt} \in \overline{\text{conv}}(F(x)), \tag{7}$$

this is that all solutions of (7) can be approximated by solutions from (6) with any given accuracy (as small as desired).

**Theorem 3.8.** *[71] Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be non-empty, compact and Lipschitz on the ball $B(x_0, a)$. Then any solution $\Phi^t(x_0)$ of (7) is defined on some interval $\mathcal{I}_{(t_0,x_0)}$ and for any given $\varepsilon > 0$, there exist a solution $\Psi^t(x_0)$ of (6) defined on the same interval $\mathcal{I}_{(t_0,x_0)}$ such that $\|\Psi^t(x_0) - \Phi^t(x_0)\| \leq \varepsilon, \forall t \in \mathcal{I}_{(t_0,x_0)}$.*

Fillipov regularization procedure: In many practical situations, $f(x)$ (the right-hand side of (5)) is defined on $\mathcal{X}\backslash\mathcal{N}$. If $f$ is continuous on $\mathcal{X}\backslash\mathcal{N}$, then it is useful to consider the following multi-valued function:

$$F(x) = \bigcap_{\varepsilon > 0} \overline{\text{conv}}(f(B(x, \varepsilon) \setminus \mathcal{N})), \tag{8}$$

which has all the good required properties for solution existence (non-empty, compact, convex and upper semi-continuous) and satisfies for all $t$ and all $x \in \mathcal{X}\backslash\mathcal{N}$ : $F(x) = \{f(x)\}$. Within the framework of variable structure systems (for example when dealing with sliding mode control), the function $f$ is not defined on a manifold $\mathcal{S} = \{x \in \mathcal{X} : s(x) = 0\}$ (where $s$ is a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}$ such that $\frac{\partial s}{\partial x} \neq 0$ for $x \in \mathcal{S}$) :

$$f(x) = \begin{cases} f^+(x), \text{ if } s(x) > 0, \\ f^-(x), \text{ if } s(x) < 0, \end{cases} \tag{9}$$

where $f^+$ and $f^-$ are continuous functions, which can be continuously prolonged to $\mathcal{S}$. Thus, we have to consider (8), which is obtained as follows:

$$F(x) = \begin{cases} \{f(x)\}, \text{ if } s(x) \neq 0, \\ \overline{\text{conv}}\,\{f^+(x), f^-(x)\}\,, \text{ if } s(x) = 0. \end{cases} \tag{10}$$

In that case, defining $\text{Pr}_{\text{normal}}$ to be the projection on the normal to the manifold $s(x) = 0$ oriented from $\mathcal{X}^-$ to $\mathcal{X}^+$, and the projection of the vector fields $f^+(x)$ and $f^-(x)$ as:

$$f_n^+(x) = \underset{\text{normal}}{\text{Pr}} \lim_{s \to 0^+} f^+(x), \tag{11}$$

$$f_n^-(x) = \underset{\text{normal}}{\text{Pr}} \lim_{s \to 0^-} f^-(x), \tag{12}$$

we obtain the following well known result of A.F. Filippov about uniqueness of solution.

**Theorem 3.9.** *[71] Let us consider system (5), with $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^- \cup \mathcal{S}$ and $\mathcal{N} = \mathcal{S}$. Assume that there exists $k \in \mathbb{R}^+$, for all $x \in \mathcal{X}^+ \cup \mathcal{X}^-$ such that:*

$$\left|\frac{\partial f_i}{\partial x_j}\right| \leq k, (i, j = 1 \ldots n).$$

*Let $s$ be a twice differentiable function, such that $f_n^+$ and $f_n^-$ are continuous with respect to $x$, for $x \in \mathcal{S}$. Let $h = f_n^+ - f_n^-$ be continuously differentiable. If, in any point of the manifold $\mathcal{S}$, at least one of the following inequalities $f_n^+ < 0$ or $f_n^- > 0$ holds, then, for any initial condition in the set $\mathcal{X}$, there exists a unique (forward) solution of (5), which depends uniquely on the initial conditions.*

In general, for DI, the uniqueness of solutions is meaningless: in the previous theorem 3.9 “a unique (forward) solution” means that if two solutions at time $t_0$ have the same value then it will be the same for all $t \geq t_0$ for which they are defined. Such a result (similar to other previously given theorems) has an immediate consequence: indeed when we have simultaneously $f_n^+ < 0$ and $f_n^- > 0$ then the solution to (5), for $x \in \mathcal{S}$ is given by:

$$\begin{cases} x \in \mathcal{S}, \\ \dot{x} = f_0(x), \end{cases} \tag{13}$$

with $f_0(x) \in \overline{\text{conv}}\,\{f^+(x), f^-(x)\} \cap T_x\mathcal{S}$ where $T_x\mathcal{S}$ is the tangent space to $\mathcal{S}$ at $x$. Finally the sliding dynamics is given by:

$$\frac{dx}{dt} = \left[\frac{\langle \frac{\partial s}{\partial x}, f^- \rangle}{\langle \frac{\partial s}{\partial x}, f^- - f^+ \rangle}\right] f^+ - \left[\frac{\langle \frac{\partial s}{\partial x}, f^+ \rangle}{\langle \frac{\partial s}{\partial x}, f^- - f^+ \rangle}\right] f^-. \tag{14}$$

Indeed, the convexified vector field is $f_0(x) = \alpha f^+(x) + (1-\alpha) f^-(x)$ and satisfies $f_0 \in T_x\mathcal{S} \Longleftrightarrow \langle \frac{\partial s}{\partial x}, f_0 \rangle = 0$, which leads to $\alpha = \frac{\langle \frac{\partial s}{\partial x}, f^-(t,x) \rangle}{\langle \frac{\partial s}{\partial x}, (f^-(t,x) - f^+(t,x)) \rangle}$.

Let us mention that, close to the manifold $s = 0$, the following holds:

$$f_n^+(x) < 0 \text{ and } f_n^-(x) > 0 \Leftrightarrow s\dot{s} < 0. \tag{15}$$

This last condition is known as the *sliding condition* [2].

Systems with perturbations: In the case of perturbed systems with discontinuous right-hand side of the form:

$$\frac{dx(t)}{dt} = f(x(t), d(t)), x \in \mathcal{X}, \tag{16}$$

where $f$ is defined and continuous except on a set of zero measure, it is necessary to replace it with a *differential inclusion*. Its construction is similar to the one introduced before and will depend on the perturbation's knowledge. For example, if $f(x, \mathrm{d}) := f(x) + \mathrm{d}$ and the perturbation is bounded $|\mathrm{d}| \leq a(x)$ (componentwise) then Filipov's construction leads to (6) with $F(x) = \{f(x)\} + [-a(x), +a(x)]$, where the notation $[-a(x), +a(x)]$ has to be understood componentwise.
Systems with control: Another interesting situation is when the system under consideration has a control variable $u$, that is a system of the form:

$$\frac{dx(t)}{dt} = f(x(t), u(x(t))), x \in \mathcal{X}, \tag{17}$$

for which the control is switching between $u^+(x)$, if $s(x) > 0$ and $u^-(x)$, if $s(x) < 0$ where $s$ is a fixed mapping defining the sliding manifold $\mathcal{S}$. Then one can use Filipov's construction or the equivalent control method introduced by V. Utkin. The equivalent control is basically ensuring the invariance of the manifold $\mathcal{S}$ (as soon as this manifold is locally attractive: the vector field is pointing toward the sliding manifold). Thus we should have $\dot{s} = 0$. In other words, the motion on the sliding manifold is obtained by considering the following ODE:

$$\dot{x}(t) = f(x(t), u_{\mathrm{eq}}(t)), x \in \mathcal{S},$$

where $u_{\mathrm{eq}}$ is the equivalent control obtained by solving the algebraic relation $\dot{s} = 0$, which is $\frac{\partial s}{\partial x} f(x, u_{\mathrm{eq}}) = 0$. However, these two methods may leads to different notions of solutions which may not coincide unless $f$ satisfies some assumptions (see [2] for more details about equivalent control method). Let us briefly review these notions on two simple examples.

**Example 3.10.** *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= 0.4x_2 + ux_1, \\
\dot{x}_2 &= -0.6x_1 + 4u^3x_1, \\
s(x) &= x_1 + x_2, \\
u &= -\mathrm{sign}(s(x)x_1),
\end{aligned}$$

*where $x_1, x_2, u \in \mathbb{R}$. First, let us check that close to the sliding manifold, the sliding condition $s\dot{s} < 0$ is satisfied: $\dot{s} = 0.4x_2 + ux_1 - 0.6x_1 + 4u^3x_1$. Which reduces to $\dot{s} \simeq x_1(-1 + u + 4u^3)$ because $x_1 + x_2 \simeq 0$ (close to the sliding manifold). Thus we have $s\dot{s} \simeq -sx_1 - 5|sx_1|$ which is negative. The equivalent control method gives: $u_{\text{eq}} = \frac{1}{2}$ since it should satisfy $\dot{s} = 0 = (-1 + u + 4u^3)x_1$ (note that the only real root of $-1 + u + 4u^3 = 0$ is $\frac{1}{2}$). Thus the equivalent dynamics (on the sliding manifold) is $\dot{x}_1 = 0.1x_1, x_1 + x_2 = 0$: the origin is locally unstable. But, application of the Filippov's method gives the following equivalent dynamics:*

$$\begin{aligned}\dot{x} &= \alpha f^+(x) + (1-\alpha)f^-(x),\\ \alpha &= \left[\frac{\langle ds, f^-(x)\rangle}{\langle ds, (f^-(x) - f^+(x))\rangle}\right],\end{aligned}$$

*where $f^+ = (0.4x_2 - x_1, -0.6x_1 - 4x_1)^T, f^- = (0.4x_2 + x_1, -0.6x_1 + 4x_1)^T$, which become $f^+ = (-1.4x_1, -4.6x_1)^T, f^- = (0.6x_1, 3.4x_1)^T$ (close to the sliding manifold). One obtain $\alpha = \frac{4}{10}$ leading to $\dot{x}_1 = -0.2x_1, x_1 + x_2 = 0$: the origin is locally asymptotically stable.*

**Example 3.11.** *Consider the sliding mode system of the form*

$$\begin{aligned}\dot{x}_1 &= u + 0.5,\\ \dot{x}_2 &= (3|u| - 2)x_2,\\ u &= -\text{sign}(x_1),\end{aligned}$$

*where $x_1, x_2, u \in \mathbb{R}$. Obviously, the equivalent control approach provides the stable sliding mode mode dynamic: $\dot{x}_2 = (3|u_{eq}| - 2)x_2 = -0.5x_2$ with $u_{eq} = -0.5$. However, Filippov's procedure gives its instability with*

$$f(x) = \alpha f^+(x) + (1-\alpha)f^-(x) = (0, x_2)^T,$$

*where $f^+(x) = (-0.5, x_2)^T$, $f^-(x) = (1.5, x_2)^T$ and $\alpha = 3/4$.*

However, the sliding dynamics obtained by using the equivalent control method agree with Filippov's dynamics in the case of affine control systems:

$$\dot{x} = f_0(x) + G(x)u(x), \quad x \in \mathbb{R}^n, \quad t > 0,$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is continuous, $u : \mathbb{R}^n \to \mathbb{R}^m$ is discontinuous control, $\frac{\partial s}{\partial x}G(x)$ is non singular close to the sliding manifold $\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = 0\}$, $s : \mathbb{R}^n \to \mathbb{R}^m$.

The reader may find additional materials on differential inclusions in [72, 73, 2, 13, 74, 71] .

## 4. Homogeneity

Homogeneity is the property whereby objects such as functions or vector fields scale in a consistent fashion with respect to a scaling operation called a dilation. The first rise of homogeneity consists in homogeneous polynomials. The symmetry properties of these polynomials were first studied by Euler and then more deeply during the nineteenth century, in view of projective geometry, algebraic geometry or in number theory. The Euler's homogeneous function theorem was the first result linking homogeneity with analysis. Following this idea, the theory of homogeneous functions uses the symmetry properties of the function, as for the homogeneous polynomials. In control theory, homogeneity appeared with Lasalle and Hahn in the 40's. This first wave of homogeneity is called the *classical homogeneity*.

The main issue with the classical homogeneity was its very restrictive field of use. For instance, consider the function $x + y^2$. Since $x$ is at the power $1$ while $y$ is at the power $2$, the quadratic part will grow two times faster than the linear one, preventing this function to be homogeneous in the classical sense. But if one scales $x$ two times faster than $y$, this obstruction is removed. Hence, a generalization of the classical homogeneity was proposed by V.I. Zubov in 50s and developed by H. Hermes in the 90's using different weights, leading to *weighted homogeneity*. Nowadays, this is the most popular definition of homogeneity.

The weighted homogeneity has permitted to extend many results at a broader class of objects. Nevertheless, this definition was still inconsistent with respect to a change of coordinates. M. Kawski in [51] (and L. Rosier independently in [69]) studied a geometric, coordinate-free setting, in which the symmetry defined by the dilation was captured into a vector field. This concept was originally introduced in 60s by V.V. Khomenuk [66]. The main purposes of homogeneity were generalized in the coordinate-free way, allowing more objects to be called homogeneous. This *geometric homogeneity* is the last step of today's homogeneity and is still an active research field.

### *4.1. Classical homogeneity*

**Definition 4.1.** *Let $n$ and $m$ be two positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous (in the classical sense) with degree $k \in \mathbb{R}$ iff $\forall \lambda > 0$ :*

$f(\lambda x) = \lambda^k f(x)$.

Let us mention some examples of homogeneous mappings.

- The function $(x_1, x_2) \mapsto \frac{x_1^3+x_2^3}{x_1^2+x_2^2}$ is homogeneous with degree 1 and continuous, but it is not linear.

- The function defined by:

$$(x_1, x_2) \mapsto \begin{cases} \frac{x_1^{1/2}+x_2^{1/2}}{x_1+x_2} & \text{if } x_1 + x_2 \neq 0 \\ 0 & \text{else} \end{cases} ,$$

  is homogeneous with degree $-\frac{1}{2}$ and not continuous.

When we take $m = n$ in the definition 4.1, we get vector fields. Let us denote by $\Phi^t(x)$ a solution of $\dot{x} = f(x)$ at time $t$ with $\Phi^0(x) = x$ (here $f(x)$ comes from (2) with $d = 0$).

- Let $A \in \mathbb{R}^{n\times n}$ and $f(x) = Ax$. Then $f$ is homogeneous with degree $1$ and we have $\Phi^t(x) = \exp(At)x$ thus $\Phi^t(\lambda x) = \lambda \Phi^t(x)$.

- Let $x$ be a scalar. The system $\dot{x} = -\mathrm{sign}(x)$ is homogeneous with degree $0$ and we have $\Phi^t(x) = \mathrm{sign}(x)\,(|x| - t)$ for $t \in [0, |x|]$ and $\Phi^t(x) = 0$ for $t > |x|$, thus $\Phi^t(\lambda x) = \lambda \Phi^{\frac{t}{\lambda}}(x)$.

**Proposition 4.2.** *[42, 18] Assume that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous with degree $k$ and admits a unique solution in forward time for each initial condition $x$. We have $\lambda \Phi^{\lambda^{k-1}t}(x) = \Phi^t(\lambda x)$.*

There exists another necessary and sufficient condition for homogeneity.vb

**Proposition 4.3 (Euler's Theorem on Homogenous Functions).** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable mapping. Then $f$ is homogeneous with degree $k$ iff for all $i \in \{1, \dots, m\}$*

$$\sum_{j=1}^{n} x_j \frac{\partial f_i}{\partial x_j}(x) = k f_i(x), \qquad \forall x \in \mathbb{R}^n.$$

Let us mention that the regularity of a homogeneous mapping $f$ is related to its degree:

- if $k < 0$ then $f$ is discontinuous at the origin;

- if $0 < k < 1$ then the Lipschitz condition is not satisfied by $f$ at $0$.

These conditions are necessary but not sufficient.

We will now be interested in homogeneous systems, e.g. systems $\dot{x} = f(x)$ where the vector field $f$ is homogeneous. Taking advantage of the symmetry of the solutions, as seen in the Proposition 4.2, we can now state stability results.

**Theorem 4.4.** *[42, 18] If the origin is a locally attractive*[4] *equilibrium of a homogeneous system, then the origin is globally asymptotically stable.*

**Theorem 4.5.** *[42] Consider a homogeneous system $\dot{x} = f(x)$ with a continuous $f$. Then the origin is globally asymptotically stable iff there exists a homogeneous and continuous function $V$, of class $C^1$ on $\mathbb{R}^n \setminus \{0\}$, s.t. $V$ and $-\dot{V}$ are positive definite.*

**Corollary 4.6.** *[18] Let $f_1, \ldots, f_p$ be continuous homogeneous vector fields with degree $k_1 < k_2 < \ldots < k_p$ and denote $f = f_1 + \cdots + f_p$. Assume moreover that $f(0) = 0$. If the origin is globally asymptotically stable under $f_1$ then the origin is locally asymptotically stable under $f$.*

See the book [18] for more details.

*4.2. Weighted homogeneity*

A generalized weight is a vector $\mathbf{r} = (r_1, \ldots, r_n)$ with $r_i > 0$. The dilation associated to the generalized weight $\mathbf{r}$ is the action of the group $\mathbb{R}_+ \setminus \{0\}$ on $\mathbb{R}^n$ given by:

$$\begin{array}{llll} \Lambda_{\mathbf{r}}: & \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n, \\ & (\lambda, x) & \longmapsto & \mathrm{diag}(\lambda^{r_i})x. \end{array}$$

For simplicity and shortness we omit the variable $\lambda$ when we write the dilation function, namely, let us denote $\Lambda_{\mathbf{r}} z := \Lambda_{\mathbf{r}}(\lambda, z)$, where $z \in \mathbb{R}^n$ and $\lambda > 0$.

**Definition 4.7.** *[44, 45] Let $\mathbf{r}$ be a generalized weight.*

- *A function $f$ is said to be $\mathbf{r}$-homogeneous with degree $m$ iff for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ we have $\lambda^{-m} f(\Lambda_{\mathbf{r}} x) = f(x)$;*

[4] In this paper we deal with the standard notions of attractivity, stability and asymptotic stability, see, for example, [18], [75].

- *A vector field $f$ is said to be $\mathbf{r}$-homogeneous with degree $m$ iff for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ we have $\lambda^{-m}\Lambda_{\mathbf{r}}^{-1} f(\Lambda_{\mathbf{r}} x) = f(x)$;*
- *A system $\dot{x} = f(x)$ is homogeneous iff $f$ is so.*

Considering functions, the classical homogeneity is in the scope of this definition. Indeed, taking $\mathbf{r} = (1, 1, \ldots, 1)$, the Definition 4.7 boils down to Definition 4.1. A mapping $g : \mathbb{R} \to \mathbb{R}$ is a homogeneous function (in the classical sense) of degree $m \in \mathbb{R}$ iff $g$ is a $(\mathbf{1})$-homogeneous function of degree $m$, or iff $g$ is a $(\mathbf{1})$-homogeneous vector field of degree $m - 1$. Let us remark also that a function or a vector field is $\mathbf{r}$-homogeneous with degree $m$ iff it is $(\alpha\mathbf{r})$-homogeneous with degree $\alpha m$ for all $\alpha > 0$.

**Example 4.8.** *Let us see some examples of weighted homogeneous objects.*

- *The function $f : x \to x_1 + x_2^2$ is $(\mathbf{2}, \mathbf{1})$-homogeneous of degree $2$;*
- *Let $\alpha_1, \ldots, \alpha_n$ be strictly positive. Consider the $n$-integrator:*

$$\begin{cases} \dot{x}_1 & = & x_2, \\ \vdots & & \vdots \\ \dot{x}_{n-1} & = & x_n, \\ \dot{x}_n & = & \sum_i k_i \lfloor x_i \rceil^{\alpha_i}. \end{cases}$$

*The system is homogeneous of degree $m$ w.r.t. $\mathbf{r} = (r_1, \ldots, r_n)$ iff the following relations hold:*

$$\begin{cases} r_i & = & r_n + (i-n)m, & \forall i \in \{1, \ldots, n\}, \\ r_i\alpha_i & = & r_n + m, & \forall i \in \{1, \ldots, n\}. \end{cases}$$

*As we have noticed previously, the weights and the degree are defined up to a multiplicative constant. Since the weights are strictly positive, we can fix $r_n = 1$. We easily see that this assumption forces $m$ to be greater than $-1$. For the sake of simplicity, we restrict ourselves to $m \in [-1, 0]$. The equations become:*

$$\begin{cases} r_i & = & 1 + (i-n)m, & \forall i \in \{1, \ldots n\}, \\ \alpha_i & = & \frac{1+m}{1+(i-n)m}, & \forall i \in \{1, \ldots n\}. \end{cases}$$

*If $m = -1$, then the vector field defining the system is discontinuous on each coordinate axis. If $m = 0$, then we recover a chain of integrators of $n^{th}$-order with a linear state feedback.*

Let us check the properties of classical homogeneity from Section 4.1 translated into the framework of weighted homogeneity.

**Proposition 4.9.** *[44] Assume that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$-homogeneous with degree $k$ and admits a unique solution in forward time for each initial condition and denote $\Phi^t(x)$ this solution at time $t \geq 0$ with initial condition $x$. We have $\Lambda_{\mathbf{r}} \Phi^{\lambda^k t}(x) = \Phi^t(\Lambda_{\mathbf{r}} x)$.*

The Proposition 4.9 corresponds to the Proposition 4.2 given for classical homogeneity.

Obviously, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$-homogeneous of degree $k$ iff each function $f_i : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $k + r_i$.

Let us see now the Euler theorem.

**Proposition 4.10.** *[44] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then $f$ is $\mathbf{r}$-homogeneous with degree $k$ iff for all $i \in \{1, \dots, n\}$*

$$\sum_{j=1}^{n} r_j x_j \frac{\partial f_i}{\partial x_j}(x) = (k + r_i) f_i(x), \qquad \forall x \in \mathbb{R}^n.$$

The theorems 4.4 and 4.5 remain true in the weighted homogeneity framework [44]. Let us set other results, which are fundamental in the study of finite-time stability (see Section 5 for the definition of this property).

**Theorem 4.11.** *[25] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous vector field of degree $k < 0$. If $f$ is locally attractive, then $f$ is globally finite-time stable (FTS).*

**Corollary 4.12.** *[25] Let $f_1, \dots, f_p$ be continuous homogeneous vector fields with degrees $k_1 < k_2 < \dots < k_p$ and denote $f = f_1 + \cdots + f_p$. Assume moreover that $f(0) = 0$. If the origin is globally asymptotically stable under $f_1$ then the origin is locally asymptotically stable under $f$. Moreover, if the origin is FTS under $f_1$ then the origin is FTS under $f$.*

*4.3. Geometric homogeneity and other extensions*

Consider the system $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with the following vector field:

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_1^2 \\ x_2 + x_1^2 + 2x_1 x_2 - 2x_1^3 \end{pmatrix}.$$

According to the previous definitions, it is not homogeneous. But setting $z = x_2 - x_1^2$, we get $\begin{pmatrix} \dot{x}_1 \\ \dot{z} \end{pmatrix} = \tilde{f}\begin{pmatrix} x_1 \\ z \end{pmatrix}$ with

$$\tilde{f}\begin{pmatrix} x_1 \\ z \end{pmatrix} = \begin{pmatrix} x_1 + z \\ z \end{pmatrix},$$

and in this form, the vector field is homogeneous. Since homogeneity is a kind of symmetry, it should be invariant under a change of coordinates. This motivates for a geometric, coordinate-free definition of homogeneity [66, 51, 69].

**Definition 4.13.** *A vector field $\nu : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Euler if $\nu$ is $\mathcal{C}^1$, complete*[5] *and $-\nu$ is globally asymptotically stable. We will denote $\Upsilon$ the flow of $\nu$.*

**Definition 4.14.** *Let $\nu$ be an Euler vector field. A function $f$ is said to be $\nu$-homogeneous of degree $m$ iff for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ we have:*

$$e^{-ms} f(\Upsilon^s(x)) = f(x). \tag{18}$$

*A vector field $f$ is said to be $\nu$-homogeneous of degree $m$ iff for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ we have:*

$$e^{-ms} \left(\tfrac{\partial}{\partial x}\Upsilon^s(x)\right)^{-1} f(\Upsilon^s(x)) = f(x). \tag{19}$$

Let us consider the generalized weight $\mathbf{r} = (r_1, \ldots, r_n)$, with $r_i > 0$. In a fixed basis, set $\nu(x) = r_1 x_1 \frac{\partial}{\partial x_1} + \ldots + r_n x_n \frac{\partial}{\partial x_n}$. This vector field is clearly Euler. We easily compute the flow $\Upsilon^s(x) = \mathrm{diag}(e^{r_i s})x$. Let $f$ be a vector field. According to (19) $f$ is $\nu$-homogeneous of degree $m$ iff for all $s \in \mathbb{R}$ we have:

$$e^{-ms}\mathrm{diag}(e^{-r_i s}) f(\mathrm{diag}(e^{r_i s})x) = f(x). \tag{20}$$

Setting $\lambda = e^s$, (20) is equivalent to Definition 4.7:

$$\lambda^{-m}\Lambda_{\mathbf{r}}^{-1} f(\Lambda_{\mathbf{r}} x) = f(x). \tag{21}$$

Thus we recover the weighted homogeneity, which appears to be a particular case of the geometric homogeneity in a fixed basis.

Similarly to classical and weighted homogeneity, there exist equivalent conditions for a vector field or a function to be homogeneous assuming regularity properties.

[5] A vector field is **complete** if its flow curves exist for all time.

**Theorem 4.15.** *[51] Let $\nu$ be an Euler vector field and let $f$ be a $\mathcal{C}^1$ vector field. The three following assertions are equivalent:*

- *$f$ is $\nu$-homogeneous of degree $m$;*
- *for all $s, t \in \mathbb{R}$ we have $\Phi^t \circ \Upsilon^s = \Upsilon^s \circ \Phi^{e^{ms}t}$ where $\Phi^t(x)$ denotes the flow of $f$;*
- *$[\nu, f] := \frac{\partial f}{\partial x}\nu - \frac{\partial \nu}{\partial x} f = mf$.*

*Let $v$ be a $\mathcal{C}^1$ function. Then $v$ is $\nu$-homogeneous of degree $m$ iff $\mathcal{L}_\nu v = mv$.*

This theorem is the equivalent for geometrical homogeneity of propositions 4.9 and 4.10.

All the stability results that have been presented for weighted homogeneity still hold for geometrical homogeneity: Theorems 4.4, 4.5, 4.11 and 4.12 remain true in this setting.

Another extension of homogeneity is the concept of homogeneous approximation. We will only present homogeneous approximation at $0$ and at $\infty$ [59].

**Definition 4.16.** *[59] Let $r_0$ be a generalized weight, $\mathfrak{d}_0 \in \mathbb{R}$ and $f_0$ be a function (resp. a vector field). A function (resp. a vector field) $f$ is said to be homogeneous in the $0$-limit with associated triple $(r_0, \mathfrak{d}_0, f_0)$ if*

$$\lim_{\lambda \to 0} \sup_{x \in K} \|\lambda^{-\mathfrak{d}_0} f(\Lambda_{r_0} x) - f_0(x)\| = 0,$$

*(resp. if $\lim_{\lambda \to 0} \sup_{x \in K} \|\lambda^{-\mathfrak{d}_0} \Lambda_{r_0}^{-1} f(\Lambda_{r_0} x) - f_0(x)\| = 0$), for all compact subsets $K$ of $\mathbb{R}^n \setminus 0$.*

**Definition 4.17.** *[59] Let $r_\infty$ be a generalized weight, $\mathfrak{d}_\infty \in \mathbb{R}$ and $f_\infty$ be a function (resp. a vector field). A function (resp. a vector field) $f$ is said to be homogeneous in the $\infty$-limit with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$ if*

$$\lim_{\lambda \to +\infty} \sup_{x \in K} \|\lambda^{-\mathfrak{d}_\infty} f(\Lambda_{r_\infty} x) - f_\infty(x)\| = 0,$$

*(resp. if $\lim_{\lambda \to +\infty} \sup_{x \in K} \|\lambda^{-\mathfrak{d}_\infty} \Lambda_{r_\infty}^{-1} f(\Lambda_{r_\infty} x) - f_\infty(x)\| = 0$), for all compact subsets $K$ of $\mathbb{R}^n \setminus 0$.*

**Definition 4.18.** *[59] A function or a vector field is said to be homogeneous in the bi-limit if it is homogeneous in the $0-$limit and in the $\infty$-limit simultaneously.*

Let us show how these definitions are used.

**Theorem 4.19.** *[59] Let $f$ be a vector field which is homogeneous in the $0$-limit with associated triple $(r_0, \mathfrak{d}_0, f_0)$. Assume that $f_0$ is globally asymptotically stable. Then $f$ is locally asymptotically stable.*

*Let $f$ be a vector field which is homogeneous in the $\infty$-limit with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$. Assume that $f_\infty$ is GAS. Then there exists an invariant compact subset containing the origin which is globally asymptotically stable for the system $\dot{x} = f(x)$.*

*4.4. Homogeneity and differential inclusions*

When handling discontinuous right-hand side, we usually apply the differential inclusion theory. In this context, it is natural to wonder whether homogeneity can be extended in a useful way. We propose here a definition which is a natural extension of Definition 4.14, and we show that basic properties of homogeneity are preserved in this generalized context. Some related results were obtained in [22], [76].

In this subsection, $\mathbf{r}$ is a generalized weight. We consider the autonomous differential inclusion defined by the set-valued map $F$:

$$\dot{x} \in F(x), \tag{22}$$

where $F$ is coming from (3) with $d = 0$.

**Definition 4.20.** *[76] A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is $\mathbf{r}$–homogeneous with degree $m \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$ we have:*

$$\lambda^{-m}\Lambda_{\mathbf{r}}^{-1}F(\Lambda_{\mathbf{r}}x) = F(x).$$

*The system (22) is $\mathbf{r}$–homogeneous of degree $m$ if the set-valued map $F$ is homogeneous of degree $m$.*

**Proposition 4.21.** *[76, 77] Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued $\mathbf{r}$-homogeneous mapping with degree $m$. Then for all $x \in \mathbb{R}^n$ and any solution $\Phi^t(x)$ of the system (22) with initial condition $x$ and all $\lambda > 0$, the absolute continuous curve $t \mapsto \Lambda_{\mathbf{r}}(\lambda)\Phi^{\lambda^m t}(x)$ is a solution of the system (22) with initial condition $\Lambda_{\mathbf{r}}(\lambda)x$.*

This proposition is the extension of Proposition 4.9.

Similarly to the usual setting, a lot of properties can be extended from the sphere to everywhere outside the origin.

**Proposition 4.22.** *[77] Let $F$ be a $\mathbf{r}$-homogeneous set-valued map with degree $m$. Then $F(x)$ is compact for all $x \in \mathbb{R}^n \setminus \{0\}$ iff $F(x)$ is compact for all $x \in \mathbb{S}$. The same property hold for convexity or upper semi-continuity.*

In many situations, the set-valued map $F$ comes from the Filippov regularization procedure of a discontinuous vector field $f$. Suppose that we have a vector field $f$, which is homogeneous in the sense of Definition 4.7. If we apply the regularization procedure, is the homogeneity property preserved? The answer is positive.

**Proposition 4.23.** *[77] Let $f$ be a vector field and $F$ be the associated set-valued map. Suppose $f$ is $\mathbf{r}$-homogeneous of degree $m$. Then $F$ is $\mathbf{r}$-homogeneous of degree $m$.*

In the sequel we will say that $F$ satisfies the *standard assumptions* if $F$ is upper semi-continuous, and for all $x \in \mathbb{R}^n$, $F(x)$ is not empty, compact and convex.

The following theorem asserts that a strongly globally asymptotically stable system admits a homogeneous Lyapunov function. This result is a generalization of Theorem 4.5.

**Theorem 4.24.** *[77] Let $F$ be a $\mathbf{r}$-homogeneous set-valued map with degree $m$, satisfying the standard assumptions. Then the following statements are equivalent:*

- *The system (22) is strongly[6] globally asymptotically stable.*
- *For all $k > \max(-m, 0)$, there exists a pair $(V, W)$ of continuous functions, such that:*
  1. *$V \in \mathcal{C}^\infty(\mathbb{R}^n)$, $V$ is positive definite and homogeneous with degree $k$;*
  2. *$W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, $W$ is strictly positive outside of the origin and homogeneous of degree $k + m$;*
  3. *$\max_{v \in F(x)} DV(x)v \leq -W(x)$ for all $x \neq 0$.*

From this theorem, one can deduce some useful results about finite-time stability that will be discussed in Section 5.

[6]Because of non-uniqueness of solutions of DI, we need a careful recast of the definitions of convergence, stability and even equilibrium. The revised definitions have to be split into two parts: if one solution has a given property, we will say that this property is weak; if all the solutions have this property, we will say that this property is strong.

**Corollary 4.25.** *[77] Let $F$ be a $\mathbf{r}$-homogeneous set-valued map with degree $m < 0$, satisfying the standard assumptions. Assume also that $F$ is strongly globally asymptotically stable. Then $F$ is strongly globally FTS and the settling-time function is continuous at zero and locally bounded.*

It has been shown in [20] that under the assumptions of homogeneity (with negative degree), continuity of the right-hand side and forward unicity of solutions, the settling-time function is continuous. Let us emphasize that these assumptions cannot be removed in our context, since the settling-time function is *not* continuous in general. See for example [78] or the following example.

**Example 4.26 (A counterexample to the second statement of Theorem 1 from [76]).** *Consider the system defined on $\mathbb{R}^2$ by:*

$$\dot{x} = -(\mathrm{sign}(x_1) + 2)\frac{x}{\|x\|}.$$

*This system is clearly strongly globally FTS and homogeneous with a negative degree. A simple computation shows that the settling-time function is:*

$$T(x) = \begin{cases} \|x\| & x_1 \geq 0 \\ \frac{\|x\|}{3} & x_1 < 0 \end{cases},$$

*which is discontinuous on $x_1 = 0$.*

## 5. Finite Time Stability

Note that initially the concept of “finite-time stability” has been introduced for investigation of behaviour of dynamical systems on finite intervals of time in [79, 80, 81], see also a recent survey [82]. That property is close to the practical stability on finite-time intervals (i.e. boundedness of solutions on bounded intervals of time). Another context for this notion has been proposed in 60s by E. Roxin [15] and developed in [30, 17], where a particular attention is paid to the time of convergence for trajectories to a steady state. In this paper we follow the latter concept only, see also a more complete survey in [83].

We are also interested in a particular case of the finite-time stability concept called “fixed-time stability” [84], which deals with the systems whose solutions have a common finite upper bound for time of convergence to the origin for all initial conditions. An example of such a system is given in the next section.

### 5.1. *Preliminary remarks*

To illustrate the features of finite-time stable systems consider an example. Let $\alpha \in ]0,1[$, then the system

$$\dot{x} = -\lceil x \rfloor^{\alpha},\ x \in \mathbb{R}, \tag{23}$$

for any initial condition $x_0 \in \mathbb{R}$ and $t \geq 0$ has the solution:

$$\Phi^t(x_0) = \begin{cases} s(t, x_0) & \text{if} \quad 0 \leq t \leq T(x_0) \\ 0 & \text{if} \quad t > T(x_0) \end{cases}, \tag{24}$$

with $s(\tau, x) = \text{sign}(x)\left(|x|^{1-\alpha} - \tau(1-\alpha)\right)^{\frac{1}{1-\alpha}}$ and $T(x) = \frac{|x|^{1-\alpha}}{1-\alpha}$, thus the solutions reach the origin in the finite time $T(x_0)$. For this example we can notice that:

- Finite Time Stability (FTS) is equivalent to an infinite eigenvalue assignation for the closed-loop system at the origin, therefore the right-hand side of the ordinary differential equation *cannot be locally Lipschitz at the origin*;

- there exists the *settling time function* $T(x_0)$ that determines the time for a solution to reach the equilibrium, this function depends on the initial condition of a solution.

The main issue with $T$ is its continuity at the origin. For continuous systems, the continuity of $T$ at 0 is equivalent to the continuity of $T$ everywhere [30]. The bi-limit homogeneity application allows us to have a globally bounded $T$ (in some special cases), which means that in practice one gets a fixed time of convergence to the origin for all initial conditions. For an example consider a simple system

$$\dot{x} = -\lceil x \rfloor^{1/2} - \lceil x \rfloor^{3/2},\ x \in \mathbb{R}, \tag{25}$$

which for any $x_0 \in \mathbb{R}$ and $t \geq 0$ has the solution

$$\Phi^t(x_0) = \begin{cases} v(t, x_0) & \text{if} \quad 0 \leq t \leq T(x_0) \\ 0 & \text{if} \quad t > T(x_0) \end{cases}, \tag{26}$$

where $v(\tau, x) = \tan[\arctan(\sqrt{|x|}) - 0.5\tau]^2 \text{sign}(x)$ and the settling time function $T(x) = 2\arctan(\sqrt{|x|})$. As we can conclude from these expressions, if $|x| \to +\infty$, then $T(x) \to \pi$, therefore all solutions approach the origin in a time less than $\pi$.

A special attention in this tutorial is devoted to discontinuous systems, since they constitute the main class of sliding-mode systems. If a system is discontinuous at the origin, then by an analogy with the previous examples it can be FTS, however the definition of solutions for such systems, as well as the settling-time functions, is more delicate as we will see in the next section.

*5.2. Finite-time stability definitions*

The main definitions and properties for FTS are recalled now following the theory emerged in [30, 17]. First we give the definition for the continuous autonomous system (2) with $d = 0$, where $f$ is a continuous but not necessarily a Lipschitz function at the origin (so it may happen that any solution of the system converges to zero in a finite time). Due to the non Lipschitz condition on the right-hand side of (2), the backward uniqueness of solutions may be lost, and thus we only assume forward uniqueness.

This section uses the notations $\mathcal{K}, \mathcal{CL}$ and $\mathcal{GKL}$ (see Section 2), which are usual for ISS theory.

**Definition 5.1.** *[85] The system (2) is said to be* **finite-time stable** ***(FTS)*** *at the origin (on an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *there exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$ we have $\|\Phi^t(x_0)\| \leq \delta(\|x_0\|)$ for all $t \geq 0$.*
2. *there exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_+$ such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $\Phi^t(x_0)$ is defined, unique, nonzero on $[0, T(x_0))$ and $\lim\limits_{t \to T(x_0)} \Phi^t(x_0) = 0$.*

*If $\mathcal{V} = \mathbb{R}^n$, then the system is called globally FTS. Furthermore, if the property 1) is fulfilled only, then the origin of the system (2) is said to be finite-time attractive.*

It is possible to show that if the system (2) is FTS, then it is also asymptotically stable with a continuous flow for all $x_0 \in \mathcal{V}$. In this case, at the origin the system has a unique solution $x(t, 0) = 0$ for all $t \geq 0$, thus we can extend the definition above taking $T(0) = 0$.

Now consider the DI (3) with $d = 0$, for an initial condition $x_0 \in \mathcal{V} \subset \mathbb{R}^n$ denote a corresponding solution as $\Phi^t(x_0)$, which under conditions introduced in Section 3 is defined at least locally. The set of solutions $\Phi^t(x_0)$ corresponding to the common initial condition $x_0$ will be denoted as $\mathbb{S}(x_0)$. Let $\mathbb{S} = \cup_{x_0 \in \mathcal{V}} \mathbb{S}(x_0)$ be the set of all possible solutions of the differential inclusion (3) starting in $\mathcal{V}$.

**Definition 5.2.** *[86] The system (3) is said to be* **finite-time stable** *at the origin (on an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *there exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$ we have $\|\Phi^t(x_0)\| \leq \delta(\|x_0\|)$ for all $t \geq 0$ and all $\Phi^t(x_0) \in \mathbb{S}(x_0)$.*
2. *there exists a function $T_0 : \mathcal{V}\backslash\{0\} \to \mathbb{R}_+$ such that for all $x_0 \in \mathcal{V}$ and all $\Phi^t(x_0) \in \mathbb{S}(x_0)$, $\lim_{t\to T_0(x_0)} \Phi^t(x_0) = 0$.*

*If $\mathcal{V} = \mathbb{R}^n$, then the system is called globally FTS.*

If the system (3) is finite-time stable then there exists the settling-time function of the system (3) defined by $T(x) = \sup_{\Phi^t(x)\in\mathbb{S}(x)} \inf_{\tau\geq 0:\Phi^\tau(x)=0} \tau$ such that $T(x) < +\infty$ for any $x \in \mathcal{V}$.

**Remark 5.3.** *For DI it is possible to define the weak and strong notions of FTS. The strong notion has been defined in the definition above, it deals with all solutions originated from a given initial condition. The weak notion can be defined assuming that for all $x_0 \in \mathcal{V}$ there exists at least one solution $\Phi^t(x_0) \in \mathbb{S}(x_0)$ converging to zero in a finite time.*

Finally, the fixed-time stability (FxTS) is a particular case of the FTS property [84], for brevity we will provide the definition for the DI (3) only.

**Definition 5.4.** *The system (3) is said to be* **FxTS** *at the origin if it is globally FTS and the settling-time function $T$ is bounded, i.e. $T(x) \leq T_{\max}$ for some $T_{\max} > 0$ and for all $x \in \mathbb{R}^n$.*

By its definition the FxTS systems always have the settling-time function $T$.

*5.3. Lyapunov function characterization of FTS*

First we introduce several sufficient and equivalent Lyapunov characterizations for the continuous system (2) with $d = 0$. For this purpose we will need

**Definition 5.5.** *A class $\mathcal{K}$ function $r$ belongs to* class $\mathcal{KI}$ *if $r \in \mathcal{CL}$ and there exists $\epsilon > 0$ such that:*

$$\int_0^\epsilon \frac{dz}{r(z)} < +\infty.$$

Let $V : \mathcal{V} \to \mathbb{R}_+$ be a Lyapunov function (it is continuously differentiable, positive definite and radially unbounded) and $r$ be from the class $\mathcal{KI}$, then the first condition is that for all $x \in \mathcal{V}$:

$$\dot{V}(x) \leq -r[V(x)]. \tag{27}$$

The existence of such a pair $(V, r)$ is also a necessary condition for FTS in some particular cases (see [17] and [21]).

A particular form of (27) is described as follow:

$$\dot{V}(x) \leq -aV(x)^{\alpha},\ a > 0,\ \alpha \in (0, 1). \tag{28}$$

In this case $r(s) = as^{\alpha}$ and it is obviously from the class $\mathcal{KI}$.

**Theorem 5.6.** *[27] Consider the system (2) with the forward uniqueness of solutions outside the origin, then the following properties are equivalent:*

$(i)$ *the origin of the system (2) is **FTS** with a continuous settling-time function $T$,*

$(ii)$ *there is a Lyapunov function $V$ satisfying the condition (28) and*

$$T(x) \leq \frac{V(x)^{1-\alpha}}{c(1-\alpha)}$$

$(iii)$ *there is a Lyapunov function $V$ and a class $\mathcal{KI}$ function $r$ satisfying the condition (27).*

The requirement of continuity of the settling-time function $T$, which is needed to prove existence of a Lyapunov function for an FTS system only, becomes redundant if the system (2) is homogeneous [20].

Now consider the DI (3) with $d = 0$, for which a sufficient stability condition can be formulated only. In this case the condition (28) has to be rewritten as follows

$$\sup_{\varphi \in F(x,0)} DV(x)\varphi \leq -r[V(x)], \tag{29}$$

where as before $r \in \mathcal{KI}$, thus we do not assume anymore that $V$ is continuously differentiable.

**Theorem 5.7.** *[86] Let $0 \in F(0, 0)$ in the system (3). If there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ (locally Lipschitz continuous, positive definite and radially unbounded) that verifies the condition (29), then the origin of (3) is FTS. Moreover, the settling-time function $T$ of (3) satisfies:*

$$T(x) \leq \int_0^{V(x)} \frac{dz}{r(z)}.$$

For the FxTS property the following sufficient Lyapunov conditions have been proposed in [84]:

**Theorem 5.8.** *Let there exist a Lipschitz continuous, positive definite and radially unbounded Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x \in \mathbb{R}^n$*:*

$$\sup_{\varphi \in F(x,0)} DV(x)\varphi \leq -[\alpha V^p(x) + \beta V^q(x)]^k,$$

*for some strictly positive* $\alpha$, $\beta$, $p$, $q$, $k$ *with* $pk < 1$ *and* $qk > 1$*, then the system (3) with* $d = 0$ *is FxTS and*

$$T(x) \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)}, \ ;\forall x \in \mathbb{R}^n. \tag{30}$$

As we can conclude from this result, due to the property $pk < 1$, locally around zero the function $V$ is an FTS Lyapunov function for (3). The time estimate (30) follows from an analogous estimate of Theorem 5.7 under substitution in the integral limits $V(x) = +\infty$ for $r(z) = [\alpha z^p + \beta z^q]^k$.

Now let us pass to the case where $d \neq 0$ and analyse the influence of $d$ on the FTS property.

### *5.4. Robustness of FTS*

In this subsection we will consider the problem of robustness of FTS with respect to external bounded inputs. Mainly the presentation will follow the definitions given in the paper [87], devoted to an extension of the input-to-state stable (ISS) systems theory [88] for the FTS concept. In [87] all definitions and results are introduced for continuous time-varying nonlinear systems, in this work for brevity we will give the extension only for DI (3) taking in mind [89, 90].

The definition of FTS for (3) with $d = 0$ has been given before. In this section we will be interested in two cases for $d \neq 0$: $d$ is an essentially bounded and Lebesgue measurable function of time, i.e. $d \in \mathcal{L}_\infty$, and $d \in \mathcal{D} = \{d \in \mathcal{L}_\infty : ||d||_\infty \leq D_{\max}\}$ for some $0 < D_{\max} < +\infty$. We will use a notion of extended Filippov's solutions from [89] and assume that for the chosen class of inputs ($\mathcal{D}$ or $\mathcal{L}_\infty$) the solutions in $\mathbb{S}$ are defined for all $t \geq 0$. Then we have the following list of robust stability properties.

**Definition 5.9.** *The system (3) is called* uniformly finite-time stable (UFTS) *with respect to* $d \in \mathcal{D}$ *in an open neighbourhood* $\mathcal{V}$ *of the origin if the following properties hold for all* $d \in \mathcal{D}$*:*

- uniform stability, *i.e. there exists* $\delta \in \mathcal{K}$ *such that* $\|\Phi^t(x_0)\| \le \delta(\|x_0\|)$ *for all* $t \ge 0$*, any* $x_0 \in \mathcal{V}$ *and all* $\Phi^t(x_0) \in \mathbb{S}(x_0)$*;*

- uniform finite-time attractivity, *i.e. there exists a function* $T : \mathcal{V}\backslash\{0\} \to \mathbb{R}_+$ *such that for all* $x_0 \in \mathcal{V}$ *and all* $\Phi^t(x_0) \in \mathbb{S}(x_0)$, $\lim_{t\to T(x_0)} \Phi^t(x_0) = 0$.

$T$ *is called the* uniform settling-time function *of the system (3). The system (3) with* $d \in \mathcal{D}$ *is called* globally UFTS *if* $\mathcal{V} = \mathbb{R}^n$.

A similar stability analysis of DI in the presence of inputs has been performed in [76, 22].

**Definition 5.10.** *The system (3) is called* locally finite-time input-to-state stable (finite-time ISS) *if there exist some* $\mathcal{D}$ *and* $\mathcal{V} \subset \mathbb{R}^n$, $0 \in \mathcal{V}$ *such that for all* $x_0 \in \mathcal{V}$ *and* $d \in \mathcal{D}$ *the estimate*

$$\|\Phi^t(x_0)\| \le \beta(\|x_0\|, t) + \gamma(||d||_{[0,t)})$$

*is satisfied for all* $t \ge 0$ *and* $\Phi^t(x_0) \in \mathbb{S}(x_0)$ *for some* $\beta \in \mathcal{GKL}$ *and* $\gamma \in \mathcal{K}$*. The system (3) is called* finite-time ISS *if* $\mathcal{V} = \mathbb{R}^n$ *and* $\mathcal{D} = \mathcal{L}_\infty$.

**Definition 5.11.** *The system (3) is called* finite-time integral input-to-state stable (finite-time iISS) *if for all* $x_0 \in \mathbb{R}^n$ *and* $d \in \mathcal{L}_\infty$ *the estimate*

$$\alpha(\|\Phi^t(x_0)\|) \le \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(\tau)\|)d\tau$$

*is satisfied for all* $t \ge 0$ *and* $\Phi^t(x_0) \in \mathbb{S}(x_0)$ *for some* $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{GKL}$ *and* $\gamma \in \mathcal{K}$.

The UFTS property means that the system stability is not influenced by the inputs $d$ from $\mathcal{D}$. The finite-time ISS and finite-time iISS quantify the deviations of trajectories for bounded and integrally bounded inputs respectively.

The Lyapunov characterization of uniform UFTS coincides with the one introduced for FTS before, the only additional requirement is that the condition (29) is satisfied for all $d \in \mathcal{D}$. To introduce the Lyapunov functions for finite-time ISS and finite-time iISS we will need the following property: for two functions $a_1, a_2 : \mathbb{R}^n \to \mathbb{R}_+$ the relation $a_1(x) \succ a_2(x)$ means that there exists $\epsilon > 0$ such that $a_1(x) \ge a_2(x)$ for all $\|x\| \le \epsilon$.

**Definition 5.12.** *A locally Lipschitz continuous function* $V:\mathbb{R}^n\to\mathbb{R}_+$ *is called* finite-time ISS Lyapunov function *for the system (3) if there are* $\alpha_1,\,\alpha_2,\,\alpha_3\in\mathcal{K}_\infty$ *and* $\sigma\in\mathcal{K}$ *such that for all* $x\in\mathbb{R}^n$ *and* $\mathrm{d}\in\mathbb{R}^p$

$$\alpha_1(\|x\|)\le V(x)\le\alpha_2(\|x\|),$$

$$\sup_{\varphi\in F(x,\mathrm{d})} DV(x)\varphi\le-\alpha_3(\|x\|)+\sigma(\|d\|), \tag{31}$$

*with* $\alpha_3(\|x\|)\succ aV^\alpha(x)$ *for some* $a>0$ *and* $0<\alpha<1$.

As in [89, 90] an equivalent finite-time ISS Lyapunov function definition can be used instead of (31):

$$\|x\|\ge\chi(\|\mathrm{d}\|)\to\sup_{\varphi\in F(x,\mathrm{d})} DV(x)\varphi\le-\alpha_3(\|x\|)$$

for all $x\in\mathbb{R}^n$ and $\mathrm{d}\in\mathbb{R}^p$ with some $\alpha_3\in\mathcal{K}_\infty$, $\chi\in\mathcal{K}$.

**Definition 5.13.** *A locally Lipschitz continuous function* $V:\mathbb{R}^n\to\mathbb{R}_+$ *is called* finite-time iISS Lyapunov function *for the system (3) if there are* $\alpha_1,\,\alpha_2\in\mathcal{K}_\infty$, $\sigma\in\mathcal{K}$ *and a positive definite continuous function* $\alpha_3:\mathbb{R}_+\to\mathbb{R}_+$ *such that for all* $x\in\mathbb{R}^n$ *and* $\mathrm{d}\in\mathbb{R}^p$

$$\alpha_1(\|x\|)\le V(x)\le\alpha_2(\|x\|),$$

$$\sup_{\varphi\in F(x,\mathrm{d})} DV(x)\varphi\le-\alpha_3(\|x\|)+\sigma(\|\mathrm{d}\|),$$

*with* $\alpha_3(\|x\|)\succ aV^\alpha(x)$ *for some* $a>0$ *and* $0<\alpha<1$.

The definition of the finite-time ISS Lyapunov function is given for the global case, the local one can be obtained for $x\in\mathcal{V}$ and $d\in\mathcal{D}$. The main result of this subsection is as follows.

**Theorem 5.14.** *[91] If for the system (3) there exists a finite-time ISS (finite-time iISS) Lyapunov function, then it is finite-time ISS (finite-time iISS).*

The proof follows from the well known results on ISS/iISS and [87, 89, 90].

All definitions and the result of this subsection are also valid for ODEs (2), where the set $\mathbb{S}(x_0)$ for any $x_0\in\mathbb{R}^n$ is composed by the unique solution $\Phi^t(x_0)$, and all formulations above have to be simplified accordingly.

Finally we would like to describe the link between the robust stability and homogeneity, which connects the material presented in Sections 4 and 5. In the UFTS case, if the degree $k$ of homogeneity of the DI (3) equals to $-\min_{1\leq i\leq n} r_i$, then there exists a set $\mathcal{D} \subset \mathcal{L}_\infty$ such that the system (3) is UFTS with respect to an additive input $d \in \mathcal{D}$. For ODE (2) and ISS property, some results have been proposed in [92, 93]. An extension of those results for (2) with inclusion of iISS property has been recently proposed by the authors [94, 95]. Define $\tilde{f}(x,d) = [f(x,d)^T\ 0_p]^T \in \mathbb{R}^{n+p}$, it is an extended auxiliary vector field for the system (2).

**Theorem 5.15.** *[94, 95] Let the vector field $\tilde{f}$ be homogeneous with the weights $\mathbf{r} = [r_1,\dots,r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1,\dots,\tilde{r}_p] \geq 0$ with a degree $k \geq -\min_{1\leq i\leq n} r_i$, i.e. $f(\Lambda_r x, \Lambda_{\tilde{r}}\mathrm{d}) = \lambda^k \Lambda_r f(x,\mathrm{d})$. Assume that the system (2) is globally asymptotically stable for $d = 0$, then the system (2) is*

ISS *if $\tilde{r}_{\min} > 0$, where $\tilde{r}_{\min} = \min_{1\leq j\leq p} \tilde{r}_j$,*

iISS *if $\tilde{r}_{\min} = 0$ and $k \leq 0$.*

Interestingly to note, that FTS and iISS have a similar restriction on the degree of homogeneity: it has to be negative (non positive for iISS). For example, according to this theorem, if for some locally Lipschitz continuous and homogeneous $f$ (with the degree $k$ and the weights $\mathbf{r}$) we have $f(x,\mathrm{d}) = f(x) + \mathrm{d}$, i.e. $\mathrm{d}$ is an additive disturbance, then the system (2) is ISS if $k > -\min_{1\leq i\leq n} r_i$, and it is iISS for $k = -\min_{1\leq i\leq n} r_i$. If $f(x,\mathrm{d}) = f(x+\mathrm{d})$ and $\mathrm{d}$ is a measurement noise, then the system is always ISS. An extension of this result for the DI (3) needs some additional conditions.

Denote an extended discontinuous function $\tilde{F}(x,\mathrm{d}) = [F(x,\mathrm{d})^T\ 0_p]^T$.

**Theorem 5.16.** *[91] Let the discontinuous function $\tilde{F}$ be homogeneous with the weights $\mathbf{r} = [r_1,\dots,r_n] > 0$, $\tilde{\mathbf{r}} = [\tilde{r}_1,\dots,\tilde{r}_p] \geq 0$ with a degree $k \geq -\min_{1\leq i\leq n} r_i$, i.e. $F(\Lambda_r x, \Lambda_{\tilde{r}}\mathrm{d}) = \lambda^k \Lambda_r F(x,\mathrm{d})$. Assume that the system (3) is globally asymptotically stable for $d = 0$. Let also*

$$\|F(y,\mathrm{d}) - F(y,0)\|_H \leq \sigma(\|\mathrm{d}\|), \quad \forall y \in S_r,\ \sigma(s) = \begin{cases} c\, s^{\varrho_{\min}} & \textit{if } s \leq 1 \\ c\, s^{\varrho_{\max}} & \textit{if } s > 1 \end{cases}$$

*for some $c > 0$* and *$\varrho_{\max} \geq \varrho_{\min} > 0$. Then the system (3) is*

ISS *if $\tilde{r}_{\min} > 0$, where $\tilde{r}_{\min} = \min_{1\leq j\leq p} \tilde{r}_j$,* iISS *if* $\tilde{r}_{\max}\varrho_{\min} - \mu \leq \nu \leq \tilde{r}_{\min} = 0$*, where* $\tilde{r}_{\max} = \max_{1\leq j\leq p} \tilde{r}_j$*.*

The conditions of Theorem 5.16 mainly repeats the conditions of Theorem 5.15. If $k < 0$, then under conditions of Theorem 5.16 the system (3) is finite-time ISS/finite-time iISS.

As we can conclude from these results, for homogeneous system (2) its robustness (ISS or iISS property) is a function of its degree of homogeneity. Thus to verify robustness of the system FTS with respect to an external input it is enough to compute its degree of homogeneity and perform some other algebraic operations, which is a big advantage of homogeneity.

## 6. Sliding mode control

The word "sliding mode" was introduced in the paper of Nikol'ski, 1934 [1], in the context of relay control systems. It may happen that the relay control as a function of the system state switches at high (theoretically infinite) frequency. This motion of the closed loop system was called *sliding mode* [2]. Later evolution of the sliding mode control theory and its applications had required more detailed specification of the sliding motion, which may have different order of "smoothness". The modern control theory calls this extended concept by the high-order sliding mode. There exist some definitions of the high-order sliding mode [14], [96], [97]. Since this survey essentially treats finite-time convergence of the sliding mode control systems we will follow definitions that explicitly request this property for sliding manifold (see, for example, [96], [97]).

**Definition 6.1.** *Let* $s : \mathbb{R}^n \to \mathbb{R}$ *be a* $\mathcal{C}^r$ *function, the Lie derivatives* $\mathcal{L}^k_{f(x,0)}s(x)$*,* $k = 1, 2, ..., r-1$ *are continuous functions of the system state* $x \in \mathbb{R}^n$ *and* $\mathcal{L}^r_{f(x,0)}s(x)$ *is the discontinuous one.*

*The surface*

$$\mathcal{S}_r := \left\{x \in \mathbb{R}^n : \mathcal{L}^i_{f(x,0)}s(x) = 0, i = 0, 1, ..., r-1\right\} \tag{32}$$

*is said to be* ***the*** $r$***-th order UFTS sliding surface*** *of the system* (2) *iff it is nonempty, invariant and uniformly finite-time stable with respect to* $d \in \mathcal{D}$ *for some* $\mathcal{D}$ *(see Definition 5.9).*

*The motion of the system* (2) *on the surface* $\mathcal{S}_r$ *is called by* ***the*** $r$***-th order UFTS sliding mode****.*

If $s$ is a vector-valued function, the order of sliding mode can be introduced for each component. The most of sliding mode control systems considered in the literature satisfy Definition 6.1. An alternative concept of sliding modes with finite-time convergence was introduced in [98, 99, 100] and called *terminal sliding mode*. Below we study ISS and iISS properties of sliding mode systems, which satisfy Definition 6.1.

For simplicity and shortness let us denote the $r$-th order UFTS sliding mode by $r$-SM.

### *6.1. First order sliding mode control*

Consider the nonlinear system that is affine with respect to control

$$\dot{x} = f(x, d(t)) + g(x)u, \tag{33}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the vector of control inputs, $d(t) \in \mathbb{R}^p$ is the vector of input disturbances, the vector-valued function $f : \mathbb{R}^{n+p} \to \mathbb{R}^n$ and the matrix-valued function $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be continuous.

In general, the sliding surface is defined by a $\mathcal{C}^1$ vector-valued function $s : \mathbb{R}^n \to \mathbb{R}^m$:

$$\mathcal{S}_1 = \{x \in \mathbb{R}^n : s(x) = 0\}. \tag{34}$$

#### *6.1.1. The 1-SM design*

Let the matrix $\mathcal{L}_g s(x)$ be invertible for all $x \in \mathbb{R}^n$ and the function $f$ satisfies the inequality

$$\left\| \frac{\partial s}{\partial x}(f(x, \mathrm{d}) - f(x, 0)) \right\| \leq k_1 + k_2\|\mathrm{d}\| \quad \text{for all } x \in \mathbb{R}^n \text{ and all } \mathrm{d} \in \mathbb{R}^p, \tag{35}$$

where $k_1, k_2 \in \mathbb{R}_+$.

If we select the sliding mode control according the standard design procedure [2, 12]:

$$\begin{aligned} u(x) = -(\mathcal{L}_g s(x))^{-1} \left(\mathcal{L}_{f(x,0)} s(x) - k\mathrm{sign}(s(x))\right), \\ k_0 = k_1 + k_2\delta + p, \quad \delta > 0, p > 0, \end{aligned} \tag{36}$$

then for the closed-loop system (33), (36) the set $\mathcal{S}_1$ is invariant and UFTS with respect to $d \in \mathcal{D} := \{d \in \mathcal{L}_\infty : \|d\|_\infty \leq \delta\}$. Indeed, the equation for the sliding variable has the form

$$\dot{s} = \frac{\partial s}{\partial x}(f(x, d(t)) - f(x, 0)) - k\mathrm{sign}(s), \tag{37}$$

and the Lyapunov function $V(s) = 0.5s^T s$ has the following estimate of time derivative:

$$\dot{V} \leq -p\sqrt{V},$$

where $p$ is a positive number defined by (36). Theorem 5.6 implies the required UFTS property.

*6.1.2. ISS and iISS properties of 1-SM*

Consider the following differential inclusion

$$\dot{s} \in -k_0\overline{\text{sign}}(s) + (k_1 + k_2\|d(t)\|)P \tag{38}$$

where $P$ is a unit box in $\mathbb{R}^m$, i.e. $P := [-1, 1]^m \subset \mathbb{R}^m$, and

$$\overline{\text{sign}}(s) = (\overline{\text{sign}}(s_1), ..., \overline{\text{sign}}(s_m))^T,$$

$$\overline{\text{sign}}(s_i) = \begin{cases} \{1\} & \text{for} \quad s_i > 0, \\ \{-1\} & \text{for} \quad s_i < 0, \\ [-1, 1] & \text{for} \quad s_i = 0, \end{cases} \tag{39}$$

is the set-valued extension of the sign function.

The DI (38) is homogenous with negative degree. Theorem 5.16 with $r = [1, ..., 1]^T \in \mathbb{R}^m$, $\tilde{r} = [0, ..., 0]$, $k = -1$, $\mu = 2$, $\varrho^{\min} = \varrho^{\max} = 1$ implies iISS property of (38). Obviously, if the inequality (35) holds then any solution of (37) is a solution of (38). Therefore, the system (37) is also iISS.

Remark that the system (37) is ISS with respect to measurement noises. Indeed, for the system

$$\dot{s} \in -k_0\overline{\text{sign}}(s + d(t)) + k_1 P, \quad d \in \mathcal{D}$$

the Lyapunov function $V(s_i) = \frac{1}{2}s_i^2, i = 1, 2, ..., m$ gives $\dot{V}(s_i) < 0$ for $|s_i| > |\mathrm{d}_i|$ implying ISS (see Theorem 5.14).

*6.1.3. Main features of 1-SM control*

The choice of the sliding surface defined by $s(x) = 0$ allows us to select some *a priori* required closed-loop dynamics. Usually, this simplifies the analysis and design of the control system. However, in order to realize the first order sliding mode the switching function $s$ must have a relative degree one with respect to the control input $u$ [10].

The sliding mode control is one of the oldest robust control schemes (since 1960s). Theoretically it is insensitive with respect to matching uncertainties and disturbances. Unfortunately, the payment for this insensibility is chattering phenomenon, which is expressed by high-frequency unmodeled oscillations of a real-life system with sliding mode control. In practice, the chattering could physically destroy the control system. That is why the high-gain linear control with saturation is frequently used in real-life implementations instead of the discontinuous first order sliding mode control [2].

*6.2. High order sliding mode control*

Higher order sliding mode controllers were introduced by A. Levant (former L. Levantovskii) in 1985 (see references in [14]). These algorithms extend all the good properties of standard sliding modes to systems with higher relative degree. They also help to reduce the chattering effect [101, 102, 103]. In contrast to standard chattering reduction technique based on smoothing of the discontinuous control [2], the higher order sliding mode approach suggests to smooth the sliding motion providing the finite-time convergence to zero for the sliding variable $s$ together with some derivatives $\dot{s}$, ..., $s^{(r-1)}$ (see Definition 6.1).

Consider the single input control system of the form

$$\dot{x} = f(x, d(t)) + g(x, d(t))u, \tag{40}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $d(t) \in \mathbb{R}^p$, $f : \mathbb{R}^{n+p} \to \mathbb{R}^n$, $g : \mathbb{R}^{n+p} \to \mathbb{R}^n$. Let the $\mathcal{C}^r$ function $s : \mathbb{R}^n \to \mathbb{R}$ be the output of the control system. The control objective is to constrain the system trajectories to evolve onto the sliding surface $\mathcal{S}_r$ defined by (32).

Assume that the relative degree of $s$ with respect to control input $u$ is equal to $r$ and $s, \dot{s}, ..., s^{(r-1)}$ are not depended explicitly on $d$.

In this case the differential equation for the output dynamics has the form

$$s^{(r)} = a(x, d(t)) + b(x, d(t))u, \tag{41}$$

where $a : \mathbb{R}^{n+p} \to \mathbb{R}$ and $b : \mathbb{R}^{n+p} \to \mathbb{R}$. The output and its derivatives up to $r - 1$ order are assumed to be measured and used for control purposes, i.e. $u = u(s, \dot{s}, ..., s^{(r-1)})$.

Under the following assumptions

$$|a(x, \mathrm{d})| \leq C \quad \text{and} \quad 0 < b_{\min} \leq b(x, \mathrm{d}) \leq b_{\max}, \quad \forall x \in \mathbb{R}^n, \tag{42}$$

the output control system (41) can be extended to the inclusion

$$s^{(r)} \in [-C, C] + [b_{\min}, b_{\max}]\overline{u}, \tag{43}$$

where $\overline{u}$ is the set-valued extension of the discontinuous control $u$ obtained in accordance with Filippov definition of the solution for systems with discontinuous right-hand sides (see Section 3). The $r$-SM control algorithms developed for the system (41) typically guarantee the same properties for the extended inclusion (43) [70].

*6.2.1. Finite-time 2-SM algorithms*

Equation (41) with $r = 2$ can be rewritten as follows

$$\ddot{s} = a(x, d(t)) + b(x, d(t))u, \tag{44}$$

where $|a(x, d(t))| < C$ and $0 < b_{\min} \leq b(x, d(t)) \leq b_{\max}$ for all $x$ and all $d \in \mathcal{D}$.

The ”standard” second order sliding mode controllers for the system (44) have the forms:

- **Twisting controller [14]:**

$$u(s, \dot{s}) = -k_1 \mathrm{sign}(s) - k_2 \mathrm{sign}(\dot{s}), \tag{45}$$

  where $0 < k_2 + C/b_{\min} < k_1 < ((b_{\min} + b_{\max})k_2 - 2C)/(b_{\max} - b_{\min})$.

- **Nested controller [104], [105]:**

$$u(s, \dot{s}) = -\alpha \mathrm{sign}\left(z\right), \quad z = \dot{s} + \beta \lceil s \rfloor^{1/2}, \tag{46}$$

  where $\alpha, \beta > 0$.

- **Sub-optimal controller [101]:**

$$u(s(\cdot), \dot{s}(\cdot)) = -U \mathrm{sign}(s(t) - \lambda s(t^*(t, \dot{s}(\cdot)))), \tag{47}$$

$$t^*(t, \dot{s}(\cdot)) = \begin{cases} 0 & \text{if} \quad \dot{s}(\tau)) \neq 0, \forall \tau \in [0, t] \\ \sup_{\forall \tau \leq t : \dot{s}(\tau) = 0} \tau & \text{otherwise} \end{cases}$$

  where $s(t) := s(x(t))$, $0 < \lambda < 1, U > \frac{C}{b_{\min}}, \frac{1}{1-\lambda} > \frac{1}{k} + k, k := \frac{b_{\max}U + C}{b_{\min}U - C}$.

Denote $y_1 = s$, $y_2 = \dot{s}$ and present the extended differential inclusion for the system (44) in the form

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 \in [-C, C] + [b_{\min}, b_{\max}]\overline{u}(y_1, y_2) \end{cases} \tag{48}$$

where $\overline{u}$ is one of the controllers (45)-(47) with the sign function replaced by its set-valued extension $\overline{\mathrm{sign}}$ (see (39) or Section 3 for the details). It is easy to see that the system (48) with controllers (45)-(47) is **r-homogeneous** of degree $-1$, where $\mathbf{r} = (2, 1)$.

The geometrical approach to proof the finite-time convergence for the system (48) with controllers (45)-(47) can be found in papers [14, 101]. The UFTS analysis based on Lyapunov function method was presented for these systems in [106].

The unified Lyapunov function candidate for the relay second order sliding mode systems has the form

$$V(y_1, y_2) = q\left(p\sqrt{\left|y_1 - \frac{y_2^2}{2(\gamma+\mu u)}\right|} - \frac{y_2}{\gamma+\mu u}\right). \tag{49}$$

This Lyapunov function candidate is **homogeneous**:

$$V(\lambda^2 y_1, \lambda y_2) = \lambda^2 V(y_1, y_2), \quad \forall\lambda > 0, \forall y_1, y_2 \in \mathbb{R}.$$

The formula (49) defines the Lyapunov function for the system (48) with controller (45) if

$$\gamma := C\mathrm{sign}(y_2) \text{ and } \mu := \frac{1}{2}b_{\min}\left(1+\mathrm{sign}\left(y_1y_2\right)\right) + \frac{1}{2}b_{\max}\left(1-\mathrm{sign}\left(y_1y_2\right)\right),$$

$$p := \frac{\sqrt{2k/|\gamma+\mu u|}\mathrm{sign}(y_1y_2)}{\sqrt{|\gamma+\mu u|}-\sqrt{k}}, q := \frac{1}{p}, b_{\max}(k_1-k_2)+C < k < b_{\min}(k_1+k_2)-C.$$

For the system (48) with the nested controller (46) we need to consider two cases:

1) If $\beta^2 \leq 2(\alpha - C)$ then all trajectories of the closed-loop system converge to the first order sliding surface $y_2 + \beta\sqrt{|y_1|}\mathrm{sign}(y_1) = 0$ in a finite time and after then they slide to the origin. The Lyapunov function providing a finite-time stability of this sliding surface has the form (49) with parameters:

$$q := 1, \quad \gamma := C\mathrm{sign}(y_2 + \beta\sqrt{|y_1|}\mathrm{sign}(x)), \quad \mu := b_{\min},$$

$$p := \frac{-\mathrm{sign}(\varphi)}{(\gamma+\mu u)\sqrt{|1/\beta^2 - \frac{\mathrm{sign}(\varphi)}{2(\gamma+\mu u)}|}}, \quad \varphi = \varphi(y_1, y_2) := y_1 - \frac{y_2^2}{2(\gamma+\mu u)}.$$

2) If $2\nu^+ < \beta^2 < \frac{4\nu^+\nu^-}{\nu^+-\nu^-}$ where $\nu^- := b_{\min}\alpha - C > 0$ and $\nu^+ := b_{\max}\alpha + C$, then all trajectories of the system (48), (46) converge to the origin avoiding any other sliding motions similarly to the twisting case [107]. For this case the Lyapunov function again has the form (49) with parameters

$$\gamma := C\mathrm{sign}(y_2) \text{ and } \mu := 0.5b_{\min}(1+\mathrm{sign}[y_2z]) + 0.5b_{\max}(1-\mathrm{sign}[y_2z]),$$

$$p := \frac{\mathrm{sign}(y_2)/(\gamma+\mu u)}{\sqrt{|1/\beta^2 + 0.5\mathrm{sign}(y_2)/(\gamma+\mu u)| - k}} \text{ and } q = \frac{1}{p},$$

where $\sqrt{\frac{1}{2\nu^-} - \frac{1}{\beta^2}} < k < \sqrt{\frac{1}{2\nu^+} + \frac{1}{\beta^2}}$.

In order to study ISS properties of 2-SM systems let us assume $\|a(x, \mathrm{d})\| \leq C + \|\mathrm{d}\|$, where $C \in \mathbb{R}_+$ and $\mathrm{d} \in \mathbb{R}^p$. In this case the extended differential inclusion becomes

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 \in (C + \|d(t)\|)[-1, 1] + [b_{\min}, b_{\max}]\overline{u}(y_1, y_2) \end{cases} \tag{50}$$

It is easy to see that this system with twisting controller and nested controller in the case 2) satisfies the conditions of Theorem 5.16 with $r = [2, 1], \tilde{r} = [1, 0], k = -1, c = 1, \varrho^{\min} = \varrho^{\max} = 1$, implying *iISS property* for the corresponding closed-loop systems.

Note that the system (48) with the suboptimal 2-SM algorithm (47) is, in fact, a *functional* DI. So, formally we cannot use the framework given in Section 3 for analysis of this system. However, the paper [106] shows that the Lyapunov functional of the form (49) can be utilized in order to prove finite-time stability of the origin of the system (48) with the suboptimal 2-SM algorithm (47). Homogeneity of the closed-loop system and the Lyapunov functional can also be shown.

### 6.3. Fixed-time 2-SM controls

The extended concept of finite-time stability related to global boundedness of the settling time function was called *fixed-time stability* (FxTS) (see, Section 5). If $b_{\min} = b_{\max} = 1$ then the *fixed-time second order sliding mode control* can be selected in the form [84]:

$$u(s, \dot{s}) = -\frac{\alpha_1 + 3\beta_1 s^2 + 2C}{2}\mathrm{sign}(z) - \left(\alpha_2 \lceil z \rfloor^2 + \beta_2 \lceil z \rfloor^4\right)^{1/3}, \tag{51}$$

$$z := z(s, \dot{s}) = \dot{s} + \left\lceil \lceil \dot{s} \rfloor^2 + \alpha_1 s + \beta_1 s^3 \right\rfloor^{1/2},$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$. For the system (48) with the controller (51) the non-smooth Lyapunov function $V_{FxTS}(s, \dot{s}) = |z(s, \dot{s})|$,

$$DV_{FxTS} \leq -\left(\alpha_2 |z|^2 + \beta_2 |z|^4\right)^{1/3}$$

proves the uniform FxTS property for the sliding surface $z(s, \dot{s}) = 0$ (see, Theorem 5.8). The reduced order dynamic equation has the form

$$\dot{s} = -\left\lceil \frac{\alpha_1}{2}s + \frac{\beta_1}{2}s^3 \right\rfloor^{1/2}.$$

It implies fixed-time convergence of the sliding variable $s$ to zero together with $\dot{s}$.

The system (48) with the controller (51) is **homogeneous** in the bi-limit:

$$r_0 = (2,1), \quad F_0 = \begin{pmatrix} y_2 \\ -[C,C] - \frac{\alpha_1+2C}{2}\overline{\text{sign}}\left(y + \lceil\lceil y_2\rfloor^2 + \alpha_1 y_1\rfloor^{1/2}\right) \end{pmatrix},$$

$$r_\infty = (2,3)\,, \quad F_\infty = \begin{pmatrix} y_2 \\ -\beta_2^{\frac{1}{3}}\left\lceil y_2 + \lceil\lceil y_2\rfloor^2 + \beta_1 y_1^3\rfloor^{1/2}\right\rfloor^{4/3} \end{pmatrix}.$$

The Lyapunov function $V_{FxTS}$ is also **homogeneous** in the bi-limit with the same weights and $V^0_{FxTS} = \left|\dot{s} + \lceil\lceil\dot{s}\rfloor^2 + \alpha_1 s\rfloor^{1/2}\right|$, $V^\infty_{FxTS} = \left|\dot{s} + \lceil\lceil\dot{s}\rfloor^2 + \beta_1 s^3\rfloor^{1/2}\right|$.

*6.3.1. Finite-time 2-SM control for systems with relative degree one*

Consider the output control system of the form:

$$\dot{s} = a(x, d(t)) + u, \tag{52}$$

where $a : \mathbb{R}^{n+p} \to \mathbb{R}$ is an unknown function such that $\left\|\frac{d}{dt}a(x(t), d(t))\right\| \leq L$. It is assumed that $d(t)$ is differentiable with a bounded derivative.

Let $u$ has the form of the so-called *super-twisting controller* (STC) [108, 70]

$$u(s(t)) = -\alpha\lceil s(t)\rfloor^{1/2} - \beta\int_0^t \text{sign}(s(\tau))d\tau, \tag{53}$$

where $\alpha > 0$ and $\beta > 0$.

Denoting $y_1 = s$ and $y_2 = a(x,d) - \beta\int_0^t \text{sign}(s(\tau))d\tau$ we can extend the closed-loop system (52)-(53) to the following differential inclusion

$$\begin{cases} \dot{y}_1 = -\alpha\lceil y_1\rfloor^{1/2} + y_2, \\ \dot{y}_2 \in -\beta\overline{\text{sign}}(y_1) + [-L, L], \end{cases} \tag{54}$$

where $\overline{\text{sign}}$ is defined by (39), $\alpha > 0$ and $\beta > 3L + 2(L/\alpha)^2$. The system (54) is **r**-homogeneous with degree $-1$ for $\mathbf{r} = (2,1)$. In [109] the non-smooth Lyapunov function

$$V_{STC}(y_1, y_2) = \begin{pmatrix} \lceil y_1\rfloor^{1/2} \\ y_2 \end{pmatrix}^T P \begin{pmatrix} \lceil y_1\rfloor^{1/2} \\ y_2 \end{pmatrix}, \quad P > 0, P \in \mathbb{R}^{2\times 2} \tag{55}$$

was presented for uniform FTS analysis of the origin of the system (54). This Lyapunov function is also **homogeneous** with the same weights:

$$V_{STC}(\lambda^2 y_1, \lambda y_2) = \lambda^2 V(y_1, y_2), \quad \forall\lambda > 0, \forall y_1, y_2 \in \mathbb{R}.$$

Remark that the super-twisting control (STC) is a continuous function of time. This gives an advantage to STC over the relay 1-SM algorithms, that is related to reduction of chattering effects [102, 103].

Assume that

$$a(x(t), d(t)) = a_1(x(t), d(t)) + a_2(x(t), d(t)),$$
$$\|a_1(x(t), d(t))\| \leq \|d_1(t)\| \text{ and } \|\tfrac{d}{dt}a_2(x(t), d(t))\| \leq L + \|d_2(t)\|,$$

where $d(t) = (d_1(t)\ d_2(t))^T, d_1(t) \in \mathbb{R}^{p_1}, d_2(t) \in \mathbb{R}^{p_2}, p = p_1 + p_2$ and $L \in \mathbb{R}_+$. In this case for $y_1 = s$ and $y_2 = a_2(x, d_2(t)) - \beta \int_0^t \operatorname{sign}(s(\tau))d\tau$ we have the following extended differential inclusion:

$$\left\{ \begin{array}{l} \dot{y}_1 \in -\alpha\lceil y_1(t)\rfloor^{1/2} + y_2 + \|d_1(t)\|[-1, 1], \\ \dot{y}_2 \in -\beta\overline{\operatorname{sign}}(y_1) + (L + \|d_2(t)\|)[-1, 1]. \end{array} \right. \tag{56}$$

This system satisfies conditions of Theorem 5.16 with $r = [2, 1], \tilde{r} = [1, 0], k = -1, c = 1, \varrho^{\min} = \varrho^{\max} = 1$, implying *iISS property* of the super-twisting system for $\alpha > 0$ and $\beta > 3L + 2(L/\alpha)^2$. Note that the same theorem guarantees ISS property with respect to the disturbance input $d_1$, if it is assumed $d_2 = 0$.

*6.3.2. Fixed-time 2-SM control for systems with relative degree one*

The *uniform super-twisting controller* (USTC) presented in [110] has the form:

$$u_{USTC} = -\alpha_1\lceil s(t)\rfloor^{\frac{1}{2}} - \alpha_2\lceil s(t)\rfloor^{\frac{3}{2}} - \int_0^t \beta_1\lceil s(\tau)\rfloor^0 + \beta_2 s(\tau) + \beta_3\lceil s(\tau)\rfloor^2 d\tau, \tag{57}$$

where $\alpha_1 > 0$, $\beta_1 > \frac{1}{2}\left(\frac{\alpha_1^2}{4} + \frac{4L^2}{\alpha_1^2}\right)$, $\alpha_2 \geq 0$, $\beta_2 = 4\alpha_2\beta_1/\alpha_1$, $\beta_3 = 3\alpha_2^2\beta_1/\alpha_1^2$.

The closed-loop system (52), (57) in variables $y_1 = s$ and $y_2 = a(x(t), d(t)) - \int_0^t \beta_1\lceil s(\tau)\rfloor^0 + \beta_2 s(\tau) + \beta_3\lceil s(\tau)\rfloor^2 d\tau$ can be extended to the following differential inclusion (DI):

$$\left\{ \begin{array}{l} \dot{y}_1 = -\alpha_1\lceil y_1\rfloor^{\frac{1}{2}} - \alpha_2\lceil y_1\rfloor^{\frac{3}{2}} + y_2, \\ \dot{y}_2 \in -\beta_1\overline{\operatorname{sign}}(y_1) - \beta_2 y_1 - \beta_3\lceil y_1\rfloor^2 + [-L, L]. \end{array} \right. \tag{58}$$

This system is **homogeneous** in the bi-limit with

$$r_0 = (2, 1), F_0 = \begin{pmatrix} -\alpha_1 y_1^{\left[\frac{1}{2}\right]} + y_2 \\ -[L, L] - \beta_1 y_1^{[0]} \end{pmatrix}, r_\infty = (2, 3), F_\infty = \begin{pmatrix} -\alpha_2 y_1^{\left[\frac{3}{2}\right]} + y_2 \\ -\beta_3 y_1^{[2]} \end{pmatrix}.$$

The Lyapunov function, which shows uniform FxTS property of the system (58) was presented in [110]:

$$V_{USTC}(x,y) = \begin{pmatrix} \alpha_1\lceil y_1\rfloor^{\frac{1}{2}} + \alpha_2\lceil y_1\rfloor^{\frac{3}{2}} \\ y_2 \end{pmatrix}^T P \begin{pmatrix} \alpha_1\lceil y_1\rfloor^{\frac{1}{2}} + \alpha_2\lceil y_1\rfloor^{\frac{3}{2}} \\ y_2 \end{pmatrix}. \quad (59)$$

This function is also **homogeneous** in the bi-limit with the same weights. The corresponding homogeneous approximations of $V_{USTC}$ can be obtained from (59) by replacing $\alpha_1 = 0$ for $0$-limit and $\alpha_2 = 0$ for $\infty$-limit.

*6.3.3. High-order algorithms*

The high-order version of the nested control algorithm for the system (41) has the form [104], [105], [70]:

$$u = -\alpha\Phi_{r-1,r}(s, \dot{s}, \ldots, s^{(r)}), \quad (60)$$

where $\alpha > 0$ and $\Phi_{r-1,r}$ is defined recursively by:

$$\Phi_{0,r} = \mathrm{sign}(s), \quad \Phi_{i,r} = \mathrm{sign}(s^{(i)} + \beta_i M_{i,r}\Phi_{i-1,r}),$$

where $\beta_i > 0$ and

$$M_{1,r} = |s|^{\frac{r-1}{r}}, \quad M_{i,r} = \left(|s|^{\frac{p}{r}} + |\dot{s}|^{\frac{p}{r-1}} + ... + \left|s^{(i-1)}\right|^{\frac{p}{r-i+1}}\right)^{\frac{r-i}{p}},$$

where $p > 0$ is the least common multiple of $1, 2, ..., r$. Obviously that for $r = 2$ we have the 2-SM nested algorithm studied above $u = -\alpha\mathrm{sign}(\dot{s} + \beta_1|s|^{\frac{1}{2}}\mathrm{sign}(s))$.

The quasi continuous $r$-SM algorithm for the system (41) has the form [70]:

$$u = -\lambda\Psi_{r-1,r}(s, \dot{s}, \ldots, s^{(r)}), \quad (61)$$

where $\lambda > 0$ and $\Psi_{r-1,r}$ is defined by the following recursive scheme

$$\begin{aligned} &\varphi_{0,r} = s, && N_{0,r} = |s|, && \Psi_{0,r} = \frac{\varphi_{0,r}}{N_{0,r}}, \\ &\varphi_{i,r} = s^{(i)} + \beta_i N_{i-1,r}^{\frac{r-i}{r-i+1}}\Psi_{i-1,r}, && N_{i,r} = |s^{(i)}| + \beta_i N_{i-1,r}^{\frac{r-i}{r-i+1}}\Psi_{i-1,r}, && \Psi_{i,r} = \frac{\varphi_{i,r}}{N_{i,r}}, \end{aligned}$$

where $\beta_i \in \mathbb{R}$ are control parameters.

Both presented controllers are **homogeneous** [70] with the weights $\mathbf{r} = (r, r-1, \ldots, 1)$. Similarly to the second order sliding mode control systems it can be easily established the iISS property of the presented high-order sliding mode control systems with respect to matched disturbances.

| **Control algorithm** | **Type of homogeneity** | **Lyapunov function is found** | **iISS property is proven** |
|---|---|---|---|
| Relay 1-SM | standard | yes | yes |
| Super-twisting | weighted | yes | yes |
| Uniform super-twisting | in the bi-limit | yes | not yet |
| Twisting | weighted | yes | yes |
| 2-SM nested | weighted | yes | yes |
| 2-SM fixed-time | in the bi-limit | yes | not yet |
| r-SM nested | weighted | no | yes |
| Quasi-continuous | weighted | no | yes |

Table 1: Properties of sliding mode controllers

The problem of implementation of the presented control schemes is related to the control parameters selection to guarantee finite-time attractivity of the sliding surface. The strong Lyapunov functions are still not designed for the r-th order sliding mode control systems, however application of homogeneity approach allows us to establish the robustness properties.

### *6.4. Summary*

In this section the standard sliding mode controllers have been considered. It was shown that all these controllers are homogeneous in some sense. It is also known that sliding mode control algorithms are insensitive with respect to bounded matched disturbances, but iISS property with respect to such type of disturbances was proven only for some of them. The summary of the controllers' properties is presented in Table 1.

## 7. Conclusion

The paper gives the main ingredients for a possible route to a complete theory of design and analysis for higher order sliding mode: homogeneity, finite-time stability and its robustness with respect to perturbations in the context of input-to-state stability. Some analysis is performed from this point of view for different existing higher order sliding mode controllers from the literature. The rest of the story can be written and concentrated on the design tools.

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