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*Splitting a tournament into two subtournaments with
given minimum outdegree*

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Splitting a tournament into two subtournaments with given minimum outdegree

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Abstract: A (k_1, k_2) -outdegree-splitting of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2 , respectively. We show that there exists a minimum function f_T such that every tournament of minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting, and $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$. We also show a polynomial-time algorithm that finds a (k_1, k_2) -outdegree-splitting of a tournament if one exists, and returns ‘no’ otherwise. We give better bound on f_T and faster algorithms when $k_1 = 1$.

Key-words: tournament, splitting, outdegree

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Partager un tournoi en deux sous-tournois de degré sortant minimum prescrits

Résumé : Un (k_1, k_2) -partage d'un digraphe D est une partition (V_1, V_2) de son ensemble de sommets telle que $D[V_1]$ et $D[V_2]$ soient de degré sortant minimum au moins k_1 et k_2 , respectivement. Nous établissons l'existence d'une fonction (minimum) f_T telle que tout tournoi de degré sortant minimum au moins $f_T(k_1, k_2)$ a un (k_1, k_2) -partage, et que $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$. Nous donnons également un algorithme en temps polynomial qui trouve un (k_1, k_2) -partage d'un tournoi s'il en existe un et renvoie 'non' sinon. Nous donnons de meilleures bornes sur f_T et des algorithmes plus rapides pour $k_1 = 1$.

Mots-clés : tournoi, partage, degré sortant

1 Introduction

Let D be a digraph. For a vertex $v \in V(D)$ the *outdegree* of v , denoted by $d_D^+(v)$, is the number of arcs directed away from v . The minimum outdegree over all vertices of D is denoted by $\delta^+(D)$. We drop D in $d_D^+(v)$ and $\delta^+(D)$ if it is clear from the context.

A (k_1, k_2) -*outdegree-splitting* of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2 , respectively. A digraph admitting a (k_1, k_2) -outdegree-splitting is said to be (k_1, k_2) -*outdegree-splittable*.

Problem 1 (Alon [1]). Is there a function f such that every digraph with minimum outdegree $f(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting?

The existence of the corresponding function f for the undirected analogue is easy and has been observed by many authors. Stiebitz [12] even proved the following tight result: if the minimum degree of an undirected graph G is $d_1 + d_2 + \dots + d_k$, where each d_i is a non-negative integer, then the vertex set of G can be partitioned into k pairwise disjoint sets V_1, \dots, V_k , so that for all i , the induced subgraph on V_i has minimum degree at least d_i . This is clearly tight, as shown by an appropriate complete graph.

Problem 1 is equivalent to the following:

Problem 2. Is there a function $f'(k_1, k_2)$ such that every digraph with minimum outdegree $f'(k_1, k_2)$ has two disjoint (induced) subdigraphs, one of them with minimum outdegree k_1 and the other with minimum outdegree k_2 ?

This follows from the following proposition.

Proposition 3. *Let D be a digraph with minimum outdegree at least $k_1 + k_2 - 1$. If D contains two disjoint subdigraphs D_1 and D_2 such that $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$, then D has a (k_1, k_2) -outdegree-splitting.*

Proof. Consider two disjoint digraphs D_1 and D_2 with $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$ such that $V(D_1) \cup V(D_2)$ is maximum. Suppose for a contradiction that $S = V(D) \setminus (V(D_1) \cup V(D_2))$ is not empty. Then every vertex $s \in S$ has at most $k_1 - 1$ outneighbours in D_1 otherwise $D_1 + s$ and D_2 contradict the maximality of D_1 and D_2 . Hence every vertex of S has at least $k_1 + k_2 - 1 - (k_1 - 1) = k_2$ outneighbours in $D - D_1$. It follows that $D - D_1$ has minimum degree k_2 . So D_1 and $D - D_1$ contradicts the maximality of D_1 and D_2 . \square

Corollary 4. $f(k_1, k_2) \leq \max\{f'(k_1, k_2), k_1 + k_2 - 1\}$.

This implies in particular that $f(1, 1) = f'(1, 1) = 3$. Indeed Thomassen [13] showed that every digraph of minimum outdegree at least 3 has two disjoint cycles. (In this paper, paths and cycles are always directed.)

This is a special case of Bermond-Thomassen Conjecture [3]:

Conjecture 5 (Bermond and Thomassen [3]). *Every digraph with $\delta^+ \geq 2k - 1$ contains k disjoint cycles.*

Note that Alon [1] proved that if $\delta^+ \geq 64k$ there are k disjoint cycles.

A *tournament* is a digraph such that for every two distinct vertices u, v there is exactly one arc with ends $\{u, v\}$ (so, either the arc uv or the arc vu but not both).

In this paper, we settle Problem 1 for tournaments.

Theorem 6. *Every tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$ has a (k_1, k_2) -outdegree-splitting.*

To prove Theorem 6, we shall prove the following theorem.

Theorem 7. *Every tournament with minimum outdegree at least k has a subtournament with minimum outdegree k and order at most $k^2/2 + 3k/2 + 1$.*

We can then easily derive Theorem 6.

Proof of Theorem 6. Let T be a tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$. By Theorem 7, there exists a subtournament T_1 with minimum outdegree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + 1$. Let $T_2 = T - T_1$. Then $\delta^+(T_2) \geq \delta^+(T) - |V(T_1)| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2) -outdegree-splitting. \square

In fact, we prove a more general statement than Theorem 7 (Theorem 17). This enables us to prove the following generalization of Theorem 6.

Theorem 8. *Let $m = \max\{k_1^2/2 + 3k_1/2 + k_2 + u_1 + 1, k_1 + u_2\}$, let T be a tournament of minimum outdegree at least m and let U_1 and U_2 be two disjoint subsets of $V(T)$ of cardinality u_1 and u_2 respectively. Then there is a (k_1, k_2) -outdegree-splitting (V_1, V_2) of T such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.*

The bound of Theorem 6 is certainly not tight. Theorem 6 asserts that every tournament with minimum outdegree 4 has a $(1, 1)$ -outdegree-splitting, but we know that having outdegree 3 is sufficient.

Problem 9. What is the minimum integer $f_T(k_1, k_2)$ such that every tournament with minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting?

Theorem 6 implies that $f_T(1, k) \leq k + 3$. We describe examples implying $f_T(1, k) \geq k + 2$, and we conjecture that this lower bound is the exact value.

Conjecture 10. *For any positive integer k , $f_T(1, k) = k + 2$.*

In Section 4, we establish this conjecture for $k \in \{2, 3, 4\}$, that is, we prove $f_T(1, 2) = 4$, $f_T(1, 3) = 5$, and $f_T(1, 4) = 6$.

Next we consider problems of deciding whether a digraph admits a (k_1, k_2) -outdegree-splitting.

OUTDEGREE SPLITTING

Input: A digraph D and two positive integers k_1 and k_2 .

Question: Does D admit a (k_1, k_2) -outdegree-splitting?

Particular cases of this problem are when k_1 and k_2 are fixed integers and not part of the input. Hence for every fixed k_1, k_2 , we have the following problem.

(k_1, k_2) -OUTDEGREE-SPLITTING

Input: A digraph D .

Question: Does D admit a (k_1, k_2) -outdegree-splitting?

Theorem 11. *$(1, 1)$ -OUTDEGREE-SPLITTING is polynomial-time solvable.*

Proof. Let us describe a polynomial-time algorithm solving $(1, 1)$ -OUTDEGREE SPLITTING.

If the input digraph D has a vertex with outdegree 0, then the answer is ‘no’ because this vertex has outdegree 0 in any subdigraph of D containing it. Henceforth we may assume that $\delta^+(D) \geq 1$.

It is well-known that a digraph with outdegree at least 1 contains a cycle. Therefore, Proposition 3 implies that a digraph with minimum outdegree at least 1 admits a $(1, 1)$ -outdegree-splitting if and only if it contains two disjoint cycles. Thus it is enough to decide whether D contains two disjoint cycles.

But deciding whether a digraph contains two disjoint cycles can be done in polynomial time as shown by McCuaig [7]. (See also [9].) \square

In Section 5, we consider the restriction of these problems to tournaments.

TOURNAMENT OUTDEGREE-SPLITTING

Input: A tournament T and two positive integers k_1 and k_2 .

Question: Does T admit a (k_1, k_2) -outdegree-splitting?

TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING

Input: A tournament T .

Question: Does T admit a (k_1, k_2) -outdegree-splitting?

TOURNAMENT $(1, 1)$ -OUTDEGREE-SPLITTING is a particular case of $(1, 1)$ -OUTDEGREE-SPLITTING, and thus is polynomial-time solvable. In Theorem 31, we show that, more generally, for any k_1, k_2 , **TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING** can be solved in $O(n^{k^2/2+3k/2+3})$ time. We then describe a faster algorithm solving **TOURNAMENT $(1, k_2)$ -OUTDEGREE-SPLITTING**. It runs in $O(n^3)$ time for $k_2 \geq 2$ and in $O(n^2)$ time for $k_2 = 1$. In view of these results, it is natural to ask the following.

Problem 12. Is **TOURNAMENT OUTDEGREE-SPLITTING** fixed-parameter tractable with (k_1, k_2) as a parameter? In other words, can we solve **TOURNAMENT OUTDEGREE SPLITTING** in $F(k_1, k_2)P(n)$ time, where F is an arbitrary computable function and P is a polynomial in the order n of the input tournament?

Finally, in Section 6, we present some possible directions for further research.

2 Definitions and folklore on tournaments

The *score sequence* of a tournament T , denoted by $s(T)$, is the non-decreasing sequence of outdegrees of its vertices. Landau [6] characterized the non-decreasing sequences of integers that are score sequences.

Theorem 13 (Landau [6]). *A non-decreasing sequence of non-negative integers (s_1, s_2, \dots, s_n) is a score sequence if and only if:*

$$(i) \quad s_1 + s_2 + \dots + s_i \geq \binom{i}{2}, \text{ for } i = 1, 2, \dots, n-1, \text{ and}$$

$$(ii) \quad s_1 + s_2 + \dots + s_n = \binom{n}{2}.$$

Condition (ii) in the above theorem implies directly the following proposition.

Proposition 14. *Every tournament of order $2k$ has minimum degree less than k .*

Corollary 15. $f_T(k_1, k_2) \geq k_1 + k_2 + 1$.

Proof. Let T be a $(k_1 + k_2)$ -regular tournament of order $2k_1 + 2k_2 + 1$. In every bipartition (V_1, V_2) of $V(T)$, either $|V_1| \leq 2k_1$ or $|V_2| \leq 2k_2$. Thus, by Proposition 14, either $\delta^+(T[V_1]) < k_1$ or $\delta^+(T[V_2]) < k_2$. \square

An ℓ -*cycle* is a cycle of length ℓ . A tournament T is *transitive* if it contains no cycles. The score sequence of a transitive tournament of order n is $(0, 1, \dots, n-1)$.

We denote by $tt_3(T)$ the number of transitive subtournaments of order 3 in T and by $c_3(T)$ number of 3-cycles in T . Since a tournament of order 3 is either a transitive tournament or a 3-cycle, we have

$$tt_3(T) + c_3(T) = \binom{|T|}{3}.$$

Now if v is a vertex, the number of transitive subtournaments of order 3 with source v is $\binom{d^+(v)}{2}$. Hence

$$tt_3(T) = \sum_{v \in V(T)} \binom{d^+(v)}{2}.$$

A digraph D is *strongly connected* or *strong* if there is a path from u to v for every $u, v \in V(D)$. A digraph D is *k-strong* if $D - X$ is strong for every $X \subset V(D)$ where $|X| \leq k - 1$. A (strong) *component* of D is a strong subdigraph of D which is maximal by inclusion.

Let T be a tournament. Let T_1, T_2, \dots, T_m be the components of T . Then $(V(T_1), V(T_2), \dots, V(T_m))$ is a partition of $V(T)$ and without loss of generality, we may suppose that $T_i \rightarrow T_j$ whenever $i < j$. In this case we say that $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$ is the *decomposition* of T . Component T_1 is said to be the *initial* component of T and T_m its *terminal* component.

A vertex is *pancyclic* in a digraph D if, for every $3 \leq \ell \leq |D|$, it is contained in an ℓ -cycle. To contain a pancyclic vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to Camion's theorem [4]. Moon [8] showed that this condition is sufficient.

Theorem 16 (Moon [8]). *Every vertex of a strong tournament is pancyclic.*

We sometimes use the results of this section without referring to them.

3 Small subtournament of minimum outdegree k

We now prove Theorem 7. In fact, we prove a more general theorem whose particular case with $U = \emptyset$ is Theorem 7.

Theorem 17. *Let T be a tournament with minimum outdegree at least k and $U \subseteq V(T)$ be a subset of vertices. There is a subtournament T' of T with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$.*

Proof. For every p , we prove the result for all sets U of size p by induction on $|V(T)|$, the result holding trivially if $|V(T)| \leq p + k^2/2 + 3k/2 + 1$.

Let T be a tournament of order at least $p + k^2/2 + 3k/2 + 2$ with minimum outdegree at least k and U a set of p vertices of T . Let S be the set of vertices of degree k in T . There are $k|S|$ arcs with their tail in S . Among them $|S|(|S| - 1)/2$ are in S and the remaining ones have their heads out of S . Hence $|N^+(S)| \leq |S| + k|S| - |S|(|S| - 1)/2$. Now the polynomial $P(x) = (k+3/2)x - x^2/2$ increases on $[0, k+3/2]$ and decreases on $[k+3/2, +\infty[$. Moreover $P(k+1) = P(k+2) = k^2/2 + 3k/2 + 1$. Consequently, $|N^+(S)| \leq k^2/2 + 3k/2 + 1$.

Since $|V(T)| \geq p + k^2/2 + 3k/2 + 2$, there is a vertex v which is not in $N^+(S) \cup U$. Thus $T - v$ has minimum outdegree at least k and by induction $T - v$ (and thus also T) has a subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$. \square

The bound $k^2/2 + 3k/2 + 1$ is Theorem 7 is tight in the following sense.

Proposition 18. *For every non-negative integer k , and for every $n \geq k^2/2 + 3k/2 + 1$, there is a tournament $T(n, k)$ of order n and a set $W \subset V(T)$ of order $n - k^2/2 + 3k/2 + 1$ such that for every $U \subset W$, every subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ has order at least $|U| + k^2/2 + 3k/2 + 1$.*

Proof. Consider the disjoint union of a strong tournament S of order $k + 1$ and a transitive tournament TT of order $k(k + 1)/2$. Set $V(S) = \{s_1, \dots, s_{k+1}\}$. Partition $V(TT)$ into $k + 1$ sets A_1, \dots, A_{k+1} such that $|A_i| = k - d_S^+(s_i)$. This is possible since $\sum_{i=1}^{k+1} d_S^+(s_i) = k(k + 1)/2$, so $\sum_{i=1}^{k+1} (k - d_S^+(s_i)) = |V(TT)|$. Now for each

i , add the arc $s_i a$ for all $a \in A_i$, and all the arcs bs_i for all $b \in V(TT) \setminus A_i$. The resulting tournament R has order $k^2/2 + 3k/2 + 1$ and minimum outdegree k .

Let R' be a subtournament of R with outdegree at least k .

It must contain a vertex of S , because all subtournaments of TT are transitive. But each element s of S has outdegree exactly k in R , so if $s \in V(R')$, then $N_R^+(s) \subset V(R')$. Since S is strong, it has a hamiltonian cycle by Camion's Theorem, and so $V(S) \subset V(R')$. But by construction, every vertex in R is dominated by a vertex in S , and thus must be in R' . Hence $R = R'$.

Set $p = n - k^2/2 + 3k/2 + 1$. Let $T(n, k)$ be a tournament obtained from the disjoint union of R and the transitive tournament TT_p of order p by adding all arcs from TT_p towards R . Then, for any set $U \subset V(TT_p)$, every subtournament of $T(n, k)$ with minimum outdegree k containing U must also contain $V(R)$ and thus has order at least $|U| + k^2/2 + 3k/2 + 1$. \square

We can then easily derive Theorem 8 from Theorem 17.

Proof of Theorem 8. Let T be a tournament of minimum outdegree at least m . The tournament $T - U_2$ has minimum outdegree at least k_1 because $|U_2| = u_2$. Thus, by Theorem 17, there exists a subtournament T_1 of $T - U_2$ with minimum degree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + u_1 + 1$ such that $U_1 \subseteq V(T_1)$. Set $V_1 = V(T_1)$, $T_2 = T - T_1$, and $V_2 = V(T_2)$. By definition, $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Now $\delta^+(T_2) \geq \delta^+(T) - |V_1| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2) -outdegree-splitting. \square

3.1 Outdegree-critical tournaments

Theorem 7 can be rephrased in terms of k -outdegree-critical tournament. A tournament T is said to be k -outdegree-critical if it has minimum outdegree k and all its proper subtournaments have outdegree less than k . Theorem 7 implies that all k -outdegree-critical tournaments have bounded size. Hence a natural problem is the following.

Problem 19. Describe the k -outdegree-critical tournaments.

The unique 1-outdegree-critical tournament is the 3-cycle.

We now show that the 2-outdegree-critical tournaments are those depicted in Figure 1.

Theorem 20. Every tournament with minimum outdegree 2 has a subtournament isomorphic to one of those depicted in Figure 1.

Proof. By induction on $|V(T)|$, the result holding trivially when $|V(T)| < 5$.

Let T be a tournament of order at least 5 with minimum outdegree 2. Every vertex v has an inneighbour u such that $d^+(u) = 2$, for otherwise $T - v$ has minimum outdegree at least 2 and by induction $T - u$ (and thus also T) has a subtournament with minimum outdegree 2 and with order 5 or 6.

Let S be the set of vertices of outdegree 2 in T . By the previous remark, S is not empty and $T[S]$ has minimum indegree at least 1. Hence $T[S]$ contains a 3-cycle $C = (x_1, x_2, x_3)$. For $i = 1, 2, 3$, let y_i be the outneighbour of x_i in $V(T) \setminus \{x_1, x_2, x_3\}$. If all y_i are distinct, then each y_i dominates $\{x_1, x_2, x_3\} \setminus \{x_i\}$, and so $T[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is one of the tournaments A_6, B_6, C_6 and D_6 . If $y_1 = y_2 = y_3$, let z_1 and z_2 be the two outneighbours of y_1 . These two vertices dominate $\{x_1, x_2, x_3\}$, so $T[\{x_1, x_2, x_3, y_1, z_1, z_2\}]$ is isomorphic to E_6 . If $y_1 = y_2 \neq y_3$, then y_3 dominates x_1 and x_2 , and y_1 dominates x_3 . If y_1 dominates y_3 , then $T[\{x_1, x_2, x_3, y_1, y_3\}]$ is isomorphic to R_5 . If y_1 is dominated by y_3 , let z be an outneighbour of y_1 distinct from x_3 . The vertex z dominates $\{x_1, x_2\}$, so $T[\{x_1, x_2, x_3, y_1, y_3, z\}]$ is isomorphic to C_6 or D_6 . \square

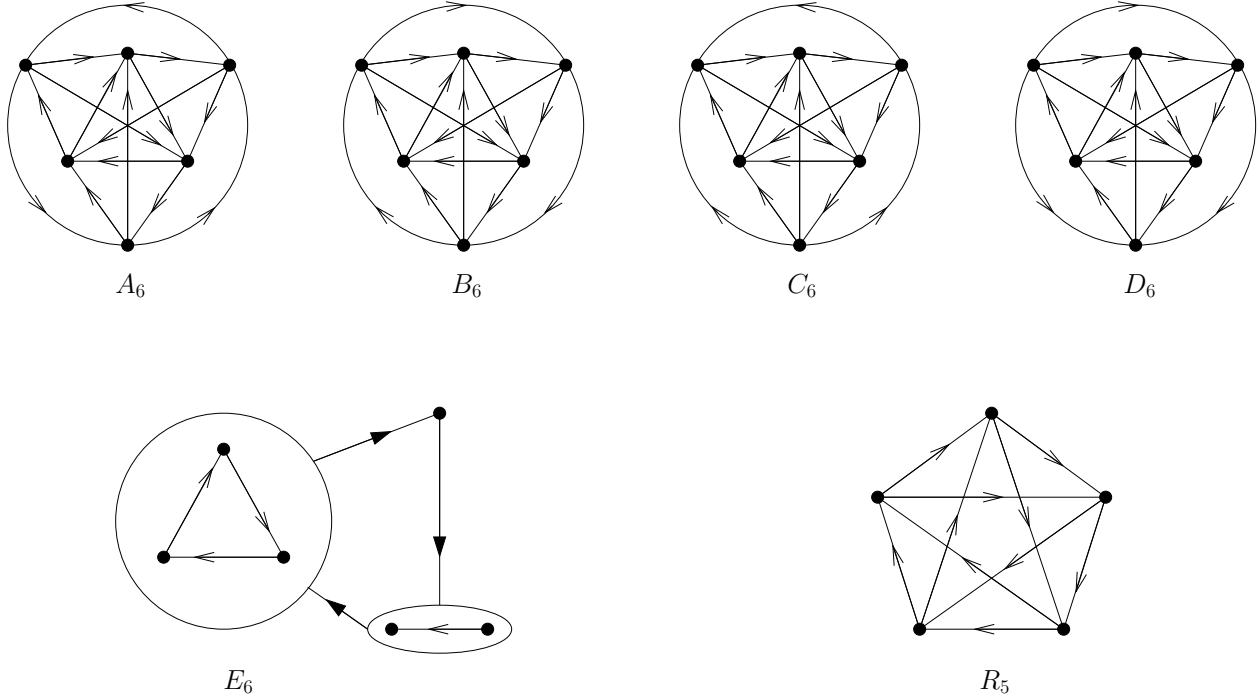


Figure 1: The 2-outdegree-critical tournaments.

4 $(1, k)$ -outdegree-splitting of tournaments

4.1 Improved upper bound for $f_T(1, k)$

A 3-cycle C in a tournament T is said to be k -good if $\delta^+(T - C) \geq k$. Clearly, if C is a k -good 3-cycle, then $(V(C), V(T - C))$ is a $(1, k)$ -splitting of T .

Lemma 21. *Let k be an integer and let T be a strong tournament with minimum outdegree at least $k + 2$. Let S be the set of vertices with outdegree $k + 2$ in T . If T has no k -good 3-cycle, then the following hold.*

- (i) *Every arc is dominated by a vertex in S .*
- (ii) *For every vertex v , the subtournament $T[N^-(v) \cap S]$ has minimum indegree 1 and at least five vertices.*
- (iii) $|V(T)| \leq \frac{1}{10}(k + 7)(k + 8)$.

Proof. Suppose that T contains no k -good 3-cycle. A 3-cycle C in T is S -dominated if there is a vertex $x \in S$ dominating C . Clearly, a 3-cycle in T is k -good if and only if it is not S -dominated. Hence all 3-cycles are S -dominated.

(i) Let uv be an arc. Since T is strong, there is a 3-cycle C containing u by Theorem 16. This cycle is dominated by a vertex $s \in S$. If s dominates v , then s dominates the arc uv . If not, then uvs is a 3-cycle. This cycle is dominated by a vertex $s' \in S$, which thus dominates uv .

(ii) Let v be a vertex of T . By (i), v is dominated by a vertex in S , so $N^-(v) \cap S$ is not empty. For any vertex $s \in N^-(v) \cap S$, the arc sv is dominated by a vertex $s' \in S$, which is distinct from s . Hence $T[N^-(v) \cap S]$ has indegree at least 1 and thus contains a 3-cycle $s_1s_2s_3$. This 3-cycle is dominated by a vertex $s \in S$.

Assume first $s \rightarrow v$. By (i) the arc sv is dominated by a vertex s' of S . Clearly $s' \notin \{s_1, s_2, s_3\}$, because s' dominates $s_1s_2s_3$. Hence s_1, s_2, s_3, s, s' are five vertices in $N^-(v) \cap S$.

Assume now that $v \rightarrow s$. Then ss_1v is a 3-cycle which is dominated by a vertex s' . This vertex is in $N^-(v) \cap S$ and is distinct from s_2, s_3 because it dominates s . Furthermore, by (i) there is a vertex t of S dominating $s'v$. If $t \notin \{s_1, s_2, s_3\}$, then s_1, s_2, s_3, s', t are five vertices in $N^-(v) \cap S$. So we may assume that $t \in \{s_1, s_2, s_3\}$ and, without loss of generality, $t = s_2$. Now, there is a vertex s'' dominating the 3-cycle ss_2v . This vertex is distinct from s_1, s_3 because it dominates s , and is distinct from s' because it dominates s_2 . Hence, s_1, s_2, s_3, s', s'' are five vertices in $N^-(v) \cap S$.

(iii) By (ii), every vertex has at least four inneighbours in S . Thus

$$|V(T)| = |N^+(S)| \leq |S| + \frac{1}{5} \left((k+2)|S| - \binom{|S|}{2} \right).$$

But the polynomial $Q(x) = x + \frac{1}{5} \left((k+2)x - \binom{x}{2} \right) = \frac{1}{10}x(2k+15-x)$ increases on $[0, k+15/2]$ and decreases on $[k+15/2, +\infty[$ and $Q(k+7) = Q(k+8) = \frac{1}{10}(k+7)(k+8)$. Consequently, $|V(T)| \leq \frac{1}{10}(k+7)(k+8)$. \square

Theorem 22. *Let k be an integer in $\{1, 2, 3, 4\}$. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k -good 3-cycle.*

Proof. It is sufficient to prove the result for strong tournaments. Indeed if T is not strong, then its terminal component T' has also outdegree at least $k+2$. Moreover, every 3-cycle that is k -good in T' is also k -good in T .

Henceforth, we may assume that T is strong. Let S be the set of vertices with outdegree $k+2$ in T .

- Assume $k \in \{1, 2\}$. Then every vertex of S has outdegree at most 4 in $T[S]$, so $T[S]$ has a vertex with indegree at most 4. Thus, by Lemma 21-(ii), T has a 1-good 3-cycle.
- Assume $k = 3$. Since $\delta^+(T) \geq 5$, then $|V(T)| \geq 11$. By Lemma 21-(iii), we have the result if $|V(T)| > 11$. Henceforth we may assume $|V(T)| = 11$, so T is 5-regular. Hence $tt_3(T) = \sum_{v \in V(T)} \binom{5}{2} = 110$. Thus $c_3(T) = \binom{11}{3} - 110 = 55$. Now a tournament of order 5 contains at most five 3-cycles, and it contains exactly five if and only if it is R_5 the 2-regular tournament on 5-vertices. If all the 3-cycles are dominated, the outneighbourhood of every vertex induces an R_5 . But then a vertex u dominates at most two inneighbours of any other vertex v . Now if T had no k -good 3-cycles, then by Lemma 21-(ii), for every vertex v the subtournament $T[N^-(v)]$ would have a 3-cycle, which cannot be dominated and thus is k -good, a contradiction.
- Assume $k = 4$. Since $\delta^+(T) \geq 6$, then $|V(T)| \geq 13$. By Lemma 21-(iii), we have the result if $|V(T)| > 13$. Henceforth we may assume $|V(T)| = 13$, so T is 6-regular. It is possible to test all 6-regular graphs on 13 vertices using a simple computer program and verify that each of them has at least one good 3-cycle. The source code of the computer program is available at <http://kam.mff.cuni.cz/~bernard/pub/6-regular.cpp>*

\square

Corollary 23. *For $k \in \{1, 2, 3, 4\}$, $f_T(1, 2) = k+2$.*

Proof. Let $k \in \{1, 2, 3, 4\}$. Theorem 22 implies $f_T(1, k) \leq k+2$ and Corollary 15 yields $f_T(1, k) \geq k+2$. \square

We believe that Theorem 22 can be extended to all values of k .

Conjecture 24. *Let k be a positive integer. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k -good 3-cycle.*

*We plan to post the code on arXiv for a more permanent storage.

A first step to prove this conjecture is the following.

Conjecture 25. *Let k be a positive integer. If T is a $(k+2)$ -regular tournament, then T contains a k -good 3-cycle.*

If true Conjecture 24 would be best possible.

Proposition 26. *Let k be a positive integer. For any $n \geq 3k + 3$, there is a tournament of order n with minimum outdegree $k + 1$ that does not admit any $(1, k)$ -outdegree-splitting.*

Proof. Let $n \geq 3k + 3$. Let T be a tournament of order n whose vertex set can be partitioned into $(X_1, X_2, \{x\})$ such that $X_1 \rightarrow X_2$, $X_2 \rightarrow x$, $x \rightarrow X_1$, $T[X_1]$ is a transitive tournament of order $n - 2k - 2$, and $T[X_2]$ is a k -regular tournament.

Clearly, $\delta^+(T) = k + 1$. Let us now prove that T has no $(1, k)$ -outdegree-splitting.

Suppose for a contradiction that T admits a $(1, k)$ -outdegree-splitting (V_1, V_2) . The set V_2 must contain a vertex in X_2 because $T[X_1 \cup \{x\}]$ is transitive. The subtournament $T[V_1]$ contains a 3-cycle C . This cycle either contains x or is contained in C_1 .

- If C contains x , then $C = xx_1x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. But $T[X_2]$ is k -regular, so it is strong. Thus there is a vertex u of $V_2 \cap X_2$ dominating a vertex in $V_1 \cap X_2$. Thus u has outdegree at most $k - 1$ in $T[V_2]$, a contradiction.
- If C is contained in C_1 , then $|V_2 \cap X_2| \leq 2k - 2$. Therefore $T[V_2 \cap X_2]$ has a vertex u with outdegree less than $k - 1$. This vertex u has outdegree less than k in $T[V_2]$, a contradiction.

□

4.2 Existence of k -good 3-cycles

A result of Song [11] states that every 2-strong tournament of order at least 6 can be split into a 3-cycle and a strong subtournament unless it is P_7 , the Paley tournament of order 7. Since P_7 is 3-regular, it has a 1-good 3-cycle by Theorem 22. Therefore we obtain the following.

Theorem 27. *Every 2-strong tournament of order at least 6 has a 1-good 3-cycle and thus admits a $(1, 1)$ -outdegree-splitting.*

In fact, having a 1-good 3-cycle is equivalent to having a $(1, 1)$ -outdegree-splitting.

Proposition 28. *Let T be a tournament. Then T has a $(1, 1)$ -outdegree-splitting if and only if it has a 1-good 3-cycle C .*

Proof. As we already observed, if C is a 1-good 3-cycle, then T has a $(1, 1)$ -outdegree-splitting.

Conversely, suppose that T admits a $(1, 1)$ -outdegree-splitting (V_1, V_2) . Then for $i = 1, 2$, $T[V_i]$ contains a 3-cycle C_i . Let S_2 be the largest set such that $V_2 \subseteq S_2 \subseteq V(T - C_1)$ and $\delta^+(T[S_2]) \geq 1$. If $S_2 = V(T - C_1)$, then C_1 is k -good. If not, then let $R = V(T - C_1) \setminus S_2$. By definition, $S_2 \rightarrow R$. Thus $\delta^+(T - C_2) \geq 1$, and C_2 is k -good. □

Unfortunately, Proposition 28 cannot be generalized for larger value of k in the sense that there are tournaments with a $(1, k)$ -splitting and no k -good 3-cycles. Furthermore, there are such tournaments with minimum outdegree $k + 1$; this shows that the condition of having minimum outdegree $k + 2$ in Conjecture 24 is best possible.

Proposition 29. *Let k be an integer greater than 1. There exists a tournament of order at $3k + 3$ with minimum outdegree $k + 1$ such that T has a $(1, k)$ -splitting but no k -good 3-cycles.*

Proof. Let T be a tournament whose vertex set can be partitioned into $(X_1, X_2, X_3, \{x\})$ such that $X_1 \rightarrow X_2$, $X_1 \cup X_2 \rightarrow X_3$, $X_3 \rightarrow x$, $x \rightarrow X_1 \cup X_2$, $T[X_1]$ is a transitive tournament of order $k - 2$, and $T[X_2]$ is a 3-cycle and $T[X_3]$ is a k -regular tournament.

Clearly, $(X_1 \cup X_2 \cup \{x\}, X_3)$ is a $(1, k)$ -splitting of T .

Let us now prove that no 3-cycle is k -good. There are three kinds of 3-cycles: $T[X_2]$, 3-cycles contained in $T[X_3]$, and 3-cycles of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$.

- $T[X_2]$ is not k -good, because x has outdegree less than k in $T - X_2$.
- If C is a 3-cycle in $T[X_3]$, then $T[X_3] - C$ has at most $2k - 2$ vertices and thus contains a vertex v of outdegree less than $k - 1$. Therefore v has outdegree less than k in $T - C$. So C is not k -good.
- If C is a 3-cycle of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$, then every inneighbour v of z in $T[X_3]$ has outdegree less than k in $T - C$. So C is not k -good.

□

Problem 30. For any fixed $k \geq 2$, are there infinitely many strong tournaments with minimum outdegree $k + 1$ that have a $(1, k)$ -splitting but no k -good 3-cycles?

5 Finding outdegree splittings in tournaments

Theorem 31. For every positive integers k_1 and k_2 , TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING is polynomial-time solvable.

Proof. Let $g(k) = k^2/2 + 3k/2 + 1$. Let T be a tournament of order n . If T has a (k_1, k_2) -outdegree-splitting (V_1, V_2) , then V_1 contains a subset S_1 of size at most $g(k_1)$ such that $\delta^+(T[S_1]) \geq k_1$.

The algorithm considers all subsets S_1 of order at most $g(k_1)$. For each of them, we first check if $\delta^+(T[S_1]) \geq k_1$. If no, we proceed to the next subtournament. If yes, we check if there is a (k_1, k_2) -outdegree-splitting (V_1, V_2) such that $S_1 \subseteq V_1$ using a procedure `extend`(S_1). If this procedure, returns ‘yes’, then we also return ‘yes’. If not we proceed to the next subtournament.

The procedure `extend`(S_1) proceeds as follows. If $S_1 = V(T)$, return ‘no’. If $T - S_1$ has minimum outdegree at least k_2 , we return $(S_1, V(T) \setminus S_1)$. Otherwise, pick a vertex x of $V(T) \setminus S_1$ having outdegree less than k_2 in $T - S_1$ and return `extend`($S_1 \cup \{x\}$).

The procedure `extend` runs in $O(n^2)$ -time. (We only need to make $O(n)$ updates on the score sequence). At worse, we run it for each subset S_1 of size at most $g(k_1)$. There are $O(n^{g(k_1)})$ such subsets. Hence the algorithm runs in $O(n^{g(k_1)+2})$ time. □

The running time of the algorithm given in the proof of Theorem 31 is certainly not optimal. When $k_1 = 1$, running time is $O(n^5)$. We now give a faster algorithm, that runs in $O(n^3)$ time for $k_1 = 1$ and $k_2 \geq 2$ and in $O(n^2)$ time for $k_1 = k_2 = 1$. This algorithm is also faster than the one described in Theorem 11.

The key ingredients are the following three statements. The first one is an immediate extension of Proposition 3 with an identical proof, which translates into a $O(n^2)$ -time algorithm.

Proposition 32. Let D be a digraph of order n . If D contains two disjoint digraphs D_1, D_2 such that $\delta^+(D_i) = k_i$ for $i = 1, 2$ and $d_D^+(v) \geq k_1 + k_2 - 1$ for all $v \in V(D - (D_1 \cup D_2))$, then D admits a (k_1, k_2) -outdegree-splitting. Moreover such a (k_1, k_2) -outdegree-splitting can be found in $O(n^2)$ time.

The second one is an algorithmic version of Theorem 17.

Proposition 33. *Let T be a tournament with minimum outdegree at least k . One can find in $O(n^3)$ time a subtournament T' of T with minimum outdegree k such that $|V(T')| \leq k^2/2 + 3k/2 + 1$.*

Proof. By the proof of Theorem 17, if $|V(T)| > k^2/2 + 3k/2 + 1$, then it contains a vertex x such that $T - x$ has minimum outdegree at least k . Such a vertex can be found in $O(n^2)$ time, by finding the set S of vertices with outdegree k , and taking x not in $S \cup N^+(S)$. We then recursively apply the procedure to $T - x$. As we reduce the order of the tournament at most n times, we find the desired subtournament T' in $O(n^3)$ time. \square

Lemma 34. *Let T be a tournament and v a vertex of T . If T has a $(1, k)$ -outdegree-splitting (V_1, V_2) with $v_1 \in V_1$, then there is a 3-cycle C_1 in $T[V_1]$ such that $v \in V(C_1)$ or $V(C_1) \subseteq N^+(v)$.*

Proof. Let $N_1 = N^+(v) \cap V_1$. If $T[N_1]$ has a cycle, then it is the desired 3-cycle. Otherwise, $T[N_1]$ is a transitive tournament. Now the sink w of $T[N_1]$ has an outneighbour u in $T[V_1]$, which is necessarily an inneighbour of v , by definition of N_1 . Therefore uvw is the desired 3-cycle. \square

Theorem 35. (i) TOURNAMENT $(1, 1)$ -OUTDEGREE-SPLITTING can be solved in $O(n^2)$ time;
(ii) for all $k \geq 2$, TOURNAMENT $(1, k)$ -OUTDEGREE-SPLITTING can be solved in $O(n^3)$ time.

Proof. (i) Let us describe a procedure $(1, 1)$ -split(T) that given a tournament T returns ‘yes’ if it admits a $(1, 1)$ -outdegree-splitting, and returns ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.
1. If $\delta^+(T) = 0$, then the tournament T has no $(1, 1)$ -outdegree-splitting, and we return ‘no’.
2. If $\delta^+(T) \geq 3$, the answer is ‘yes’, by Corollary 23.
3. If $\delta^+(T) \in \{1, 2\}$, let v be a vertex of degree 1 or 2 in T . Without loss of generality, one may look for a $(1, 1)$ -outdegree-splitting (V_1, V_2) of T such that $v \in V_1$. For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - \{u, v, w\}$ contains a 3-cycle. If yes for at least one choice of $\{u, v, w\}$, the answer is ‘yes’ by Proposition 3 since $\delta^+(T) \geq k$. If not, then return ‘no’. This is valid by Lemma 34.

Given its score sequence, checking if a tournament of order n contains a 3-cycle can be done in $O(n)$ by checking whether the score sequence is distinct from $(0, 1, 2, \dots, n - 1)$, the score sequence of the transitive tournament. Since the score sequence of $T - \{u, v, w\}$ can be obtained in linear time from the list of outdegrees of T , checking if $T - \{u, v, w\}$ contains a cycle can be done in $O(n)$ time.

Now since v has degree at most 2, the procedure considers at most $2(n - 1)$ subtournaments $T - \{u, v, w\}$. Therefore $(1, 1)$ -split runs in $O(n^2)$ time.

(ii) Let us describe a procedure $(1, k)$ -split(T) that given a tournament T returns ‘yes’ if T admits a $(1, k)$ -outdegree-splitting, and return ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.
1. If $\delta^+(T) = 0$, then the tournament T has no $(1, k)$ -outdegree-splitting, and we return ‘no’.

2. If $1 \leq \delta^+(T) \leq k - 1$, let U_1 be the set of vertices of degree less than k in T . Clearly, for any $(1, 2)$ -outdegree-splitting (V_1, V_2) of T , $U_1 \subseteq V_1$. Let v be a vertex of U_1 . For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - (U_1 \cup \{u, v, w\})$ contains a subtournament of minimum outdegree k using the procedure `Outdegree- k -Subtournament` described below. If yes for at least one choice of $\{u, v, w\}$, the answer is ‘yes’ by Proposition 32 since all vertices of $V(T) \setminus U_1$ have outdegree at least k in T . If not, then return ‘no’. This is valid by Lemma 34.
3. If $\delta^+(T) \geq k$, then we first find a subtournament T' of T with $\delta(T') \geq k$ and $|V(T')| \leq k^2/2 + 3k/2 + 1$. If $T - T'$ contains a 3-cycle, then T admits a $(1, k)$ -outdegree-splitting by Proposition 3, and so we return ‘yes’. If not then $T - T'$ is a transitive tournament and all 3-cycles of T intersect T' and therefore there are at most $(k^2/2 + 3k/2 + 1)n^2$ of them. For each 3-cycle C , we check with `Outdegree- k -Subtournament` whether $T - C$ contains a subtournament of minimum outdegree k . If yes, for one of them, then we return ‘yes’ because there is a $(1, k)$ -outdegree-splitting by Proposition 3. If not, then we return ‘no’.

Remark 36. In the above procedure, one can shorten Step 3 if $\delta^+(T) \geq k + 2$. In this case, by Corollary 23, we can directly return ‘yes’.

The procedure `Outdegree- k -Subtournament`(T) takes as an input the tournament T as well as its list of outdegrees and a list L of vertices in the transitive tournament $T - T'$ ordered in increasing order of their outdegrees. Observe that the list of outdegrees is already computed when `degree-($1, k$)-split` call this procedure and the order of $T - T'$ can be computed just once after computing T' . First, we alter the list of outdegrees by keeping the outdegrees for vertices in T' but for vertices in $T - T'$ we count only outneighbours in T' . At each step, `Outdegree- k -Subtournament` first checks $V(T)$ and returns ‘no’ if $V(T) = \emptyset$, otherwise it tries to find a vertex v with $d^+(v) < k$. Notice that possible candidates for v are only vertices in T' and the first k vertices in L . If there is no such vertex v , it returns ‘yes’. Otherwise it removes v and tries again. If $v \in V(T')$, then it decreases the outdegree of all inneighbours of v and if $v \notin V(T')$, then it decreases outdegrees only for inneighbours from $V(T')$. The total time spent on a vertex $v \in V(T')$ is $O(n)$, which gives $O(V(T')n) = O(n)$ in total. The total time spent on a vertex $v \notin V(T')$ is $O(1)$, which gives $O(n)$ in total. Therefore, `Outdegree- k -Subtournament` runs in $O(n)$ time.

Now Step 1 runs in constant time. In Step 2, there are at most $k+1$ candidates for w , and thus `Outdegree- k -Subtournament` is called less than $(k + 1)n$ times. Therefore Step 2 runs in $O(n^3)$ time. Step 3 first finds a small subtournament T' with outdegree k , which can be done in $O(n^3)$ time by Proposition 33. Then it runs $O(n^2)$ times `Outdegree- k -Subtournament`. Therefore Step 3 runs in $O(n^3)$ time.

Overall `($1, k$)-split` runs in $O(n^3)$ time. □

The procedure `($1, k$)-split`(T) can be modified to find a $(1, k)$ -outdegree-splitting if it exists, using Proposition 32 instead of Proposition 3.

In contrast, the procedure `($1, 1$)-split`(T) cannot be instantly modified into a procedure that finds a $(1, 1)$ -outdegree-splitting if it exists. However, using a similar approach, we now describe such a procedure.

Theorem 37. *One can find a $(1, 1)$ -outdegree-splitting of a tournament in $O(n^2)$ time.*

Proof. Let us describe a procedure `($1, 1$)-findsplit`(T) that returns a $(1, 1)$ -outdegree-splitting of the tournament T if it admits one, and return ‘no’, otherwise.

We first compute the outdegree of every vertex and we determine $\delta^+(T)$.

If T contains a vertex of outdegree 0, then we return ‘no’. If $\delta^+(T) \geq 4$, then we pick a vertex x and find a 3-cycle C containing x . Such a cycle can be found in $O(n^2)$ by testing if there is an arc from $N^+(x)$ to $N^-(x)$. We return $(V(C), V(T - C))$. This is valid since $\delta^+(T - C) \geq \delta^+(T) - |V(C)| \geq 1$.

If $\delta^+(T) \leq 3$, we choose a vertex v such that $d^+(v) \in \{1, 2, 3\}$. If $T[N^+(v)]$ induces a 3-cycle, then we check whether $T - N^+(v)$ contains a cycle C . If yes, we extend $(T[N^+(v)], C)$ into a $(1, 1)$ -outdegree-splitting by Proposition 32. If not, for every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check if $T - \{u, v, w\}$ contains a cycle $C(uvw)$. If yes for at least one choice of $\{u, w\}$, then we extend $(uvw, C(uvw))$ into a $(1, 1)$ -outdegree-splitting by Proposition 32 and we return ‘no’ otherwise. This is valid by Lemma 34.

Since there are at most three candidates for w , there are $O(n)$ cases to check. Therefore $(1, 1)$ -findsplit runs in $O(n^2)$ time. \square

Remark 38. The proof of Proposition 28 yields a $O(n^2)$ -time procedure to find a 1-good 3-cycle given a $(1, 1)$ -outdegree-splitting. Combining this procedure with $(1, 1)$ -findsplit, we obtain a $O(n^2)$ -time algorithm that finds a 1-good 3-cycle in a tournament if it exists, and returns ‘no’ otherwise.

6 Further research

6.1 Splittable score sequences

Being $(1, 1)$ -outdegree-splittable is not determined by the score sequence. For example, the two tournaments depicted Figure 2 have score sequences $(2, 2, 2, 2, 3, 4)$ but the one to the left has no $(1, 1)$ -outdegree-splitting (See Proposition 26) while the one to the right admits the $(1, 1)$ -outdegree-splitting $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$.

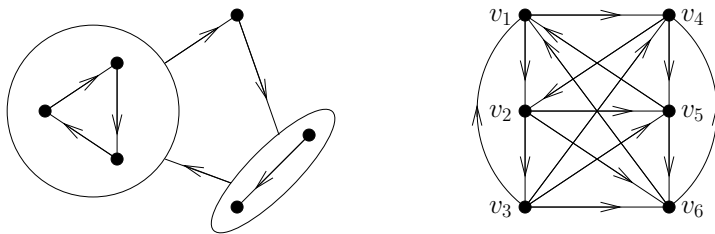


Figure 2: Non- $(1, 1)$ -outdegree-splittable and $(1, 1)$ -outdegree-splittable tournaments with the same score sequence

However there are score sequences s such that all tournaments with score sequence s are $(1, 1)$ -outdegree-splittable. Such score sequences are said to be $(1, 1)$ -outdegree-splittable. For example, Theorem 22 implies that (s_1, \dots, s_n) is $(1, 1)$ -outdegree-splittable.

Problem 39. Which score sequences are $(1, 1)$ -outdegree-splittable?

6.2 Erdős-Posa property for digraphs with minimum outdegree k

McCuaig’s algorithm [7] relies on the theorem stating that a digraph D has either two disjoint cycles or a set S of at most three vertices such that $D - S$ is acyclic. More generally, Reed et al. [9] showed that cycles in digraphs have the Erdős-Posa property.

Theorem 40 (Reed et al. [9]). *For every positive integer n , there exists an integer $t(n)$ such that for every digraph D , either D has a n pairwise-disjoint cycles, or there exists a set T of at most $t(n)$ vertices such that $D - T$ is acyclic.*

It is then natural to ask whether digraphs with maximum outdegree k have the the Erdős-Posa property.

Problem 41. Let k be a fixed integer. For every positive integer n , does there exist an integer $t_k(n)$ such that for every digraph D , either D has a n pairwise-disjoint subdigraphs with minimum outdegree k , or there exists a set T of at most $t_k(n)$ vertices such that $\delta^+(D - T) < k$?

6.3 Strong connectivity and outdegree-splitting with prescribed vertices

Any $f_T(k_1, k_2)$ -strong tournament has minimum outdegree at least $f_T(k_1, k_2)$ and thus admits a (k_1, k_2) -outdegree-splitting. Therefore, it is natural to ask the following.

Problem 42. What is the minimum integer $h_T(k_1, k_2)$ such that every $h_T(k_1, k_2)$ -strong tournament T of order at least $2k_1 + 2k_2 + 2$ contains a (k_1, k_2) -outdegree-splitting?

The condition $|V(T)| \geq 2k_1 + 2k_2 + 2$ in the above problem is just to avoid the small tournaments that cannot have any (k_1, k_2) -outdegree-splitting for cardinality reasons. Clearly, $h_T(k_1, k_2) \leq f_T(k_1, k_2)$. But it is very likely that $h_T(k_1, k_2)$ is smaller than $f_T(k_1, k_2)$. As mentioned in the beginning of Subsection 4.2, a result of Song [11] implies that $h_T(1, 1) \leq 2$ (In fact $h_T(1, 1) = 2$ because a 1-strong tournament T with a vertex v such that $T - v$ is a transitive tournament has clearly no $(1, 1)$ -outdegree-splitting.) whereas $f_T(1, 1) = 3$.

One might also ask similar questions for outdegree-splitting with prescribed vertices (as in Theorem 8). Bang-Jensen et al. [2] proved that if T is a tournament of order 8 and xy an arc in T such that $T \setminus xy$ is 2-strong, then T contains an outdegree-1-splitting (V_x, V_y) with $x \in V_x$ and $y \in V_y$.

References

- [1] N. Alon, Disjoint cycles, *J. Combinatorial Theory, Ser. B* **68** (1996), 167-178.
- [2] J. Bang-Jensen, Y. Guo, and A. Yeo. Complementary cycles containing prescribed vertices in tournaments. *Discrete Math.* 214 (2000), no. 1-3, 77-87.
- [3] J.-C. Bermond and C. Thomassen, Cycles in digraphs—a survey, *J. Graph Theory* **5** (1981), no. 1, 1-43.
- [4] P. Camion, Chemins et circuits hamiltoniens des graphes complets, *C.R. Acad. Sci. Paris* **249** (1959), 2151-2152.
- [5] G. Chen, R.J. Gould, and H. Li, Partitioning vertices of a tournament into independent cycles, *J. combin. Theory Ser B, Vol 83*, no. 2 (2001) 213-220.
- [6] H. G. Landau. On dominance relations and the structure of animal societies. III. The condition for a score structure. *Bulletin of Mathematical Biophysics* **15** (1953), no. 2. 143-148.
- [7] Intercyclic digraphs. *Graph Structure Theory*, (Neil Robertson and Paul Seymour, eds.), *AMS Contemporary Math.*, 147:203-245, 1993.
- [8] J. W. Moon, *Topics on Tournaments*, (Holt, Rinehart and Winston, New York) (1968).
- [9] B. Reed, N. Robertson, P.D. Seymour, and R. Thomas. Packing directed circuits. *Combinatorica*, 16(4):535-554, 1996.
- [10] K.B. Reid, Three problems on tournaments, *Graph Theory and Its Applications*, East. and West. Ann. New York Acad. Sci. 576 (1989), 466-473.
- [11] Z. Song. Complementary cycles of all lengths in tournaments *J. Combin. Theory Ser. B*, **57** (1993), 18-25.
- [12] M. Stiebitz, Decomposing graphs under degree constraints, *J. Graph Theory* **23** (1996), 31-324.
- [13] C. Thomassen, Disjoint cycles in digraphs, *Combinatorica* **3** (1983), no. 3-4, 393-396.



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