

Splitting a tournament into two subtournaments with given minimum outdegree

Frédéric Havet, Bernard Lidicky

► **To cite this version:**

Frédéric Havet, Bernard Lidicky. Splitting a tournament into two subtournaments with given minimum outdegree. [Research Report] RR-8469, INRIA. 2014. hal-00943152

HAL Id: hal-00943152

<https://hal.inria.fr/hal-00943152>

Submitted on 7 Feb 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Splitting a tournament into two subtournaments with
given minimum outdegree*

Frédéric Havet — Bernard Lidický

N° 8469

February 2014

Thème COM

A large blue rectangle occupies the lower half of the page. On the left side of this rectangle is a large, light gray, stylized letter 'R'. To the right of the 'R', the words 'Rapport de recherche' are written in a white, serif font. A horizontal white line is positioned below the text.

*Rapport
de recherche*

Splitting a tournament into two subtournaments with given minimum outdegree

Frédéric Havet* , Bernard Lidický†

Thème COM — Systèmes communicants
Équipe-Projet Coati

Rapport de recherche n° 8469 — February 2014 — 15 pages

Abstract: A (k_1, k_2) -outdegree-splitting of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2 , respectively. We show that there exists a minimum function f_T such that every tournament of minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting, and $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$. We also show a polynomial-time algorithm that finds a (k_1, k_2) -outdegree-splitting of a tournament if one exists, and returns ‘no’ otherwise. We give better bound on f_T and faster algorithms when $k_1 = 1$.

Key-words: tournament, splitting, outdegree

* Projet Coati, I3S (CNRS, UNSA) and INRIA Sophia Antipolis, Frederic.Havet@cnrs.fr, Partially supported by ANR Blanc STINT ANR-13-BS02-0007. Part of this work was done while F. Havet was on leave at Simon Fraser University, PIMS, UMI 3069, CNRS

† Department of Mathematics, University of Illinois, Urbana, IL, USA, lidicky@illinois.edu. Research is partially supported by NSF grant DMS-1266016.

Partager un tournoi en deux sous-tournois de degré sortant minimum prescrits

Résumé : Un (k_1, k_2) -partage d'un digraphe D est une partition (V_1, V_2) de son ensemble de sommets telle que $D[V_1]$ et $D[V_2]$ soient de degré sortant minimum au moins k_1 et k_2 , respectivement. Nous établissons l'existence d'une fonction (minimum) f_T telle que tout tournoi de degré sortant minimum au moins $f_T(k_1, k_2)$ a un (k_1, k_2) -partage, et que $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$. Nous donnons également un algorithme en temps polynomial qui trouve un (k_1, k_2) -partage d'un tournoi s'il en existe un et renvoie 'non' sinon. Nous donnons de meilleures bornes sur f_T et des algorithmes plus rapides pour $k_1 = 1$.

Mots-clés : tournoi, partage, degré sortant

1 Introduction

Let D be a digraph. For a vertex $v \in V(D)$ the *outdegree* of v , denoted by $d_D^+(v)$, is the number of arcs directed away from v . The minimum outdegree over all vertices of D is denoted by $\delta^+(D)$. We drop D in $d_D^+(v)$ and $\delta^+(D)$ if it is clear from the context.

A (k_1, k_2) -*outdegree-splitting* of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2 , respectively. A digraph admitting a (k_1, k_2) -outdegree-splitting is said to be (k_1, k_2) -*outdegree-splittable*.

Problem 1 (Alon [1]). Is there a function f such that every digraph with minimum outdegree $f(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting?

The existence of the corresponding function f for the undirected analogue is easy and has been observed by many authors. Stiebitz [12] even proved the following tight result: if the minimum degree of an undirected graph G is $d_1 + d_2 + \dots + d_k$, where each d_i is a non-negative integer, then the vertex set of G can be partitioned into k pairwise disjoint sets V_1, \dots, V_k , so that for all i , the induced subgraph on V_i has minimum degree at least d_i . This is clearly tight, as shown by an appropriate complete graph.

Problem 1 is equivalent to the following:

Problem 2. Is there a function $f'(k_1, k_2)$ such that every digraph with minimum outdegree $f'(k_1, k_2)$ has two disjoint (induced) subdigraphs, one of them with minimum outdegree k_1 and the other with minimum outdegree k_2 ?

This follows from the following proposition.

Proposition 3. *Let D be a digraph with minimum outdegree at least $k_1 + k_2 - 1$. If D contains two disjoint subdigraphs D_1 and D_2 such that $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$, then D has a (k_1, k_2) -outdegree-splitting.*

Proof. Consider two disjoint digraphs D_1 and D_2 with $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$ such that $V(D_1) \cup V(D_2)$ is maximum. Suppose for a contradiction that $S = V(D) \setminus (V(D_1) \cup V(D_2))$ is not empty. Then every vertex $s \in S$ has at most $k_1 - 1$ outneighbours in D_1 otherwise $D_1 + s$ and D_2 contradict the maximality of D_1 and D_2 . Hence every vertex of S has at least $k_1 + k_2 - 1 - (k_1 - 1) = k_2$ outneighbours in $D - D_1$. It follows that $D - D_1$ has minimum degree k_2 . So D_1 and $D - D_1$ contradicts the maximality of D_1 and D_2 . \square

Corollary 4. $f(k_1, k_2) \leq \max\{f'(k_1, k_2), k_1 + k_2 - 1\}$.

This implies in particular that $f(1, 1) = f'(1, 1) = 3$. Indeed Thomassen [13] showed that every digraph of minimum outdegree at least 3 has two disjoint cycles. (In this paper, paths and cycles are always directed.)

This is a special case of Bermond-Thomassen Conjecture [3]:

Conjecture 5 (Bermond and Thomassen [3]). *Every digraph with $\delta^+ \geq 2k - 1$ contains k disjoint cycles.*

Note that Alon [1] proved that if $\delta^+ \geq 64k$ there are k disjoint cycles.

A *tournament* is a digraph such that for every two distinct vertices u, v there is exactly one arc with ends $\{u, v\}$ (so, either the arc uv or the arc vu but not both).

In this paper, we settle Problem 1 for tournaments.

Theorem 6. *Every tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$ has a (k_1, k_2) -outdegree-splitting.*

To prove Theorem 6, we shall prove the following theorem.

Theorem 7. *Every tournament with minimum outdegree at least k has a subtournament with minimum outdegree k and order at most $k^2/2 + 3k/2 + 1$.*

We can then easily derive Theorem 6.

Proof of Theorem 6. Let T be a tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$. By Theorem 7, there exists a subtournament T_1 with minimum outdegree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + 1$. Let $T_2 = T - T_1$. Then $\delta^+(T_2) \geq \delta^+(T) - |V(T_1)| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2) -outdegree-splitting. \square

In fact, we prove a more general statement than Theorem 7 (Theorem 17). This enables us to prove the following generalization of Theorem 6.

Theorem 8. *Let $m = \max\{k_1^2/2 + 3k_1/2 + k_2 + u_1 + 1, k_1 + u_2\}$, let T be a tournament of minimum outdegree at least m and let U_1 and U_2 be two disjoint subsets of $V(T)$ of cardinality u_1 and u_2 respectively. Then there is a (k_1, k_2) -outdegree-splitting (V_1, V_2) of T such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.*

The bound of Theorem 6 is certainly not tight. Theorem 6 asserts that every tournament with minimum outdegree 4 has a $(1, 1)$ -outdegree-splitting, but we know that having outdegree 3 is sufficient.

Problem 9. What is the minimum integer $f_T(k_1, k_2)$ such that every tournament with minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2) -outdegree-splitting?

Theorem 6 implies that $f_T(1, k) \leq k + 3$. We describe examples implying $f_T(1, k) \geq k + 2$, and we conjecture that this lower bound is the exact value.

Conjecture 10. *For any positive integer k , $f_T(1, k) = k + 2$.*

In Section 4, we establish this conjecture for $k \in \{2, 3, 4\}$, that is, we prove $f_T(1, 2) = 4$, $f_T(1, 3) = 5$, and $f_T(1, 4) = 6$.

Next we consider problems of deciding whether a digraph admits a (k_1, k_2) -outdegree-splitting.

OUTDEGREE SPLITTING

Input: A digraph D and two positive integers k_1 and k_2 .

Question: Does D admit a (k_1, k_2) -outdegree-splitting?

Particular cases of this problem are when k_1 and k_2 are fixed integers and not part of the input. Hence for every fixed k_1, k_2 , we have the following problem.

(k_1, k_2) -OUTDEGREE-SPLITTING

Input: A digraph D .

Question: Does D admit a (k_1, k_2) -outdegree-splitting?

Theorem 11. $(1, 1)$ -OUTDEGREE-SPLITTING is polynomial-time solvable.

Proof. Let us describe a polynomial-time algorithm solving $(1, 1)$ -OUTDEGREE SPLITTING.

If the input digraph D has a vertex with outdegree 0, then the answer is ‘no’ because this vertex has outdegree 0 in any subdigraph of D containing it. Henceforth we may assume that $\delta^+(D) \geq 1$.

It is well-known that a digraph with outdegree at least 1 contains a cycle. Therefore, Proposition 3 implies that a digraph with minimum outdegree at least 1 admits a $(1, 1)$ -outdegree-splitting if and only if it contains two disjoint cycles. Thus it is enough to decide whether D contains two disjoint cycles.

But deciding whether a digraph contains two disjoint cycles can be done in polynomial time as shown by McCuaig [7]. (See also [9].) \square

In Section 5, we consider the restriction of these problems to tournaments.

TOURNAMENT OUTDEGREE-SPLITTING

Input: A tournament T and two positive integers k_1 and k_2 .

Question: Does T admit a (k_1, k_2) -outdegree-splitting?

TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING

Input: A tournament T .

Question: Does T admit a (k_1, k_2) -outdegree-splitting?

TOURNAMENT $(1, 1)$ -OUTDEGREE-SPLITTING is a particular case of $(1, 1)$ -OUTDEGREE-SPLITTING, and thus is polynomial-time solvable. In Theorem 31, we show that, more generally, for any k_1, k_2 , **TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING** can be solved in $O(n^{k^2/2+3k/2+3})$ time. We then describe a faster algorithm solving **TOURNAMENT $(1, k_2)$ -OUTDEGREE-SPLITTING**. It runs in $O(n^3)$ time for $k_2 \geq 2$ and in $O(n^2)$ time for $k_2 = 1$. In view of these results, it is natural to ask the following.

Problem 12. Is **TOURNAMENT OUTDEGREE-SPLITTING** fixed-parameter tractable with (k_1, k_2) as a parameter? In other words, can we solve **TOURNAMENT OUTDEGREE SPLITTING** in $F(k_1, k_2)P(n)$ time, where F is an arbitrary computable function and P is a polynomial in the order n of the input tournament?

Finally, in Section 6, we present some possible directions for further research.

2 Definitions and folklore on tournaments

The *score sequence* of a tournament T , denoted by $s(T)$, is the non-decreasing sequence of outdegrees of its vertices. Landau [6] characterized the non-decreasing sequences of integers that are score sequences.

Theorem 13 (Landau [6]). *A non-decreasing sequence of non-negative integers (s_1, s_2, \dots, s_n) is a score sequence if and only if:*

$$(i) \quad s_1 + s_2 + \dots + s_i \geq \binom{i}{2}, \text{ for } i = 1, 2, \dots, n-1, \text{ and}$$

$$(ii) \quad s_1 + s_2 + \dots + s_n = \binom{n}{2}.$$

Condition (ii) in the above theorem implies directly the following proposition.

Proposition 14. *Every tournament of order $2k$ has minimum degree less than k .*

Corollary 15. $f_T(k_1, k_2) \geq k_1 + k_2 + 1$.

Proof. Let T be a $(k_1 + k_2)$ -regular tournament of order $2k_1 + 2k_2 + 1$. In every bipartition (V_1, V_2) of $V(T)$, either $|V_1| \leq 2k_1$ or $|V_2| \leq 2k_2$. Thus, by Proposition 14, either $\delta^+(T[V_1]) < k_1$ or $\delta^+(T[V_2]) < k_2$. \square

An ℓ -*cycle* is a cycle of length ℓ . A tournament T is *transitive* if it contains no cycles. The score sequence of a transitive tournament of order n is $(0, 1, \dots, n-1)$.

We denote by $tt_3(T)$ the number of transitive subtournaments of order 3 in T and by $c_3(T)$ number of 3-cycles in T . Since a tournament of order 3 is either a transitive tournament or a 3-cycle, we have

$$tt_3(T) + c_3(T) = \binom{|T|}{3}.$$

Now if v is a vertex, the number of transitive subtournaments of order 3 with source v is $\binom{d^+(v)}{2}$. Hence

$$tt_3(T) = \sum_{v \in V(T)} \binom{d^+(v)}{2}.$$

A digraph D is *strongly connected* or *strong* if there is a path from u to v for every $u, v \in V(D)$. A digraph D is *k-strong* if $D - X$ is strong for every $X \subset V(D)$ where $|X| \leq k - 1$. A (strong) *component* of D is a strong subdigraph of D which is maximal by inclusion.

Let T be a tournament. Let T_1, T_2, \dots, T_m be the components of T . Then $(V(T_1), V(T_2), \dots, V(T_m))$ is a partition of $V(T)$ and without loss of generality, we may suppose that $T_i \rightarrow T_j$ whenever $i < j$. In this case we say that $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$ is the *decomposition* of T . Component T_1 is said to be the *initial* component of T and T_m its *terminal* component.

A vertex is *pancyclic* in a digraph D if, for every $3 \leq \ell \leq |D|$, it is contained in an ℓ -cycle. To contain a pancyclic vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to Camion's theorem [4]. Moon [8] showed that this condition is sufficient.

Theorem 16 (Moon [8]). *Every vertex of a strong tournament is pancyclic.*

We sometimes use the results of this section without referring to them.

3 Small subtournament of minimum outdegree k

We now prove Theorem 7. In fact, we prove a more general theorem whose particular case with $U = \emptyset$ is Theorem 7.

Theorem 17. *Let T be a tournament with minimum outdegree at least k and $U \subseteq V(T)$ be a subset of vertices. There is a subtournament T' of T with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$.*

Proof. For every p , we prove the result for all sets U of size p by induction on $|V(T)|$, the result holding trivially if $|V(T)| \leq p + k^2/2 + 3k/2 + 1$.

Let T be a tournament of order at least $p + k^2/2 + 3k/2 + 2$ with minimum outdegree at least k and U a set of p vertices of T . Let S be the set of vertices of degree k in T . There are $k|S|$ arcs with their tail in S . Among them $|S|(|S| - 1)/2$ are in S and the remaining ones have their heads out of S . Hence $|N^+(S)| \leq |S| + k|S| - |S|(|S| - 1)/2$. Now the polynomial $P(x) = (k+3/2)x - x^2/2$ increases on $[0, k+3/2]$ and decreases on $[k+3/2, +\infty[$. Moreover $P(k+1) = P(k+2) = k^2/2 + 3k/2 + 1$. Consequently, $|N^+(S)| \leq k^2/2 + 3k/2 + 1$.

Since $|V(T)| \geq p + k^2/2 + 3k/2 + 2$, there is a vertex v which is not in $N^+(S) \cup U$. Thus $T - v$ has minimum outdegree at least k and by induction $T - v$ (and thus also T) has a subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$. \square

The bound $k^2/2 + 3k/2 + 1$ is Theorem 7 is tight in the following sense.

Proposition 18. *For every non-negative integer k , and for every $n \geq k^2/2 + 3k/2 + 1$, there is a tournament $T(n, k)$ of order n and a set $W \subset V(T)$ of order $n - k^2/2 + 3k/2 + 1$ such that for every $U \subset W$, every subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ has order at least $|U| + k^2/2 + 3k/2 + 1$.*

Proof. Consider the disjoint union of a strong tournament S of order $k + 1$ and a transitive tournament TT of order $k(k + 1)/2$. Set $V(S) = \{s_1, \dots, s_{k+1}\}$. Partition $V(TT)$ into $k + 1$ sets A_1, \dots, A_{k+1} such that $|A_i| = k - d_S^+(s_i)$. This is possible since $\sum_{i=1}^{k+1} d_S^+(s_i) = k(k + 1)/2$, so $\sum_{i=1}^{k+1} (k - d_S^+(s_i)) = |V(TT)|$. Now for each

i , add the arc $s_i a$ for all $a \in A_i$, and all the arcs bs_i for all $b \in V(TT) \setminus A_i$. The resulting tournament R has order $k^2/2 + 3k/2 + 1$ and minimum outdegree k .

Let R' be a subtournament of R with outdegree at least k .

It must contain a vertex of S , because all subtournaments of TT are transitive. But each element s of S has outdegree exactly k in R , so if $s \in V(R')$, then $N_R^+(s) \subset V(R')$. Since S is strong, it has a hamiltonian cycle by Camion's Theorem, and so $V(S) \subset V(R')$. But by construction, every vertex in R is dominated by a vertex in S , and thus must be in R' . Hence $R = R'$.

Set $p = n - k^2/2 + 3k/2 + 1$. Let $T(n, k)$ be a tournament obtained from the disjoint union of R and the transitive tournament TT_p of order p by adding all arcs from TT_p towards R . Then, for any set $U \subset V(TT_p)$, every subtournament of $T(n, k)$ with minimum outdegree k containing U must also contain $V(R)$ and thus has order at least $|U| + k^2/2 + 3k/2 + 1$. \square

We can then easily derive Theorem 8 from Theorem 17.

Proof of Theorem 8. Let T be a tournament of minimum outdegree at least m . The tournament $T - U_2$ has minimum outdegree at least k_1 because $|U_2| = u_2$. Thus, by Theorem 17, there exists a subtournament T_1 of $T - U_2$ with minimum degree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + u_1 + 1$ such that $U_1 \subseteq V(T_1)$. Set $V_1 = V(T_1)$, $T_2 = T - T_1$, and $V_2 = V(T_2)$. By definition, $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Now $\delta^+(T_2) \geq \delta^+(T) - |V_1| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2) -outdegree-splitting. \square

3.1 Outdegree-critical tournaments

Theorem 7 can be rephrased in terms of k -outdegree-critical tournament. A tournament T is said to be k -outdegree-critical if it has minimum outdegree k and all its proper subtournaments have outdegree less than k . Theorem 7 implies that all k -outdegree-critical tournaments have bounded size. Hence a natural problem is the following.

Problem 19. Describe the k -outdegree-critical tournaments.

The unique 1-outdegree-critical tournament is the 3-cycle.

We now show that the 2-outdegree-critical tournaments are those depicted in Figure 1.

Theorem 20. Every tournament with minimum outdegree 2 has a subtournament isomorphic to one of those depicted in Figure 1.

Proof. By induction on $|V(T)|$, the result holding trivially when $|V(T)| < 5$.

Let T be a tournament of order at least 5 with minimum outdegree 2. Every vertex v has an inneighbour u such that $d^+(u) = 2$, for otherwise $T - v$ has minimum outdegree at least 2 and by induction $T - u$ (and thus also T) has a subtournament with minimum outdegree 2 and with order 5 or 6.

Let S be the set of vertices of outdegree 2 in T . By the previous remark, S is not empty and $T[S]$ has minimum indegree at least 1. Hence $T[S]$ contains a 3-cycle $C = (x_1, x_2, x_3)$. For $i = 1, 2, 3$, let y_i be the outneighbour of x_i in $V(T) \setminus \{x_1, x_2, x_3\}$. If all y_i are distinct, then each y_i dominates $\{x_1, x_2, x_3\} \setminus \{x_i\}$, and so $T[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is one of the tournaments A_6, B_6, C_6 and D_6 . If $y_1 = y_2 = y_3$, let z_1 and z_2 be the two outneighbours of y_1 . These two vertices dominate $\{x_1, x_2, x_3\}$, so $T[\{x_1, x_2, x_3, y_1, z_1, z_2\}]$ is isomorphic to E_6 . If $y_1 = y_2 \neq y_3$, then y_3 dominates x_1 and x_2 , and y_1 dominates x_3 . If y_1 dominates y_3 , then $T[\{x_1, x_2, x_3, y_1, y_3\}]$ is isomorphic to R_5 . If y_1 is dominated by y_3 , let z be an outneighbour of y_1 distinct from x_3 . The vertex z dominates $\{x_1, x_2\}$, so $T[\{x_1, x_2, x_3, y_1, y_3, z\}]$ is isomorphic to C_6 or D_6 . \square

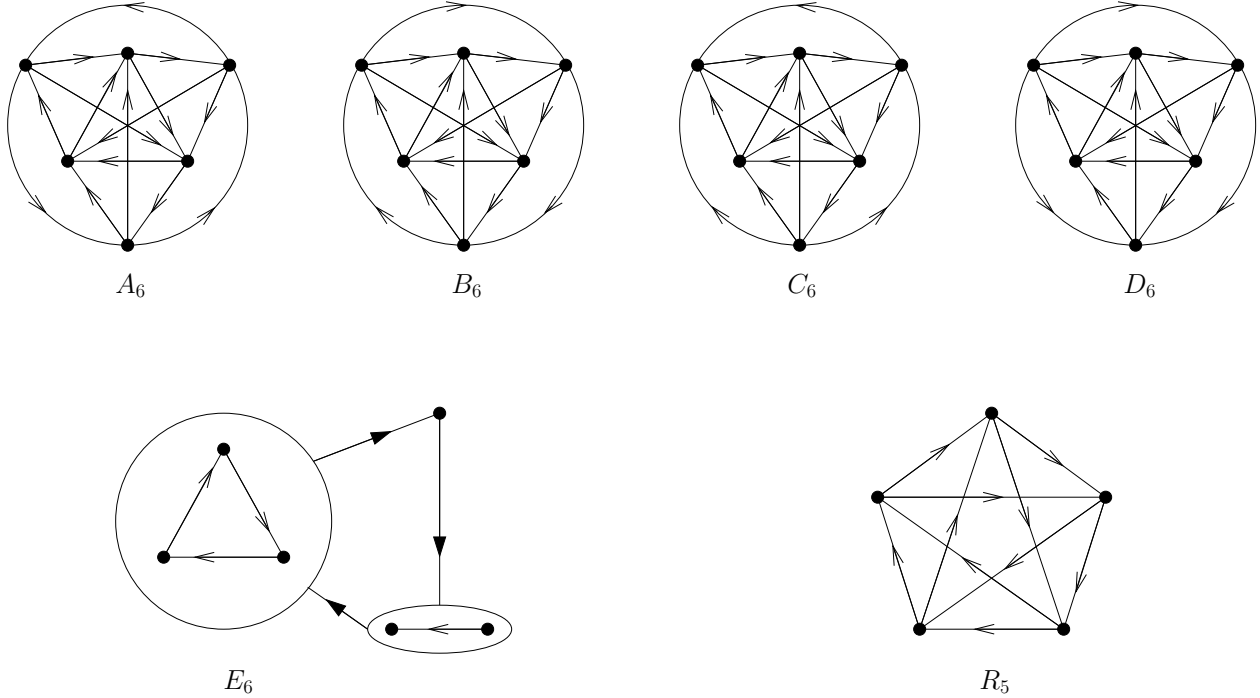


Figure 1: The 2-outdegree-critical tournaments.

4 $(1, k)$ -outdegree-splitting of tournaments

4.1 Improved upper bound for $f_T(1, k)$

A 3-cycle C in a tournament T is said to be k -good if $\delta^+(T - C) \geq k$. Clearly, if C is a k -good 3-cycle, then $(V(C), V(T - C))$ is a $(1, k)$ -splitting of T .

Lemma 21. *Let k be an integer and let T be a strong tournament with minimum outdegree at least $k + 2$. Let S be the set of vertices with outdegree $k + 2$ in T . If T has no k -good 3-cycle, then the following hold.*

- (i) *Every arc is dominated by a vertex in S .*
- (ii) *For every vertex v , the subtournament $T[N^-(v) \cap S]$ has minimum indegree 1 and at least five vertices.*
- (iii) $|V(T)| \leq \frac{1}{10}(k + 7)(k + 8)$.

Proof. Suppose that T contains no k -good 3-cycle. A 3-cycle C in T is S -dominated if there is a vertex $x \in S$ dominating C . Clearly, a 3-cycle in T is k -good if and only if it is not S -dominated. Hence all 3-cycles are S -dominated.

(i) Let uv be an arc. Since T is strong, there is a 3-cycle C containing u by Theorem 16. This cycle is dominated by a vertex $s \in S$. If s dominates v , then s dominates the arc uv . If not, then uvs is a 3-cycle. This cycle is dominated by a vertex $s' \in S$, which thus dominates uv .

(ii) Let v be a vertex of T . By (i), v is dominated by a vertex in S , so $N^-(v) \cap S$ is not empty. For any vertex $s \in N^-(v) \cap S$, the arc sv is dominated by a vertex $s' \in S$, which is distinct from s . Hence $T[N^-(v) \cap S]$ has indegree at least 1 and thus contains a 3-cycle $s_1s_2s_3$. This 3-cycle is dominated by a vertex $s \in S$.

Assume first $s \rightarrow v$. By (i) the arc sv is dominated by a vertex s' of S . Clearly $s' \notin \{s_1, s_2, s_3\}$, because s' dominates $s_1s_2s_3$. Hence s_1, s_2, s_3, s, s' are five vertices in $N^-(v) \cap S$.

Assume now that $v \rightarrow s$. Then ss_1v is a 3-cycle which is dominated by a vertex s' . This vertex is in $N^-(v) \cap S$ and is distinct from s_2, s_3 because it dominates s . Furthermore, by (i) there is a vertex t of S dominating $s'v$. If $t \notin \{s_1, s_2, s_3\}$, then s_1, s_2, s_3, s', t are five vertices in $N^-(v) \cap S$. So we may assume that $t \in \{s_1, s_2, s_3\}$ and, without loss of generality, $t = s_2$. Now, there is a vertex s'' dominating the 3-cycle ss_2v . This vertex is distinct from s_1, s_3 because it dominates s , and is distinct from s' because it dominates s_2 . Hence, s_1, s_2, s_3, s', s'' are five vertices in $N^-(v) \cap S$.

(iii) By (ii), every vertex has at least four inneighbours in S . Thus

$$|V(T)| = |N^+(S)| \leq |S| + \frac{1}{5} \left((k+2)|S| - \binom{|S|}{2} \right).$$

But the polynomial $Q(x) = x + \frac{1}{5} \left((k+2)x - \binom{x}{2} \right) = \frac{1}{10}x(2k+15-x)$ increases on $[0, k+15/2]$ and decreases on $[k+15/2, +\infty[$ and $Q(k+7) = Q(k+8) = \frac{1}{10}(k+7)(k+8)$. Consequently, $|V(T)| \leq \frac{1}{10}(k+7)(k+8)$. \square

Theorem 22. *Let k be an integer in $\{1, 2, 3, 4\}$. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k -good 3-cycle.*

Proof. It is sufficient to prove the result for strong tournaments. Indeed if T is not strong, then its terminal component T' has also outdegree at least $k+2$. Moreover, every 3-cycle that is k -good in T' is also k -good in T .

Henceforth, we may assume that T is strong. Let S be the set of vertices with outdegree $k+2$ in T .

- Assume $k \in \{1, 2\}$. Then every vertex of S has outdegree at most 4 in $T[S]$, so $T[S]$ has a vertex with indegree at most 4. Thus, by Lemma 21-(ii), T has a 1-good 3-cycle.
- Assume $k = 3$. Since $\delta^+(T) \geq 5$, then $|V(T)| \geq 11$. By Lemma 21-(iii), we have the result if $|V(T)| > 11$. Henceforth we may assume $|V(T)| = 11$, so T is 5-regular. Hence $tt_3(T) = \sum_{v \in V(T)} \binom{5}{2} = 110$. Thus $c_3(T) = \binom{11}{3} - 110 = 55$. Now a tournament of order 5 contains at most five 3-cycles, and it contains exactly five if and only if it is R_5 the 2-regular tournament on 5-vertices. If all the 3-cycles are dominated, the outneighbourhood of every vertex induces an R_5 . But then a vertex u dominates at most two inneighbours of any other vertex v . Now if T had no k -good 3-cycles, then by Lemma 21-(ii), for every vertex v the subtournament $T[N^-(v)]$ would have a 3-cycle, which cannot be dominated and thus is k -good, a contradiction.
- Assume $k = 4$. Since $\delta^+(T) \geq 6$, then $|V(T)| \geq 13$. By Lemma 21-(iii), we have the result if $|V(T)| > 13$. Henceforth we may assume $|V(T)| = 13$, so T is 6-regular. It is possible to test all 6-regular graphs on 13 vertices using a simple computer program and verify that each of them has at least one good 3-cycle. The source code of the computer program is available at <http://kam.mff.cuni.cz/~bernard/pub/6-regular.cpp>*

\square

Corollary 23. *For $k \in \{1, 2, 3, 4\}$, $f_T(1, 2) = k+2$.*

Proof. Let $k \in \{1, 2, 3, 4\}$. Theorem 22 implies $f_T(1, k) \leq k+2$ and Corollary 15 yields $f_T(1, k) \geq k+2$. \square

We believe that Theorem 22 can be extended to all values of k .

Conjecture 24. *Let k be a positive integer. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k -good 3-cycle.*

*We plan to post the code on arXiv for a more permanent storage.

A first step to prove this conjecture is the following.

Conjecture 25. *Let k be a positive integer. If T is a $(k+2)$ -regular tournament, then T contains a k -good 3-cycle.*

If true Conjecture 24 would be best possible.

Proposition 26. *Let k be a positive integer. For any $n \geq 3k + 3$, there is a tournament of order n with minimum outdegree $k + 1$ that does not admit any $(1, k)$ -outdegree-splitting.*

Proof. Let $n \geq 3k + 3$. Let T be a tournament of order n whose vertex set can be partitioned into $(X_1, X_2, \{x\})$ such that $X_1 \rightarrow X_2$, $X_2 \rightarrow x$, $x \rightarrow X_1$, $T[X_1]$ is a transitive tournament of order $n - 2k - 2$, and $T[X_2]$ is a k -regular tournament.

Clearly, $\delta^+(T) = k + 1$. Let us now prove that T has no $(1, k)$ -outdegree-splitting.

Suppose for a contradiction that T admits a $(1, k)$ -outdegree-splitting (V_1, V_2) . The set V_2 must contain a vertex in X_2 because $T[X_1 \cup \{x\}]$ is transitive. The subtournament $T[V_1]$ contains a 3-cycle C . This cycle either contains x or is contained in C_1 .

- If C contains x , then $C = xx_1x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. But $T[X_2]$ is k -regular, so it is strong. Thus there is a vertex u of $V_2 \cap X_2$ dominating a vertex in $V_1 \cap X_2$. Thus u has outdegree at most $k - 1$ in $T[V_2]$, a contradiction.
- If C is contained in C_1 , then $|V_2 \cap X_2| \leq 2k - 2$. Therefore $T[V_2 \cap X_2]$ has a vertex u with outdegree less than $k - 1$. This vertex u has outdegree less than k in $T[V_2]$, a contradiction.

□

4.2 Existence of k -good 3-cycles

A result of Song [11] states that every 2-strong tournament of order at least 6 can be split into a 3-cycle and a strong subtournament unless it is P_7 , the Paley tournament of order 7. Since P_7 is 3-regular, it has a 1-good 3-cycle by Theorem 22. Therefore we obtain the following.

Theorem 27. *Every 2-strong tournament of order at least 6 has a 1-good 3-cycle and thus admits a $(1, 1)$ -outdegree-splitting.*

In fact, having a 1-good 3-cycle is equivalent to having a $(1, 1)$ -outdegree-splitting.

Proposition 28. *Let T be a tournament. Then T has a $(1, 1)$ -outdegree-splitting if and only if it has a 1-good 3-cycle C .*

Proof. As we already observed, if C is a 1-good 3-cycle, then T has a $(1, 1)$ -outdegree-splitting.

Conversely, suppose that T admits a $(1, 1)$ -outdegree-splitting (V_1, V_2) . Then for $i = 1, 2$, $T[V_i]$ contains a 3-cycle C_i . Let S_2 be the largest set such that $V_2 \subseteq S_2 \subseteq V(T - C_1)$ and $\delta^+(T[S_2]) \geq 1$. If $S_2 = V(T - C_1)$, then C_1 is k -good. If not, then let $R = V(T - C_1) \setminus S_2$. By definition, $S_2 \rightarrow R$. Thus $\delta^+(T - C_2) \geq 1$, and C_2 is k -good. □

Unfortunately, Proposition 28 cannot be generalized for larger value of k in the sense that there are tournaments with a $(1, k)$ -splitting and no k -good 3-cycles. Furthermore, there are such tournaments with minimum outdegree $k + 1$; this shows that the condition of having minimum outdegree $k + 2$ in Conjecture 24 is best possible.

Proposition 29. *Let k be an integer greater than 1. There exists a tournament of order at $3k + 3$ with minimum outdegree $k + 1$ such that T has a $(1, k)$ -splitting but no k -good 3-cycles.*

Proof. Let T be a tournament whose vertex set can be partitioned into $(X_1, X_2, X_3, \{x\})$ such that $X_1 \rightarrow X_2$, $X_1 \cup X_2 \rightarrow X_3$, $X_3 \rightarrow x$, $x \rightarrow X_1 \cup X_2$, $T[X_1]$ is a transitive tournament of order $k - 2$, and $T[X_2]$ is a 3-cycle and $T[X_3]$ is a k -regular tournament.

Clearly, $(X_1 \cup X_2 \cup \{x\}, X_3)$ is a $(1, k)$ -splitting of T .

Let us now prove that no 3-cycle is k -good. There are three kinds of 3-cycles: $T[X_2]$, 3-cycles contained in $T[X_3]$, and 3-cycles of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$.

- $T[X_2]$ is not k -good, because x has outdegree less than k in $T - X_2$.
- If C is a 3-cycle in $T[X_3]$, then $T[X_3] - C$ has at most $2k - 2$ vertices and thus contains a vertex v of outdegree less than $k - 1$. Therefore v has outdegree less than k in $T - C$. So C is not k -good.
- If C is a 3-cycle of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$, then every inneighbour v of z in $T[X_3]$ has outdegree less than k in $T - C$. So C is not k -good.

□

Problem 30. For any fixed $k \geq 2$, are there infinitely many strong tournaments with minimum outdegree $k + 1$ that have a $(1, k)$ -splitting but no k -good 3-cycles?

5 Finding outdegree splittings in tournaments

Theorem 31. For every positive integers k_1 and k_2 , TOURNAMENT (k_1, k_2) -OUTDEGREE-SPLITTING is polynomial-time solvable.

Proof. Let $g(k) = k^2/2 + 3k/2 + 1$. Let T be a tournament of order n . If T has a (k_1, k_2) -outdegree-splitting (V_1, V_2) , then V_1 contains a subset S_1 of size at most $g(k_1)$ such that $\delta^+(T[S_1]) \geq k_1$.

The algorithm considers all subsets S_1 of order at most $g(k_1)$. For each of them, we first check if $\delta^+(T[S_1]) \geq k_1$. If no, we proceed to the next subtournament. If yes, we check if there is a (k_1, k_2) -outdegree-splitting (V_1, V_2) such that $S_1 \subseteq V_1$ using a procedure `extend`(S_1). If this procedure, returns ‘yes’, then we also return ‘yes’. If not we proceed to the next subtournament.

The procedure `extend`(S_1) proceeds as follows. If $S_1 = V(T)$, return ‘no’. If $T - S_1$ has minimum outdegree at least k_2 , we return $(S_1, V(T) \setminus S_1)$. Otherwise, pick a vertex x of $V(T) \setminus S_1$ having outdegree less than k_2 in $T - S_1$ and return `extend`($S_1 \cup \{x\}$).

The procedure `extend` runs in $O(n^2)$ -time. (We only need to make $O(n)$ updates on the score sequence). At worse, we run it for each subset S_1 of size at most $g(k_1)$. There are $O(n^{g(k_1)})$ such subsets. Hence the algorithm runs in $O(n^{g(k_1)+2})$ time. □

The running time of the algorithm given in the proof of Theorem 31 is certainly not optimal. When $k_1 = 1$, running time is $O(n^5)$. We now give a faster algorithm, that runs in $O(n^3)$ time for $k_1 = 1$ and $k_2 \geq 2$ and in $O(n^2)$ time for $k_1 = k_2 = 1$. This algorithm is also faster than the one described in Theorem 11.

The key ingredients are the following three statements. The first one is an immediate extension of Proposition 3 with an identical proof, which translates into a $O(n^2)$ -time algorithm.

Proposition 32. Let D be a digraph of order n . If D contains two disjoint digraphs D_1, D_2 such that $\delta^+(D_i) = k_i$ for $i = 1, 2$ and $d_D^+(v) \geq k_1 + k_2 - 1$ for all $v \in V(D - (D_1 \cup D_2))$, then D admits a (k_1, k_2) -outdegree-splitting. Moreover such a (k_1, k_2) -outdegree-splitting can be found in $O(n^2)$ time.

The second one is an algorithmic version of Theorem 17.

Proposition 33. *Let T be a tournament with minimum outdegree at least k . One can find in $O(n^3)$ time a subtournament T' of T with minimum outdegree k such that $|V(T')| \leq k^2/2 + 3k/2 + 1$.*

Proof. By the proof of Theorem 17, if $|V(T)| > k^2/2 + 3k/2 + 1$, then it contains a vertex x such that $T - x$ has minimum outdegree at least k . Such a vertex can be found in $O(n^2)$ time, by finding the set S of vertices with outdegree k , and taking x not in $S \cup N^+(S)$. We then recursively apply the procedure to $T - x$. As we reduce the order of the tournament at most n times, we find the desired subtournament T' in $O(n^3)$ time. \square

Lemma 34. *Let T be a tournament and v a vertex of T . If T has a $(1, k)$ -outdegree-splitting (V_1, V_2) with $v_1 \in V_1$, then there is a 3-cycle C_1 in $T[V_1]$ such that $v \in V(C_1)$ or $V(C_1) \subseteq N^+(v)$.*

Proof. Let $N_1 = N^+(v) \cap V_1$. If $T[N_1]$ has a cycle, then it is the desired 3-cycle. Otherwise, $T[N_1]$ is a transitive tournament. Now the sink w of $T[N_1]$ has an outneighbour u in $T[V_1]$, which is necessarily an inneighbour of v , by definition of N_1 . Therefore uvw is the desired 3-cycle. \square

Theorem 35. (i) TOURNAMENT $(1, 1)$ -OUTDEGREE-SPLITTING can be solved in $O(n^2)$ time;
(ii) for all $k \geq 2$, TOURNAMENT $(1, k)$ -OUTDEGREE-SPLITTING can be solved in $O(n^3)$ time.

Proof. (i) Let us describe a procedure $(1, 1)$ -split(T) that given a tournament T returns ‘yes’ if it admits a $(1, 1)$ -outdegree-splitting, and returns ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.
1. If $\delta^+(T) = 0$, then the tournament T has no $(1, 1)$ -outdegree-splitting, and we return ‘no’.
2. If $\delta^+(T) \geq 3$, the answer is ‘yes’, by Corollary 23.
3. If $\delta^+(T) \in \{1, 2\}$, let v be a vertex of degree 1 or 2 in T . Without loss of generality, one may look for a $(1, 1)$ -outdegree-splitting (V_1, V_2) of T such that $v \in V_1$. For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - \{u, v, w\}$ contains a 3-cycle. If yes for at least one choice of $\{u, v, w\}$, the answer is ‘yes’ by Proposition 3 since $\delta^+(T) \geq k$. If not, then return ‘no’. This is valid by Lemma 34.

Given its score sequence, checking if a tournament of order n contains a 3-cycle can be done in $O(n)$ by checking whether the score sequence is distinct from $(0, 1, 2, \dots, n - 1)$, the score sequence of the transitive tournament. Since the score sequence of $T - \{u, v, w\}$ can be obtained in linear time from the list of outdegrees of T , checking if $T - \{u, v, w\}$ contains a cycle can be done in $O(n)$ time.

Now since v has degree at most 2, the procedure considers at most $2(n - 1)$ subtournaments $T - \{u, v, w\}$. Therefore $(1, 1)$ -split runs in $O(n^2)$ time.

(ii) Let us describe a procedure $(1, k)$ -split(T) that given a tournament T returns ‘yes’ if T admits a $(1, k)$ -outdegree-splitting, and return ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.
1. If $\delta^+(T) = 0$, then the tournament T has no $(1, k)$ -outdegree-splitting, and we return ‘no’.

2. If $1 \leq \delta^+(T) \leq k - 1$, let U_1 be the set of vertices of degree less than k in T . Clearly, for any $(1, 2)$ -outdegree-splitting (V_1, V_2) of T , $U_1 \subseteq V_1$. Let v be a vertex of U_1 . For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - (U_1 \cup \{u, v, w\})$ contains a subtournament of minimum outdegree k using the procedure `Outdegree-k-Subtournament` described below. If yes for at least one choice of $\{u, v, w\}$, the answer is ‘yes’ by Proposition 32 since all vertices of $V(T) \setminus U_1$ have outdegree at least k in T . If not, then return ‘no’. This is valid by Lemma 34.
3. If $\delta^+(T) \geq k$, then we first find a subtournament T' of T with $\delta(T') \geq k$ and $|V(T')| \leq k^2/2 + 3k/2 + 1$. If $T - T'$ contains a 3-cycle, then T admits a $(1, k)$ -outdegree-splitting by Proposition 3, and so we return ‘yes’. If not then $T - T'$ is a transitive tournament and all 3-cycles of T intersect T' and therefore there are at most $(k^2/2 + 3k/2 + 1)n^2$ of them. For each 3-cycle C , we check with `Outdegree-k-Subtournament` whether $T - C$ contains a subtournament of minimum outdegree k . If yes, for one of them, then we return ‘yes’ because there is a $(1, k)$ -outdegree-splitting by Proposition 3. If not, then we return ‘no’.

Remark 36. In the above procedure, one can shorten Step 3 if $\delta^+(T) \geq k + 2$. In this case, by Corollary 23, we can directly return ‘yes’.

The procedure `Outdegree-k-Subtournament`(T) takes as an input the tournament T as well as its list of outdegrees and a list L of vertices in the transitive tournament $T - T'$ ordered in increasing order of their outdegrees. Observe that the list of outdegrees is already computed when `degree-(1, k)-split` call this procedure and the order of $T - T'$ can be computed just once after computing T' . First, we alter the list of outdegrees by keeping the outdegrees for vertices in T' but for vertices in $T - T'$ we count only outneighbours in T' . At each step, `Outdegree-k-Subtournament` first checks $V(T)$ and returns ‘no’ if $V(T) = \emptyset$, otherwise it tries to find a vertex v with $d^+(v) < k$. Notice that possible candidates for v are only vertices in T' and the first k vertices in L . If there is no such vertex v , it returns ‘yes’. Otherwise it removes v and tries again. If $v \in V(T')$, then it decreases the outdegree of all inneighbours of v and if $v \notin V(T')$, then it decreases outdegrees only for inneighbours from $V(T')$. The total time spent on a vertex $v \in V(T')$ is $O(n)$, which gives $O(V(T')n) = O(n)$ in total. The total time spent on a vertex $v \notin V(T')$ is $O(1)$, which gives $O(n)$ in total. Therefore, `Outdegree-k-Subtournament` runs in $O(n)$ time.

Now Step 1 runs in constant time. In Step 2, there are at most $k+1$ candidates for w , and thus `Outdegree-k-Subtournament` is called less than $(k + 1)n$ times. Therefore Step 2 runs in $O(n^3)$ time. Step 3 first finds a small subtournament T' with outdegree k , which can be done in $O(n^3)$ time by Proposition 33. Then it runs $O(n^2)$ times `Outdegree-k-Subtournament`. Therefore Step 3 runs in $O(n^3)$ time.

Overall `(1, k)-split` runs in $O(n^3)$ time. □

The procedure `(1, k)-split`(T) can be modified to find a $(1, k)$ -outdegree-splitting if it exists, using Proposition 32 instead of Proposition 3.

In contrast, the procedure `(1, 1)-split`(T) cannot be instantly modified into a procedure that finds a $(1, 1)$ -outdegree-splitting if it exists. However, using a similar approach, we now describe such a procedure.

Theorem 37. *One can find a $(1, 1)$ -outdegree-splitting of a tournament in $O(n^2)$ time.*

Proof. Let us describe a procedure `(1, 1)-findsplit`(T) that returns a $(1, 1)$ -outdegree-splitting of the tournament T if it admits one, and return ‘no’, otherwise.

We first compute the outdegree of every vertex and we determine $\delta^+(T)$.

If T contains a vertex of outdegree 0, then we return ‘no’. If $\delta^+(T) \geq 4$, then we pick a vertex x and find a 3-cycle C containing x . Such a cycle can be found in $O(n^2)$ by testing if there is an arc from $N^+(x)$ to $N^-(x)$. We return $(V(C), V(T - C))$. This is valid since $\delta^+(T - C) \geq \delta^+(T) - |V(C)| \geq 1$.

If $\delta^+(T) \leq 3$, we choose a vertex v such that $d^+(v) \in \{1, 2, 3\}$. If $T[N^+(v)]$ induces a 3-cycle, then we check whether $T - N^+(v)$ contains a cycle C . If yes, we extend $(T[N^+(v)], C)$ into a $(1, 1)$ -outdegree-splitting by Proposition 32. If not, for every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check if $T - \{u, v, w\}$ contains a cycle $C(uvw)$. If yes for at least one choice of $\{u, w\}$, then we extend $(uvw, C(uvw))$ into a $(1, 1)$ -outdegree-splitting by Proposition 32 and we return ‘no’ otherwise. This is valid by Lemma 34.

Since there are at most three candidates for w , there are $O(n)$ cases to check. Therefore $(1, 1)$ -findsplit runs in $O(n^2)$ time. \square

Remark 38. The proof of Proposition 28 yields a $O(n^2)$ -time procedure to find a 1-good 3-cycle given a $(1, 1)$ -outdegree-splitting. Combining this procedure with $(1, 1)$ -findsplit, we obtain a $O(n^2)$ -time algorithm that finds a 1-good 3-cycle in a tournament if it exists, and returns ‘no’ otherwise.

6 Further research

6.1 Splittable score sequences

Being $(1, 1)$ -outdegree-splittable is not determined by the score sequence. For example, the two tournaments depicted Figure 2 have score sequences $(2, 2, 2, 2, 3, 4)$ but the one to the left has no $(1, 1)$ -outdegree-splitting (See Proposition 26) while the one to the right admits the $(1, 1)$ -outdegree-splitting $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$.

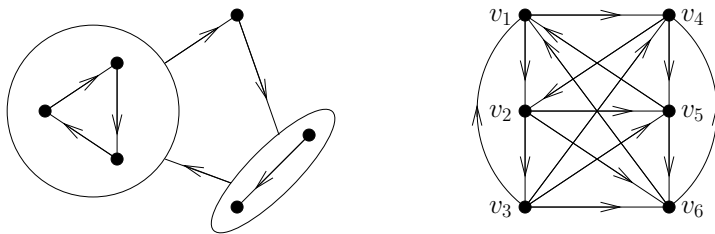


Figure 2: Non- $(1, 1)$ -outdegree-splittable and $(1, 1)$ -outdegree-splittable tournaments with the same score sequence

However there are score sequences s such that all tournaments with score sequence s are $(1, 1)$ -outdegree-splittable. Such score sequences are said to be $(1, 1)$ -outdegree-splittable. For example, Theorem 22 implies that (s_1, \dots, s_n) is $(1, 1)$ -outdegree-splittable.

Problem 39. Which score sequences are $(1, 1)$ -outdegree-splittable?

6.2 Erdős-Posa property for digraphs with minimum outdegree k

McCuaig’s algorithm [7] relies on the theorem stating that a digraph D has either two disjoint cycles or a set S of at most three vertices such that $D - S$ is acyclic. More generally, Reed et al. [9] showed that cycles in digraphs have the Erdős-Posa property.

Theorem 40 (Reed et al. [9]). *For every positive integer n , there exists an integer $t(n)$ such that for every digraph D , either D has a n pairwise-disjoint cycles, or there exists a set T of at most $t(n)$ vertices such that $D - T$ is acyclic.*

It is then natural to ask whether digraphs with maximum outdegree k have the the Erdős-Posa property.

Problem 41. Let k be a fixed integer. For every positive integer n , does there exist an integer $t_k(n)$ such that for every digraph D , either D has a n pairwise-disjoint subdigraphs with minimum outdegree k , or there exists a set T of at most $t_k(n)$ vertices such that $\delta^+(D - T) < k$?

6.3 Strong connectivity and outdegree-splitting with prescribed vertices

Any $f_T(k_1, k_2)$ -strong tournament has minimum outdegree at least $f_T(k_1, k_2)$ and thus admits a (k_1, k_2) -outdegree-splitting. Therefore, it is natural to ask the following.

Problem 42. What is the minimum integer $h_T(k_1, k_2)$ such that every $h_T(k_1, k_2)$ -strong tournament T of order at least $2k_1 + 2k_2 + 2$ contains a (k_1, k_2) -outdegree-splitting?

The condition $|V(T)| \geq 2k_1 + 2k_2 + 2$ in the above problem is just to avoid the small tournaments that cannot have any (k_1, k_2) -outdegree-splitting for cardinality reasons. Clearly, $h_T(k_1, k_2) \leq f_T(k_1, k_2)$. But it is very likely that $h_T(k_1, k_2)$ is smaller than $f_T(k_1, k_2)$. As mentioned in the beginning of Subsection 4.2, a result of Song [11] implies that $h_T(1, 1) \leq 2$ (In fact $h_T(1, 1) = 2$ because a 1-strong tournament T with a vertex v such that $T - v$ is a transitive tournament has clearly no $(1, 1)$ -outdegree-splitting.) whereas $f_T(1, 1) = 3$.

One might also ask similar questions for outdegree-splitting with prescribed vertices (as in Theorem 8). Bang-Jensen et al. [2] proved that if T is a tournament of order 8 and xy an arc in T such that $T \setminus xy$ is 2-strong, then T contains an outdegree-1-splitting (V_x, V_y) with $x \in V_x$ and $y \in V_y$.

References

- [1] N. Alon, Disjoint cycles, *J. Combinatorial Theory, Ser. B* **68** (1996), 167-178.
- [2] J. Bang-Jensen, Y. Guo, and A. Yeo. Complementary cycles containing prescribed vertices in tournaments. *Discrete Math.* 214 (2000), no. 1-3, 77-87.
- [3] J.-C. Bermond and C. Thomassen, Cycles in digraphs—a survey, *J. Graph Theory* **5** (1981), no. 1, 1-43.
- [4] P. Camion, Chemins et circuits hamiltoniens des graphes complets, *C.R. Acad. Sci. Paris* **249** (1959), 2151-2152.
- [5] G. Chen, R.J. Gould, and H. Li, Partitioning vertices of a tournament into independent cycles, *J. combin. Theory Ser B, Vol 83*, no. 2 (2001) 213-220.
- [6] H. G. Landau. On dominance relations and the structure of animal societies. III. The condition for a score structure. *Bulletin of Mathematical Biophysics* **15** (1953), no. 2. 143-148.
- [7] Intercyclic digraphs. *Graph Structure Theory*, (Neil Robertson and Paul Seymour, eds.), *AMS Contemporary Math.*, 147:203-245, 1993.
- [8] J. W. Moon, *Topics on Tournaments*, (Holt, Rinehart and Winston, New York) (1968).
- [9] B. Reed, N. Robertson, P.D. Seymour, and R. Thomas. Packing directed circuits. *Combinatorica*, 16(4):535-554, 1996.
- [10] K.B. Reid, Three problems on tournaments, *Graph Theory and Its Applications*, East. and West. Ann. New York Acad. Sci. 576 (1989), 466-473.
- [11] Z. Song. Complementary cycles of all lengths in tournaments *J. Combin. Theory Ser. B*, **57** (1993), 18-25.
- [12] M. Stiebitz, Decomposing graphs under degree constraints, *J. Graph Theory* **23** (1996), 31-324.
- [13] C. Thomassen, Disjoint cycles in digraphs, *Combinatorica* **3** (1983), no. 3-4, 393-396.



Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399