ISS-Lyapunov Functions for Discontinuous Discrete-Time Systems

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Abstract

Input-to-State Stability (ISS) and the ISS-Lyapunov function have proved to be useful tools for the analysis and design of nonlinear systems in a variety of contexts. Motivated by the fact that many feedback control laws, such as model predictive control or event-based control, lead to discontinuous discrete-time dynamics, we investigate ISS-Lyapunov functions for such systems. ISS-Lyapunov functions were originally introduced in a so-called *implication form* and, in many cases, this has been shown to be equivalent to an ISS-Lyapunov function of *dissipative form*. However, for discontinuous dynamics, we demonstrate via an example that this equivalence no longer holds. We therefore propose a stronger implication form ISS-Lyapunov function and provide a complete characterization of ISS-Lyapunov functions for discrete-time systems with discontinuous dynamics.

Index Terms

Input-to-State Stability (ISS), Lyapunov Methods, Discrete-Time Systems

I. Introduction

The notion of input-to-state stability (ISS) was introduced by Sontag in [20] in order to formalize a Lyapunov type stability property of nonlinear systems taking into account persisting inputs. Soon after its introduction it was recognized as a versatile tool for analyzing stability properties of nonlinear systems and it has become one of the most influential concepts in

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nonlinear stability theory of the last decades. In this paper we will focus on the ISS property and one of its most useful characterizations: the ISS-Lyapunov function, introduced in [23].

Originally formulated in continuous time, the ISS concept and the basic results were soon adapted to discrete time systems. In this paper, we consider discrete-time nonlinear systems given by

$$x^+ = f(x, w) \tag{1}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We take as inputs those sequences that are locally bounded and we denote this space by $\mathcal{W}$. We denote solutions of (1) by $\phi : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$.

Many of the continuous-time ISS results carry over to the discrete-time setting if the discrete time dynamics are continuous, see [8]. However, this statement is no longer true when discontinuous dynamics are considered and in this paper we will not impose any regularity assumptions on $f(\cdot, \cdot)$. Besides the fact that certain models naturally lead to discontinuous dynamics, our main motivation for considering discontinuous f are controller design techniques leading to discontinuous dynamics. Indeed, even if the controlled dynamics $x^+ = g(x, u, w)$ with control input $u \in U$ is continuous, the use of a discontinuous controller $u : \mathbb{R}^n \to U$ leads to a discontinuous closed loop system $x^+ = g(x, u(x), w) =: f(x, w)$ of the form (1). Among modern controller design techniques, optimization based techniques like model predictive control (MPC) naturally lead to discontinuous feedback laws and, in the presence of state constraints, even the corresponding Lyapunov function is typically discontinuous, cf. [3], [18] or [5, Sections 8.5–8.9]. Similarly, quantized [16], [4] or event-based [2], [15] feedback laws naturally lead to discontinuous closed loop dynamics.

It was observed before that additional assumptions are required in order to make the usual ISS-Lyapunov function arguments work for discontinuous discrete time systems, see, e.g., [6, Assumptions 7 and 8]. Also, it is known that discontinuities may affect the usual inherent robustness properties of, e.g., asymptotic [12] or exponential stability [14]. It is the goal of this paper to give a comprehensive and rigorous collection of results on ISS-Lyapunov functions for discontinuous systems. Particularly, we present necessary and sufficient Lyapunov function characterizations of ISS, discuss the equivalence of different types of decay estimates for ISS Lyapunov functions, and introduce a stronger variant of an implication-form Lyapunov function that is demonstrated to be better suited to the discontinuous setting.

The paper is organized as follows. In Section II we recall the definitions of input-to-state stability (ISS) and dissipative-form ISS-Lyapunov functions and discuss the relation between these concepts as well as different decay properties of the Lyapunov functions in the discontinuous setting. In Section III we recall the standard definition of an implication-form ISS-Lyapunov function and show that in the presence of discontinuities additional conditions are needed in order to conclude ISS from the existence of these Lyapunov functions. In Section IV we present and analyze our new stronger implication-form ISS-Lyapunov function. We show that the existence of such a function is indeed equivalent to the ISS property for discontinuous systems and illustrate the usefulness of this concept by proving two propositions yielding sufficient conditions for ISS. Conclusions are presented in Section V and proofs of the main results can be found in Section VI.

II. ISS AND DISSIPATIVE-FORM ISS-LYAPUNOV FUNCTIONS

In the sequel, we will denote the class of continuous positive definite functions $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\mathcal{P}$. We will also make use of the standard function classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$, and $\mathcal{KL}$ (see [7] or [10]).

Definition 2.1: The system (1) is input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$|\phi(k, x, w)| \leq \max\left\{\beta(|x|, k), \sup_{i \in \mathbb{Z}_{[0,k-1]}} \gamma(|w(i)|)\right\} \tag{2}$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.2: An ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{3}$$

$$V(f(x, w)) - V(x) \leq -\alpha(|x|) + \sigma(|w|). \tag{4}$$

An implicit constraint in this definition is that $\alpha(|x|) \leq V(x)$ for all $x \in \mathbb{R}^n$.

In [8, Lemma 3.5] it was shown that the existence of an ISS-Lyapunov function implies ISS. Though the authors of [8] consider systems described by continuous dynamics and also continuous ISS-Lyapunov functions, the assumed regularity plays no part in the proof of [8, Lemma 3.5].

Lemma 2.3: [8, Lemma 3.5] If there exists an ISS-Lyapunov function for (1) then the system is ISS.

In the remainder of this section, we discuss several ways to modify the assumption that $\alpha \in \mathcal{K}_\infty$ in Definition 2.2. We start by observing that this assumption can be weakened to $\alpha \in \mathcal{K}$ provided an additional compatibility condition between α and σ holds.

Proposition 2.4: Suppose $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \sigma \in \mathcal{K}$ satisfy (3) and (4). If $\sup \alpha > \sup \sigma$, then system (1) is ISS.
This follows from results on changing supply functions for ISS systems presented in [22] for continuous time and in [17] for discrete-time. We provide the proof in Section VI-B.

Next we investigate the special case in which $\alpha(|x|)$ in (4) can be replaced by $\lambda V(x)$ for some $\lambda \in (0,1)$. Since this formulation implies exponential decay of V for $w \equiv 0$, a V with this property is called exponential ISS-Lyapunov function.

Definition 2.5: An exponential ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that there exist $\lambda \in [0,1)$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, (3) holds and

$$V(f(x,w)) \leq \lambda V(x) + \sigma(|w|). \tag{5}$$

Despite the lack of any regularity assumptions on either the system dynamics (1) or the ISS-Lyapunov function (3)-(4), we may prove the following converse to Lemma 2.3 which shows that ISS always implies the existence of an exponential ISS-Lyapunov function.

Theorem 2.6: Fix $\lambda \in (0,1)$ and suppose system (1) is ISS. Then there exists an exponential ISS-Lyapunov function for (1) with the decrease rate of (5) given by λ.

This result differs from [8, Theorem 1] in three respects. The first is that the obtained ISS-Lyapunov function does not possess any guaranteed regularity properties. This is inevitable since f does not possess any regularity properties, either. The second is that we observe that an ISS-Lyapunov function exists for any desired decrease rate $\lambda \in (0,1)$. The third is that, while the proof is similar, the proof of [8, Theorem 1] relies on the equivalence of ISS-Lyapunov functions of the form (3)-(4), what we will call dissipative-form ISS-Lyapunov functions, and the implication-form ISS-Lyapunov functions defined in the next section. In Example 3.7 we will demonstrate that this equivalence does not hold if the system dynamics (1) are not continuous.

Using the above results, we can demonstrate the following discrete-time version of [19, Proposition 8]:

Theorem 2.7: If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfying (3)-(4), then, for any $\lambda \in (0,1)$, there exists a function $\widehat{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ and $\hat{\sigma} \in \mathcal{K}$ so that

$$\hat{\alpha}_1(|x|) \leq \widehat{V}(x) \leq \hat{\alpha}_2(|x|)$$

$$\widehat{V}(f(x,w)) \leq \lambda \widehat{V}(x) + \hat{\sigma}(|w|)$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

Proof: By Lemma 2.3, an ISS-Lyapunov function implies that (1) is ISS. Theorem 2.6 then yields that, for any $\lambda \in (0,1)$ there exists an exponential ISS-Lyapunov function. ■

We will see later in Remark 4.6 that $\widehat{V}$ can in fact be explicitly derived from V as $\widehat{V} = \hat{\alpha}(V)$ for some $\hat{\alpha} \in \mathcal{K}_\infty$.

III. Implication-Form ISS-Lyapunov Functions

As an alternate to the "dissipative-form" decrease condition (4), the following "implication-form" ISS-Lyapunov function has frequently been used in the literature:

Definition 3.1: An implication-form ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, and $\hat{\rho} \in \mathcal{P}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, (3) holds and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\rho}(|x|). \tag{6}$$

As before, there is an implicit constraint that $\hat{\rho}(|x|) \leq V(x)$.

Having a merely positive definite decrease rate $\hat{\rho}$ as in (6) is not necessarily convenient for calculations. In many cases, rather than (6) it is useful to have the following class-$\mathcal{K}_\infty$ decrease rate: let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ satisfy (3) and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|). \tag{7}$$

A further potentially useful refinement is the following exponential decrease rate: let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\hat{\lambda} \in (0,1)$, and $\chi \in \mathcal{K}$ satisfy (3) and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \leq \hat{\lambda} V(x). \tag{8}$$

We can demonstrate the following relationship between these three implication-form ISS-Lyapunov functions.

Theorem 3.2: The following are equivalent:

(i) There exists a $\hat{\rho} \in \mathcal{P}$ and an implication-form ISS-Lyapunov function V satisfying (6);

(ii) There exists an $\hat{\alpha} \in \mathcal{K}_\infty$ and an implication-form ISS-Lyapunov function $\widehat{V}$ satisfying (7);

(iii) For any given $\hat{\lambda} \in (0,1)$ there exists an implication-form ISS-Lyapunov function $\widetilde{V}$ satisfying (8).

Moreover, for V satisfying (i) there exist $\hat{\alpha}$ and $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\widehat{V}$ in (ii) and $\widetilde{V}$ in (iii) can be written in the form $\widehat{V} = \hat{\alpha}(V)$ and $\widetilde{V} = \tilde{\alpha}(V)$.
The equivalence of (i) and (ii) was stated in [8, Remark 3.3] and the proof follows as in [9, Lemma 2.8]. The equivalence of (ii) and (iii) follows an argument in the proof of [11, Theorem 6]. The complete proof is provided in Section VI-A.

We now turn to the relationship between dissipative-form ISS-Lyapunov functions (4) and implication-form ISS-Lyapunov functions (7). While, by Theorem 3.2, we could as easily refer to decrease conditions given by (6) or (8), our interest here is in demonstrating *quantitative* equivalences, and hence we will examine the decrease rates given by (4) and (7). We first observe that a dissipative form ISS-Lyapunov function is always an implication form ISS-Lyapunov function.

Proposition 3.3: If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfying (3) and (4) then V satisfies (7) with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} = \frac{1}{2}\alpha \in \mathcal{K}_\infty$.

Proof: We rewrite (4) as

$$V(f(x,w)) - V(x) \;\; \leq \;\; -\frac{1}{2}\alpha(|x|) - \frac{1}{2}\alpha(|x|) + \sigma(|w|). \tag{9}$$

Therefore, with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} \doteq \frac{1}{2}\alpha \in \mathcal{K}_\infty$ we immediately see that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\frac{1}{2}\alpha(|x|). \tag{10}$$

■

Remark 3.4: We observe that we can trade off the decrease rate, $\hat{\alpha}$ and the input-dependent level set defined by χ. In particular, for any $\rho \in \mathcal{P}$ and $\varphi \in \mathcal{K}_\infty$ such that $\rho(s) + \varphi(s) \leq \alpha(s)$, for all $s \in \mathbb{R}_{\geq 0}$ we see that V satisfies (6) with $\hat{\rho} \doteq \rho$ and $\chi \doteq \varphi^{-1} \circ \sigma$.

The converse of Proposition 3.3, i.e., moving from an implication-form ISS-Lyapunov function to a dissipative-form ISS-Lyapunov function requires additional assumptions. We present two sufficient conditions.

Proposition 3.5: Assume system (1) satisfies the ISS-estimate (2). If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ satisfying (3) and (7), then V satisfies (4) with $\alpha \doteq \min\{\hat{\alpha}, \alpha_1\}$ and $\sigma(s) \doteq \alpha_2(\beta(\chi(s), 1) + \gamma(s))$ for all $s \in \mathbb{R}_{\geq 0}$.

The proof makes explicit use of the ISS-estimate to ensure that the potential increase in the ISS-Lyapunov function is bounded for states below the level set defined by $\chi(|w|)$. This is similar to the final argument at the end of the proof of Theorem 2.6 and we thus omit the details.

We observe that Proposition 3.3 and Proposition 3.5 do not rely on the regularity of either the system dynamics or the ISS-Lyapunov functions. We also note that the assumption of ISS immediately yields a dissipative form ISS-Lyapunov function (by Theorem 2.6) and consequently, the only novelty of Proposition 3.5 is that the specific given implication-form ISS-Lyapunov function $V(\cdot)$ is also a dissipative-form ISS-Lyapunov function.

Rather than assuming that system (1) is ISS to show the result of Proposition 3.5, we may assume continuity of both the ISS-Lyapunov function and the system dynamics. This was already stated in [8, Remark 3.3] without proof.

Proposition 3.6: Assume $f(\cdot,\cdot)$ is continuous. If there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\hat{\alpha}, \chi \in \mathcal{K}$ satisfying (3) and (7), then V satisfies (4) with $\alpha \doteq \hat{\alpha}$ and

$$\sigma(r) \doteq \max\{V(f(x,w)) - V(x) + \alpha \circ \chi(|w|) : |w| \leq r, |x| \leq \chi(r)\}. \tag{11}$$

Proof: The proof follows the same argument as in [23, Remark 2.4].

That the maximum in (11) is well-defined follows from the fact that the function being maximized is a continuous function by the assumptions on $V(\cdot)$ and $f(\cdot,\cdot)$ and the fact that the domain being maximized over is compact. It is then straightforward to see that

$$\begin{aligned} \sigma(|w|) &= \sup_{|w|=r} (V(f(x,w)) - V(x) + \alpha \circ \chi(|w|) \\ &\geq V(f(x,w)) - V(x) + \alpha(|x|), \end{aligned}$$

yielding the desired result. ∎

This proposition shows that, under the assumption of continuous system dynamics, the existence of a continuous ISS-Lyapunov function satisfying (3) and (7) implies ISS. The following example shows that in the absence of continuous system dynamics this implication no longer holds.

Example 3.7: Consider the system

$$x^+ = f(x,w) = \nu(w)\kappa(x) \tag{12}$$

where

$$\kappa(x) \doteq \begin{cases} 0 & , \quad x = 0 \\ \frac{1}{|x|} & , \quad |x| \in (0,1) \\ \frac{1}{2|x|} & , \quad |x| \geq 1 \end{cases} \tag{13}$$

and

$$\nu(w) \doteq \begin{cases} 0 & , \quad w = 0 \\ \frac{1}{2}|w|^2 & , \quad |w| \in (0,1) \\ 1 & , \quad |w| \geq 1. \end{cases} \tag{14}$$

Take $V(x) \doteq |x|$ so that both the upper and lower bounds of (3) can be trivially taken as $|x|$. We observe that if $|x| \geq |w|$ then for $|x| \in (0,1)$

$$|f(x,w)| = \frac{|w|^2}{2|x|} \leq \frac{|x|^2}{2|x|} = \frac{|x|}{2}$$

and for $|x| \geq 1$

$$|f(x,w)| = \nu(w)\frac{1}{2|x|} \leq \frac{1}{2|x|} \leq \frac{|x|}{2}.$$

Therefore, with $\alpha(s) \doteq \frac{1}{2}s$ we see that

$$|x| \geq |w| \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\alpha(|x|).$$

However, it is straightforward to see that the system (12) is not ISS. Take $w \equiv 1$ and any initial condition $x \in (0,1)$. Then we see that

$$\phi(2k+1,x) = 2^{2k}\frac{1}{x}, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

In other words, every other time step the solution increases so that the ISS estimate (2) can never be satisfied.

We note that, as an interim step in the proof of Theorem 2.6, we demonstrate that ISS implies the existence of an implication-form ISS-Lyapunov function satisfying (3) and (8). However, the above example demonstrates that the converse is not true. Hence, this indicates that neither the implication-form of (8), nor the equivalent forms demonstrated by Theorem 3.2, are useful when one allows discontinuous system dynamics since (3) and (7) do not imply ISS of (1). This motivates a new definition for implication-form ISS-Lyapunov functions in the following section.

IV. An alternative implication form ISS-Lyapunov function

As we have seen, in the discontinuous setting the existence of an ISS-Lyapunov function in the implication form (3), (7) does not imply ISS and is not equivalent to the existence of an ISS-Lyapunov function in dissipation form (3), (4). In this section, we propose the following stronger alternative to the implication (6) which fixes these problems.

Definition 4.1: A strong implication-form ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\hat{\varphi}, \chi \in \mathcal{K}$, and $\hat{\rho} \in \mathcal{P}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, V satisfies (3) and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\rho}(|x|) \tag{15}$$

$$|x| < \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \leq \hat{\varphi}(|w|). \tag{16}$$

This definition is motivated by the ISS Lyapunov functions in implication form in continuous time, which will always satisfy the second implication on time intervals on which w is constant.

Similar to Theorem 3.2, we can demonstrate that the strong implication-form ISS-Lyapunov function (3)-(15)-(16) implies the existence of a strong implication-form ISS-Lyapunov function with (15) replaced with $\hat{\alpha} \in \mathcal{K}_\infty$ such that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|) \tag{17}$$

or with $\hat{\lambda} \in (0,1)$ such that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \leq \hat{\lambda} V(x). \tag{18}$$

Theorem 4.2: The following are equivalent:

(i) There exists a $\hat{\rho} \in \mathcal{P}$ and a strong implication-form ISS-Lyapunov function V satisfying (15);

(ii) There exists an $\hat{\alpha} \in \mathcal{K}_\infty$ and a strong implication-form ISS-Lyapunov function $\widehat{V}$ satisfying (17);

(iii) For any given $\hat{\lambda} \in (0,1)$ there exists a strong implication-form ISS-Lyapunov function $\widetilde{V}$ satisfying (18).

Moreover, for V satisfying (i) there exist $\hat{\alpha}$ and $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\widehat{V}$ in (ii) and $\widetilde{V}$ in (iii) can be written in the form $\widehat{V} = \hat{\alpha}(V)$ and $\widetilde{V} = \tilde{\alpha}(V)$.

The proof follows exactly as that for Theorem 3.2, where $\varphi \in \mathcal{K}$ is scaled along with V. The details are provided in Section VI-A.

The next theorem shows that the strong implication-form ISS-Lyapunov function (16)-(17) is equivalent to the dissipative-form ISS-Lyapunov function (4). This then overcomes the gap observed between a dissipative-form ISS-Lyapunov function and the classical implication-form ISS-Lyapunov function (7) when considering discontinuous systems and ISS-Lyapunov functions. An immediate consequence is that the existence of a strong implication form ISS-Lyapunov function is equivalent to the system being ISS. This is formulated in a subsequent corollary.

Theorem 4.3: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function satisfying (3) for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

(i) If V together with functions $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfies (4) then V also satisfies (17) with $\hat{\alpha} \doteq \alpha/2$, $\chi \doteq \alpha^{-1} \circ 2\sigma$ and satisfies (16) with $\hat{\varphi} \doteq \gamma \circ \chi + \sigma$, where $\gamma \in \mathcal{K}_\infty$ is arbitrary with $\gamma \geq \alpha_2 - \alpha$.

(ii) If V together with functions $\hat{\alpha} \in \mathcal{K}_\infty$ and $\chi, \hat{\varphi} \in \mathcal{K}$ satisfies (17) and (16), then V also satisfies (4) with $\alpha \doteq \min\{\hat{\alpha}, \alpha_1\}$ and $\sigma = \hat{\varphi}$.

Proof: (i) We rewrite (4) as

$$V(f(x,w)) - V(x) \;\; \leq \;\; -\frac{1}{2}\alpha(|x|) - \frac{1}{2}\alpha(|x|) + \sigma(|w|). \tag{19}$$

Therefore, with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} \doteq \frac{1}{2}\alpha \in \mathcal{K}_\infty$ we immediately see that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\frac{1}{2}\alpha(|x|) = -\hat{\alpha}(|x|) \tag{20}$$

and

$$\begin{aligned} |x| < \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \;\; &\leq \;\; V(x) - \alpha(|x|) + \sigma(|w|) \\ &\leq \;\; \alpha_2(|x|) - \alpha(|x|) + \sigma(|w|) \\ &\leq \;\; \gamma(|x|) + \sigma(|w|) \;\; \leq \;\; \gamma(\chi(|w|)) + \sigma(|w|) \\ &= \;\; \hat{\varphi}(|w|). \end{aligned} \tag{21}$$

(ii) If $|x| \geq \chi(|w|)$ then we get

$$V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|) \leq -\alpha(|x|) + \sigma(|w|). \tag{22}$$

In case $|x| < \chi(|w|)$, using (3) we obtain

$$V(f(x,w)) - V(x) \leq \hat{\varphi}(|w|) - \alpha_1(|x|) \leq -\alpha(|x|) + \sigma(|w|). \tag{23}$$

■

Hence, the stronger implication form (17) is equivalent to the dissipation form (4). As a consequence, the existence of a strong implication-form ISS-Lyapunov function is equivalent to ISS, as stated in the following corollary.

Corollary 4.4: System (1) is ISS if and only if there exists a strong implication-form ISS-Lyapunov function in the sense of Definition 4.1.

Proof: If the system is ISS, then by Theorem 2.6 there exists a dissipative-form ISS-Lyapunov function which by Theorem 4.3 is also a strong implication-form ISS-Lyapunov function. Conversely, if there exists a strong implication-form ISS-Lyapunov function then by Theorem 4.3 this is also a dissipative-form ISS-Lyapunov function which by Lemma 2.3 implies ISS of system (1). ■

We note that we can prove a result similar to Theorem 4.3 for the relationship between dissipative-form and strong implication-form ISS Lyapunov functions when we have an exponential decrease.

Theorem 4.5: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function satisfying (3) for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

(i) If V together with $\lambda \in (0,1)$ and $\sigma \in \mathcal{K}$ satisfies (5), then V also satisfies (16) and (18) with $\hat{\lambda} \doteq \lambda + \varepsilon$ satisfying $\lambda + \varepsilon < 1$, $\chi \doteq \alpha_1^{-1}(\frac{1}{\varepsilon}\sigma)$, and $\hat{\varphi} \doteq \lambda\alpha_2 \circ \chi + \sigma$.

(ii) If V together with $\hat{\lambda} \in (0,1)$, $\chi, \hat{\varphi} \in \mathcal{K}$ satisfies (16) and (18), then V also satisfies (5) with $\lambda \doteq \hat{\lambda}$ and $\sigma \doteq \hat{\varphi}$.

Proof: (i) Since $\lambda \in (0,1)$ there exists $\varepsilon > 0$ such that $\lambda + \varepsilon \in (0,1)$. We may then rewrite (5) as

$$\begin{aligned} V(f(x,w)) &\leq (\lambda+\varepsilon)V(x) - \varepsilon V(x) + \sigma(|w|) \\ &\leq \hat{\lambda}V(x) - \varepsilon\alpha_1(|x|) + \sigma(|w|) \end{aligned}$$

which yields the implication (18). The implication (16) follows from the upper bound on V and the condition $|x| < \chi(|w|)$ as

$$V(f(x,w)) \leq \lambda\alpha_2(|x|) + \sigma(|w|) \leq \lambda\alpha_2 \circ \chi(|w|) + \sigma(|w|).$$

The proof of (i) is immediate by inspection. ■

Remark 4.6: Theorems 4.2, 4.3 and 4.5 imply that we can explicitly choose the function $\widehat{V}$ in Theorem 2.7 in the form $\widehat{V} = \hat{\alpha}(V)$ for some $\hat{\alpha} \in \mathcal{K}_\infty$. Indeed, by Theorem 4.3 the dissipative-form ISS-Lyapunov function V in the assumption of Theorem 2.7 is also a strong implication-form ISS-Lyapunov function. Theorem 4.2 then shows that by rescaling with $\hat{\alpha} \in \mathcal{K}_\infty$ we can turn this V into an exponential strong implication-form ISS-Lyapunov function which, by Theorem 4.5(ii), is also an exponential ISS-Lyapunov function in dissipative form.

Theorem 4.3 can be used in order to pass from the (weak) implication form (7) to the dissipation form (4) under a weaker continuity assumption than in Proposition 3.6. Particularly, we only require continuity of f at $w = 0$ (uniformly in x) and continuity of V at $x = 0$. Note that the continuity of V at $x = 0$ is a consequence of the bounds (3).

Proposition 4.7: Let V be a function satisfying (3) and (7) for appropriate $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$. Assume that f is continuous in $w = 0$ uniformly in x in the following sense:

For each $r > 0$ there is $\gamma_r \in \mathcal{K}_\infty$ such that for all $|x| \leq r$, $|w| \leq r$ the inequality

$$|f(x,w) - f(x,0)| \leq \gamma_r(|w|)$$

holds.

Then there exists $\hat{\varphi} \in \mathcal{K}$ so that V satisfies (16) and thus also (4).

Proof: First, consider $w \equiv 0$. Then we observe that

$$\begin{aligned} |f(x,0)| \;&\leq\; \alpha_1^{-1}(V(f(x,0))) \;\leq\; \alpha_1^{-1}\left(V(x) - \hat{\alpha}(|x|)\right) \\ &\leq\; \alpha_1^{-1}\left(\alpha_2(|x|) - \hat{\alpha}(|x|)\right). \end{aligned} \tag{24}$$

Since by (7) $\hat{\alpha}(|x|) \leq V(x) \leq \alpha_2(|x|)$ for all $x \in \mathbb{R}^n$, with equality if and only if $x = 0$, the function $\alpha_1^{-1}(\alpha_2(s) - \hat{\alpha}(s))$ is positive definite. Define $\bar{\gamma} \in \mathcal{K}_\infty$ by

$$\bar{\gamma}(s) \doteq \max\left\{s, \alpha_1^{-1}(\alpha_2(s) - \hat{\alpha}(s))\right\}, \quad \forall s \in \mathbb{R}_{\geq 0} \tag{25}$$

so that

$$|f(x,0)| \leq \bar{\gamma}(|x|), \quad \forall x \in \mathbb{R}^n. \tag{26}$$

Now, if for all $r > 0$ we define

$$\hat{\gamma}(r) \doteq \sup\{|f(x,w) - f(x,0)| \,:\, |x| \leq r,\, |w| \leq r\},$$

then for all $r_1 \geq r$ we obtain $\hat{\gamma}(r) \leq \gamma_{r_1}(r)$ which implies $\hat{\gamma}(r) \to 0$ as $r \to 0$. Moreover, $\hat{\gamma}(r)$ is finite for all $r > 0$. Hence, we may overbound $\hat{\gamma}$ with a function $\gamma \in \mathcal{K}_\infty$.

It is now sufficient to show that there exists $\varphi \in \mathcal{K}_\infty$ such that the implication in (16) holds. To this end, let $|x| < \chi(|w|)$. Then we have

$$\begin{aligned} |f(x,w)| &= |f(x,w) - f(x,0) + f(x,0)| \\ &\leq \gamma(\max\{|w|, \chi(|w|)\}) + \bar{\gamma}(|x|) \\ &\leq \gamma(\max\{|w|, \chi(|w|)\}) + \bar{\gamma}(\chi(|w|)) =: \tilde{\gamma}(|w|) \end{aligned}$$

implying

$$V(f(x,w)) \leq \alpha_2(|f(x,w)|) \leq \alpha_2(\tilde{\gamma}(|w|)).$$

This shows the desired inequality with $\varphi(r) \doteq \alpha_2(\tilde{\gamma}(|w|))$. ■

We note that the map $f(x,w) = \nu(w)\kappa(x)$ in (12) of Example 3.7 does not satisfy the required continuity property of Proposition 4.7. To see this, we first observe that

$$|f(x,w) - f(x,0)| = |f(x,w)|.$$

Choose $r = 1$ and any $\gamma_1 \in \mathcal{K}_\infty$. Then, with $w = 1$, we see that

$$|f(x,1)| = \frac{1}{|x|}, \quad \forall x \in (-1,1)\backslash\{0\}$$

so that $|f(x,1)| > \gamma_1(1)$ for some $x \in (0,1)$.

Using a strong implication-form ISS-Lyapunov function allows us to prove a variant of Proposition 2.4.

Proposition 4.8: Suppose $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \sigma \in \mathcal{K}$ satisfy (3) and (4). If there exists $\rho \in \mathcal{P}$ such that $\alpha(s) = \rho(s) + \sigma(s)$ for all $s \in \mathbb{R}_{\geq 0}$ then V satisfies (15) and (16) with $\chi \doteq \mathrm{id}$, $\hat{\rho} \doteq \rho$, and $\hat{\varphi} \doteq \gamma + \sigma$ where $\gamma \in \mathcal{K}$ is such that $\gamma > \alpha_2 - \alpha$, and hence (1) is ISS.

Proof: By assumption, we have

$$V(f(x,w)) - V(x) \leq -\rho(|x|) - \sigma(|x|) + \sigma(|w|). \tag{27}$$

Therefore

$$|x| \geq |w| \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\rho(|x|) \tag{28}$$

and

$$\begin{aligned} |x| < |w| \quad \Rightarrow \quad V(f(x,w)) &\leq V(x) - \alpha(|x|) + \sigma(|w|) \\ &\leq \alpha_2(|x|) - \alpha(|x|) + \sigma(|w|) \\ &\leq \gamma(|w|) + \sigma(|w|). \end{aligned} \tag{29}$$

Therefore, by Theorem 4.2 there exists a strong implication form ISS-Lyapunov function satisfying (17) so that Theorem 4.3 implies the existence of a dissipative form ISS-Lyapunov function and Lemma 2.3 yields ISS of (1). ■

We note that the assumptions of Propositions 2.4 and 4.8 do not imply each other. Clearly, $\sup\alpha > \sup\sigma$ does not imply $\alpha(r) > \sigma(r)$ for all $r > 0$. Conversely, one checks that the functions

$$\sigma(s) = \frac{s}{1+s} \quad \text{and} \quad \alpha(s) = \sigma(s) + \rho(s) \text{ with } \rho(s) = \min\left\{s, \frac{1}{2}\frac{1}{1+s}\right\}$$

satisfy the assumption of 4.8 although $\sup\alpha = 1 = \sup\sigma$. From a quantitative point of view, Proposition 4.8 provides a stronger statement than Proposition 2.4, as it maintains the functions of the given ISS-Lyapunov function without requiring a rescaling of the ISS-Lyapunov function.

V. Conclusions

In this paper we have provided a complete characterization of ISS-Lyapunov functions for discrete-time systems with discontinuous dynamics. In contrast to the original definition of an ISS-Lyapunov function in [23], we here observed that an implication-form ISS-Lyapunov function does not necessarily imply ISS of the system (1). In order to counter this difficulty, we proposed an alternative strong implication-form ISS-Lyapunov function and demonstrated that this ISS-Lyapunov function satisfies many of the desirable properties that hold in a more classical setting such as equivalence to an ISS-Lyapunov function in dissipative form and that this strong implication-form ISS-Lyapunov function is both necessary and sufficient for ISS of discrete-time systems with discontinuous dynamics.

In addition, we have presented results on the nonlinear scaling of such strong implication-form ISS-Lyapunov functions and have demonstrated that it is always possible to move between decrease rates that are given by positive definite functions, functions of class-$\mathcal{K}_\infty$, or even an exponential decrease. In all cases, we have explicitly shown how these various functions are related to each other in a quantitative manner.

VI. PROOFS

A. Proof of Theorems 3.2 and 4.2

In this appendix we will prove Theorem 3.2. Theorem 4.2 follows exactly the same argument with appropriate scalings on the bound (16) and we include these scalings in remarks in what follows. We observe that for both Theorems the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are trivial.

1) Positive Definite to $\mathcal{K}_\infty$: $(i) \Rightarrow (ii)$: We start from an ISS-Lyapunov function with a positive definite decrease rate; i.e., $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, and $\rho \in \mathcal{P}$ satisfying (3) and (6).

For $\rho \in \mathcal{P}$, [1, Lemma IV.1] ([10, Lemma 12]) yields $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{L}$ so that

$$\rho(s) \geq \alpha(s)\sigma(s), \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{30}$$

Using the bounds (3) we see that, for all $x \in \mathbb{R}^n$

$$\begin{aligned} |x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) &\leq -\rho(|x|) \\ &\leq -\alpha(|x|)\sigma(|x|) \\ &\leq -\alpha \circ \alpha_2^{-1}(V(x))\sigma \circ \alpha_1^{-1}(V(x)) \\ &= -\hat{\rho}(V(x)) \end{aligned} \tag{31}$$

where $\rho(s) \doteq \alpha_2^{-1}(s)\sigma \circ \alpha_1^{-1}(s)$ for all $s \in \mathbb{R}_{\geq 0}$ is positive definite.

From here, we follow [9, Lemma 2.8]. Let $\bar{\alpha} \in \mathcal{K}_\infty$ be such that

$$\bar{\alpha}\left(\frac{s}{2}\right)\hat{\rho}(s) \geq s, \quad \forall s \geq 1 \tag{32}$$

and define $\hat{\alpha} \in \mathcal{K}_\infty$ by

$$\hat{\alpha}(s) \doteq s + \int_0^s \bar{\alpha}(r)dr, \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{33}$$

We observe that $\hat{\alpha} \in \mathcal{K}_\infty$ and

$$\hat{\alpha}'(s) = 1 + \bar{\alpha}(s), \quad \forall s \in \mathbb{R}_{>0} \tag{34}$$

so that $\hat{\alpha}'$ is strictly increasing.

Define $\widehat{V}(x) \doteq \hat{\alpha}(V(x))$ for all $x \in \mathbb{R}^n$ and observe that with the $\mathcal{K}_\infty$ functions $\hat{\alpha}_1 \doteq +\hat{\alpha} \circ \alpha_1$ and $\hat{\alpha}_2 \doteq \hat{\alpha} \circ \alpha_2$ we have

$$\hat{\alpha}_1(|x|) \leq \widehat{V}(x) \leq \hat{\alpha}_2(|x|). \tag{35}$$

Remark 6.1: In order to prove $(i) \Rightarrow (ii)$ in Theorem 4.2, additionally let $\hat{\varphi} \in \mathcal{K}$ be given by $\hat{\varphi} \doteq \hat{\alpha} \circ \varphi$ so that (16) implies, for $|x| < \chi(|w|)$,

$$\widehat{V}(f(x,w)) = \hat{\alpha}(V(f(x,w))) \leq \hat{\alpha} \circ \varphi(|w|) = \hat{\varphi}(|w|). \tag{36}$$

To simplify the notation, we use $\widehat{V}^+ \doteq \widehat{V}(f(x,w))$, $\widehat{V} \doteq \widehat{V}(x)$, $V^+ \doteq V(f(x,w))$, and $V \doteq V(x)$. In what follows we assume $|x| \geq \chi(|w|)$.

Since $\hat{\alpha}$ is differentiable, the mean value theorem yields the existence of $\theta \in (0,1)$ so that

$$\hat{\alpha}(V^+) - \hat{\alpha}(V) = \hat{\alpha}'(V^+ + \theta(V - V^+))(V^+ - V). \tag{37}$$

Note that, as a consequence of (31), $V^+ - V \leq 0$.

We first restrict attention to $V \geq 1$ and consider two cases. First, we assume $V^+ \leq \frac{V}{2}$ and note that $\hat{\alpha}'(s) \geq 1$ for all $s \in \mathbb{R}_{\geq 0}$. Then

$$\widehat{V}^+ - \widehat{V} \leq V^+ - V \leq -\frac{V}{2}. \tag{38}$$

Now suppose that $V^+ \geq \frac{V}{2}$. In this case, using $V - V^+ \geq 0$ and (34), we have

$$\hat{\alpha}'(V^+ + \theta(V - V^+)) \geq \hat{\alpha}'(V^+) \geq \hat{\alpha}'\left(\frac{V}{2}\right) > \bar{\alpha}\left(\frac{V}{2}\right). \tag{39}$$

Therefore, for $V \geq 1$, using (39), (31), and (32) we obtain

$$\widehat{V}^+ - \widehat{V} \leq \bar{\alpha}\left(\frac{V}{2}\right)(V^+ - V) \leq -\bar{\alpha}\left(\frac{V}{2}\right)\hat{\rho}(V) \leq -V. \tag{40}$$

Combining (38) and (40) we see that, for $V \geq 1$,

$$\widehat{V}^+ - \widehat{V} \leq -\frac{V}{2}. \tag{41}$$

For $V \leq 1$, we note that by definition (33) and (31) we have

$$\begin{aligned} \widehat{V}^+ - \widehat{V} &= V^+ + \int_0^{V^+} \bar{\alpha}(r)dr - V - \int_0^{V} \bar{\alpha}(r)dr \\ &\leq V^+ - V \leq -\hat{\rho}(V). \end{aligned} \tag{42}$$

Take $\breve{\alpha} \in \mathcal{K}_\infty$ so that

$$\begin{aligned} \breve{\alpha}(s) &\leq \hat{\rho}(s), \quad s \in [0,1] \\ \breve{\alpha}(s) &\leq \frac{s}{2}, \quad s \geq 1. \end{aligned}$$

Finally, let $\alpha \in \mathcal{K}_\infty$ be defined as $\alpha \doteq \breve{\alpha} \circ \alpha_1$ so that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) - \widehat{V}(x) \leq -\breve{\alpha}(V(x)) \leq -\breve{\alpha} \circ \alpha_1(|x|) = -\alpha(|x|). \tag{43}$$

2) $\mathcal{K}_\infty$ to Exponential: $(ii) \Rightarrow (iii)$: Since every $\mathcal{K}_\infty$-function is also positive definite, we can follow the first part of the proof to conclude (35) and (43) which imply

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) - \widehat{V}(x) \leq -\alpha(|x|) \leq -\alpha \circ \hat{\alpha}_2^{-1}(\widehat{V}(x)). \tag{44}$$

Define $\mu \in \mathcal{K}_\infty$ by

$$\mu(s) \doteq \min\left\{\alpha \circ \hat{\alpha}_2^{-1}(s), \frac{s}{2}\right\} \tag{45}$$

and note that $\mathrm{id} - \mu \in \mathcal{K}_\infty$ and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) \leq \widehat{V}(x) - \mu(\widehat{V}(x)) = (\mathrm{id} - \mu)(\widehat{V}(x)). \tag{46}$$

Select any $\lambda \in (0,1)$. Then [10, Corollary 1] yields $\hat{\mu} \in \mathcal{K}_\infty$ so that

$$\hat{\mu} \circ (\mathrm{id} - \mu)(s) = \lambda \hat{\mu}(s), \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{47}$$

Define $\widetilde{V} \doteq \hat{\mu}(\widehat{V})$ and note that, with $\mathcal{K}_\infty$ functions $\bar{\alpha}_1 \doteq \hat{\mu} \circ \hat{\alpha}_1$ and $\bar{\alpha}_2 \doteq \hat{\mu} \circ \hat{\alpha}_2$,

$$\bar{\alpha}_1(|x|) \leq \widetilde{V}(x) \leq \bar{\alpha}_2(|x|). \tag{48}$$

Furthermore,

$$\begin{aligned} |x| \geq \chi(|w|) \quad \Rightarrow \quad \widetilde{V}(f(x,w)) &= \hat{\mu}(\widehat{V}(f(x,w))) \\ &\leq \hat{\mu} \circ (\mathrm{id} - \mu)(\widehat{V}(x)) = \lambda \hat{\mu}(\widehat{V}(x)) \\ &= \lambda \widetilde{V}(x). \end{aligned} \tag{49}$$

Finally, the form $\widetilde{V} = \tilde{\alpha}(V)$ follows by combining both parts of the proof and setting $\tilde{\alpha} = \hat{\mu} \circ \hat{\alpha}$.

Remark 6.2: To additionally demonstrate $(ii) \Rightarrow (iii)$ in Theorem 4.2, define $\bar{\varphi} \in \mathcal{K}$ by $\bar{\varphi} \doteq \hat{\mu} \circ \hat{\varphi}$ so that, for $|x| < \chi(|w|)$,

$$\widetilde{V}(f(x,w)) = \hat{\mu}(\widehat{V}(f(x,w))) \leq \hat{\mu} \circ \hat{\varphi}(|w|) = \bar{\varphi}(|w|). \tag{50}$$

B. Proof of Proposition 2.4

The proof closely follows [17, Lemma 1]. The condition $\sup \alpha > \sup \sigma$ implies there exists $c > 1$ so that

$$\sup \alpha > c \sup \sigma. \tag{51}$$

Define

$$\varphi(s) \doteq \int_0^s \hat{\alpha}(r) dr. \tag{52}$$

We observe that $\varphi \in \mathcal{K}_\infty$ and, for $\widehat{V} \doteq \varphi(V)$

$$\widehat{V}^+ - \widehat{V} = \varphi(V^+) - \varphi(V) = \hat{\alpha}(V^+ - \theta(V^+ - V))(V^+ - V) \tag{53}$$

for some $\theta \in (0,1)$.

Consider $V^+ \leq \frac{V}{2}$. Then we may write

$$\begin{aligned} \varphi(V^+) - \varphi(V) &\leq \varphi\left(\frac{V}{2}\right) - \varphi(V) \\ &\leq \hat{\alpha}\left(\frac{V}{2}\right)\left(\frac{V}{2} - V\right) \\ &\leq -\hat{\alpha}\left(\frac{V}{2}\right)\frac{V}{2} + \sigma(|w|). \end{aligned} \tag{54}$$

Now consider $V^+ > \frac{V}{2}$ and examine two cases. The first case is when $\frac{1}{c}\alpha(V) > \sigma(|w|)$. Then

$$\begin{aligned} \varphi(V^+) - \varphi(V) &\leq \hat{\alpha}(V^+)(V^+ - V) \\ &\leq \hat{\alpha}(V^+)\left(-\alpha(V) + \frac{1}{c}\alpha(V)\right) \\ &\leq -\hat{\alpha}\left(\frac{V}{2}\right)\left(1 - \frac{1}{c}\right)\alpha(V) \\ &\leq -\hat{\alpha}\left(\frac{V}{2}\right)\left(1 - \frac{1}{c}\right)\alpha(V) + \sigma(|w|). \end{aligned} \tag{55}$$

Finally, consider $V^+ > \frac{V}{2}$ and $\frac{1}{c}\alpha(V) \leq \sigma(|w|)$. Since $c\sup\sigma < \sup\alpha$, we see that $\alpha^{-1}(s)$ exists for all $s \in [0, c\sup\sigma]$ and hence we have that

$$\hat{\alpha}(V^+) \leq \hat{\alpha}(V + \sigma(|w|)) \leq \hat{\alpha}\left(\alpha^{-1}(c\sigma(|w|)) + \sigma(|w|)\right). \tag{56}$$

Define $\gamma(s) \doteq \hat{\alpha}(\alpha^{-1}(c\sigma(s)) + \sigma(s))$. Then

$$\begin{aligned} \varphi(V^+) - \varphi(V) &\leq \hat{\alpha}(V^+)(V^+ - V) \\ &\leq \hat{\alpha}(V^+)(-\alpha(V) + \sigma(|w|)) \\ &\leq -\hat{\alpha}(V^+)\alpha(V) + \gamma(|w|)\sigma(|w|) \\ &\leq -\hat{\alpha}\left(\frac{V}{2}\right)\alpha(V) + \gamma(|w|)\sigma(|w|). \end{aligned} \tag{57}$$

Therefore, with

$$\begin{aligned} \bar{\alpha}(s) &\doteq \min\left\{\hat{\alpha}\left(\frac{s}{2}\right)\frac{s}{2},\ \hat{\alpha}\left(\frac{s}{2}\right)\left(1 - \frac{1}{c}\right)\alpha(s),\ \hat{\alpha}\left(\frac{s}{2}\right)\alpha(s)\right\} \\ \bar{\sigma}(s) &\doteq \max\{\sigma(s),\ \gamma(s)\sigma(s)\} \end{aligned} \tag{58}$$

and noting that $\bar{\alpha} \in \mathcal{K}_\infty$ and $\bar{\sigma} \in \mathcal{K}$,

$$\widehat{V}(f(x,w)) - \widehat{V}(x) \le -\bar{\alpha}(V(x)) + \bar{\sigma}(|w|) \le -\bar{\alpha} \circ \alpha_1(|x|) + \bar{\sigma}(|w|)$$

so that (1) is ISS.

C. *Proof of Theorem 2.6*

Our proof relies on a converse Lyapunov theorem for difference inclusions. We denote the set of solutions to the difference inclusion

$$x^+ \in F(x), \quad x \in \mathbb{R}^n \tag{59}$$

defined by the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and from an initial condition $x \in \mathbb{R}^n$ by $\mathcal{S}(x)$. A solution $\phi \in \mathcal{S}(x)$ is a function $\phi : \mathbb{Z}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(0,x) = x$ and $\phi(k+1,x) \in F(\phi(k,x))$ for all $k \in \mathbb{Z}_{\ge 0}$.

Definition 6.3: The difference inclusion (59) is said to be $\mathcal{KL}$-stable if there exists $\beta \in \mathcal{KL}$ so that

$$|\phi(k,x)| \le \beta(|x|,k), \quad \forall x \in \mathbb{R}^n,\ \phi \in \mathcal{S}(x),\ k \in \mathbb{Z}_{\ge 0}. \tag{60}$$

Theorem 6.4: If the difference inclusion (59) is $\mathcal{KL}$-stable then, for any given $\lambda \in (0,1)$ there exists an exponential-decrease Lyapunov function; i.e., there exist functions $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{61}$$

$$V(\phi(1,x)) \le \lambda V(x) \tag{62}$$

for all $x \in \mathbb{R}^n$ and $\phi(1,x) \in F(x)$.

Proof: The proof follows that of [13, Theorem 2.7] where, here, we need not worry about regularity of the Lyapunov function.

Given $\beta \in \mathcal{KL}$ and $\lambda \in (0,1)$, Sontag's lemma on $\mathcal{KL}$-estimates [21, Proposition 7] yields $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(\beta(s,k)) \le \alpha_2(s)\lambda^k, \quad \forall s \in \mathbb{R}_{\ge 0},\ k \in \mathbb{Z}_{\ge 0}. \tag{63}$$

For all $x \in \mathbb{R}^n$, define

$$V(x) \doteq \sup_{k \in \mathbb{Z}_{\ge 0}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k}. \tag{64}$$

Then

$$V(x) \geq \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(0,x)|)\lambda^0 = \alpha_1(|x|)$$

and

$$V(x) \leq \sup_{k \in \mathbb{Z}_{\geq 0}} \alpha_1(\beta(|x|,k))\lambda^{-k} \leq \sup_{k \in \mathbb{Z}_{\geq 0}} \alpha_2(|x|)\lambda^k \lambda^{-k} = \alpha_2(|x|)$$

so that $V(x)$ satisfies the desired upper and lower bounds (61). The desired decrease condition follows as

$$\begin{aligned} V(\phi(1,x)) &= \sup_{k \in \mathbb{Z}_{\geq 0}} \sup_{\psi \in \mathcal{S}(\phi(1,x))} \alpha_1(|\psi(k,\phi(1,x))|)\lambda^{-k} \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 0}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} \;=\; \lambda V(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. ■

In order to demonstrate that ISS implies the existence of an ISS-Lyapunov function, we follow the standard argument as in [23] and [8]. Denote the closed unit ball in $\mathbb{R}^m$ by $\mathcal{B}^m$. We show that there exists a $\mu \in \mathcal{K}_\infty$ such that the differential inclusion defined by

$$x(k+1) \in f(x(k), \mu(|x(k)|)\mathcal{B}^m) \tag{65}$$

is $\mathcal{KL}$-stable, allowing us to appeal to Theorem 6.4 to obtain an ISS-Lyapunov function. We denote the solution set of (65) from an initial condition $x \in \mathbb{R}^n$ by $\mathcal{S}_\mu(x)$.

Proposition 6.5: [13, Proposition 2.2.] The following are equivalent:

1) The difference inclusion $x(k+1) \in F(x(k))$ is $\mathcal{KL}$-stable.
2) The following hold:
   a) (Uniform stability): There exists $\gamma \in \mathcal{K}_\infty$ so that, for each $x \in \mathbb{R}^n$, all solutions $\phi \in \mathcal{S}(x)$ satisfy
   $$|\phi(k,x)| \leq \gamma(|x|), \quad \forall k \in \mathbb{Z}_{\geq 0}.$$
   b) (Uniform global attractivity): For each $r, \varepsilon \in \mathbb{R}_{>0}$, there exists $K(r,\varepsilon) > 0$ so that, for each $x \in \mathbb{R}^n$, all solutions $\phi \in \mathcal{S}(x)$ satisfy
   $$|x| \leq r, \quad k \in \mathbb{Z}_{\geq K(r,\varepsilon)} \quad \Rightarrow \quad |\phi(k,x)| \leq \varepsilon.$$

Lemma 6.6: If (1) is ISS then there exists $\mu \in \mathcal{K}_\infty$ such that the difference inclusion (65) is $\mathcal{KL}$-stable.

Proof: Without loss of generality, we assume that $\gamma \in \mathcal{K}$ from (2) satisfies $\gamma(r) \geq r$. Define $\alpha, \mu \in \mathcal{K}_\infty$ as

$$\alpha(s) \doteq \max\left\{\gamma(\beta(s,0)), \gamma\left(\tfrac{1}{2}s\right)\right\}, \quad \mu(s) \doteq \tfrac{1}{2}\gamma^{-1}\left(\tfrac{1}{4}\alpha^{-1}(s)\right)$$

for all $s \in \mathbb{R}_{\geq 0}$.

Claim 6.7: For any $x \in \mathbb{R}^n$ and $\phi \in \mathcal{S}_\mu(x)$ we have

$$\gamma \circ \mu(|\phi(k,x)|) \leq \tfrac{1}{2}|x|, \quad \forall k \in \mathbb{Z}_{\geq 0}. \tag{66}$$

Proof: By the definition of α we have $|x| \leq \beta(|x|,0) \leq \alpha(|x|)$ so that

$$\gamma \circ \mu(|x|) \leq \tfrac{1}{4}\alpha^{-1}(|x|) \leq \tfrac{1}{4}|x|. \tag{67}$$

Let

$$k_1 \doteq \min\left\{k \in \mathbb{Z}_{\geq 0} : \gamma \circ \mu(|\phi(k,x)|) > \tfrac{1}{2}|x|\right\}$$

and note that (67) implies $k_1 \in \mathbb{Z}_{\geq 1}$. In order to obtain a contradiction, assume $k_1 < \infty$. Then (66) holds for all $k \in \mathbb{Z}_{[0,k_1-1]}$. Therefore, $\gamma(|\mu(|\phi(k,x)|)\mathcal{B}^m|) \leq \frac{1}{2}|x|$ for all $\phi \in \mathcal{S}_\mu(x)$ and $k \in \mathbb{Z}_{[0,k_1-1]}$. Applying $\gamma \in \mathcal{K}_\infty$ to both sides of the ISS-estimate (2) in conjunction with this fact yields

$$\gamma(|\phi(k,x)|) \leq \max\left\{\gamma(\beta(|x|,0)), \gamma\left(\tfrac{1}{2}|x|\right)\right\} = \alpha(|x|), \tag{68}$$

for all $\phi \in \mathcal{S}_\mu(x)$, $k \in \mathbb{Z}_{[0,k_1-1]}$.

Then, using the definition of μ, the ISS-estimate (2), and (68), we have

$$\begin{aligned}
\gamma \circ \mu(|\phi(k_1,x)|) &\leq \tfrac{1}{4}\alpha^{-1}(|\phi(k_1,x)|) \\
&\leq \tfrac{1}{4}\max\left\{\alpha^{-1}(\beta(|x|,k)) \max_{j \in \mathbb{Z}_{[0,k_1-1]}} \alpha^{-1} \circ \gamma(|\phi(j,x)|)\right\} \\
&\leq \tfrac{1}{4}\max\{|x|,|x|\} = \tfrac{1}{4}|x|
\end{aligned} \tag{69}$$

which contradicts the definition of k_1 and hence proves the claim. ■

We now prove $\mathcal{KL}$-stability of difference inclusion (65) by proving uniform stability and uniform global attractivity and then appeal to Proposition 6.5.

Uniform stability follows using (2), (66), and the fact that $\gamma(s) \geq s$ as

$$\begin{aligned}
|\phi(k,x)| &\leq \max\left\{\beta(|x|,k), \max_{i\in\mathbb{Z}_{[0,k-1]}} \gamma(|\mu(|\phi(i,x)|)\mathcal{B}^m)\right\} \\
&\leq \max\left\{\beta(|x|,0), \max_{i\in\mathbb{Z}_{[0,k-1]}} \gamma(|\mu(|\phi(i,x)|))\right\} \\
&\leq \max\left\{\beta(|x|,0), \tfrac{1}{2}|x|\right\} \leq \max\left\{\beta(|x|,0), \tfrac{1}{2}\gamma(|x|)\right\} \\
&= \alpha(|x|). \qquad (70)
\end{aligned}$$

To establish uniform global attractivity, as above we note that, for all $x \in \mathbb{R}^n$, $\phi \in \mathcal{S}_\mu(x)$, and $k \in \mathbb{Z}_{\geq 0}$,

$$|\phi(k,x)| \leq \max\left\{\beta(|x|,k), \tfrac{1}{2}|x|\right\}.$$

Since $\beta \in \mathcal{KL}$, for each $r \in \mathbb{R}_{\geq 0}$ there exists a finite $T(r) \in \mathbb{Z}_{\geq 1}$ so that $\beta(r,k) \leq \frac{1}{2}r$ for all $k \in \mathbb{Z}_{\geq T(r)}$. Therefore, for all $|x| \leq r$ we have $|\phi(k,x)| \leq \frac{1}{2}r$ for all $\phi \in \mathcal{S}_\mu(x)$ and $k \in \mathbb{Z}_{\geq T(r)}$.

Fix any $\varepsilon \in \mathbb{R}_{>0}$ and let $k \in \mathbb{Z}_{\geq 1}$ be such that $2^{-k}r \leq \varepsilon$. Define $r_1 \doteq r$, $r_i \doteq \frac{1}{2}r_{i-1}$ for all $i \in \mathbb{Z}_{\geq 2}$, and $K(r,\varepsilon) \doteq \sum_{i=1}^{k} T(r_i)$. Then

$$|\phi(k,x)| \leq 2^{-k}r \leq \varepsilon, \quad \forall |x| \leq r, \quad \forall \phi \in \mathcal{S}_\mu(x), \quad \forall k \in \mathbb{Z}_{\geq K(r,\varepsilon)}.$$

Therefore, the difference inclusion (65) is $\mathcal{KL}$-stable. ■

We now complete the proof of Theorem 2.6. Since (1) is ISS, the difference inclusion (65) is $\mathcal{KL}$-stable, and by Theorem 6.4, for any $\lambda \in \mathbb{R}_{(0,1)}$, there exist functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (61) and (62) hold for the difference inclusion given by (65). This then implies that

$$|w| \leq \mu(|x|) \quad \Rightarrow \quad V(\phi(1,x,w)) \leq \lambda V(x) \qquad (71)$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and $\phi(1,x,w) = f(x,w)$.

It remains to show that the function V satisfies (5). Let $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ come from the ISS-estimate (2) and $\alpha_2 \in \mathcal{K}_\infty$ the upper bound in (61). Define $\sigma \in \mathcal{K}$ by

$$\sigma(s) \doteq \alpha_2(\beta(\mu^{-1}(s),1) + \gamma(s)), \quad \forall s \in \mathbb{R}_{\geq 0}.$$

For $|w| > \mu(|x|)$ we have

$$\begin{aligned}
V(f(x,w)) &\leq \alpha_2(|f(x,w)|) \leq \alpha_2\left(\beta(|x|,1) + \gamma(|w|)\right) \\
&\leq \alpha_2\left(\beta(\mu^{-1}(|w|),1) + \gamma(|w|)\right) = \sigma(|w|). \qquad (72)
\end{aligned}$$

Together with (71) we then have

$$V(f(x,w)) \leq \lambda V(x) + \sigma(|w|)$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. ■

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