Rotation-invariant measures for population study in HARDI

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Target audience– We present a novel family of rotational-invariant measures for High Angular Resolution Diffusion Imaging[1] (HARDI) data. This research is motivated by need for the developing new biomarkers and thereby facilitating population-based studies in HARDI.

Purpose– From a spherical harmonics (SH) representation of the angular information in diffusion MRI, we derive rotational-invariant measures computed as homogeneous polynomials of the SH coefficients. When the SH is truncated to order $L = 4$, we find a total of 12 such invariants, which are expected to be sensitive to subtle changes in white matter structure, not captured by tensor fractional anisotropy (FA) or generalized fractional anisotropy (GFA). The method is generalizable to any spherical function, we report here the derivation and results based on the apparent diffusion coefficient[2] (ADC) profile.

Methods– *Notations and problem statement.* From the coefficients, $\mathbf{x} = x_{lm}$ of the truncated SH expansion of the ADC profile, we propose to search for all homogeneous polynomials, $P[\mathbf{a}](\mathbf{x})$ (where $\mathbf{a}$ represents the vector of coefficients) that are invariant per rotation. We make use of the Wigner matrix, $\mathbf{W}_\mathbf{R}$, associated to the rotation matrix $\mathbf{R}$. So for a given polynomial rank, *t*, we want to find all vectors of polynomial coefficients $\mathbf{a} \in \mathcal{R}^D$, such that $\forall \mathbf{R} \in \mathcal{SO}(3)$, $\forall \mathbf{x} \in \mathcal{R}^N$, $P[\mathbf{a}](\mathbf{W}_\mathbf{R} \cdot \mathbf{x}) = P[\mathbf{a}](\mathbf{x})$. The dimensions for a given truncation order, *L*, and a given polynomial order, *t*, are $N = (L + 1)(L + 2)/2$, and $D = \binom{t+N-1}{N-1}$.

Homogeneous polynomial and linear transform. In order to solve the problem above, we try to derive the coefficients of the polynomial $P[\mathbf{a}](\mathbf{W}_\mathbf{R} \cdot \mathbf{x})$ from the vector of coefficients $\mathbf{a}$. Albeit not reported here due to lack of space, we can find a linear transform, $\mathbf{T}(\mathbf{W}_\mathbf{R})$, such that $\forall \mathbf{R} \in \mathcal{SO}(3)$, $\forall \mathbf{x} \in \mathcal{R}^N$, $P[\mathbf{a}](\mathbf{W}_\mathbf{R} \cdot \mathbf{x}) = P[\mathbf{T}(\mathbf{W}_\mathbf{R}) \cdot \mathbf{a}](\mathbf{x})$. This means that applying a linear transform to the argument of the polynomial is equivalent to applying an equivalent transform to the coefficients of the polynomial. Then the problem resolves into finding $\mathbf{a} \in \mathcal{R}^D$ such that $\forall \mathbf{R} \in \mathcal{SO}(3)$, $\mathbf{a} = \mathbf{T}(\mathbf{W}_\mathbf{R}) \cdot \mathbf{a}$.

Sufficient condition. Using arguments based on Euler angles decomposition and density of $\mathcal{N}$ in $[0, 2\pi] \mod 2\pi$, we can show that it is sufficient to solve the problem for the two particular rotations, $\mathbf{R}_x(1)$ and $\mathbf{R}_z(1)$, of 1.0 rad about x and z axes, respectively. The problem reduces to a large linear system of equations; we use efficient solver for sparse systems[3] to find a solution. The set of solutions is further pruned to keep only a set of linearly and functionally independent polynomials.

Results– The linear system gives 12 rotational invariant measures, after removing functionally dependent solutions. This is consistent with the dimension of the space of real symmetric SH: for $L = 4$, $N = 15$, and the 3 degrees of freedom of a rotation. These invariants are reported on Fig. 1.b. We fit an ADC profile in SH, $L = 4$, to images of a healthy subject scanned on a 3T Siemens 3T Verio™ scanner, 64 evenly distributed gradient directions, $b = 3000$ s/mm² and 2 non-weighted diffusion-weighted images. The maps reported on Fig. 1.b show a range of different contrasts, significantly different from FA and GFA, which suggests that they bring novel information on the "shape" of the ADC profile.

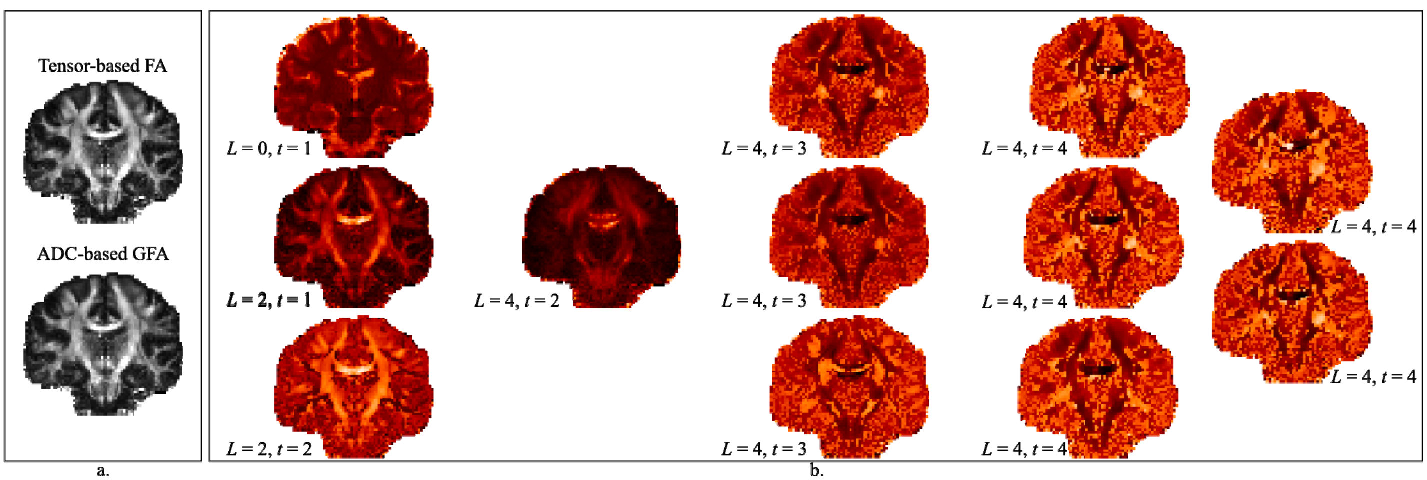

Figure 1. a. FA and GFA, for reference. b. The 12 ADC-based polynomial invariants, for increasing truncation order, L, and increasing polynomial rank, *t*.

Conclusions– We proposed a novel method to compute rotationally invariant measures in HARDI. The method is general and applies to the ADC, the ODF, or any other angular function represented by its SH coefficients. We believe that these measures will bring additional information on the angular complexity of diffusion functions, complementary to the indices already in use, based on tensor or on microstructure features. The results are complementary to recently published related methods[4,5]. The large number of measures will be used to create population classifiers, paving the way for pathology-specific biomarkers.

References– **1. D. S. Tuch et al.**, Magn Reson Med 48(4), pp. 577-582, 2002. **2. M. Descoteaux et al.**, Magn Reson Med 56(2), pp.395-41,, 2006. **3. R. Lehoucq et al.**, SIAM 6, 1998. **4. A. Ghosh et al.**, ISBI, pp. 26-29, 2012. **5. E. Schwab et al.**, IPMI, pp. 705-717, 2013.