Branching processes for the fragmentation equation

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Branching processes and stochastic fragmentation equation

Lucian Beznea[1], Madalina Deaconu[2], and Oana Lupaşcu[3]

Abstract. We investigate branching properties of the solution of a stochastic differential equation of fragmentation (SDEF) and we properly associate a continuous time càdlàg Markov process on the space $S^{\downarrow}$ of all fragmentation sizes, introduced by J. Bertoin. A binary fragmentation kernel induces a specific class of integral type branching kernels and taking as base process the solution of the initial (SDEF), we construct a branching process corresponding to a rate of loss of mass greater than a given strictly positive size d. It turns out that this branching process takes values in the set of all finite configurations of sizes greater than d. The process on $S^{\downarrow}$ is then obtained by letting d tend to zero. A key argument for the convergence of the branching processes is given by the Bochner-Kolmogorov theorem. The construction and the proof of the path regularity of the Markov processes are based on several newly developed potential theoretical tools, in terms of excessive functions and measures, compact Lyapunov functions, and some appropriate absorbing sets.

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1 Introduction

We study branching properties of the solution of a stochastic differential equation of fragmentation. Recall that the basic property of a measure-valued branching process is the following: if we consider two independent versions X and X' of the process, started respectively from two measures μ and μ', then $X + X'$ and the process started from $\mu + \mu'$ are equal in distribution. In studying the time evolution of fragmentation phenomena, it is supposed that "fragments split independently of each other", so, a branching property is fulfilled; cf. [4]. More specific, a main tool for defining the fragmentation chains are the branching Markov chains.

A different stochastic approach for studying the fragmentation (and coagulation) phenomena was developed in [21, 22, 27]: the evolution of the size of a typical particle in the system during a fragmentation process may be described by the solution of a stochastic differential equation, called *stochastic differential equation of fragmentation* (SDEF).

In this paper we associate a continuous time Markov branching process to an (SDEF), describing the time evolution of the fragments greater than a strictly positive size d. The

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model for the time evolution of all fragments (of arbitrary small size) is then constructed as a limit of a sequence of branching processes, corresponding to a fixed sequence of sizes $(d_n)_{n\geq 1}$ decreasing to zero. It is a continuous time Markov process on the state $S^{\downarrow}$ of all fragmentation sizes, considered by J. Bertoin. This process should be compared with the *stochastic coalescent process*, induced by Smoluchowski's coagulation equation in [36], page 95.

As a byproduct we emphasize integral type branching kernels on the space of all finite configurations of an interval $[d, 1]$, associated to the given fragmentation kernel and corresponding to the rate of loss of mass (in sense of [27]) greater than a fixed size d. These branching kernels lead to relevant examples of branching processes and it is possible to write down the nonlinear evolution equations satisfied by the associated cumulant semigroups.

The paper is organized as follows.

In the next section we present the fragmentation equation and the stochastic differential equation associated to it, following mainly [27]. A binary fragmentation kernel F is fixed, we state some hypotheses, give the basic definitions, and an example. Corollary 2.2 points out that in the case when the uniqueness of the solution holds, the solution of the stochastic differential equation of fragmentation induces a standard (Markov) process with state space the interval $[0, 1]$, its transition function being a C_0-semigroup on $\mathcal{C}([0, 1])$.

In Section 3 we first prove some properties of the real-valued Markov processes, produced by the procedure presented in Section 2, from the fragmentation kernel F truncated to sizes greater than d_n. In particular, the interval $[d_n, d_{n-1})$ becomes an absorbing set and therefore it is possible to restrict the process to this set. Putting together all these restrictions we obtain the base process of a forthcoming branching process on the set $\widehat{E_n}$ of all finite configurations of the set $E_n := [d_n, 1]$, $n \geq 1$. We show in Section 4 (Proposition 4.6) that the associated sequence of transition functions is a projective system and then, applying the Bochner-Kolmogorov theorem, we obtain a transition function on $S^{\downarrow}$ (Proposition 4.7). The already mentioned branching kernels associated to the given fragmentation kernel F, necessary for constructing the branching processes, are also introduced in Section 4.

The results on the existence of the branching processes (with state spaces $\widehat{E_n}$, $n \geq 1$) and of the fragmentation process (with state space $S^{\downarrow}$) are proved in Section 5, Proposition 5.1 and Theorem 5.2. A fragmentation property of the Markov process with state space $S^{\downarrow}$ is proved in Corollary 5.4. We apply the main result from [11] and a method developed in [12], using a Ray type compactification technique. A key point in proving the existence of the fragmentation process and its path regularity is the fact that there are excessive functions having compact level sets (see (5.13) and Remark 5.3).

The paper is completed by two appendices. Appendix (A) gives briefly some necessary complements on the potential theory associated to a right (Markov) processes: the entry time, the reduced function, excessive and strongly supermedian functions, absorbing sets, the restriction to an absorbing set of a resolvent and of a process, excessive measures. Appendix (B) presents the proofs of two results from Section 4.

2 Fragmentation equation and SDE

In this section we introduce the stochastic differential equation associated to the fragmentation equation.

The stochastic model. We consider a model which describes the fragmentation phenomenon for an infinite particle system. Each particle is characterized by its size and, at some random times, it can split into two particles by conserving mass. Let us denote by $c(t,x)$ the concentration of particles of size x at time t in the system. The evolution in time of $c(t,x)$ is governed by the fragmentation equation:

$$(2.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}c(t,x) & = \displaystyle\int_x^1 F(x,y-x)c(t,y)\mathrm{d}y - \frac{1}{2}c(t,x)\int_0^x F(y,x-y)\mathrm{d}y \\ & \qquad\qquad\qquad\qquad\qquad \text{for all } t \geq 0 \text{ and } x \in [0,1], \\ c(0,x) & = c_0(x) \text{ for all } x \in [0,1]. \end{cases}$$

In equation (2.1), F is the fragmentation kernel, that is $F : (0,1]^2 \longrightarrow \mathbb{R}_+$ is a symmetric function and $F(x,y)$ represents the rate of fragmentation of a particle of size $x+y$ into two particles of size x and y. We can suppose that the size of the initial particle is one.

In the first line of (2.1), the first term on the right hand side is counting the creation of particles of size x, due to the fragmentation of particles of larger size, say y, with $y > x$, into two parts x and $y-x$. The second term counts for the particles of size x which disappears after splitting into two smaller particles of size y and $x-y$, for $y < x$.

We aim to introduce a pure jump Markov process on $\mathbb{R}_+$ denoted by $(X_t)_{t\geq 0}$ whose law is the solution, in some sense, to the equation (2.1). This process will describe the evolution of the size of a typical particle in the system. The stochastic approach of the coagulation/fragmentation models goes up to the works of Deaconu, Fournier, and Tanré [21, 22] where mainly the coagulation part was considered. This phenomenon is more complex as it leads to non-linear equations. Later on, Fournier and Giet [27] considered the coagulation/fragmentation model and obtained existence results for the case of an infinite total rate of fragmentation and also they allow existence of particles of mass zero. Other studies on pure fragmentation case were made by Bertoin [2, 3], allowing multiple fragmentation and also erosion. Haas [28] studied the appearance or not of mass-zero particles.

We follow here mainly the structure of [27] for the pure fragmentation phenomena that we aim to link to the branching processes.

The main point that allows a probabilistic approach of (2.1) is given by the conservation of mass property. This writes on the form $\int_0^1 xc(t,x)\mathrm{d}x = 1$ and means that $p(t,x) = xc(t,x)$, $x \in [0,1]$, is a probability distribution for every fixed t. The aim is to describe the process having this distribution.

We start by stating some hypotheses on the fragmentation kernel.

Hypothesis

(H_1) The *fragmentation kernel* $F : (0,1]^2 \longrightarrow \mathbb{R}_+$ is a continuous symmetric map. Moreover, F is continuous from $[0,1]^2$ to $\mathbb{R}_+ \cup \{+\infty\}$. Let us define the function

$$\psi(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x y(x-y)F(y,x-y)\mathrm{d}y & \text{for } x > 0, \\ 0 & \text{for } x = 0, \end{cases}$$

which is supposed continuous on $[0,1]$. $\psi(x)$ represents the *rate of loss of mass of particles of mass* x. For each $\varepsilon \in (0,1)$ we have

$$(2.2)\qquad \lim_k \sup_{x\geq\varepsilon} \psi_k(x) = 0,$$

where

$$\psi_k(x) := \frac{1}{x}\int_0^x y(x-y)F(y,x-y)\mathbb{1}_{\{F(y,x-y)\geq k\}}\mathrm{d}y, \;\; k\in\mathbb{N}. \tag{2.3}$$

Notion of solution

Assume that (H_1) holds. A family $(Q_t)_{t\geq 0}$ of probability measures on $[0,1]$ is solution of (2.1) if the following conditions is fulfilled :

$$\langle Q_t,\phi\rangle = \langle Q_0,\phi\rangle + \int_0^t \langle Q_s,\mathcal{F}\phi\rangle \mathrm{d}s, \;\text{ for all } \phi\in\mathcal{C}^1([0,1]) \text{ and } t\geq 0, \tag{2.4}$$

where we denote $\langle Q_t,\phi\rangle = \int_0^1 \phi(y)Q_t(\mathrm{d}y)$ and for any $x\in[0,1]$ we define

$$\mathcal{F}\phi(x) = \int_0^x [\phi(x-y)-\phi(x)]\frac{x-y}{x}F(y,x-y)\mathrm{d}y. \tag{2.5}$$

Note that under hypothesis (H_1), $\lim\limits_{x\to 0_+}\mathcal{F}\phi(x)=0$.

In order to construct the pure jump stochastic process associated to (2.4) we introduce a probability space $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\geq 0},\mathbb{P})$. Consider also $\mathbb{D}([0,+\infty),[0,1])$ the space of càdlàg functions from $[0,+\infty)$ into $[0,1]$, endowed with the Skorokhod topology.

Under the hypothesis (H_1), let Q_0 be a probability measure on $[0,1]$. We say that X is a solution of the stochastic differential equation of fragmentation (abbreviated (SDEF)) if the following conditions hold :

1. $X=(X_t)_{t\geq 0}$ is an adapted process on $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\geq 0},\mathbb{P})$ whose paths belong to $\mathbb{D}([0,+\infty),[0,1])$.
2. $\mathcal{L}(X_0)=Q_0$.
3. For all $T\geq 0, \mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^{p+1}\right] < +\infty$.
4. There exists a Poisson measure $N(\mathrm{d}s,\mathrm{d}y,\mathrm{d}u)$ adapted to $(\mathcal{G}_t)_{t\geq 0}$ on $[0,+\infty)\times[0,1)\times[0,1)$ respectively with intensity measure $\mathrm{d}s\mathrm{d}y\mathrm{d}u$ such that the following stochastic differential equation holds :

$$X_t = X_0 - \int_0^t\int_0^1\int_0^1 y\mathbb{1}_{\{y\in(0,X_{s-})\}}\mathbb{1}_{\{u\leq \frac{X_{s-}-y}{X_{s-}}F(y,X_{s-}-y)\}}N(\mathrm{d}s,\mathrm{d}y,\mathrm{d}u). \tag{2.6}$$

The process X can be seen as the size of a sort of typical particle. This means that at some random instants the typical particle breaks into two smaller particles : we thus subtract y from X for some $y\in(0,X_{s-})$ at rate $F(y,X_{s-}-y)\frac{X_{s-}-y}{X_{s-}}$.

By Theorem 3.2 from [27] there exists a solution of (SDEF) under hypothesis (H_1).

Proposition 2.1. *Assume (H_1) is fulfilled and let Q_0 be a probability on $[0,1]$. Consider X a solution to $(SDEF)$ and for each $t\geq 0$ let $Q_t=\mathcal{L}(X_t)$. Then the family $\{Q_t\}_{t\geq 0}$ is a solution to* (2.4).

Remark. *If Q_t, $t \geq 0$, has a density with respect to the Lebesgue measure on $[0,1]$ and if we set $c(t,x) := \frac{\mathrm{d}Q_t}{x\mathrm{d}x}$, then $c(t,x)$ is a solution of (2.1); see [23].*

We give an example of a fragmentation kernel which satisfies the hypothesis (H_1).

Example. An example of a fragmentation kernel $F : [0,1]^2 \longrightarrow \mathbb{R}_+$ which satisfies the hypothesis (H_1) is $F(x,y) = x+y$. The rate of loss of mass of particles of mass x is

$$\psi(x) = \frac{1}{x}\int_0^x y(x-y)F(y,x-y)\mathrm{d}y = \frac{x^3}{6}, \quad x > 0.$$

Clearly ψ is continuous on $[0,1]$ and the fragmentation equation (2.1) becomes

$$\begin{cases} \dfrac{\partial}{\partial t}c(t,x) & = \displaystyle\int_x^1 yc(t,y)\mathrm{d}y - \frac{x^2}{2}c(t,x) \ \text{ for all } t, x \geq 0, \\ c(0,x) & = c_0(x) \text{ for all } x \geq 0. \end{cases}$$

Observe that the mass conservation condition $\int_0^1 xc(t,x) = 1$ is equivalent with $c(t,0) = t+2$, $t \geq 0$.

The uniqueness of the solution for the coagulation/fragmentation equation was studied by Banasiak and Lamb in a series of papers [1, 33]. Their approach is based on the semigroup theory. For the pure fragmentation case, under polynomially bounded fragmentation conditions they obtain uniqueness. For the discrete mass case, in the coagulation/fragmentation context, the uniqueness was also studied by Jourdain [31, 32] by using a probabilistic interpretation.

We emphasize now the Markov process induced by the solution X of the stochastic differential equation of fragmentation, in the case when the uniqueness of the solution holds.

For each $x \in [0,1]$ let $X_x = (X_{x,t})_{t\geq 0}$ be the solution of the stochastic differential equation of fragmentation with the initial distribution δ_x, i.e., $Q_0 = \delta_x$.

Corollary 2.2. *Assume that for each $x \in [0,1]$, taking $Q_0 = \delta_x$, the equation (2.4) has a unique solution $(Q_{t,x})_{t\geq 0}$ and the function $[0,1] \ni x \longmapsto \langle Q_{t,x}, \phi\rangle$ is continuous for each $\phi \in \mathcal{C}^1([0,1])$ and $t > 0$. Then the family of kernels $(Q_t)_{t\geq 0}$ on $[0,1]$, defined as*

$$Q_t f(x) := \langle Q_{t,x}, f\rangle, \ \ f \in p\mathcal{B}([0,1]), x \in [0,1],$$

induces a C_0-semigroup on $\mathcal{C}([0,1])$ and consequently it is the transition function of a standard (Markov) process $X^0 = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t^0, P^x)$ with state space $[0,1]$. In addition, the following assertions hold.

(i) For all $t \geq 0$ and $x \in [0,1]$ $(X_{x,t}, \mathbb{P})$ and (X_t^0, P^x) have the same distribution.

(ii) For every $t > 0$ we have a.s. $X_t^0 \leq X_0^0$.

Proof. The semigroup property of $(Q_t)_{t\geqslant 0}$ is rather a straight-forward consequence of the uniqueness. Indeed, we have to show that $Q_{t'+t}\phi = Q_t(Q_{t'}\phi)$, so, it is enough to prove that the mapping $s \longmapsto Q_{t'+s,x}$ verifies the equation (2.4) (with $Q_0 = \delta_x$, $x \in [0,1]$) and $Q_{t'}\phi$ instead of ϕ. We have

$$\begin{aligned} \langle Q_{t'+t,x}\phi\rangle & = \phi(x) + \int_0^{t'}\langle Q_{s,x}, \mathcal{F}\phi\rangle \mathrm{d}s + \int_0^t \langle Q_{t'+s,x}, \mathcal{F}\phi\rangle \mathrm{d}s \\ & = Q_{t'}\phi(x) + \int_0^t \langle Q_{t'+s,x}, \mathcal{F}\phi\rangle \mathrm{d}s. \end{aligned}$$

We claim that $(Q_t)_{t\geq 0}$ is a C_0-semigroup on $\mathcal{C}([0,1])$. Indeed, if $\phi \in \mathcal{C}^1([0,1])$ then $\mathcal{F}\phi(x) \leq ||\phi'||_\infty \psi(x)$, $x \in [0,1]$, and by (2.4) $||Q_t\phi - \phi||_\infty \leq t||\phi'||_\infty||\psi||_\infty$ for all $t > 0$. The assertion follows by the density of $\mathcal{C}^1([0,1])$ in $\mathcal{C}([0,1])$. The existence of the standard process X^0 having $(Q_t)_{t\geq 0}$ as transition function is now a consequence of a main result on Feller processes, see e.g., [17], Theorem (9.4).

Assertion (i) is clear since by Proposition 2.1, for each $x \in [0,1]$, we have $\mathcal{L}(X_{x,t}) = Q_{t,x} = \mathcal{L}(X_t^0)$. (ii) is a consequence of (i), observing that from (2.6) for each $x \in [0,1]$ and $t > 0$ we get a.s. $X_{x,t} \leq X_{x,0}$. $\square$

Remark. *Let* $(\mathsf{L}, \mathcal{D}(\mathsf{L}))$ *be the infinitesimal generator of the* C_0*-semigroup* $(Q_t)_{t\geq 0}$ *from Corollary 2.2. Then* $\mathcal{C}^1([0,1]) \subset \mathcal{D}(\mathsf{L})$ *and the restriction of* L *to* $\mathcal{C}^1([0,1])$ *is the operator* $\mathcal{F}$ *given by* (2.5). *In particular, for every* $\phi \in \mathcal{C}^1([0,1])$ *and every probability* ν *on* $[0,1]$ *the process*

$$\phi(X_t^0) - \int_0^t \mathcal{F}\phi(X_s^0)\mathrm{d}s,\ t \geq 0,$$

is an $(\mathcal{F}_t)_{t\geq 0}$*-martingale under* $P^\nu := \int P^x\nu(\mathrm{d}x)$. *(The martingale property is a version of a result from [27], page 1313.)*

Indeed, observe first that if $\phi \in \mathcal{C}^1([0,1])$ then $\mathcal{F}\phi \in \mathcal{C}([0,1])$ (as in [27], page 1314). Consequently, from the C_0-continuity of $(Q_t)_{t\geq 0}$ we deduce that the function $\alpha(s) := ||Q_s\mathcal{F}\phi - \mathcal{F}\phi||_\infty$, $s \in \mathbb{R}_+$, is continuous and therefore, using (2.4),

$$\lim_{t\to 0}\left|\left|\frac{Q_t\phi - \phi}{t} - \mathcal{F}\phi\right|\right|_\infty \leq \lim_{t\to 0}\frac{1}{t}\int_0^t \alpha(s)\mathrm{d}s = 0.$$

We conclude that $\phi \in \mathcal{D}(\mathsf{L})$ and $\mathsf{L}\phi = \mathcal{F}\phi$. The claimed martingale property is a straightforward consequence of the Markov property of the process X^0.

3 Markov processes induced by fragmentation kernels

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$ be a right (Markov) process with state space E, a Borel subset of $[0,1]$, and let $(P_t)_{t\geq 0}$ be its transition function,

$$P_tf(x) = E^x(f \circ X_t), x \in E, f \in p\mathcal{B}(E);$$

$\mathcal{B}(E)$ denotes the Borel σ-algebra of E and $p\mathcal{B}(E)$ (resp. $bp\mathcal{B}(E)$) the set of all positive numerical (resp. bounded) $\mathcal{B}(E)$-measurable functions on E. Assume that X is conservative (i.e., $P_t1 = 1$) and for $x \in E$ and $t \geq 0$ let $P_{t,x}$ be the probability measure on $(E, \mathcal{B})$ induced by the kernel P_t,

$$P_{t,x}(A) := P_t(1_A)(x) \text{ for all } A \in \mathcal{B}(E).$$

Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the resolvent family of X,

$$U_\alpha f(x) = E^x\int_0^\infty e^{-\alpha t} f \circ X_t\mathrm{d}t = \int_0^\infty e^{-\alpha t}P_tf(x)\mathrm{d}t, f \in p\mathcal{B}(E), \alpha > 0.$$

A set $A \in \mathcal{B}(E)$ is called *absorbing* provided that $R_\beta^{E\setminus A}1 = 0$ on A; see (A1) in Appendix (A). It is easy to see that the property of a set to be absorbing does not depend on β and that every absorbing set is finely open. Recall that the *fine topology* is the smallest topology on E

making continuous all $\mathcal{U}_\beta$-excessive functions. Note that the absorbing sets we consider are not necessary finely closed; see e.g. [38] and (3.1) from [15].

If A is absorbing and $x \in A$ then $P^x(X_t \in A) = 1$ for all $t \geq 0$, i.e., the probability measure $P_{t,x}$ is carried by A, or equivalently $P_t(1_{E\setminus A}) = 0$ on A for all $t \geq 0$.

We fix a sequence $(d_n)_{n\geq 1} \subseteq (0,1)$ strictly decreasing to zero and for each $n \geq 1$ define

$$E_n := [d_n, 1].$$

The main hypotheses are the following.

(H_2) For each $n \geq 1$ there exists a conservative right Markov process X^n with state space E_n and transition function $(P_t^n)_{t\geq 0}$ such that

$$P_{t,x}^{n+1} = P_{t,x}^n \text{ for all } n \geq 1, t \geq 0, \text{ and } x \in E_n.$$

(H_3) For each $n \geq 1$ the set

$$E_n' := [d_{n+1}, d_n)$$

is absorbing in E_{n+1} with respect to the resolvent $\mathcal{U}^{n+1} = (U_\alpha^{n+1})_{\alpha>0}$ of X^{n+1}.

Remark. *(i) The compatibility between P_t^{n+1} and P_t^n stated in (H_2) expresses the fact that the Markov process X^n is induced by a fragmentation in particles with "size" bigger than d_n.*

(ii) With the above interpretation, condition (H_3) is natural: if a particle is already smaller than d_n, then it is not possible to produce further "fragments" with bigger size.

A first consequence of the hypotheses (H_2) and (H_3) is the following.

(3.1) The set E_n is absorbing in E_{n+1} (with respect to the resolvent $\mathcal{U}^{n+1}$).

Indeed, if $x \in E_n$ then by (H_2), $P_{t,x}^{n+1}(E_n) = P_{t,x}^n(E_n) = 1$, hence $P_{t,x}^{n+1}(E_n') = 0$ for all $t > 0$,

$$U_\alpha^{n+1}(1_{E_n'}) = \int_0^\infty e^{-\alpha t} P_t^{n+1}(1_{E_n'})\mathrm{d}t = 0 \text{ on } E_n.$$

So, the function $v := U_\alpha^{n+1}(1_{E_n'})$ is $\mathcal{U}_\alpha^{n+1}$-excessive and vanishes on E_n. Since by (H_3) the set E_n' is absorbing in E_{n+1} it is a finely open subset of E_{n+1} and thus $U_\alpha^{n+1}(1_{E_n'}) > 0$ on E_n' and we conclude that $E_n = [v = 0]$, therefore it is absorbing by $(A1.4)$ from Appendix (A).

Let F be a fragmentation kernel as in Section 2 and for $n \geq 1$ define

$$F_n(x,y) := 1_{(d_n,1]}(x \wedge y \wedge |x-y|)F(x,y), \; x, y \in E := [0,1].$$

Assume further that hypothesis (H_1) is fulfilled by the fragmentation kernel F. We claim that

(3.2) condition (H_1) is verified by F_n for all n.

Indeed, the continuity of the corresponding rate of loss of mass follows by dominate convergence while (2.2) is fulfilled because $F_n \leq F$.

By (3.2) and Proposition 2.1 for every $n \geq 1$ and each probability Q_0 on $[0,1]$ there exist solutions to the (SDEF) and to (2.4), where in (2.6) F is replaced by F_n. Assume that:

(3.3) Taking $Q_0 = \delta_x$, the equation (2.4) has a unique solution $(Q^n_{t,x})_{t\geq 0}$ and the function $[0,1] \ni x \longmapsto \langle Q^n_{t,x}, \phi\rangle$ is continuous for each $\phi \in \mathcal{C}^1(E)$ and $t > 0$.

By (3.3) and Corollary 2.2 there exists a standard process $X^{0,n}$ with state space E and transition function $(Q^n_t)_{t\geq 0}$, where $Q^n_t f(x) := \langle Q^n_{t,x}, f\rangle$, $f \in p\mathcal{B}(E)$, $x \in E$, and $(Q^n_t)_{t\geq 0}$ is a C_0-semigroup on $\mathcal{C}(E)$.

Assertion (ii) of Corollary 2.2 implies that for all $n \geq 1$

$$X^{0,n}_t \leq X^{0,n}_0 \text{ a.s. for each } t > 0. \tag{3.4}$$

Let $\mathcal{U}^{0,n} = (U^{0,n}_\alpha)_{\alpha>0}$ be the resolvent of $(Q^n_t)_{t\geq 0}$, $U^{0,n}_\alpha = \int_0^\infty e^{-\alpha t} Q^n_t \mathrm{d}t$, $\alpha > 0$.

Proposition 3.1. *The following assertion hold for $n \geq 1$ and with respect to the resolvent $\mathcal{U}^{0,n}$ on E.*

(i) For all $x \leq d_n$ and $t \geq 0$ we have $Q^n_{t,x} = \delta_x$.

(ii) The set $E_n = [d_n, 1]$ is an absorbing subset of E.

(iii) For each $x \in [0,1)$ the sets $[0,x]$ and $[0,x)$ are absorbing subsets of E. In particular, the set $E'_{n-1} = [d_n, d_{n-1})$ is absorbing, where $E'_0 := E_1$.

Proof. If $\phi \in \mathcal{C}^1(E)$ let

$$\mathcal{F}_n\phi(x) := \int_0^x [\phi(x-y) - \phi(x)] \frac{x-y}{x} F_n(y, x-y)\mathrm{d}y, \ x \in E.$$

If $x \leq d_n$ then $F_n(y, x-y) = 0$ for all y, hence $\mathcal{F}_n\phi(x) = 0$ and by (2.4) $Q^n_{t,x} = \delta_x$, hence (i) holds. If in addition $supp\, \phi \subset [0, d_n]$ then $\mathcal{F}_n\phi = 0$ and again by (2.4) $Q^n_t\phi = \phi$. Consequently, $Q^n_t(1_{[0,d_n)}) = 1_{[0,d_n)}$ for all $t \geq 0$, $U^{0,n}_\alpha(1_{[0,d_n)}) = \frac{1}{\alpha} 1_{[0,d_n)}$, $E_n = [U^{0,n}_\alpha(1_{[0,d_n)}) = 0]$, therefore (ii) also holds.

Assertion (iii) is a consequence of (3.4). Indeed, observe first that the right continuity of the trajectories implies that for all $x \in E$ we have P^x-a.s.

$$X^{0,n}_t \leq X^{0,n}_0 \text{ for all } t \geq 0$$

and consequently for each $x \in E$

$$X^{0,n}_t \leq x \text{ for all } t \geq 0\,, P^x - \text{a.s.}$$

It follows that $D_{(x,1]} = \infty$, P^x-a.s. and by $(A1.2)$ we conclude that the set $[0,x]$ is absorbing.

The set $[0,x)$ is also absorbing by $(A1.3)$, since $[0,x) = \bigcup_n [0, (x - \frac{1}{n})^+]$. $\square$

Because by assertion (iii) of Proposition 3.1 the set E'_{n-1} is absorbing with respect to the resolvent $\mathcal{U}^{0,n}$, $n \geq 1$, we may consider the restriction $\widetilde{X^{0,n}}$ of $X^{0,n}$ to E'_{n-1}; see $(A1.5)$ in Appendix (A). It is a conservative standard (Markov) process with state space E'_{n-1} and let $(\widetilde{P}^n_t)_{t\geq 0}$ be its transition function. By $(A1.6)$ we have

$$\widetilde{P}^n_t(f|_{E'_{n-1}}) = Q^n_t f \text{ on } E'_{n-1} \text{ for each } f \in p\mathcal{B}(E) \text{ and } t \geq 0.$$

Let now $n \geq 1$ be fixed. Then $E_n = \bigcup_{k=1}^n E'_{k-1}$ and we consider the conservative standard process X^n with state space E_n, which behaves as $\widetilde{X^{0,k}}$ on E'_{k-1}, $1 \leq k \leq n$. The transition function $(P_t^n)_{t\geq 0}$ of X^n is the following

$$P_t^n(f|_{E_n}) = \widetilde{P}_t^k(f|_{E'_{k-1}}) = Q_t^k f \text{ on } E'_{k-1}, 1 \leq k \leq n, f \in p\mathcal{B}(E), t \geq 0. \tag{3.5}$$

Proposition 3.2. *Conditions (H_2) and (H_3) are fulfilled by X^n, $n \geq 1$, and*

$$X_t^n \leq X_0^n \text{ a.s. for each } t > 0.$$

Proof. (H_3) holds since the process X^n is by construction such that all the sets E'_k, $k \leq n-1$, are absorbing in E_n. If $x \in E_n$ then there exists $k \leq n$ such that $x \in E'_{k-1}$ and by (3.5) for $f \in p\mathcal{B}(E)$ and $t \geq 0$ we have

$$P_{t,x}^{n+1}(f|_{E_{n+1}}) = Q_t^k f(x) = P_{t,x}^n(f|_{E_n})$$

and consequently (H_2) also holds. The claimed inequality follows from (3.4). □

4 Branching kernels and transition functions on the space of fragmentation sizes

For a Borel subset E of $[0,1]$ define the set $\widehat{E}$ of finite positive measures on E as

$$\widehat{E} := \left\{ \sum_{k \leq k_0} \delta_{x_k} : k \in \mathbb{N}^*, x_k \in E \text{ for all } 1 \leq k \leq k_0 \right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero measure. We identify $\widehat{E}$ with the union of all symmetric m-th powers $E^{(m)}$ of E:

$$\widehat{E} = \bigcup_{m \geq 0} E^{(m)},$$

where $E^{(0)} := \{0\}$; see, e.g., [29, 13, 11]. The set $\widehat{E}$ is called the *space of finite configurations of E* and it is endowed with the topology of disjoint union of topological spaces and the corresponding Borel σ-algebra $\mathcal{B}(\widehat{E})$; see [25].

Let $M(E)$ be the set of all positive finite measures on E. For a function $f \in p\mathcal{B}(E)$ we consider the mappings $l_f : M(E) \longrightarrow \overline{\mathbb{R}}_+$ and $e_f : M(E) \longrightarrow [0,1]$, defined as

$$l_f(\mu) := \langle \mu, f \rangle := \int f \mathrm{d}\mu, \;\; \mu \in M(E), \quad e_f := \exp(-l_f).$$

Consider the σ-algebra $\mathcal{M}(E)$ on $M(E)$ generated by $\{l_f : f \in bp\mathcal{B}(E)\}$. $\widehat{E}$ becomes a $\mathcal{M}(E)$-measurable subset of $M(E)$ and the trace of $\mathcal{M}(E)$ on $\widehat{E}$ is $\mathcal{B}(\widehat{E})$.

If p_1, p_2 are two finite measures on $\widehat{E}$, then their convolution $p_1 * p_2$ is the finite measure on $\widehat{E}$ defined for every $F \in p\mathcal{B}(\widehat{E})$ by

$$\int_{\widehat{E}} p_1 * p_2(\mathrm{d}\nu) F(\nu) := \int_{\widehat{E}} p_1(\mathrm{d}\nu_1) \int_{\widehat{E}} p_2(\mathrm{d}\nu_2) F(\nu_1 + \nu_2).$$

If $\varphi \in p\mathcal{B}(E)$, define the *multiplicative function* $\widehat{\varphi} : \widehat{E} \longrightarrow \mathbb{R}_+$ as

$$\widehat{\varphi}(\mathbf{x}) = \begin{cases} \prod_{k:x_k>0} \varphi(x_k), & \text{if } \mathbf{x} = (x_k)_{k\geq 1} \in \widehat{E}, \mathbf{x} \neq \mathbf{0}, \\ 1, & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Observe that each multiplicative function $\widehat{\varphi}$, $\varphi \in p\mathcal{B}(E)$, $\varphi \leq 1$, is the restriction to $\widehat{E}$ of an exponential function on $M(E)$,

$$\widehat{\varphi} = e_{-\ln \varphi}.$$

In the harmonic analysis on configuration spaces the multiplicative function $\widehat{\varphi}$ is called *coherent state*; see, e.g., [25].

Remark 4.1. *Since the family $\mathcal{A} = \{e_f : f \in pb\mathcal{B}(E)\}$ is multiplicative, separates the points of $M(E)$, and $\sigma(\mathcal{A}|_{\widehat{E}}) = \mathcal{B}(\widehat{E})$, the following assertions hold for two finite measures p_1, p_2 on $\widehat{E}$:*
(i) $p_1 = p_2$ if and only if $p_1(\widehat{\varphi}) = p_2(\widehat{\varphi})$ for all $\varphi \in p\mathcal{B}(E), \varphi \leq 1$.
(ii) $p_1 * p_2(\widehat{\varphi}) = p_1(\widehat{\varphi})p_2(\widehat{\varphi})$ for all $\varphi \in p\mathcal{B}(E), \varphi \leq 1$.

Recall that a bounded kernel N on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ is called *branching kernel* if

$$N_{\mu+\nu} = N_\mu * N_\nu \text{ for all } \mu, \nu \in \widehat{E},$$

where N_μ denotes the measure on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ such that $\int g \mathrm{d} N_\mu = Ng(\mu)$ for all $g \in p\mathcal{B}(\widehat{E})$. Note that if N is a branching kernel on $\widehat{E}$ then $N_0 = \delta_0 \in M(E)$.

(4.1) If $B : p\mathcal{B}(\widehat{E}) \longrightarrow p\mathcal{B}(E)$ is a sub-Markovian kernel (resp. a Markovian kernel) then there exists a unique sub-Markovian (resp. Markovian) branching kernel $\widehat{B}$ on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ such that

$$\widehat{B}\widehat{\varphi} = \widehat{B\widehat{\varphi}} \text{ for all } \varphi \in p\mathcal{B}(E), \varphi \leq 1.$$

Sketch of the proof of (4.1). The kernel $\widehat{B}$ is defined as:

(4.2)
$$\widehat{B}_{\mathbf{x}} := \begin{cases} B_{x_1} * \ldots * B_{x_n}, & \text{if } \mathbf{x} = \delta_{x_1} + \ldots + \delta_{x_n},\ x_1, \ldots, x_n \in E, \\ \delta_{\mathbf{0}}, & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Examples of branching kernels.

1. Let $(q_m)_{m\geq 1} \subseteq p\mathcal{B}(E)$ be such that $\sum_{m\geq 1} q_m = 1$. One can consider the Markovian kernel $B : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$ defined as

$$Bh(x) := \sum_{m\geq 1} q_m(x) h_m(x, \ldots, x), \quad h \in bp\mathcal{B}(\widehat{E}),$$

where $h_m := h|_{E^{(m)}}$. By (4.1) there exists a branching kernel $\widehat{B}$ on $\widehat{E}$ such that

$$\widehat{B}\widehat{\varphi}|_E = \sum_{m\geq 1} q_m \varphi^m \text{ for each } \varphi \in p\mathcal{B}(E), \varphi \leq 1.$$

2. **Branching kernels induced by binary fragmentation kernels.** Let $F : [0,1]^2 \longrightarrow \mathbb{R}$ be a symmetric function. Recall that (cf. [27], page 1303) $F(x, y)$ may be seen as the rate of fragmentation of particles of mass $x + y$ into particles of mass x, y. F is called *fragmentation kernel.*

Let $n \geq 1$ and define the function $\psi_n^0 : E_n \longrightarrow \mathbb{R}_+$ as

$$\psi_n^0(x) := \int_{d_n}^{x} \frac{y(x-y)}{x} F(y, x-y)\mathrm{d}y, \ x \in E_n.$$

Observe that ψ_n^0 is continuous on E_n and $\lim\limits_{x \searrow d_n} \psi_n^0(x) = 0.$

Analogously with the interpretation given in [27] (where $d_n = 0$), $\psi_n^0(x)$ represents the *rate of loss of mass of particles of mass x greater than $d_n > 0$.* This truncation ψ_n^0 of ψ should be compared with ψ_n defined in (2.3), which is also a truncation of ψ but it refers rather to the large values of the fragmentation kernel F.

If $d > 0$ and $g \in bp\mathcal{B}([d, 1])$ define the function $g_d \in bp\mathcal{B}([d, 1])$ as

$$g_d(y) := g(d)1_{[0,d)}(y) + g(y)1_{[d,1]}(y), \ y \in [0, 1].$$

Define further the kernel $B_d : p\mathcal{B}(\widehat{[d, 1]}) \longrightarrow p\mathcal{B}([d, 1])$ as

$$B_d h(x) := \frac{6}{x^3} \int_0^x y(x-y)h_d(y, y)\mathrm{d}y, \ x \in [d, 1],$$

where $h_d(\cdot, \cdot)$ denotes the restriction of $h \in bp\mathcal{B}(\widehat{[d, 1]})$ to $[d, 1]^{(2)} = \{\delta_{x_1} + \delta_{x_2} : x_1, x_2 \in [d, 1]\}$. Clearly, the kernel B_d is Markovian (since $B_d 1(x) = \dfrac{6}{x^3} \int_0^x y(x-y)\mathrm{d}y = 1$) and for each $x \geq d$ the probability measure $B_{d,x}$ is carried by $[d, 1]^{(2)}$.

Let $n \geq 1$ and consider the kernel $B^n : bp\mathcal{B}(\widehat{E}_n) \longrightarrow bp\mathcal{B}(E_n)$ defined by

$$B^n h := \sum_{k=1}^{n} 1_{E'_{k-1}} B_{d_k} h, \ h \in bp\mathcal{B}(\widehat{E}_n),$$

where $E'_0 := E_1 = [d_1, 1]$. In particular, $B^1 = B_{d_1}$, for each $n \geq 1$ the kernel B^n is Markovian, and the probability measure B^n_x, $x \in E_n$, is carried by $E_n^{(2)} = \{\delta_{x_1} + \delta_{x_2} : x_1, x_2 \in E_n\}$. The main property of the kernel B^n is the linear dependence between the image of the fragmentation kernel through B^n and the rate of loss of mass, namely

$$B^n F_x(x) = c_n(x)F(d_n, 0) + \frac{6}{x^2}\psi_n^0(x), \ x \in E'_{n-1},$$

where $c_n(x) := \frac{6}{x^3} \int_0^{d_n} y(x-y)\mathrm{d}y$, $F_x(x, y) := F(y, x-y)$, $x, y \in E_n$, and F_x is regarded as a function on $\widehat{E_n}$, having a non-zero component only on $E_n^{(2)}$.

Branching processes. We assume that there exists a branching Markov process with state space $\widehat{E}$, associated with the branching kernel $\widehat{B}$ on $\widehat{E}$; see [11] for results on the existence of such a branching process. Recall that a right (Markov) process with state space $\widehat{E}$ is called *branching process* provided that its transition function is formed by branching kernels; the probabilistic interpretation of this analytic branching property was mentioned in the beginning of the Introduction. For further developments see the classical works [29, 30, 39], the lecture notes [20] and [34], the monograph [35], and the articles [26] and [5].

Let $(\widehat{P}_t)_{t\geq 0}$ be the transition function of the above branching process and $\widehat{\mathcal{U}} = (\widehat{U_\alpha})_{\alpha>0}$ its resolvent of kernels on $\widehat{E}$.

Lemma 4.2. *If a subset A of E is absorbing with respect to $\mathcal{U}$ then $\widehat{A}$ is an absorbing subset of $\widehat{E}$ with respect to $\widehat{\mathcal{U}}$.*

Proof. By [7] one can see that

(4.3) A subset of E is absorbing with respect to $\mathcal{U}$ if and only if it is absorbing with respect to $\mathcal{U}_\beta$ for some $\beta > 0$.

Applying Proposition 4.8 from [11], if $\beta > 0$ there exists $\beta' > 0$ such that for every $v \in b\mathcal{E}(\mathcal{U}^0_\beta)$, where $\mathcal{U}^0$ is the resolvent on E obtained from $\mathcal{U}$ by a convenient perturbation with a kernel (cf. Propostion 4.5 from [11]), the function $1 - e_v$ is $\widehat{\mathcal{U}_{\beta'}}$ -excessive. Using now $(A1.1)$ one can see that if v is strongly supermedian with respect to $\mathcal{U}^0_\beta$, then the function $1 - e_v$ is strongly supermedian with respect to $\widehat{\mathcal{U}_{\beta'}}$.

Let A be absorbing with respect to $\mathcal{U}$. By (4.3) one can show that A is absorbing with respect to $\mathcal{U}^0$ and let v be a strongly supermedian function with respect to $\mathcal{U}^0_\beta$ such that $A = [v = 0]$. Consequently, $\int v \mathrm{d}\mu = 0$ for all $\mu \in \widehat{A}$ and $\int v \mathrm{d}\mu > 0$ if $\mu \notin \widehat{A}$. We conclude that $\widehat{A} = [e_v = 1]$ and since $1 - e_v$ is strongly supermedian with respect to $\mathcal{U}_{\beta'}$, again by (4.3) it follows that the set $\widehat{A}$ is absorbing with respect to $\widehat{\mathcal{U}}$. $\square$

Let $(\widehat{P^n_t})_{t\geq 0}$ be the transition function on $\widehat{E_n}$, $n \geq 1$, induced by $(P^n_t)_{t\geq 0}$ and by the kernel B^n associated to a fragmentation kernel F.

Proposition 4.3. *If $\mathbf{x} \in \widehat{E_n}$ and $t > 0$ then $\widehat{P^{n+1}_{t,\mathbf{x}}} = \widehat{P^n_{t,\mathbf{x}}}$.*

Proof. If $\mathbf{x} = \mathbf{0}$ then by (4.2) $\widehat{P^{n+1}_{t,\mathbf{0}}} = \delta_{\mathbf{0}} = \widehat{P^n_{t,\mathbf{0}}}$. Hence we may assume further that $\mathbf{x} \neq \mathbf{0}$. By Remark 4.1 (i) we have to prove that $\widehat{P^{n+1}_t}(\widehat{1_{E_n}\varphi})(\mathbf{x}) = \widehat{P^n_t}\widehat{\varphi}(\mathbf{x})$ for all $\varphi \in p\mathcal{B}(E_n)$, $\varphi \leq 1$, and $\mathbf{x} \in \widehat{E_n}$. Let h^n_t be the absolutely monotonic map such that $\widehat{P^n_t}\widehat{\varphi} = \widehat{h^n_t(\varphi)}$; see [11]. So, we have to show that $h^{n+1}_t(1_{E_n}\varphi) = h^n_t(\varphi)$ on E_n. By Proposition 4.1 from [11] $h^{n+1}_t(1_{E_n}\varphi) =: h'_t$ is the unique solution of the equation

$$h'_t(x) = {}^cP^{n+1}_t(1_{E_n}\varphi)(x) + c\int_0^t {}^cP^{n+1}_{t-u}(B^{n+1}\widehat{h'_u})(x)\mathrm{d}u,\ t \geqslant 0,\ x \in E_{n+1},$$

where ${}^cP^n_t f := e^{-ct}P^n_t f$ with $0 < c < 2$. By (H_2) we have on E_n: ${}^cP^{n+1}_t f = {}^cP^{n+1}_t(1_{E_n} f) = {}^cP^n_t(f|_{E_n})$ for all $f \in bp\mathcal{B}(E_{n+1})$. On the other hand the following equality also holds on E_n: $B^{n+1}\widehat{h'_u} = B^n\widehat{h'_u}$. From the above considerations h'_t verifies on E_n the equation

$$h'_t = {}^cP^n_t\varphi + c\int_0^t {}^cP^n_{t-u}(B^n\widehat{h'_u})\mathrm{d}u,\ t \geqslant 0.$$

Since $h^n_t(\varphi)$ is also a solution of this equation, again by Proposition 4.1 from [11] we conclude that $h'_t = h^n_t(\varphi)$ on E_n. $\square$

For $n \geq 1$ let $\mathcal{L}^n$ be the infinitesimal generator of the right Markov process X^n with state space E_n and transition function $(P_t^n)_{t\geq 0}$, given by hypothesis (H_2). Let further $h_t^n, t \geq 0$, be the absolutely monotonic map such that $\widehat{P_t^n}\widehat{\varphi} = \widehat{h_t^n(\varphi)}, \varphi \in p\mathcal{B}(E^n), \varphi \leq 1$; see the proof of Proposition 4.3. Consider also the associated cumulant semigroup $(V_t^n)_{t\geq 0}$,

$$V_t^n f := -\ln h_t^n(e^{-f}), f \in bp\mathcal{B}(E_n), t \geq 0;$$

see Corollary 4.3 from [11]. In particular, $\widehat{P_t^n}(e_f) = e_{V_t^n f}$ for all $f \in bp\mathcal{B}(E_n)$.

Remark 4.4. *(i) By Remark 4.2 (ii) from [11], the integral equation verified by $(h_t^n)_{t\geq 0}$ is formally equivalent to*

$$\frac{d}{dt}h_t^n = (\mathcal{L}^n - c)h_t^n + cB^n\widehat{h_t^n},\ t \geqslant 0.$$

(ii) Let $n = 1$. Since $h_t^1 = e^{-V_t^1 f}$ and $B^1 = B_{d_1}$, one can deduce from (i) that the cumulant semigroup $(V_t^1)_{t\geq 0}$ is formally the solution of the following evolution equation

$$\begin{aligned}\frac{d}{dt}v_t(x) \quad &= -e^{v_t(x)}\mathcal{L}(e^{-v_t})(x) + \\ & c\left[1 - \frac{d_1^2(3x-2d_1)}{x^3}e^{v_t(x)-2v_t(d_1)} - \frac{6}{x^3}\int_{d_1}^{x} y(x-y)e^{v_t(x)-2v_t(y)}\mathrm{d}y\right],\end{aligned}$$

for $x \geq d_1, t \geq 0$, where $v_t(x) := V_t^1 f(x)$. The above equation may be compared with the one satisfied by the cumulant semigroup of the discrete branching process in the case $\mathcal{L} = \Delta$ and with the first example of a branching kernel, given by a sequence $(q_n)_{n\geq 1} \subset bp\mathcal{B}(E)$; see [11], Remark 4.4 (iii). Recall also that in this case the equation of the cumulant semigroup $(V_t)_{t\geq 0}$ is: $\frac{d}{dt}V_t f = \Delta V_t f - (V_t f)^2, t \geq 0$, see, e.g. [26] and [5].

The space of fragmentation sizes. Following [4] we consider as state space the set $S^{\downarrow}$ of decreasing numerical sequences bounded above from 1 and with limit 0,

$$S^{\downarrow} := \{\mathbf{x} = (x_k)_{k\geq 1} \subseteq [0,1] : (x_k)_{k\geq 1} \text{ decreasing}, \lim_k x_k = 0\}.$$

Recall that a sequence $\mathbf{x}$ from $S^{\downarrow}$ may be considered as "the sizes of the fragments resulting from the split of some block with unit size" (cf. [4], page 16).

Let further

$$S := \{\mathbf{x} = (x_k)_{k\geq 1} \in S^{\downarrow} : \exists\, k_0 \in \mathbb{N}^* \text{ s.t. } x_{k_0} > 0 \text{ and } x_k = 0 \text{ for all } k > k_0\}.$$

$$S_i^{\downarrow} := \{\mathbf{x} = (x_k)_{k\geq 1} \subseteq (0,1] : (x_k)_{k\geq 1} \text{ decreasing}, \lim_k x_k = 0\}.$$

We have

$$S^{\downarrow} = S \cup S_i^{\downarrow} \cup \{\mathbf{0}\} \text{ and } S \cap S_i^{\downarrow} = \emptyset,$$

where $\mathbf{0}$ denotes the zero constant sequence.

Recall that $E_n = [d_n, 1]$, $n \geq 1$, where $(d_n)_{n\geq 1} \subseteq (0,1)$ is a sequence strictly decreasing to zero and let

$$S_n := \left\{ \sum_{k\leq k_0} \delta_{x_k} : k_0 \in \mathbb{N}^* ;\ x_k \in E_n,\ \text{for all } 1 \leq k \leq k_0 \right\}.$$

It is convenient to identify a sequence $\mathbf{x} = (x_k)_{k\geq 1}$ from $S^{\downarrow}$ with the σ-finite measure $\mu_{\mathbf{x}}$ on $[0,1]$, defined as

$$\mu_{\mathbf{x}} := \begin{cases} \displaystyle\sum_{k:x_k>0} \delta_{x_k} & , \text{ if } \mathbf{x} \neq \mathbf{0}, \\ \mathbf{0} & , \text{ if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Consequently, the mapping $\mathbf{x} \longmapsto \mu_{\mathbf{x}}$ identifies S with $\bigcup_{n\geq 1} S_n$. For $\mathbf{x} \in S^{\downarrow}$ we write $\mathbf{x} = \mu_{\mathbf{x}}$ where it is necessary to emphasize the identification of the sequence $\mathbf{x}$ with the measure $\mu_{\mathbf{x}}$. To each set of finite measures S_n on E_n we add the zero measure $\mathbf{0} = \mu_{\mathbf{0}}$,

$$S_n^0 := S_n \cup \{\mathbf{0}\},\ n \geq 1.$$

So,

$$\widehat{E_n} = S_n^0 \ \text{ for all } n \geq 1.$$

Define the mapping $\alpha_n : S^{\downarrow} \longmapsto S_n^0$ as

$$\alpha_n(\mathbf{x}) := \mu_{\mathbf{x}}|_{E_n},\ \mathbf{x} = \mu_{\mathbf{x}} \in S^{\downarrow}.$$

We have $\alpha_n(\mathbf{0}) = \mathbf{0}$ and $\alpha_n|_{S_n^0} = \mathrm{Id}_{S_n^0}$. Define

$$S_\infty := \{(\mathbf{x}^n)_{n\geq 1} \in \prod_{n\geq 1} S_n^0 :\ \mathbf{x}^n = \alpha_n(\mathbf{x}^m) \text{ for all } m > n \geq 1\}.$$

In the next two propositions we identify first $S^{\downarrow}$ and S_∞. We show then that the (branching) transition functions we constructed on each $\widehat{E_n}$, $n \geq 1$, induces a projective system of probability measures; for the proofs see (B1) and (B2) in Appendix (B).

Proposition 4.5. *The mapping $i : S^{\downarrow} \longmapsto S_\infty$, defined as*

$$i(\mathbf{x}) := (\alpha_n(\mathbf{x}))_{n\geq 1},\ \mathbf{x} \in S^{\downarrow},$$

is a bijection.

Proposition 4.6. *Let $\mathbf{x} \in S^{\downarrow}$ and $\mathbf{x}_n := \alpha_n(\mathbf{x}) \in \widehat{E_n}, n \geq 1$. If $t > 0$ then the sequence of probability measures $(\widehat{P}_{t,\mathbf{x}_n})_{n\geq 1}$ is projective with respect to $(\widehat{E_n}, \alpha_n)_{n\geq 1}$, that is*

$$\widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}} \circ \alpha_n^{-1} = \widehat{P^n_{t,\mathbf{x}_n}} \ \textit{for all } n \geq 1.$$

Proposition 4.7. *Assume that conditions (H_2) and (H_3) hold. Then there exists a Markovian transition function $(\widehat{P}_t)_{t\geq 0}$ on $(S^{\downarrow}, \mathcal{B}(S^{\downarrow}))$ such that for each $\mathbf{x} \in S^{\downarrow}$ and $n \geq 1$ we have*

$$\widehat{P_{t,\mathbf{x}}} \circ \alpha_n^{-1} = \widehat{P^n_{t,\mathbf{x}_n}},$$

where $\mathbf{x}_n := \alpha_n(\mathbf{x})$.

Proof. We apply the Bochner-Kolmogorov theorem (cf. Theorem 53 from [24] and also [19]), which is a more general version of Kolmogorov theorem on the existence of the limit of a projective sequence of measure spaces, assuming no continuity of the mappings; for the topological case (with continuity conditions) see [16], Theorem 2.1 in [18] and [37].

By Proposition 4.6 the system $(\widehat{E_n}, \widehat{P^n_{t,\mathbf{x}_n}}, \alpha_n)_{n\geq 1}$ is projective. We already mentioned that each $\widehat{E_n}$ is endowed with the canonical topological structures and the corresponding Borel σ-algebra $\mathcal{B}(\widehat{E_n})$. Therefore there exists a unique probability measure $\widehat{P_{t,\mathbf{x}}}$ on S_∞ such that the claimed equality holds. Note that by Propostion 4.5 the map $i : S^{\downarrow} \longrightarrow S_\infty$ is a bijection and by Proposition 1.1 from [19] it identifies $S^{\downarrow}$ and S_∞ as measurable spaces. The uniqueness property implies that the family of kernels $(\widehat{P}_t)_{t\geq 0}$ is a transition function on $S^{\downarrow}$. □

5 Fragmentation and branching processes on finite configurations

Let X^n, $n \geq 1$, be the Markov processes constructed in Section 3 from the fragmentation kernel F. In particular, by (3.2) and Proposition 3.2 conditions (H_1), (H_2), and (H_3) are fulfilled by F_n and the processes X^n. We also consider the branching kernels B^n, $n \geq 1$, associated to F.

Proposition 5.1. *Let $n \geq 1$ and $(\widehat{P^n_t})_{t\geq 0}$ be the transition function on $\widehat{E_n}$ induced by $(P^n_t)_{t\geq 0}$ and by the branching kernel B^n. Then there exists a branching standard (Markov) process with state space $\widehat{E_n}$, having $(\widehat{P^n_t})_{t\geq 0}$ as transition function.*

Proof. Note that by (H_3) and (3.1) E_k and E'_k are absorbing subsets of E_n for all $k = 1, \ldots, n-1$. Consequently, we have

(5.1) If $v_0, v_1, \ldots, v_{n-1} \in \mathcal{E}(\mathcal{U}^n_\beta), \beta > 0$, then the function $v := \sum_{k=0}^{n-1} 1_{E'_k} v_k$ is also $\mathcal{U}^n_\beta$-excessive; recall that $E'_0 = [d_1, 1]$.

Consider the vector space $\mathcal{C}_n$ defined as

$$\mathcal{C}_n := \{f : [d_n, 1] \longrightarrow \mathbb{R} \, : \, f|_{E'_k} \in \mathcal{C}(E'_k) \text{ s.t. } \lim_{y \searrow d_k} f(y) \in \mathbb{R}, \forall\, k = 0, \ldots, n-1\}$$

and let $\mathcal{A}_n$ denotes the closure in the supremum norm of the linear space $[b\mathcal{E}(\mathcal{U}^n_\beta)]$ spanned by the bounded $\mathcal{U}^n_\beta$-excessive functions. $\mathcal{A}_n$ does not depend on $\beta > 0$; see e.g. Remark 2.1 from [11].

We claim that

$$\mathcal{C}_n \subseteq \mathcal{A}_n. \tag{5.2}$$

To prove it, we start with a notation: if $k \in \{0, \ldots, n-1\}$ and $f \in \mathcal{C}_n$, we consider the function $f_k \in \mathcal{C}(E_n)$ defined as

$$f_k(x) = \begin{cases} f(x) & , \text{ if } x \in E'_k, \\ f(d_{k+1}) & , \text{ if } d_n \leq x < d_{k+1}, \\ \lim_{y \searrow d_k} f(y), & \text{if } d_k \leq x \leq 1. \end{cases}$$

The C_0-continuity of the semigroups $(Q_t^k)_{t\geq 0}$ and (3.5) imply that

$$\lim_{\alpha\to\infty} \alpha U_\alpha^n f_k = f$$

uniformly on $E'_k, 0 \leq k \leq n-1$. We have $U_\alpha^n f_k \in [b\mathcal{E}(\mathcal{U}_\beta^n)]$ for all α and k. By (5.1) the function

$$v_\alpha := \sum_{k=0}^{n-1} 1_{E'_k} \alpha U_\alpha^n f_k$$

belongs to $[b\mathcal{E}(\mathcal{U}_\beta^n)]$ for all $\alpha > 0$. We conclude that $\lim\limits_{\alpha\to\infty} v_\alpha = f$ uniformly on $E_n = \bigcup_{k=0}^{n-1} E'_k$, hence $f \in \mathcal{A}_n$.

We show now that

$$P_t^n(\mathcal{C}_n) \subset \mathcal{C}_n,\ t \geq 0, \text{ and } \lim_{t\searrow 0} ||P_t^n f - f||_\infty = 0 \text{ for all } f \in \mathcal{C}_n, \tag{5.3}$$

i.e., $(P_t^n)_{t\geq 0}$ is a C_0-semigroup of contractions on $\mathcal{C}_n$. Indeed, if $f \in \mathcal{C}_n$ then again by the C_0-continuity of the semigroups $(Q_t^n)_{t\geq 0}$ and (3.5) we have $P_t^n f|_{E'_k} = Q_t^n f_k|_{E'_k}$ and $Q_t^n f_k \in \mathcal{C}(E_n)$, so, $\lim\limits_{t\searrow 0} P_t^n f = f$ uniformly on E'_k for all $k = 0, \ldots, n-1$, $\lim\limits_{t\searrow 0} ||P_t^n f - f||_\infty = 0$.

Observe that the integral form of the kernels B_{d_k}, occurring in the definition of the kernel B^n, implies that

$$\text{if } \varphi \in p\mathcal{B}(E_n), \varphi \leq 1, \text{ then } B^n\widehat{\varphi} \in \mathcal{C}_n. \tag{5.4}$$

Let K_n be the kernel on E_n defined as

$$K_n f := \frac{c}{c+2} B^n(l_f),\ f \in bp\mathcal{B}(E_n).$$

Because the probability measure B_x^n is carried by $E_n^{(2)}, l_1|_{E_n^{(2)}} = 2$, and $c \leq 2$, we deduce that K_n is a sub-Markovian kernel on E_n. We may consider the perturbation with the kernel K_n of the semigroup $(P_t^n)_{t\geq 0}$, that is, the sub-Markovian semigroup of kernels $(Q_t^{0,n})_{t\geq 0}$ on E_n such that for each $f \in pb\mathcal{B}(E_n)$ the function $r_t := Q_t^{0,n} f$ is the solution of the integral equation

$$r_t = {}^cP_t^n f + c\int_0^t {}^cP_{t-u}^n(K_n r_u)\mathrm{d}u,\ t \geq 0;$$

see Proposition 4.5 from [11]. By (5.3) and (5.4) we have

$$(Q_t^{0,n})_{t\geq 0} \text{ is a } C_0\text{-semigroup of contractions on } \mathcal{C}_n. \tag{5.5}$$

Let $\mathcal{U}^0 = (U_\alpha^0)_{\alpha>0}$ be the resolvent of kernels on E_n associated with $(Q_t^{0,n})_{t\geq 0}$.

We can prove now that the following condition $(\star)$ holds for $\mathcal{C}_n$ (which is a subset of $\mathcal{A}_n$ by (5.2)).

$(\star)$ There exists a countable subset $\mathcal{F}_0$ of $b\mathcal{E}(\mathcal{U}_\beta^0)$ which is additive, $0 \in \mathcal{F}_0$, and separates the finite measures on E_n, such that $\{e^{-u} : u \in \mathcal{F}_0\} \subseteq \mathcal{C}_n$, $P_t^n\varphi, P_t^n(B^n\widehat{\varphi}) \in \mathcal{C}_n$ for all $\varphi \in \mathcal{C}_n$, $0 \leq \varphi \leq 1$, and $t > 0$.

Indeed, the existence of the countable set $\mathcal{F}_0$ follows using Ray cones techniques (see Proposition 2.2 (ii) from [15]). Note that one can take $\mathcal{F}_0 \subset \mathcal{C}_n$ and (5.5) ensures that it separates the finite measures on E_n. Since $\mathcal{C}_n$ is a Banach algebra, it follows that $\{e^{-u} : u \in \mathcal{F}_0\} \subset \mathcal{C}_n$. By (5.3) and (5.4) we conclude that condition $(\star)$ is verified. The existence of the claimed standard process with state space $\widehat{E_n}$ and having $(\widehat{P_t^n})_{t\geq 0}$ as transition function follows now by Theorem 4.10 from [11]. $\square$

We can state now the main result on the existence of an associated Markov process on $S^{\downarrow}$. We endow $S^{\downarrow}$ with the topology induced by the identification with S^{∞} (endowed with the product topology) given by Proposition 1.1 from [19].

Theorem 5.2. *There exists a right (Markov) process with state space $S^{\downarrow}$, having càdlàg trajectories, and with $(\widehat{P}_t)_{t\geq 0}$ (given by Proposition 4.7) as transition function.*

Proof. By Proposition 4.7 we have for all $n \geq 1$

$$\widehat{P}_t(f \circ \alpha_n) = \widehat{P_t^n} f \circ \alpha_n, \ f \in bp\mathcal{B}(\widehat{E_n}), t \geq 0. \tag{5.6}$$

Let further $\widehat{\mathcal{U}} = (\widehat{U_\alpha})_{\alpha>0}$ be the resolvent of kernels of the semigroup $(\widehat{P}_t)_{t\geq 0}$. From (5.6) we get

$$\widehat{U_\alpha}(f \circ \alpha_n) = \widehat{U_\alpha^n} f \circ \alpha_n \ \text{ for all } f \in bp\mathcal{B}(\widehat{E_n}) \text{ and } \alpha > 0. \tag{5.7}$$

As a consequence of (5.7) the following properties holds:

$$\text{If } v \in \mathcal{E}(\widehat{\mathcal{U}_\beta^n}) \text{ then } v \circ \alpha_n \in \mathcal{E}(\widehat{\mathcal{U}_\beta}). \tag{5.8}$$

The assertion follows by Hunt approximation Theorem.

(5.9) If ξ is a $\widehat{\mathcal{U}_\beta}$-excessive measure then $\xi \circ \alpha_n^{-1}$ is a $\widehat{\mathcal{U}_\beta^n}$-excessive measure. If in addition ξ is a potential $\widehat{\mathcal{U}_\beta}$-excessive measure, $\xi = \mu \circ \widehat{U_\beta}$, then $\xi \circ \alpha_n^{-1}$ is also a potential $\widehat{\mathcal{U}_\beta^n}$-excessive measure, $\xi = (\mu \circ \alpha_n^{-1}) \circ \widehat{U_\beta^n}$. (For the definition of the excessive measure see (A2) in Appendix (A).)

We have

$$\mathcal{B}(S^{\downarrow}) = \sigma\left(\bigcup_{n\geq 1}\{f \circ \alpha_n \ : \ f \in bp\mathcal{B}(\widehat{E_n})\}\right). \tag{5.10}$$

Since clearly $\mathcal{B}(\widehat{E_n}) = \sigma(b\mathcal{E}(\widehat{\mathcal{U}_\beta^n}))$, by (5.8) we get

$$\mathcal{B}(S^{\downarrow}) = \sigma(b\mathcal{E}(\widehat{\mathcal{U}_\beta})). \tag{5.11}$$

We claim that

(5.12) all the points of $S^{\downarrow}$ are non-branch points with respect to $\widehat{\mathcal{U}_\beta}$, i.e., if u, v are two $\widehat{\mathcal{U}_\beta}$-excessive functions then $\inf(u, v)$ is also a $\widehat{\mathcal{U}_\beta}$-excessive function.

Indeed, if $u = \widehat{U_\beta}(f\circ\alpha_n), v = \widehat{U_\beta}(g\circ\alpha_n)$ with $f,g\in bp\mathcal{B}(\widehat{E_n})$, then by Proposition 5.1 the function $w_n = \inf(\widehat{U^n_\beta}f, \widehat{U^n_\beta}g)$ is $\widehat{\mathcal{U}^n_\beta}$-excessive and by (5.7) we have $\inf(u,v) = \inf(\widehat{U^n_\beta}f\circ\alpha_n, \widehat{U^n_\beta}g\circ\alpha_n) = w_n\circ\alpha_n$. By (5.8) we conclude that $\inf(u,v)\in\mathcal{E}(\widehat{\mathcal{U}_\beta})$. Using (5.10) and Lemma 1.2.10 from [7] it follows (by a monotone class argument) that all the points of $S^\downarrow$ are non-branch points.

Since (5.12) and (5.11) hold, by Theorem 2.1 from [12], to obtain that $(\widehat{P}_t)_{t\geq 0}$ is the transition function of a càdlàg Markov process with state space $S^\downarrow$, it remains to show that the following two conditions are fulfilled:

(5.13) For all $\mathbf{x}\in S^\downarrow$ there exists a $\widehat{\mathcal{U}_\beta}$-excessive function $v_\mathbf{x}$ such that $v_\mathbf{x}(\mathbf{x})<\infty$ and the set $[v_\mathbf{x}\leq k]$ is relatively compact for all $k\geq 1$; such a function $v_\mathbf{x}$ is called *compact Lyapunov function*.

(5.14) There exists a countable subset $\mathcal{F}$ of $[b\mathcal{E}(\widehat{\mathcal{U}_\beta})]$, generating the topology of $S^\downarrow, 1\in\mathcal{F}$, and there exists $u_0\in\mathcal{E}(\widehat{\mathcal{U}_\beta}), u_0<\infty$, such that if ξ,η are two $\widehat{\mathcal{U}_\beta}$-excessive measures with $L_\beta(\xi+\eta,u_0)<\infty$ and $L_\beta(\xi,\varphi) = L_\beta(\eta,\varphi)$ for all $\varphi\in\mathcal{F}$, then $\xi=\eta$.

Here L_β is the energy functional with respect to $\widehat{\mathcal{U}_\beta}$; see (A2) in Appendix (A).

Let $\mathbf{x}\in S^\downarrow$. If $n\geq 1$, since the branching process $\widehat{X^n_t}$ with state space $\widehat{E_n}$ is càdlàg, there exists $w^n_\mathbf{x}\in\mathcal{E}(\widehat{\mathcal{U}^n_\beta})$ such that $w^n_\mathbf{x}(\alpha_n(\mathbf{x})) = \frac{1}{2^n}$ and the sets $[w^n_\mathbf{x}\leq k], k\geq 1$, are relatively compact subsets of $\widehat{E_n}$; let M^n_k be the closure of $[w^n_\mathbf{x}\leq k]$ in $\widehat{E_n}$ and $v^n_\mathbf{x} := w^n_\mathbf{x}\circ\alpha_n$. By (5.8) we have $(v^n_\mathbf{x})_{n\geq 1}\subseteq\mathcal{E}(\widehat{\mathcal{U}^n_\beta})$. We take $v_\mathbf{x} := \sum\limits_{n\geq 1} v^n_\mathbf{x}$. Then $v_\mathbf{x}\in\mathcal{E}(\widehat{\mathcal{U}_\beta}), v_\mathbf{x}(\mathbf{x}) = 1$, and $[v_\mathbf{x}\leq k]\subseteq\bigcap\limits_{n\geq 1}\alpha_n^{-1}(M^n_k) =: M$. Because $\alpha_n(M)$ is a compact subset of $\widehat{E_n}$ for all $n\geq 1$ we conclude that M is a compact subset of $S^\downarrow$. So, $v_\mathbf{x}$ has compact level sets and consequently (5.13) holds.

We prove now (5.14). Observe first that for each $n\geq 1, v\in\mathcal{E}(\widehat{\mathcal{U}^n_\beta})$, and $\widehat{\mathcal{U}_\beta}$-excessive measure ξ

$$L_\beta(\xi, v\circ\alpha_n) = L^n_\beta(\xi\circ\alpha_n^{-1}, v), \tag{5.15}$$

where L^n_β denotes the energy functional with respect to $\widehat{\mathcal{U}^n_\beta}$ and we recall that by (5.8) we have $v\circ\alpha_n\in\mathcal{E}(\widehat{\mathcal{U}_\beta})$ and by (5.9) the measure $\xi\circ\alpha_n^{-1}$ is $\widehat{\mathcal{U}^n_\beta}$-excessive. To check (5.15) we may assume that $v = \widehat{U^n_\beta}f$ with $f\in bp\mathcal{B}(\widehat{E_n})$. Using (5.7) we then obtain:

$$L_\beta(\xi, v\circ\alpha_n) = L_\beta(\xi, \widehat{U_\beta}(f\circ\alpha_n)) = \xi\circ\alpha_n^{-1}(f) = L^n_\beta(\xi\circ\alpha_n^{-1}, v).$$

From [11], the proof of Theorem 4.10, for each $n\geq 1$ (5.14) holds for the resolvent $\widehat{\mathcal{U}^n_\beta}$ on $\widehat{E_n}$. Let $\mathcal{F}_n\subset[b\mathcal{E}(\widehat{\mathcal{U}^n_\beta})]$ be the corresponding set of functions generating the topology of $\widehat{E_n}$. Consequently, if we consider the family $\mathcal{F} := \bigcup\limits_{n\geq 1}\{f\circ\alpha_n \ : \ f\in\mathcal{F}_n\}$ then $\mathcal{F}$ generated the topology of $S^\downarrow$. (5.8) implies $\mathcal{F}\subseteq[b\mathcal{E}(\widehat{\mathcal{U}_\beta})]$ and we take $u_0 := \widehat{U_\beta}1$. Let ξ,η be two finite $\widehat{\mathcal{U}_\beta}$-excessive measures such that $L_\beta(\xi,\varphi) = L_\beta(\eta,\varphi)$ for all $\varphi\in\mathcal{F}$. Then by (5.15)

$$L^n_\beta(\xi\circ\alpha_n^{-1}, f) = L^n_\beta(\eta\circ\alpha_n^{-1}, f) \text{ for all } n\geq 1 \text{ and } f\in\mathcal{F}_n.$$

From (5.14) applied for the resolvent $\widehat{\mathcal{U}^n_\beta}$ we get $\xi \circ \alpha_n^{-1} = \eta \circ \alpha_n^{-1}$ for all $n \geq 1$ and by (5.10) we conclude that $\xi = \eta$. So, (5.14) also holds for $\widehat{\mathcal{U}^n_\beta}$ on $S^\downarrow$. □

Remark 5.3. (i) *A main argument in proving Theorem 5.2 was the existence of the compact Lyapunov functions with respect to the resolvent of kernels $\widehat{\mathcal{U}}$ on $S^\downarrow$ (see the above condition* (5.13)*). This method was initially used for finding martingale solutions of stochastic partial differential equations on Hilbert spaces (cf. [9]) but it turned out to be efficient in other situations too, e.g., to prove existence results for measure-valued branching processes (see [5, 12, 11]).*

(ii) *According to Remark 2.2* (i) *from [12], condition* (5.13) *is necessary in order to obtain a process with càdlàg trajectories and it is equivalent with the tightness property of the associated Choquet capacities; for details see also [10, 14, 15].*

Let $(\widehat{X_t})_{t\geq 0}$ be the Markov process with state space $S^\downarrow$, having the transition function $(\widehat{P}_t)_{t\geq 0}$ (see Theorem 5.2). In the sequel if $\mathbf{y} = (y_k)_{k\geq 1} \in S^\downarrow$ and $x \in [0,1]$, we write $\mathbf{y} \leq x$ provided that $y_k \leq x$ for all $k \geq 1$.

The next corollary emphasize a fragmentation property of $(\widehat{X_t})_{t\geq 0}$.

Corollary 5.4. *For each $x \in [0,1]$, $\mathbf{y} \in S^\downarrow$, $\mathbf{y} \leq x$, and $t \geq 0$*

$$\widehat{X_t} \leq x \quad P^{\mathbf{y}}\text{-}a.s.$$

Proof. Using Proposition 4.7 we have

$$P_t^{\mathbf{y}}(\widehat{X_t} \leq x) = \lim_n P^{\mathbf{y}}(d_n \leq \widehat{X_t} \leq x) = \lim_n \widehat{P_{t,\mathbf{y}}}(\alpha_n^{-1}([d_n, x])) =$$

$$\lim_n \widehat{P^n_{t,y_n}}([d_n, x]) = \lim_n P^{y_n}(X_t^n \leq x) \geq \lim_n P^{y_n}(X_t^n \leq X_0^n) = 1,$$

where the above inequality holds because P^{y_n}-a.s $X_0^n = y_n \leq x$. □

Appendix (A): complements on Markov processes

(A1) The restriction to an absorbing set. We assume further that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of a right (Markov) process X with state space E.

A function $v \in p\mathcal{B}(E)$ is called $\mathcal{U}$-excessive provided that $\alpha U_\alpha v \leq v$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha U_\alpha v = v$ point-wise. Denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive functions.

If $\beta > 0$, consider the β-level subprocess of X, recall that its transition function is $(e^{-\beta t}P_t)_{t\geq 0}$ and has $\mathcal{U}_\beta := (U_{\beta+\alpha})_{\alpha>0}$ as associated resolvent. For $u \in \mathcal{E}(\mathcal{U}_\beta)$ and every subset A of E we consider the function

$$R^A_\beta u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) : \ v \geq u \text{ on } A\},$$

called the β-order *reduced function* of u on A. It is known that if $A \in \mathcal{B}(E)$ then $R^A_\beta u$ is universally $\mathcal{B}(E)$-measurable and if moreover A is finely open and $u \in p\mathcal{B}(E)$ then $R^A_\beta u \in p\mathcal{B}(E)$.

The reduced function $R^A_\beta u$ is no longer an $\mathcal{U}_\beta$-excessive function, however it is strongly supermedian; recall that a positive universally $\mathcal{B}(E)$-measurable function v is called *strongly* *supermedian* (with respect to $\mathcal{U}_\beta$) provided that $\int v \mathrm{d}\mu \leq \int v \mathrm{d}\nu$ for every two finite measures μ, ν on E such that $\mu \circ U_\beta \leq \nu \circ U_\beta$; for details see [7].

($A1.1$) Let v be a positive, $\mathcal{B}(E)$-measurable function (or only a nearly Borel measurable function). By [6, 8, 7] the following assertions are equivalent.

(i) The function v is strongly supermedian with respect to $\mathcal{U}_\beta$.

(ii) $R_\beta^M v \leq v$ for every $M \in \mathcal{B}(E)$ (or only for every Ray compact subset M of E).

(iii) There exists a family $\mathcal{F}$ of $\mathcal{U}_\beta$-excessive functions such that $v = \inf \mathcal{F}$.

(iv) We have $v = \inf\{u \in \mathcal{E}(\mathcal{U}_\beta) : u \geq v\}$.

Recall that if $A \in \mathcal{B}(E)$ then the *entry time* of A is the stopping time $D_A : \Omega \longrightarrow [0, \infty]$, defined as $D_A(\omega) := \inf\{t \geq 0 : X_t(\omega) \in A\}$, $\omega \in \Omega$. The following identification (essentially due to G. A. Hunt; see e.g. [24]) of the entry operators and the reduced function on a set holds for all $A \in \mathcal{B}(E)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$:

$$R_\beta^A u(x) = E^x(e^{-\beta D_A} u(X_{D_A})).$$

($A1.2$) Using the above formula on the reduced function, one can check that the following assertions are equivalent for a set $A \in \mathcal{B}(E)$:

(i) The set A is absorbing, i.e., $R_\beta^{E \setminus A} 1 = 0$ on A.

(ii) We have P^x-a.s. $D_{E \setminus A} = \infty$ for every $x \in A$.

(iii) There exists a strongly supermedian function v such that $A = [v = 0]$.

($A1.3$) If $(A_k)_k$ is a sequence of absorbing sets then $\bigcup_k A_k$ is also absorbing.

($A1.4$) *Examples.* If v is a $\mathcal{U}_\beta$-excessive function then the sets $[v < \infty]$ and $[v = 0]$ are absorbing. Indeed, since $1 \leq \frac{1}{n} v$ on the set $[v = \infty]$ for every $n \geq 1$, it follows that $R_\beta^{[v=\infty]} 1 = 0$ on $[v < \infty]$. Since every $\mathcal{U}_\beta$-excessive function is strongly supermedian, by ($A1.1$) we get that the set $[v = 0]$ is absorbing.

The following properties hold for an absorbing set A.

Restriction of the resolvent. If A is absorbing then $U_\beta(1_{E \setminus A}) = 0$ on A and therefore we may consider the *restriction* $\mathcal{U}' = (U'_\alpha)_{\alpha > 0}$ *of* $\mathcal{U}$ *to* A, i.e., the sub-Markovian resolvent of kernels on $(A, \mathcal{B}(A))$, defined as:

$$U'_\alpha f := U_\alpha \bar{f}|_A \text{ for all } f \in p\mathcal{B}(A),$$

where $\bar{f} \in p\mathcal{B}(E)$ is such that $\bar{f}|_A = f$.

Restriction of the excessive function. A function $u \in p\mathcal{B}(A)$ is $\mathcal{U}'_\beta$-excessive if and only if there exists a function $\bar{u} \in \mathcal{E}(\mathcal{U}_\beta)$ such that $u = \bar{u}|_A$.

($A1.5$) *Restriction of the process.* If A is absorbing then the restriction of $\mathcal{U}$ to A is the resolvent of a conservative right (Markov) process with state space A, called the *restriction of* X *to* A and we denote it by $\widetilde{X}$: $\widetilde{\Omega} := \{\omega \in \Omega : X_t(\omega) \in A \text{ for all } t \geq 0\}$, $\widetilde{P}^x := P^x|_{\widetilde{\Omega}}$ for all $x \in A$, and $\widetilde{X}_t(\omega) := X_t(\omega)$ if $\omega \in \widetilde{\Omega}$ (see, e.g., (12.30) in [38]). The main observation is that the transition function $(\widetilde{P}_t)_{t\geq 0}$ of $\widetilde{X}$ is precisely the restriction to A of the transition function $(P_t)_{t\geq 0}$ of X, that is

$$(A1.6) \qquad \widetilde{P}_t(f|_A) = P_t f \text{ on } A \text{ for all } f \in p\mathcal{B}(E), t \geq 0.$$

(A2) Excessive measures, the energy functional. Let $\mathcal{U} = (U_\alpha)_{\alpha \geqslant 0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B}(E))$, such that the σ-algebra $\mathcal{B}(E)$ is generated by $\mathcal{E}(\mathcal{U}_\beta)$ and all the points of E are non-branch points with respect to $\mathcal{U}_\beta$, $\beta > 0$.

A positive σ-finite measure ξ on E is called *$\mathcal{U}$-excessive*, provided that $\xi \circ \alpha U_\alpha \leqslant \xi$ for all $\alpha > 0$. Let $\mathsf{Exc}(\mathcal{U})$ be the set of all $\mathcal{U}$-excessive measures on E and recall that if $\xi \in \mathsf{Exc}(\mathcal{U})$ then $\xi \circ \alpha U_\alpha \nearrow \xi$ as $\alpha \to \infty$. Let $\mathsf{Pot}(\mathcal{U})$ be the set of all *potential* $\mathcal{U}$-excessive measures: if $\xi \in \mathsf{Exc}(\mathcal{U})$ then $\xi \in \mathsf{Pot}(\mathcal{U})$ if $\xi = \nu \circ U$, where ν is a σ-finite on E.

The *energy functional* $L_\beta : \mathsf{Exc}(\mathcal{U}_\beta) \times \mathcal{E}(\mathcal{U}_\beta) \longrightarrow \overline{\mathbb{R}}_+$ is defined as

$$L_\beta(\xi, u) := \sup\{\nu(u) : \mathsf{Pot}(\mathcal{U}_\beta) \ni \nu \circ U_\beta \leqslant \xi\}.$$

Appendix (B)

(B1) Proof of Proposition 4.5. Observe first that: $\mathbf{x} = \mathbf{0}$ if and only if $\alpha_n(\mathbf{x}) = \mathbf{0}$ for all $n \geq 1$. Consequently, if $\mathbf{x} \in S^\downarrow$ and $i(\mathbf{x}) = i(\mathbf{0})$ then $\mathbf{x} = \mathbf{0}$. Let now $\mathbf{x}, \mathbf{y} \in S^\downarrow, \mathbf{x} \neq \mathbf{0} \neq \mathbf{y}$. Therefore $\mu_{\mathbf{x}}$ and $\mu_{\mathbf{y}}$ are measures on $(0,1]$ and if $i(\mathbf{x}) = i(\mathbf{y})$ then $\mu_{\mathbf{x}}|_{E_n} = \mu_{\mathbf{y}}|_{E_n}$ for all $n \geq 1$ and we conclude that $\mu_{\mathbf{x}} = \mu_{\mathbf{y}}$ and $\mathbf{x} = \mathbf{y}$.

If $\mathbf{x} \in S^\downarrow$ and $1 \leq n < m$ then $\alpha_n(\mathbf{x}) = \mu_{\mathbf{x}}|_{E_n} = \alpha_n(\alpha_m(\mathbf{x}))$ and thus $i(\mathbf{x}) \in S_\infty$.

Let now $(\mathbf{x}^n)_{n\geq 1} \in \prod_{n\geq 1} S_n^0$ be such that $\alpha_n(\mathbf{x}^m) = \mathbf{x}^n$ for all $m > n \geq 1$. If $\mathbf{x}^n = \mathbf{0}$ for all $n \geq 1$ then clearly $i(\mathbf{0}) = (\mathbf{x}^n)_{n\geq 1}$. Suppose that there exists $n \geq 1$ such that $\mathbf{x}^n \neq \mathbf{0}$. Then there exists $n_0 \in \mathbb{N}^*$ such that $\mathbf{x}^n = \mathbf{0}$ if $n < n_0$ and $\mathbf{x}^n \neq \mathbf{0}$ if $n \geq n_0$. For each $n \geq n_0$ let $k(n) \in \mathbb{N}^*$ be such that $x^n_{k(n)} > 0$ and $x^n_k = 0$ for $k > k(n)$, where $\mathbf{x}^n = (x^n_k)_{k\geq 1} \in S^0_n$. We have

$$(B1.1) \qquad x^m_k = x^n_k \text{ if } m > n \text{ and } k \leq k(n),$$

the sequence $(k(n))_{n\geq n_0} \subseteq \mathbb{N}^*$ is increasing and let $k_\infty := \sup_n k(n) \in \mathbb{N}^* \cup \{\infty\}$. If $k_\infty < \infty$ then let $n_1 \geq 1$ be such that $k_\infty = k(n_1)$. Then $\mathbf{x}^n = \mathbf{x}^{n_1}$ for all $n \geq n_1$ and $i(\mathbf{x}^{n_1}) = (\mathbf{x}^n)_{n\geq 1}$. If $k_\infty = \infty$ then define $\mathbf{x} = (x_k)_{k\geq 1}$ as $x_k := x^n_k$, where for $k \in \mathbb{N}^*$ we take n such that $k \leq k(n)$. By $(B1.1)$ x_k is well defined (it does not depend on n) and $(x_k)_{k\geq 1}$ is a decreasing sequence. In addition, if $n' > n$ is such that $k(n') > k(n)$, then from $\alpha_n(\mathbf{x}^{n'}) = \mathbf{x}^n$ we get $\mu_{\mathbf{x}^{n'}} \neq \mu_{\mathbf{x}^n}$ and therefore $x_{k(n')} = x^{n'}_{k(n')} < d_n$. It follows that $\lim_k x_k = 0$, so, $\mathbf{x} \in S^\downarrow$ and $i(\mathbf{x}) = (\mathbf{x}^n)_{n\geq 1}$. We conclude that i is surjective. $\square$

(B2) Proof of Propostion 4.6. We have to show that if $M \in \mathcal{B}(\widehat{E_n})$ then

$$(B2.1) \qquad \widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}}(\alpha_n^{-1}(M)) = \widehat{P^n_{t,\mathbf{x}_n}}(M) \text{ for all } n \geq 1.$$

Observe that $\alpha_n^{-1}(M) = \widehat{E'_n} \oplus M$. By (H_3), (3.1), and from Lemma 4.2 it follows that $\widehat{E_n}$ and $\widehat{E'_n}$ are both absorbing subsets of $\widehat{E_{n+1}}$. Therefore

$$(B2.2) \qquad \widehat{P^{n+1}_{t,\mathbf{x}_n}}(\widehat{E_{n+1}} \setminus \widehat{E_n}) = 0,$$

and

$$(B2.3) \qquad \widehat{P^{n+1}_{t,\mathbf{x}'_n}}(\widehat{E_{n+1}} \setminus \widehat{E'_n}) = 0 \text{ if } \mathbf{x}'_n \in \widehat{E'_n}.$$

If $\mathbf{x}_n = \mathbf{0}$ then $\mathbf{x}_{n+1} \in \widehat{E'_n}$ and by (4.2) $\widehat{P^{n+1}_{t,\mathbf{0}}} = \delta_{\mathbf{0}} = \widehat{P^n_{t,\mathbf{0}}}$. Therefore, using $(B2.3)$, $\widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}}(\alpha_n^{-1}(M)) = \delta_{\mathbf{0}}(M) = \widehat{P^n_{t,\mathbf{x}_n}}(M)$. So, in the sequel we may assume that $\mathbf{x}_n \neq \mathbf{0}$.

Suppose that $\alpha_{n+1}(\mathbf{x}) \neq \alpha_n(\mathbf{x})$. Consider $\mathbf{x}'_n \in \widehat{E_{n+1}} \setminus \widehat{E_n}$, $\mathbf{x'}_n \neq \mathbf{0}$, such that $\mu_{\mathbf{x}_{n+1}} = \mu_{\mathbf{x}_n} + \mu_{\mathbf{x}'_n}$. Because $\widehat{P^{n+1}_t}$ is a branching kernel we have $\widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}} = \widehat{P^{n+1}_{t,\mathbf{x}'_n}} * \widehat{P^{n+1}_{t,\mathbf{x}_n}}$. Consequently, using $(B2.3)$ and $(B2.2)$,

$$\begin{aligned}
\widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}}(\alpha_n^{-1}(M)) &= \widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}}(\widehat{E'_n} \oplus M) \\
&= \int\int 1_{\widehat{E'_n}\oplus M}(\xi+\nu)\widehat{P^{n+1}_{t,\mathbf{x}'_n}}(\mathrm{d}\xi)\widehat{P^{n+1}_{t,\mathbf{x}_n}}(\mathrm{d}\nu) \\
&= \int_{\widehat{E'_n}} \widehat{P^{n+1}_{t,\mathbf{x}'_n}}(\mathrm{d}\xi) \int_{\widehat{E_n}} \widehat{P^{n+1}_{t,\mathbf{x}_n}}(\mathrm{d}\nu) 1_{\widehat{E'_n}\oplus M}(\xi+\nu) \\
&= \int_{\widehat{E'_n}} 1_{\widehat{E'_n}}(\xi)\widehat{P^{n+1}_{t,\mathbf{x}'_n}}(\mathrm{d}\xi) \int_{\widehat{E_n}} 1_M(\nu)\widehat{P^{n+1}_{t,\mathbf{x}_n}}(\mathrm{d}\nu) \\
&= \widehat{P^{n+1}_{t,\mathbf{x}_n}}(M) = \widehat{P^n_{t,\mathbf{x}_n}}(M),
\end{aligned}$$

where the last equality holds by Proposition 4.3. From the above considerations we conclude that $(B2.1)$ holds in this case.

Assume now that $\alpha_{n+1}(\mathbf{x}) = \alpha_n(\mathbf{x})$. Then $\mathbf{x}_{n+1} = \mathbf{x}_n \in \widehat{E_n}$ and using $(B2.2)$ and again Proposition 4.3 we have

$$\widehat{P^{n+1}_{t,\mathbf{x}_{n+1}}}(\alpha_n^{-1}(M)) = \widehat{P^{n+1}_{t,\mathbf{x}_n}}(\widehat{E'_n} \oplus M) = \widehat{P^{n+1}_{t,\mathbf{x}_n}}(M) = \widehat{P^n_{t,\mathbf{x}_n}}(M),$$

hence $(B2.1)$ is fulfilled. $\square$

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