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# Maximization Coloring Problems on graphs with few $P_4$ s

V. Campos\*    C. Linhares Sales\*    A. K. Maia†    R. Sampaio\*

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## Abstract

Given a graph  $G = (V, E)$ , a *greedy coloring* of  $G$  is a proper coloring such that, for each two colors  $i < j$ , every vertex of  $V(G)$  colored  $j$  has a neighbor with color  $i$ . The greatest  $k$  such that  $G$  has a greedy coloring with  $k$  colors is the *Grundy number* of  $G$ . A *b-coloring* of  $G$  is a proper coloring such that every color class contains a vertex which is adjacent to at least one vertex in every other color class. The greatest integer  $k$  for which there exists a *b-coloring* of  $G$  with  $k$  colors is its *b-chromatic number*. Determining the Grundy number and the *b-chromatic number* of a graph are NP-hard problems in general.

For a fixed  $q$ , the  $(q, q - 4)$ -graphs are the graphs for which no set of at most  $q$  vertices induces more than  $q - 4$  distinct induced  $P_4$ s. In this paper, we obtain polynomial-time algorithms to determine the Grundy number and the *b-chromatic number* of  $(q, q - 4)$ -graphs, for a fixed  $q$ . They generalize previous results obtained for cographs and  $P_4$ -sparse graphs, classes strictly contained in the  $(q, q - 4)$ -graphs.

## 1 Introduction

Let  $G = (V, E)$  be a finite undirected graph, without loops or multiple edges. A *k-coloring* of  $G$  is a surjective mapping  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  such that  $c(u) \neq c(v)$  for any edge  $uv \in E$ . The sets of vertices  $S_1, \dots, S_k$  with colors  $1, 2, \dots, k$ , respectively, that form a partition of  $V(G)$  in stable sets, are called *color classes*. The *chromatic number*  $\chi(G)$  of  $G$  is the smallest integer  $k$  such that  $G$  admits a  $k$ -coloring. It is well known that determining  $\chi(G)$  is a NP-hard problem.

Hence lots of heuristics have been developed to color a graph. One of the most basic and used is the greedy algorithm. Given an order  $v_1, v_2, \dots, v_n$  of the vertices of  $G$ , the greedy algorithm colors the vertices of  $G$  assigning to  $v_i$  the minimum positive integer that was not already assigned to its neighbors in the set  $\{v_1, \dots, v_{i-1}\}$ . Such a coloring is called a *greedy coloring*. The maximum number of colors of a greedy coloring of a graph  $G$ , over all possible orderings of the vertices of  $V(G)$ , is the *Grundy number* of  $G$  and it is denoted by  $\Gamma(G)$ .

Zaker [1] showed that, for any fixed  $k$ , one can decide in polynomial time if a given graph has Grundy number at least  $k$  (that is, deciding if  $\Gamma(G) \geq k$  is fixed parameter tractable on  $k$ ). However determining the Grundy number of a graph is NP-hard [1]. Moreover, in 2010, Havet and Sampaio [2] proved that it is NP-complete to decide if  $\Gamma(G) = \Delta(G) + 1$ . In addition, Asté et al. [3] showed that, for any constant  $c \geq 1$ , it is NP-complete to decide if  $\Gamma(G) \leq c \cdot \chi(G)$ .

Another alternative way of dealing with the coloring problem is to try to improve any coloring  $c$  of the graph by applying some strategy, obtaining from  $c$  a coloring with a smaller number of colors.

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Observe that, if  $c$  has a color class  $S_i$  such that for every vertex  $v \in S_i$ , there is at least one other color class  $S_j$  such that  $v$  does not have neighbors in  $S_j$ , we could eliminate  $S_i$  by recoloring every vertex  $v$  from  $S_i$  with the color  $j$  that does not appear in its neighborhood. A vertex  $v$  from  $S_i$  is said to be *dominant* if  $v$  is adjacent to at least one vertex in  $S_j$  for all  $j \neq i$ . It is easy to see that if every color class  $S_i \in c$  has a dominant vertex, then it is not possible to improve  $c$  by applying the above strategy.

A  $b$ -coloring of  $G$  is a coloring such that every color class contains a dominant vertex. The  $b$ -chromatic number  $\chi_b(G)$  of a graph  $G$  is the maximum number  $k$  such that there exists a  $b$ -coloring of  $G$  with  $k$  colors. Observe that the  $b$ -chromatic number of  $G$  measures the worst performance of the improvement strategy of a coloring described previously. This parameter has been introduced by R. W. Irving and D. F. Manlove [4]. They proved that determining the  $b$ -chromatic number is polynomial-time solvable for trees, but it is NP-hard for general graphs. In [5], Kratochvíl, Tuza and Voigt proved that computing the  $b$ -chromatic number is NP-hard even if  $G$  is a connected bipartite graph.

Let  $G = (V, E)$  be a graph. We say that  $G$  is a  $P_4$  if  $V(G) = \{w, x, y, z\}$  and  $E(G) = \{wx, xy, yz\}$ , that is, an induced path on four vertices. We say that  $w$  and  $z$  are the *endpoints* and  $x$  and  $y$  the *midpoints* of the  $P_4$ .

A *cograph* is a  $P_4$ -free graph and a  $P_4$ -sparse graph is a graph  $G$  such that each subset of  $G$  with five vertices induces at most one  $P_4$ . The  $P_4$ -sparse graphs, introduced in [6], generalize cographs and can be recognized in linear time [7].

Many NP-hard problems were proved to be polynomial-time solvable on cographs and  $P_4$ -sparse graphs. In particular, polynomial-time algorithms were presented to solve the problem of determining the Grundy number and the  $b$ -chromatic number for these graphs [8, 9, 10].

Babel and Olariu [11] defined a graph as  $(q, q-4)$ -graph if no set of at most  $q$  vertices induces more than  $q-4$  distinct  $P_4$ s. For example, cographs and  $P_4$ -sparse graphs are precisely  $(4, 0)$ -graphs and  $(5, 1)$ -graphs, respectively.

Our main result (Theorem 1) says that, for every fixed integer  $q > 0$ , there is a polynomial algorithm to obtain the Grundy number and the  $b$ -chromatic number of a  $(q, q-4)$ -graph.

**Theorem 1** (Main result). *Let  $q > 0$  be a fixed integer. The Grundy number and the  $b$ -chromatic number of a  $(q, q-4)$ -graph  $G$  can be computed in polynomial time.*

This paper is organized as follows. Section 2 contains structural results for  $(q, q-4)$ -graphs. Section 3 presents the results used to calculate the Grundy number of these graphs and in Section 4 we show how to determine their  $b$ -chromatic number.

## 2 Decomposing $(q, q-4)$ -graphs

A graph  $H$  is  $p$ -connected if, for every partition of  $V(G)$  into nonempty disjoint sets  $V_1$  and  $V_2$ , there exists an  $(V_1, V_2)$ -crossing  $P_4$ , that is, an induced  $P_4$  containing vertices from both  $V_1$  and  $V_2$ . A  $p$ -connected graph  $H$  is *separable* if there exists a partition of  $V(G)$  into nonempty disjoint subsets  $V_1$  and  $V_2$  such that each  $(V_1, V_2)$ -crossing  $P_4$  has its *midpoints* in  $V_1$  and its *endpoints* in  $V_2$ . We say that  $(V_1, V_2)$  is the *separation* of  $H$  and  $H_1$  and  $H_2$  are the graphs  $H[V_1]$  and  $H[V_2]$ , respectively. A maximal  $p$ -connected induced subgraph is called a  $p$ -component. Vertices which are not contained in a nontrivial  $p$ -component are called *weak*.

A *decomposition tree* of a graph  $G$  is a tree  $T_G$ , where the leaves are subsets of vertices of  $G$  and each non-leaf node  $v$  in  $T_G$ , with children  $v_1, \dots, v_l$ , represents the subgraph of  $G$ , denoted by  $G(v)$ , induced by the leaves of the subtree of  $T_G$  rooted by  $v$ . Moreover,  $v$  is labelled according to its relation with the graphs  $G(v_1), \dots, G(v_l)$ . Clearly, the intersection of the leaves must be empty and their union must be the set of vertices of  $G$ . The root node of  $T_G$  represents the original graph  $G$ .

In [12], Jamison and Olariu suggest a decomposition tree for general graphs, called *primeval decomposition tree*, which can be computed in linear time [12]. The leaves of its decomposition tree are  $p$ -connected graphs and its weak vertices, and its internal nodes are labelled *union*, *join* or  *$p$ -component*.

If the label of a node  $v$  is *union*,  $G(v)$  is the disjoint union of  $G(v_1), \dots, G(v_l)$ , that is, the set of vertices of  $G(v)$  is the union of the set of vertices of  $G(v_1), \dots, G(v_l)$  and the set of edges of  $G(v)$  is the union of the set of edges of  $G(v_1), \dots, G(v_l)$ .

If the label of a node  $v$  is *join*,  $G(v)$  is the join of  $G(v_1), \dots, G(v_l)$ , that is, the set of vertices of  $G(v)$  is the union of the set of vertices of  $G(v_1), \dots, G(v_l)$  and the set of edges of  $G(v)$  is the union of the set of edges of  $G(v_1), \dots, G(v_l)$ , in addition to all the possible edges between the vertices of  $G(v_1), \dots, G(v_l)$ .

If  $v$  is labelled  *$p$ -component*, it has two children on the tree: a separable  $p$ -component  $H$ , which is a leaf on the primeval decomposition tree and an internal node that represents the graph  $G(v) - H$ . Moreover, every vertex from  $G(v) - H$  is adjacent to every vertex in  $H_1$  and to no vertex in  $H_2$ .

A graph is a *spider* if its vertex set can be partitioned into three sets  $S$ ,  $K$  and  $R$  in such a way that  $S$  is a stable set,  $K$  is a clique, all the vertices of  $R$  are adjacent to all the vertices of  $K$  and to none of the vertices of  $S$  and there exists a bijection  $f : S \rightarrow K$  such that, for all  $s \in S$ , either the neighborhood of  $s$   $N(s) = \{f(s)\}$  (and it is a *thin spider*) or  $N(s) = K - \{f(s)\}$  (and it is a *thick spider*). We say that the spider is without head if  $R = \emptyset$ .

In [11], Babel and Olariu also proved that the primeval decomposition of a  $(q, q - 4)$ -graph has a special property: every node  $v$  on the tree labelled as  $p$ -component is such that its separable  $p$ -component  $H$  is a headless spider or it has less than  $q$  vertices. If  $H$  is the headless spider, it is easy to see that  $H_1$  is the clique and  $H_2$  is the stable set. Since every vertex from  $V(G(v) - H)$  is adjacent to every vertex in  $H_1$  and non-adjacent to every vertex in  $H_2$ , we have that  $G(v)$  is itself a spider with head, where  $G(v) - H$  is the head.

In this paper, we calculate the Grundy number and the  $b$ -chromatic number of  $(q, q - 4)$ -graphs through bottom-up traversal on their primeval decomposition tree. More specifically we solve the case in which a node  $v$  on is labelled as  $p$ -component and the  $p$ -connected component  $H$  of  $G(v)$  is a graph with less than  $q$  vertices. In the remaining non-trivial cases, we use some results in [9] and [10] to calculate  $\Gamma(G(v))$  and  $\chi_b(G(v))$ , respectively.

### 3 Greedy Coloring of $(q, q - 4)$ -graphs

As first shown in [8], if  $G$  is the disjoint union of two graphs  $G_1$  and  $G_2$ , then  $\Gamma(G) = \max\{\Gamma(G_1), \Gamma(G_2)\}$ . On the other hand, if  $G$  is the join of two graphs  $G_1$  and  $G_2$ , then  $\Gamma(G) = \Gamma(G_1) + \Gamma(G_2)$ . In [9] is shown how to determine the Grundy number for spiders.

**Lemma 2** ([9]). *Let  $G$  be a spider with partition  $(S, K, R)$  and  $n$  vertices. If  $G$  is a spider and  $\Gamma(R)$  is given, then  $\Gamma(G)$  can be determined in linear time.*

Let  $G = (V, E)$  be a graph. A subset  $M$  of  $V$  with  $1 \leq |M| \leq |V|$  is called a *module* if each vertex in  $V - M$  is either adjacent to all vertices of  $M$  or to none of them. A module  $M$  is called a *homogeneous set* if  $1 < |M| < |V|$ . The graph obtained from  $G$  by shrinking every maximal homogeneous set to one single vertex is called the *characteristic graph* of  $G$ .

A graph is called *split graph* if its vertex set has a partition  $(K, S)$  such that  $K$  induces a clique and  $S$  induces a stable set.

**Lemma 3** ([13]). *A  $p$ -connected graph  $H$  is separable if and only if its characteristic graph is a split graph.*

Note that, if  $M_1$  and  $M_2$  are two modules of a graph  $G$  such that  $M_1 \cap M_2 = \emptyset$ , then either the edges from  $\{\{v, w\} : v \in M_1, w \in M_2\}$  belong to  $G$  or  $G$  has none of such edges.

Recall Lemma 3. Clearly, if the characteristic graph of a separable  $p$ -component  $H$  with separation  $(V_1, V_2)$  is the split graph  $(K, S)$ , then every maximal homogeneous set  $M_i^1 \subseteq V_1$  shrinks to a vertex  $v_i^1$  in the clique  $K$ , and every maximal homogeneous set  $M_j^2 \subseteq V_2$  shrinks to a vertex  $v_j^2$  in the stable set  $S$ . We say that  $H[M_j^i] = H_j^i$ .

Let  $H$  be a separable  $p$ -component with separation  $(V_1, V_2)$ . Observe that  $H_1 = H[V_1]$  is the join of  $H_1^1, \dots, H_l^1$ , since, between the graphs induced by two modules in the same graph, or there exist all the edges or none between them, and  $H_1^1, \dots, H_l^1$  are the graphs induced by the strong maximal modules of  $H_1$ . So,  $\Gamma(H_1)$  is the Grundy number of the join of the graphs  $H_1^1, \dots, H_l^1$ , which is  $\sum_{i=1}^l \Gamma(H_i^1)$ . Similarly, the Grundy number of some  $H_j^2$  in  $H_2 = H[V_2]$  with its neighborhood in  $H_1$  is the Grundy number of the join of these graphs.

In [9], a relation between the Grundy number of a graph and the Grundy number of its modules is shown.

**Proposition 4.** *Let  $G, H_1, \dots, H_n$  be disjoint graphs such that  $n = |V(G)|$  and let  $V(G) = \{v_1, \dots, v_n\}$ . Let  $G'$  be the graph obtained by replacing  $v_i \in V(G)$  by  $H_i$ , in such a way that there exist all the edges between the vertices of  $H_i$  and  $H_j$ ,  $i \neq j$ , if and only if  $v_i v_j \in E(G)$ . Then for every greedy coloring of  $G'$  at most  $\Gamma(H_i)$  colors contain vertices of the induced subgraph  $G'[V(H_i)] \subseteq G'$ , for all  $i \in \{1, \dots, n\}$ .*

According to Proposition 4, a greedy coloring of a graph  $G$  restricted to its modules is a greedy coloring to them. The following result is a simple generalization of a result in [9]:

**Lemma 5.** *Let  $G$  be a graph and let  $M$  be a module of  $G$  such that  $G[M] = H$  and in a greedy coloring that generates  $\Gamma(G)$  there are  $k$  colors in  $H$ . Let  $G'$  be the graph obtained from  $G$  by replacing  $H$  by a complete graph  $K_k$ . Then,  $\Gamma(G) = \Gamma(G')$ .*

*Proof.* Let  $c$  be the coloring that generates  $\Gamma(G)$ . Let  $A = \{\alpha_1, \dots, \alpha_k\}$  be the set of colors of  $c$  appearing on  $H$ . Let the vertices of the complete graph that replaces  $H$  on  $G'$  be  $w_1, \dots, w_k$  and  $c'$  be the coloring of  $G'$  defined by  $c'(w_i) = \alpha_i$  for  $i \in \{1, \dots, k\}$  and  $c'(v) = c(v)$  for each vertex  $v \in V(G) - M$ . It is a simple matter to check that  $c'$  is a greedy coloring of  $G'$ . Hence  $\Gamma(G') \geq \Gamma(G)$ . Now let  $\{S_1, \dots, S_k\}$  be a greedy  $k$ -coloring of  $H$  and  $c'$  be a greedy  $\Gamma(G')$ -coloring of  $G'$ . It is important to see that there is a greedy  $k$ -coloring of  $H$ , by Proposition 4. Let  $B = \{\beta_1, \dots, \beta_k\}$  be the set of colors appearing on  $K_k$  with  $\beta_1 < \dots < \beta_k$ . Let  $c$  be the coloring of  $G$  which, for every  $1 \leq i \leq k$ , assigns the color  $\beta_i$  to the vertices from  $S_i$ . Clearly,  $c$  is a greedy coloring of  $G$ . So  $\Gamma(G) \geq \Gamma(G')$ .  $\square$

We denote by  $\theta_H$  an order that produces a coloring with  $\Gamma(H)$  colors for  $H$ . In particular, we denote by  $\theta_j^i$  an order that produces a coloring with  $\Gamma(H_j^i)$  colors for  $H_j^i$ . Theorem 6 is the main result of this section.

**Theorem 6.** *Let  $G$  be a  $(q, q-4)$ -graph containing a separable  $p$ -component  $H$  with separation  $(V_1, V_2)$  and at most  $q$  vertices, such that every vertex in  $R = G - H$  is adjacent to all vertices in  $H_1 = H[V_1]$  and to no vertex in  $H_2 = H[V_2]$ . Let  $H_1^1, \dots, H_l^1$  be the graphs induced by the maximal homogeneous sets of  $H_1$  and  $H_1^2, \dots, H_m^2$  the graphs induced by the maximal homogeneous sets of  $H_2$ . Given  $\chi(R)$  and  $\Gamma(R)$ , let  $G'$  be the graph obtained from  $G$  by replacing  $R$  by a complete graph  $K_{\Gamma(R)}$ . Then:*

- (a) *If  $\Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H_i^2)$ , then  $\Gamma(G) = \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ ;*
- (b) *If  $\Gamma(R) < \max_{1 \leq i \leq m} \Gamma(H_i^2)$ , then  $\Gamma(G) = \Gamma(G')$ .*

*Proof.* (a) If we give an order to the greedy algorithm starting by  $\theta_R, \theta_1^1, \dots, \theta_l^1$ , we have a greedy coloring of  $G$  with at least  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  colors, since  $R \cup H_1$  is the join of  $R, H_1^1, \dots, H_l^1$ . So, we have to prove that  $\Gamma(G) \leq \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ . Suppose by contradiction that there is a greedy coloring  $c$  of  $G$  with more than  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  colors and let  $c_{max}$  be the highest color in  $c$ . Consider the following cases:

(i) There is a vertex  $v \in R$  colored  $c_{max}$ :

Let  $c' = c(R \cup H_1)$ . All colors in  $c$  should appear in  $c'$ , since  $v$ , to be colored  $c_{max}$ , has to be adjacent to vertices colored with all colors different from  $c_{max}$ , and a vertex in  $R$  has neighbors only in  $R \cup H_1$ . So,  $c'$  has more than  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  colors. Note that  $c'$  is not a greedy coloring to  $R \cup H_1$ , because a greedy coloring to  $R \cup H_1$  has at most  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  colors, since  $R \cup H_1$  is the join of  $R, H_1^1, \dots, H_l^1$ . Thus, there is a vertex  $u \in R \cup H_1$  colored  $t$  that has no neighbor colored  $f$  in  $R \cup H_1$ , for some  $f < t$ . Such vertex should be in  $H_1$ , since all neighbors of vertices in  $R$  are in  $R \cup H_1$ . Then,  $u \in H_i^1$  has a neighbor  $w \in H_j^2$  colored  $f$ . Note that there exist all edges between  $H_i^1$  and  $H_j^2$ . Some vertex  $z \in R \cup H_1$  is also colored  $f$ . It is easy to see that  $z \notin R$ , otherwise  $u$  would have a neighbor in  $R \cup H_1$  colored  $f$ , since every vertex from  $R$  is adjacent all vertex in  $H_1$ . Lemma 3 shows that there is all possible edges between two modules of  $H_1$ . So,  $z \notin H_s^1$ , for  $s \neq i$ , because in this case also  $u$  would have a neighbor in  $R \cup H_1$  already colored  $f$ . Therefore  $z \in H_i^1$  and consequently  $z$  is adjacent to  $w$ , since there must exist all possible edges between  $H_i^1$  and  $H_j^2$ . But both are colored  $f$ , and this coloring would be improper.

(ii) There is a vertex  $v \in H_2$  colored  $c_{max}$ :

For some  $s \in \{1, \dots, m\}$ , let  $v \in H_s^2$  and  $c' = c(H_s^2 \cup N(H_s^2)), N(H_s^2)$  beeing the graphs induced by the maximal homogeneous sets of  $H_1$  such that the vertices are adjacent to the vertices of  $H_s^2$ . All colors in  $c$  should appear in  $c'$ , since  $v$  has to be adjacent to vertices colored with all colors different from  $c_{max}$  and a vertex in  $H_s^2$  has neighbors only in  $(H_s^2) \cup N(H_s^2)$ . So,  $c'$  has more than  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  colors. Note that  $\Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H_i^2)$  implies  $\Gamma(R) \geq \Gamma(H_s^2)$ . Therefore,  $\Gamma(H_s^2) + \sum_{i \in N(H_s^2)} \Gamma(H_i^1) \leq \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ . Then  $c'$  is not a greedy coloring to  $(H_s^2) \cup N(H_s^2)$ , because a greedy coloring to it has at most  $\Gamma(H_s^2) + \sum_{i \in N(H_s^2)} \Gamma(H_i^1)$  colors, since  $H_s^2 \cup N(H_s^2)$  is the join of  $H_s^2, H_i^1, \forall i \in N(H_s^2)$ . Thus, there is a vertex  $u \in H_s^2 \cup N(H_s^2)$ , colored  $t$ , that has no neighbor colored  $f$  in  $H_s^2 \cup N(H_s^2)$ , for some  $f < t$ . Such vertex should be in  $H_1$ , because all neighbors of vertices in  $H_s^2$  are in  $H_s^2 \cup N(H_s^2)$ . So,  $u \in H_i^1$ , where  $H_i^1 \in N(H_s^2)$ , has a neighbor  $w \in R \cup H_1 - N(H_s^2)$  colored  $f$ . Observe that some vertex  $z \in H_s^2 \cup N(H_s^2)$  is also colored  $f$ . It is easy to see that  $z \notin H_s^2$ . Otherwise,  $u$  would have a neighbor in  $H_s^2 \cup N(H_s^2)$  colored  $f$  since every vertex in  $H_s^2$  is adjacent to every vertex in  $N(H_s^2)$ . For the same reason,  $z \notin H_j^1$ , for  $j \neq i$  and  $j \in N(H_s^2)$ . Therefore  $z \in H_i^1$ , but there is all possible edges between  $H_i^1$  and  $R \cup H_1 - N(H_s^2)$ , what makes  $w$  and  $z$  neighbors. But both  $w$  and  $z$  are colored  $f$ , and this coloring would be improper.

(iii) There is a vertex  $v \in H_1$  colored  $c_{max}$ :

To receive a color bigger than  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ ,  $v$  must have at least  $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$  neighbors of different colors. From its neighborhood in  $R$ ,  $v$  has at most  $\Gamma(R)$  neighbors with different colors, by Proposition 4. From the neighborhood of  $v$  in  $H_i^1$ , for  $i \in \{1, \dots, l\}$ ,  $v$  has at most  $\sum_{i=1}^l \Gamma(H_i^1) - 1$  (its own color), also by Proposition 4. So, it must appear another color  $c_n$  in a vertex  $w \in H_j^2$ , where  $V(H_j^2) \in N(v)$ . Since the vertices in  $R$  have no neighborhood with  $H_2$ ,  $c_n$  must be bigger than all colors in  $R$  and  $w$  must be neighbor of vertices colored with all colors in  $R$ . All these colors must appear in  $H_j^2$ , because the neighbors from  $w$  outside  $H_j^2$  are

vertices in  $H_1$ , all neighbors from all vertices in  $R$  and, therefore, with different colors of  $R$ . We know that in  $H_j^2$  appears at most  $\Gamma(H_j^2)$  colors, what makes  $w$  to have at most  $\Gamma(H_j^2) - 1$  neighbors colored differently in  $H_j^2$ . But we know  $\Gamma(H_j^2) \leq \Gamma(R)$  implies  $\Gamma(H_j^2) - 1 < \Gamma(R)$ . So, all colors of  $R$  cannot appear on the neighborhood of  $w$ , and such vertex cannot receive a different color.

(b) Since  $\Gamma(R) < \max_{1 \leq i \leq m} \Gamma(H_i^2)$ , in a greedy  $\Gamma(G)$ -coloring of  $G$ , by Proposition 4, there are  $p < q$  colors on  $R$ . We do not know the exact value of  $p$ , but we know that  $p$  goes from  $\chi(R)$  to  $\Gamma(R)$ . By Lemma 5, we can replace  $R$  by a complete graph on  $p$  vertices and we can obtain all possible ordinations of  $V(G)$ , which are  $(q+p)!$  in total. So, we can calculate all greedy colorings for  $G$  in  $\sum_{p=\chi(R)}^{\Gamma(R)} (p+q)! \leq q(2q)! = O(1)$  steps, for a fixed  $q$ .  $\square$

## 4 $b$ -coloring of $(q, q-4)$ -graphs

In [10], Bonomo et al. presented a dynamic programming polynomial-time algorithm to compute the  $b$ -chromatic number of a  $P_4$ -sparse graph. For this, they introduced the *dominance vector* of a graph.

**Definition 7.** Let  $G$  be a graph. Given a coloring of  $G$ , a vertex  $v$  is said to be dominant if  $v$  is adjacent to at least one vertex colored within each of the colors not assigned to  $v$ . The *dominance vector*  $dom_G$  of  $G$  is such that  $dom_G[t]$  is the maximum number of distinct color classes admitting dominant vertices in any coloring of  $G$  with  $t$  colors, where  $\chi(G) \leq t \leq |V(G)|$ .

Note that a graph  $G$  admits a  $b$ -coloring with  $t$  colors if and only if  $dom_G[t] = t$ . So, the  $b$ -chromatic number  $\chi_b(G)$  is the maximum number  $t$  such that  $dom_G[t] = t$ . Thus, once calculated the dominance vector of a graph, we have its  $b$ -chromatic number. Bonomo et al. [10] proved that calculating the dominance vector is polynomial-time solvable for cographs and  $P_4$ -sparse graphs.

Lemmas 8 and 9 below from [10] show how to obtain the dominance vector for disjoint unions, joins and spiders. The calculation of  $\chi(G)$  is from [14] and [15].

**Lemma 8** (Dominance vector for union and join operations [10]). *Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs such that  $V_1 \cap V_2 = \emptyset$  and let  $t \geq \chi(G)$ . If  $G = G_1 \cup G_2$ , then  $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$  and*

$$dom_G[t] = \min\{t, dom_{G_1}[t] + dom_{G_2}[t]\}.$$

*If  $G = G_1 \vee G_2$ , let  $a = \max\{\chi(G_1), t - |V(G_2)|\}$  and  $b = \min\{|V(G_1)|, t - \chi(G_2)\}$ . Then,  $\chi(G) = \chi(G_1) + \chi(G_2)$  and*

$$dom_G[t] = \max_{a \leq j \leq b} \{dom_{G_1}[t] + dom_{G_2}[t - j]\}.$$

**Lemma 9** (Dominance vector for spiders [10]). *Let  $G$  be a spider with partition  $(S, K, R)$ , where  $k = |S| = |K| \geq 2$ . If  $R$  is empty, consider  $\chi(G[R]) = 0$  and  $dom_{G[R]}[0] = 0$ . Thus,  $\chi(G) = k + \chi(G[R])$  and*

(a) *If  $G$  is a thin spider, then*

$$dom_G[i] = \begin{cases} k + dom_{G[R]}[i - k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ k, & \text{if } i = k + |R| + 1, \\ 0, & \text{if } i > k + |R| + 1 \end{cases}$$

(b) If  $G$  is a thick spider, then

$$dom_G[i] = \begin{cases} k + dom_{G[R]}[i - k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ \min\{k, 4k - 2i + 2|R|\}, & \text{if } k + |R| + 1 \leq i \leq 2k + |R|, \\ 0, & \text{if } i > 2k + |R| \end{cases}$$

Using these lemmas, Bonomo et al. proved the theorem below.

**Theorem 10** (Bonomo et al. [10]). *The dominance vector and the  $b$ -chromatic number of a cograph or  $P_4$ -sparse graph can be computed in  $O(n^3)$  time.*

Let  $G = (V, E)$  be a graph and  $M$  be a module of  $G$ . Let  $G_M = G[M]$  and let  $N(M)$  be the neighborhood of a vertex in  $M$ . Let  $H, H_1$  and  $H_2$  be the subgraphs of  $G$  induced by  $V \setminus M, N(M)$  and  $V(H) \setminus N(M)$ , respectively. If  $H$  has less than  $q$  vertices,  $G$  is obtained by applying  $p$ -component( $q$ ) operation over  $(G_M, H = (H_1, H_2))$ .

To calculate  $dom_G[t]$ , auxiliary lemma below shows us that there exists a good coloring such that: (a) all colors appears in  $M$  or  $H_1$  or (b) vertices of  $M$  have distinct colors. Given a coloring  $c$  of  $G$  and a subgraph  $G'$  of  $G$ , let  $n(C)$  be the number of colors used in  $C$  and let  $(C, G')$  be the restriction of the coloring  $C$  to  $G'$ .

**Lemma 11.** *If  $\chi(G) \leq t \leq |V(G)|$ , then there is a proper coloring  $C$  of  $G$  with  $t$  colors that maximizes the number of color classes with dominant vertices such that  $n(C) = n(C, H_1) + n(C, G_M)$  or  $n(C, G_M) = |V(G_M)|$ .*

*Proof.* Let  $C$  be a coloring of  $G$  with  $t$  colors that maximizes the number of color classes with dominant vertices and then maximizes  $n(C, G_M)$ . Since  $M$  is a module, each vertex in  $G_M$  is adjacent to all vertices in  $H_1$ . Thus,  $n(C) \geq n(C, H_1) + n(C, G_M)$ . Suppose that  $C$  does not satisfy the lemma. Since  $n(C) > n(C, H_1) + n(C, G_M)$ , then there is a color  $c$  that appears only in vertices of  $H_2$  and thus no vertex of  $G_M$  is dominant in  $C$ . Since  $n(C, G_M) < |V(G_M)|$ , then there are two vertices  $v$  and  $v'$  of  $G_M$  that have the same color in  $C$ . Consider the coloring  $C'$  obtained from  $C$  by coloring  $v$  with color  $c$ . Note that any dominant vertex in  $C$  is also a dominant vertex in  $C'$  and thus  $C'$  also has a maximum number of color classes with dominant vertices among colorings with  $t$  colors. Note that  $n(C', G_M) > n(C, G_M)$ . Suppose again that  $C'$  does not satisfy the lemma. So, we can repeat this argument until we obtain a coloring  $C^*$  such that all vertices of  $G_M$  are colored with distinct colors. Thus,  $n(C^*, G_M) = |V(G_M)|$  as desired.  $\square$

Applying this lemma, we have four possible cases:

- (a) all colors appears in  $M$  or  $H_1$ 
  - (a.1) There is no dominant vertex in  $H_2$
  - (a.2) There is a dominant vertex in  $H_2$
- (b) Vertices of  $M$  have distinct colors
  - (b.1) There are colors in  $M$  that are not in  $H$
  - (b.2) Every color in  $M$  appears in  $H$

Case (b.2) is easy to handle because it implies that  $|M| \leq |V(H)|$ . Since we will force that  $|V(H)| \leq q$ , we can obtain all colorings of  $G$  with  $t$  colors in constant time. To deal with cases (a.1), (a.2) and (b.1), we have to define some parameters.



Let  $\mathcal{C}(t)$  be the set of all colorings of  $H$  with  $t$  colors and let  $\mathcal{C}(t, t')$  be subset of  $\mathcal{C}(t)$  with colorings of  $H$  such that  $H_1$  uses  $t'$  colors. Let  $C \in \mathcal{C}(t, t')$ . For  $H' \subseteq H$ , let  $c(C, H')$  denote the set of colors used in  $H'$ . We say that a vertex  $v$  in  $H_1$  is partially dominant if  $v$  is adjacent to at least one vertex receiving each color in  $c(C, H_1)$ . Let  $d_1(C)$  be the number of color classes of  $C$  with partially dominant vertices in  $H_1$ . Let  $d_2(C)$  be the number of color classes of  $c(C, H_2) \setminus c(C, H_1)$  with a dominant vertex. Let  $d_3(C)$  be the number of color classes in  $c(C, H_1)$  with either a dominant vertex in  $H_2$  or a partially dominant vertex in  $H_1$ . Let  $J \subseteq c(C, H_2) \setminus c(C, H_1)$ . We say that a vertex  $v$  in  $H_1$  is  $\bar{J}$ -dominant if  $v$  is adjacent to at least one vertex receiving each color in  $c(C, H) \setminus J$ . Let  $d_4(C, J)$  be the number of color classes of  $C$  with either a dominant vertex in  $H_2$  or a  $\bar{J}$ -dominant vertex in  $H_1$  and  $d_5(C, j) = \sup\{d_4(C, J) \mid J \subseteq c(C, H_2) \setminus c(C, H_1), |J| = j\}$ .

Let  $\chi(G) \leq t \leq |V|$ , let  $t_1 = \max\{t - |V(G_M)|, 0\}$ , let  $t_2 = \min\{|V(H_1)|, t - \chi(G_M)\}$ , let  $t_3 = \min\{|V(H)|, t\}$ , let  $t_4 = \min\{t - |V(G_M)|, |V(H_1)|\}$  and let

$$\begin{aligned} \tau_1(t) &= \sup_{\substack{t_1 \leq t' \leq t_2 \\ t' \leq t \leq t_3}} \{dom_{G_M}[t - t'] + d_1(C) \mid C \in \mathcal{C}(\hat{t}, t')\} \\ \tau_2(t) &= \sup_{t_1 \leq t' \leq t_2} \{\min\{t - t', d_2(C) + dom_{G_M}[t - t']\} + d_3(C) \mid C \in \mathcal{C}(t, t')\} \\ \tau_3(t) &= \sup_{\substack{t_1 \leq \hat{t} \leq t_3 \\ 0 \leq t' \leq t_4}} \{d_5(C, \hat{t} + |V(G_M)| - t) \mid C \in \mathcal{C}(\hat{t}, t')\} \end{aligned}$$

Excluding case (b.2) by forcing that  $|V(G)| > 2|V(H)|$ , we have the important lemma below.

**Lemma 12.** *If  $\chi(G) \leq t \leq |V(G)|$  and  $|V(G)| > 2|V(H)|$ , then*

$$dom_G[t] = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}.$$

*Proof.* Let  $C$  be a coloring of  $G$  with  $t$  colors that maximizes the number of color classes with dominant vertices. According to Lemma 11, suppose that either  $n(C, H_1) + n(C, G_M) = t$  or  $n(C, H_1) + n(C, G_M) < t$  and  $n(C, G_M) = |V(G_M)|$ . Let  $\hat{t} = n(C, H)$ .

The first case considered is (a) when  $n(C, H_1) + n(C, G_M) = t$ . Note that if  $v$  is a dominant vertex in  $C$ , then  $v$  is dominant in  $(C, G_M)$  if  $v \in V(G_M)$  and  $v$  is partially dominant in  $(C, H)$  if  $v \in V(H_1)$ . Let  $t' = n(C, H_1)$ . Since  $\chi(G_M) \leq n(C, G_M) = t - n(C, H_1) \leq |V(G_M)|$ , then  $t - |V(G_M)| \leq t' \leq t - \chi(G_M)$ . We also get that  $t' \leq |V(H_1)|$  and, thus,  $t_1 \leq t' \leq t_2$ .

Now, consider (a.1) that there is no dominant vertex in  $H_2$ . In this case,  $t' \leq \hat{t} \leq \min\{|V(H)|, t\}$ . We also have that the number of color classes of  $C$  with dominant vertices of colors that appear in  $H_1$  is precisely  $d_1(C, H)$  and the with dominant vertices of colors that appear in  $G_M$  is at most  $dom_{G_M}[t - t']$ . Thus, if  $n(C, H_1) + n(C, G_M) = t$  and there is no dominant vertex of  $C$  in  $H_2$ , then  $dom_G[t] \leq \tau_1(t)$ .

Now, consider (a.2) that there is at least one dominant vertex  $u$  in  $H_2$ . Since  $u$  is adjacent to every other color of  $C$  and every neighbour of  $u$  is in  $H$ , then  $\hat{t} = t$ . Note that the number of color classes of  $C$  with dominant vertices of colors that appear in  $H_1$  is precisely  $d_3(C, H)$ . The number of color classes of  $C$  with dominant vertices of colors that appear in  $G_M$  is at most  $\min\{t - t', d_2(C) + dom_G[t - t']\}$ . Thus, if  $n(C, H_1) + n(C, G_M) = t$  and there is at least one dominant vertex of  $C$  in  $H_2$ , then  $dom_G[t] \leq \tau_2(t)$ .

The second case considered is (b) when  $n(C, H_1) + n(C, G_M) < t$  and  $n(C, G_M) = |V(G_M)|$ . If (b.2)  $c(C, G_M) \subseteq c(C, H)$ , then  $n(C, G_M) = |V(G_M)|$  implies that  $|M| \leq |V(H)|$  and  $|V(G)| \leq |V(H)|$ , a contradiction. Thus, (b.1) there is a color unique to vertices in  $G_M$ . Note also that  $n(C, H_1) + n(C, G_M) < t$  implies that there is a color unique to vertices in  $H_2$ . Thus, all dominant vertices of  $C$  are in  $H_1$ . Note that  $\hat{t} \leq |V(H)|$  and  $\hat{t} \leq t$  and, thus,  $\hat{t} \leq t_3$ . We also have that  $t \leq \hat{t} + |V(G_M)|$  which implies that  $\hat{t} \geq t_1$ . Let  $J = c(C, G_M) \cap c(C, H)$ . Note that  $t = \hat{t} + |V(G_M)| - |J|$  which implies that

$|J| = \hat{t} + |V(G_M)| - t$ . Since  $J$  is a subset of  $c(C, H) \setminus c(C, H_1)$ , then  $|J| = \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$  which implies that  $t' \leq t - |V(G_M)|$ . Since  $H_1$  has at most  $V(H_1)$  colors, then  $t' \leq t_4$ . Now, note that every dominant vertex of  $C$  is a  $J$ -dominant vertex of  $H_1$  in  $(C, H)$ . Thus, the number of color classes with dominant vertices in  $C$  is  $d_4((C, H), J)$ , which is at most  $d_5((C, H), \hat{t} + |V(G_M)| - t)$ . Thus, if  $n(C, H_1) + n(C, G_M) < t$  and  $|V| > 2|V(H)|$ , then  $dom_G[t] \leq \tau_3(t)$ .

We can conclude from the previous paragraphs that if  $|V| > 2|V(H)|$ , then  $dom_G[t] \leq \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}$ . To conclude this proof, it remains to prove that  $dom_G[t] \geq \tau_i(t)$ , for  $i \in \{1, 2, 3\}$ . Let  $C_H$  be a coloring of  $H$  with  $\hat{t}$  colors and  $t' = n(C_H, H_1)$ . We break into cases depending on  $C_H$  being related to each of the parameters  $\tau_i(t)$ . To do so, let  $C_M$  be a coloring of  $G_M$  with  $t - t'$  colors and  $dom_{G_M}[t - t']$  color classes with dominant vertices and  $C'_M$  be a coloring of  $G_M$  with  $|V(G_M)|$  colors.

Suppose that  $t_1 \leq t' \leq t_2$ . If  $\hat{t} \leq t$ , then rename the colors in  $c(C_H, H_2) \setminus c(C_H, H_1)$  to colors in the set  $c(C_M)$  and let  $C$  be the coloring of  $G$  obtained by piecing together this coloring with  $C_M$ . Note that  $C$  has precisely  $t$  colors and there are  $dom_{G_M}[t - t']$  color classes with dominant vertices in colors of  $c(C, G_M)$  and  $d_1(C_H)$  color classes with dominant vertices in colors of  $c(C, H_1)$ . Since  $c(C, G_M) \cap c(C, H_1) = \emptyset$ , then  $C$  has at least  $dom_{G_M}[t - t'] + d_1(C)$  color classes with dominant vertices. This implies that  $dom_G[t] \geq \tau_1(t)$ .

Now, suppose that  $\hat{t} = t$ . Let  $c(C_M) = \{c_1, \dots, c_{t-t'}\}$  and suppose that the color classes with indices in  $\{1, \dots, dom_{G_M}[t - t']\}$  have dominant vertices in  $C_M$ . Now let  $C'_H$  be obtained from  $C_H$  by renaming the colors in  $c(C_H, H_2) \setminus c(C_H, H_1)$  to colors in  $c(C_M)$  in such a way that the color classes with the highest indices have dominant vertices. Note that this is possible as  $c(C_H, H_2) \setminus c(C_H, H_1)$  has size precisely  $t - t'$ . Let  $C$  be obtained by piecing together the colorings  $C_M$  and  $C'_H$ . Note that  $C$  has precisely  $t$  colors,  $d_3(C_H)$  color classes in  $c(C, H_1)$  with dominant vertices and  $\min\{t - t', d_2(C) + dom_{G_M}[t - t']\}$  color classes in  $c(C, G_M)$  with dominant vertices. This implies that  $dom_G[t] \geq \tau_2(t)$ .

Now, suppose that  $0 \leq t' \leq t_4$  and  $t_1 \leq \hat{t} \leq t_3$ . Note that  $0 \leq \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$ , as  $\hat{t} \geq t_1 = t - |V(G_M)|$  and  $t' \leq t_4 \leq t - |V(G_M)|$ . Thus, let  $J$  be a subset of  $c(C_H, H_2) \setminus c(C_H, H_1)$  such that  $d_4(C_H, J) = d_5(C_H, \hat{t} + |V(G_M)| - t)$ . Let  $C'_H$  be obtained by renaming the colors of  $C_H$  in the set  $J$  to colors in  $C'_M$  so that  $|c(C'_H) \cap c(C'_M)| = |J| = \hat{t} + |V(G_M)| - t$ . Let  $C$  be obtained by piecing together the colorings  $C'_H$  and  $C'_M$ . Note that  $n(C) = n(C, H) + n(C, G_M) - |J| = \hat{t} + |V(G_M)| - |J| = t$ . This implies that  $dom_G[t] \geq \tau_3(t)$ .  $\square$

**Lemma 13.** *Let  $q > 0$  be a fixed integer, let  $H$  be a graph with less than  $q$  vertices and let  $H_1$  and  $H_2$  be induced subgraphs of  $H$  such that  $V(H_1)$  and  $V(H_2)$  are a vertex partition of  $H$ . Given a graph  $G_M$  with  $n$  vertices, let  $G$  be the graph obtained by applying  $p$ -component operation over  $(G_M, H = (H_1, H_2))$  (just join all edges between  $G_M$  and  $H_1$ ). Then, given the chromatic number  $\chi(G_M)$  and the dominance vector  $dom_M$  of  $G_M$ , we can calculate the chromatic number  $\chi(G)$  in time  $\Theta(n)$  and the dominance vector  $dom_G$  of  $G$  in time  $\Theta(n^2)$ .*

*Proof.* Since  $|V(H)| \leq q$ , where  $q$  is an integer fixed, we have that parameters  $\tau_1(t)$ ,  $\tau_2(t)$  and  $\tau_3(t)$  can be obtained in linear time (once fixed  $t'$  and  $\hat{t}$ , the value in sup can be obtained in constant time that depends only on  $q$ ). If  $|V(G)| \leq 2|V(H)| \leq 2q$ , then we can calculate  $dom_G[t]$  in constant time. If  $|V(G)| > 2|V(H)|$ , then, applying Lemma 12, we have  $dom_G[t]$  in linear time. So we can obtain the dominance vector  $dom_G$  of  $G$  in time  $\Theta(n^2)$  for all possible values of  $t$ .  $\square$

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