Maximization Coloring Problems on graphs with few P4s

Victor Campos, Claudia Linhares Sales, Ana Karolinna Maia, Rudini Sampaio

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V. Campos* C. Linhares Sales * A. K. Maia† R. Sampaio *

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Abstract

Given a graph $G = (V,E)$, a *greedy coloring* of G is a proper coloring such that, for each two colors $i < j$, every vertex of $V(G)$ colored j has a neighbor with color i. The greatest k such that G has a greedy coloring with k colors is the *Grundy number* of G. A *b-coloring* of G is a proper coloring such that every color class contains a vertex which is adjacent to at least one vertex in every other color class. The greatest integer k for which there exists a b-coloring of G with k colors is its *b-chromatic number*. Determining the Grundy number and the b-chromatic number of a graph are NP-hard problems in general.

For a fixed q, the $(q,q-4)$-*graphs* are the graphs for which no set of at most q vertices induces more than $q-4$ distinct induced P_4s. In this paper, we obtain polynomial-time algorithms to determine the Grundy number and the b-chromatic number of $(q,q-4)$-graphs, for a fixed q. They generalize previous results obtained for cographs and P_4-sparse graphs, classes strictly contained in the $(q,q-4)$-graphs.

1 Introduction

Let $G = (V,E)$ be a finite undirected graph, without loops or multiple edges. A *k-coloring* of G is a surjective mapping $c : V(G) \rightarrow \{1,2,\dots,k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The sets of vertices $S_1,\dots,S_k$ with colors $1,2,\dots,k$, respectively, that form a partition of $V(G)$ in stable sets, are called *color classes*. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G admits a k-coloring. It is well known that determining $\chi(G)$ is a NP-hard problem.

Hence lots of heuristics have been developed to color a graph. One of the most basic and used is the greedy algorithm. Given an order $v_1, v_2, \dots, v_n$ of the vertices of G, the greedy algorithm colors the vertices of G assigning to v_i the minimum positive integer that was not already assigned to its neighbors in the set $\{v_1,\dots,v_{i-1}\}$. Such a coloring is called a *greedy coloring*. The maximum number of colors of a greedy coloring of a graph G, over all possible orderings of the vertices of $V(G)$, is the *Grundy number* of G and it is denoted by $\Gamma(G)$.

Zaker [1] showed that, for any fixed k, one can decide in polynomial time if a given graph has Grundy number at least k (that is, deciding if $\Gamma(G) \geq k$ is fixed parameter tractable on k). However determining the Grundy number of a graph is NP-hard [1]. Moreover, in 2010, Havet and Sampaio [2] proved that it is NP-complete to decide if $\Gamma(G) = \Delta(G) + 1$. In addition, Asté et al. [3] showed that, for any constant $c \geq 1$, it is NP-complete to decide if $\Gamma(G) \leq c \cdot \chi(G)$.

Another alternative way of dealing with the coloring problem is to try to improve any coloring c of the graph by applying some strategy, obtaining from c a coloring with a smaller number of colors.

*ParGO Research Group, Federal University of Ceará, Campus do Pici, Bloco 910, 60455-760 - Fortaleza - Brazil. Partially supported by CNPq/Brazil and FUNCAP/Brazil. (email: [campos, linhares, rudini]@lia.ufc.br)

†COATI Project, I3S (CNRS/UNSA) & INRIA, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis Cedex, France. Partially supported by ANR Blanc STINT and CAPES/Brazil. (email: karol.maia@inria.fr)

Observe that, if c has a color class S_i such that for every vertex $v \in S_i$, there is at least one other color class S_j such that v does not have neighbors in S_j, we could eliminate S_i by recoloring every vertex v from S_i with the color j that does not appear in its neighborhood. A vertex v from S_i is said to be *dominant* if v is adjacent to at least one vertex in S_j for all $j \neq i$. It is easy to see that if every color class $S_i \in c$ has a dominant vertex, then it is not possible to improve c by applying the above strategy.

A *b-coloring* of G is a coloring such that every color class contains a dominant vertex. The *b-chromatic number* $\chi_b(G)$ of a graph G is the maximum number k such that there exists a b-coloring of G with k colors. Observe that the b-chromatic number of G measures the worst performance of the improvement strategy of a coloring described previously. This parameter has been introduced by R. W. Irving and D. F. Manlove [4]. They proved that determining the b-chromatic number is polynomial-time solvable for trees, but it is NP-hard for general graphs. In [5], Kratochvíl, Tuza and Voigt proved that computing the b-chromatic number is NP-hard even if G is a connected bipartite graph.

Let $G = (V,E)$ be a graph. We say that G is a P_4 if $V(G) = \{w,x,y,z\}$ and $E(G) = \{wx,xy,yz\}$, that is, an induced path on four vertices. We say that w and z are the *endpoints* and x and y the *midpoints* of the P_4.

A *cograph* is a P_4-free graph and a P_4*-sparse* graph is a graph G such that each subset of G with five vertices induces at most one P_4. The P_4-sparse graphs, introduced in [6], generalize cographs and can be recognized in linear time [7].

Many NP-hard problems were proved to be polynomial-time solvable on cographs and P_4-sparse graphs. In particular, polynomial-time algorithms were presented to solve the problem of determing the Grundy number and the b-chromatic number for these graphs [8, 9, 10].

Babel and Olariu [11] defined a graph as $(q,q-4)$*-graph* if no set of at most q vertices induces more than $q-4$ distinct P_4s. For example, cographs and P_4-sparse graphs are precisely $(4,0)$-graphs and $(5,1)$-graphs, respectively.

Our main result (Theorem 1) says that, for every fixed integer $q > 0$, there is a polynomial algorithm to obtain the Grundy number and the b-chromatic number of a $(q,q-4)$-graph.

Theorem 1 (Main result)**.** *Let $q > 0$ be a fixed integer. The Grundy number and the b-chromatic number of a $(q,q-4)$-graph G can be computed in polynomial time.*

This paper is organized as follows. Section 2 contains structural results for $(q,q-4)$-graphs. Section 3 presents the results used to calculate the Grundy number of these graphs and in Section 4 we show how to determine their b-chromatic number.

2 Decomposing $(q,q-4)$-graphs

A graph H is *p-connected* if, for every partition of $V(G)$ into nonempty disjoint sets V_1 and V_2, there exists an (V_1,V_2)*-crossing* P_4, that is, an induced P_4 containing vertices from both V_1 and V_2. A *p*-connected graph H is *separable* if there exists a partition of $V(G)$ into nonempty disjoint subsets V_1 and V_2 such that each (V_1,V_2)-crossing P_4 has its *midpoints* in V_1 and its *endpoints* in V_2. We say that (V_1,V_2) is the *separation* of H and H_1 and H_2 are the graphs $H[V_1]$ and $H[V_2]$, respectively. A maximal p-connected induced subgraph is called a *p-component*. Vertices which are not contained in a nontrivial p-component are called *weak*.

A *decomposition tree* of a graph G is a tree T_G, where the leaves are subsets of vertices of G and each non-leaf node v in T_G, with children $v_1,\dots,v_l$, represents the subgraph of G, denoted by $G(v)$, induced by the leaves of the subtree of T_G rooted by v. Moreover, v is labelled according to its relation with the graphs $G(v_1),\dots,G(v_l)$. Clearly, the intersection of the leaves must be empty and their union must be the set of vertices of G. The root node of T_G represents the original graph G.

In [12], Jamison and Olariu suggest a decomposition tree for general graphs, called *primeval decomposition tree*, which can be computed in linear time [12]. The leaves of its decomposition tree are p-connected graphs and its weak vertices, and its internal nodes are labelled *union*, *join* or *p-component*.

If the label of a node v is *union*, $G(v)$ is the disjoint union of $G(v_1),\dots,\ G(v_l)$, that is, the set of vertices of $G(v)$ is the union of the set of vertices of $G(v_1),\dots,G(v_l)$ and the set of edges of $G(v)$ is the union of the set of edges of $G(v_1),\dots,G(v_l)$.

If the label of a node v is *join*, $G(v)$ is the join of $G(v_1),\dots,\ G(v_l)$, that is, the set of vertices of $G(v)$ is the union of the set of vertices of $G(v_1),\dots,G(v_l)$ and the set of edges of $G(v)$ is the union of the set of edges of $G(v_1),\dots,G(v_l)$, in addition to all the possible edges between the vertices of $G(v_1),\dots,G(v_l)$.

If v is labelled *p-component*, it has two children on the tree: a separable p-component H, which is a leaf on the primeval decomposition tree and an internal node that represents the graph $G(v)-H$. Moreover, every vertex from $G(v)-H$ is adjacent to every vertex in H_1 and to no vertex in H_2.

A graph is a *spider* if its vertex set can be partitioned into three sets S, K and R in such a way that S is a stable set, K is a clique, all the vertices of R are adjacent to all the vertices of K and to none of the vertices of S and there exists a bijection $f:S\to K$ such that, for all $s\in S$, either the neighborhood of s $N(s) = \{f(s)\}$ (and it is a *thin* spider) or $N(s) = K-\{f(s)\}$ (and it is a *thick* spider). We say that the spider is without head if $R=\emptyset$.

In [11], Babel and Olariu also proved that the primeval decomposition of a $(q,q-4)$-graph has a special property: every node v on the tree labelled as p-component is such that its separable p-component H is a headless spider or it has less than q vertices. If H is the headless spider, it is easy to see that H_1 is the clique and H_2 is the stable set. Since every vertex from $V(G(v)-H)$ is adjacent to every vertex in H_1 and non-adjacent to every vertex in H_2, we have that $G(v)$ is itself a spider with head, where $G(v)-H$ is the head.

In this paper, we calculate the Grundy number and the b-chromatic number of $(q,q-4)$-graphs through bottom-up traversal on the their primeval decomposition tree. More specifically we solve the case in which a node v on is labelled as p-component and the p-connected component H of $G(v)$ is a graph with less than q vertices. In the remaining non-trivial cases, we use some results in [9] and [10] to calculate $\Gamma(G(v))$ and $\chi_b(G(v))$, respectively.

3 Greedy Coloring of $(q,q-4)$-graphs

As first shown in [8], if G is the disjoint union of two graphs G_1 and G_2, then $\Gamma(G)=\max\{\Gamma(G_1),\Gamma(G_2)\}$. On the other hand, if G is the join of two graphs G_1 and G_2, then $\Gamma(G)=\Gamma(G_1)+\Gamma(G_2)$. In [9] is shown how to determine the Grundy number for spiders.

Lemma 2 ([9])**.** *Let G be a spider with partition (S,K,R) and n vertices. If G is a spider and $\Gamma(R)$ is given, then $\Gamma(G)$ can be determined in linear time.*

Let $G=(V,E)$ be a graph. A subset M of V with $1\le |M|\le |V|$ is called a *module* if each vertex in $V-M$ is either adjacent to all vertices of M or to none of them. A module M is called a *homogeneous set* if $1<|M|<|V|$. The graph obtained from G by shrinking every maximal homogeneous set to one single vertex is called the *characteristic graph* of G.

A graph is called *split graph* if its vertex set has a partition (K,S) such that K induces a clique and S induces an stable set.

Lemma 3 ([13])**.** *A p-connected graph H is separable if and only if its characteristic graph is a split graph.*

Note that, if M_1 and M_2 are two modules of a graph G such that $M_1 \cap M_2 = \emptyset$, then either the edges from $\{\{v,w\} : v \in M_1, w \in M_2\}$ belong to G or G has none of such edges.

Recall Lemma 3. Clearly, if the characteristic graph of a separable p-component H with separation (V_1,V_2) is the split graph (K,S), then every maximal homogeneous set $M_i^1 \subseteq V_1$ shrinks to a vertex v_i^1 in the clique K, and every maximal homogeneous set $M_j^2 \subseteq V_2$ shrinks to a vertex v_j^2 in the stable set S. We say that $H[M_j^i] = H_j^i$.

Let H be a separable p-component with separation (V_1,V_2). Observe that $H_1 = H[V_1]$ is the join of $H_1^1,\ldots,H_l^1$, since, between the graphs induced by two modules in the same graph, or there exist all the edges or none between them, and $H_1^1,\ldots,H_l^1$ are the graphs induced by the strong maximal modules of H_1. So, $\Gamma(H_1)$ is the Grundy number of the join of the graphs $H_1^1,\ldots,H_l^1$, which is $\sum_{i=1}^{l}\Gamma(H_i^1)$. Similarly, the Grundy number of some H_i^2 in $H_2 = H[V_2]$ with its neighborhood in H_1 is the Grundy number of the join of these graphs.

In [9], a relation between the Grundy number of a graph and the Grundy number of its modules is shown.

Proposition 4. *Let G, $H_1,\ldots,H_n$ be disjoint graphs such that $n = |V(G)|$ and let $V(G) = \{v_1,\ldots,v_n\}$. Let G' be the graph obtained by replacing $v_i \in V(G)$ by H_i, in such a way that there exist all the edges between the vertices of H_i and H_j, $i \neq j$, if and only if $v_iv_j \in E(G)$. Then for every greedy coloring of G' at most $\Gamma(H_i)$ colors contain vertices of the induced subgraph $G'[V(H_i)] \subseteq G'$, for all $i \in \{1,\ldots,n\}$.*

According to Proposition 4, a greedy coloring of a graph G restricted to its modules is a greedy coloring to them. The following result is a simple generalization of a result in [9]:

Lemma 5. *Let G be a graph and let M be a module of G such that $G[M] = H$ and in a greedy coloring that generates $\Gamma(G)$ there are k colors in H. Let G' be the graph obtained from G by replacing H by a complete graph K_k. Then, $\Gamma(G) = \Gamma(G')$.*

Proof. Let c be the coloring that generates $\Gamma(G)$. Let $A = \{\alpha_1,\ldots,\alpha_k\}$ be the set of colors of c appearing on H. Let the vertices of the complete graph that replaces H on G' be $w_1,\ldots,w_k$ and c' be the coloring of G' defined by $c'(w_i) = \alpha_i$ for $i \in \{1,\ldots,k\}$ and $c'(v) = c(v)$ for each vertex $v \in V(G) - M$. It is a simple matter to check that c' is a greedy coloring of G'. Hence $\Gamma(G') \geq \Gamma(G)$. Now let $\{S_1,\ldots,S_k\}$ be a greedy k-coloring of H and c' be a greedy $\Gamma(G')$-coloring of G'. It is important to see that there is a greedy k-coloring of H, by Proposition 4. Let $B = \{\beta_1,\ldots,\beta_k\}$ be the set of colors appearing on K_k with $\beta_1 < \ldots < \beta_k$. Let c be the coloring of G which, for every $1 \leq i \leq k$, assigns the color β_i to the vertices from S_i. Clearly, c is a greedy coloring of G. So $\Gamma(G) \geq \Gamma(G')$. □

We denote by θ_H an order that produces a coloring with $\Gamma(H)$ colors for H. In particular, we denote by θ_j^i an order that produces a coloring with $\Gamma(H_j^i)$ colors for H_j^i. Theorem 6 is the main result of this section.

Theorem 6. *Let G be a $(q,q-4)$-graph containing a separable p-component H with separation (V_1,V_2) and at most q vertices, such that every vertex in $R = G - H$ is adjacent to all vertices in $H_1 = H[V_1]$ and to no vertex in $H_2 = H[V_2]$. Let H_1^1, …, H_l^1 be the graphs induced by the maximal homogeneous sets of H_1 and H_1^2, …, H_m^2 the graphs induced by the maximal homogeneous sets of H_2. Given $\chi(R)$ and $\Gamma(R)$, let G' be the graph obtained from G by replacing R by a complete graph $K_{\Gamma(R)}$. Then:*

(a) If $\Gamma(R) \geq \max_{1\leq i\leq m}\Gamma(H_i^2)$, then $\Gamma(G) = \Gamma(R) + \sum_{i=1}^{l}\Gamma(H_i^1)$;

(b) If $\Gamma(R) < \max_{1\leq i\leq m}\Gamma(H_i^2)$, then $\Gamma(G) = \Gamma(G')$.

Proof. (a) If we give an order to the greedy algorithm starting by θ_R, θ_1^1, ..., θ_l^1, we have a greedy coloring of G with at least $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ colors, since $R\cup H_1$ is the join of R, H_1^1, ..., H_l^1. So, we have to prove that $\Gamma(G)\leq\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$. Suppose by contradiction that there is a greedy coloring c of G with more than $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ colors and let c_{max} be the highest color in c. Consider the following cases:

(i) There is a vertex $v\in R$ colored c_{max}:

Let $c'=c(R\cup H_1)$. All colors in c should appear in c', since v, to be colored c_{max}, has to be adjacent to vertices colored with all colors different from c_{max}, and a vertex in R has neighbors only in $R\cup H_1$. So, c' has more than $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ colors. Note that c' is not a greedy coloring to $R\cup H_1$, because a greedy coloring to $R\cup H_1$ has at most $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ colors, since $R\cup H_1$ is the join of R, H_1^1, ..., H_l^1. Thus, there is a vertex $u\in R\cup H_1$ colored t that has no neighbor colored f in $R\cup H_1$, for some $f<t$. Such vertex should be in H_1, since all neighbors of vertices in R are in $R\cup H_1$. Then, $u\in H_i^1$ has a neighbor $w\in H_j^2$ colored f. Note that there exist all edges between H_i^1 and H_j^2. Some vertex $z\in R\cup H_1$ is also colored f. It is easy to see that $z\notin R$, otherwise u would have a neighbor in $R\cup H_1$ colored f, since every vertex from R is adjacent all vertex in H_1. Lemma 3 shows that there is all possible edges between two modules of H_1. So, $z\notin H_s^1$, for $s\neq i$, because in this case also u would have a neighbor in $R\cup H_1$ already colored f. Therefore $z\in H_i^1$ and consequently z is adjacent to w, since there must exist all possible edges between H_i^1 and H_j^2. But both are colored f, and this coloring would be improper.

(ii) There is a vertex $v\in H_2$ colored c_{max}:

For some $s\in\{1,\ldots,m\}$, let $v\in H_s^2$ and $c'=c(H_s^2\cup N(H_s^2))$, $N(H_s^2)$ beeing the graphs induced by the maximal homogeneous sets of H_1 such that the vertices are adjacents to the vertices of H_s^2. All colors in c should appear in c', since v has to be adjacent to vertices colored with all colors different from c_{max} and a vertex in H_s^2 has neighbors only in $(H_s^2)\cup N(H_s^2)$. So, c' has more than $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ colors. Note that $\Gamma(R)\geq\max_{1\leq i\leq m}\Gamma(H_i^2)$ implies $\Gamma(R)\geq\Gamma(H_s^2)$. Therefore, $\Gamma(H_s^2)+\sum_{i\in N(H_s^2)}\Gamma(H_i^1)\leq\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$. Then c' is not a greedy coloring to $(H_s^2)\cup N(H_s^2)$, because a greedy coloring to it has at most $\Gamma(H_s^2)+\sum_{i\in N(H_s^2)}\Gamma(H_i^1)$ colors, since $H_s^2\cup N(H_s^2)$ is the join of H_s^2, H_1^i, $\forall i\in N(H_s^2)$. Thus, there is a vertex $u\in H_s^2\cup N(H_s^2)$, colored t, that has no neighbor colored f in $H_s^2\cup N(H_s^2)$, for some $f<t$. Such vertex should be in H_1, because all neighbors of vertices in H_s^2 are in $H_s^2\cup N(H_s^2)$. So, $u\in H_i^1$, where $H_i^1\in N(H_s^2)$, has a neighbor $w\in R\cup H_1-N(H_s^2)$ colored f. Observe that some vertex $z\in H_s^2\cup N(H_s^2)$ is also colored f. It is easy to see that $z\notin H_s^2$. Otherwise, u would have a neighbor in $H_s^2\cup N(H_s^2)$ colored f since every vertex in H_s^2 is adjacent to every vertex in $N(H_s^2)$. For the same reason, $z\notin H_j^1$, for $j\neq i$ and $j\in N(H_s^2)$. Therefore $z\in H_i^1$, but there is all possible edges between H_i^1 and $R\cup H_1-N(H_s^2)$, what makes w and z neighbors. But both w and z are colored f, and this coloring would be improper.

(iii) There is a vertex $v\in H_1$ colored c_{max}:

To receive a color bigger than $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$, v must have at least $\Gamma(R)+\sum_{i=1}^{l}\Gamma(H_i^1)$ neighbors of different colors. From its neighborhood in R, v has at most $\Gamma(R)$ neighbors with different colors, by Proposition 4. From the neighborhood of v in H_i^1, for $i\in\{1,\ldots,l\}$, v has at most $\sum_{i=1}^{l}\Gamma(H_i^1)-1$(its own color), also by Proposition 4. So, it must appear another color c_n in a vertex $w\in H_j^2$, where $V(H_j^2)\in N(v)$. Since the vertices in R have no neighborhood with H_2, c_n must be bigger than all colors in R and w must be neighbor of vertices colored with all colors in R. All these colors must appear in H_j^2, because the neighbors from w outside H_j^2 are vertices in H_1, all neighbors from all vertices in R and, therefore, with different colors of R. We know that in H_j^2 appears at most $\Gamma(H_j^2)$ colors, what makes w to have at most $\Gamma(H_j^2)-1$ neighbors colored differently in H_j^2. But we know $\Gamma(H_j^2) \leq \Gamma(R)$ implies $\Gamma(H_j^2)-1 < \Gamma(R)$. So, all colors of R cannot appear on the neighborhood of w, and such vertex cannot receive a different color.

(b) Since $\Gamma(R) < \max_{1\leq i\leq m}\Gamma(H_i^2)$, in a greedy $\Gamma(G)$-coloring of G, by Proposition 4, there are $p<q$ colors on R. We do not know the exact value of p, but we know that p goes from $\chi(R)$ to $\Gamma(R)$. By Lemma 5, we can replace R by a complete graph on p vertices and we can obtain all possible ordinations of $V(G)$, which are $(q+p)!$ in total. So, we can calculate all greedy colorings for G in $\sum_{p=\chi(R)}^{\Gamma(R)}(p+q)! \leq q(2q)! = O(1)$ steps, for a fixed q. □

4 b-coloring of $(q,q-4)$-graphs

In [10], Bonomo et al. presented a dynamic programming polynomial-time algorithm to compute the b-chromatic number of a P_4-sparse graph. For this, they introduced the *dominance vector* of a graph.

Definition 7. Let G be a graph. Given a coloring of G, a vertex v is said to be dominant if v is adjacent to at least one vertex colored within each of the colors not assigned to v. The *dominance vector dom_G* of G is such that $dom_G[t]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of G with t colors, where $\chi(G) \leq t \leq |V(G)|$.

Note that a graph G admits a b-coloring with t colors if and only if $dom_G[t]=t$. So, the b-chromatic number $\chi_b(G)$ is the maximum number t such that $dom_G[t]=t$. Thus, once calculated the dominance vector of a graph, we have its b-chromatic number. Bonomo et al. [10] proved that calculating the dominance vector is polynomial-time solvable for cographs and P_4-sparse graphs.

Lemmas 8 and 9 below from [10] show how to obtain the dominance vector for disjoint unions, joins and spiders. The calculation of $\chi(G)$ is from [14] and [15].

Lemma 8 (Dominance vector for union and join operations [10])**.** *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be graphs such that $V_1\cap V_2=\emptyset$ and let $t\geq\chi(G)$. If $G=G_1\cup G_2$, then $\chi(G)=\max\{\chi(G_1),\chi(G_2)\}$ and*

$$dom_G[t] \;=\; min\{t, dom_{G_1}[t]+dom_{G_2}[t]\}.$$

If $G=G_1\vee G_2$, let $a=max\{\chi(G_1), t-|V(G_2)|\}$ and $b=\min\{|V(G_1)|, t-\chi(G_2)\}$. Then, $\chi(G)=\chi(G_1)+\chi(G_2)$ and

$$dom_G[t] \;=\; \max_{a\leq j\leq b}\{dom_{G_1}[t]+dom_{G_2}[t-j]\}.$$

Lemma 9 (Dominance vector for spiders [10])**.** *Let G be a spider with partition (S,K,R), where $k=|S|=|K|\geq 2$. If R is empty, consider $\chi(G[R])=0$ and $dom_{G[R]}[0]=0$. Thus, $\chi(G)=k+\chi(G[R])$ and*

(a) If G is a thin spider, then

$$dom_G[i] \;=\; \begin{cases} k+dom_{G[R]}[i-k], & \text{if } k+\chi(G[R])\leq i\leq k+|R|,\\ k, & \text{if } i=k+|R|+1,\\ 0, & \text{if } i>k+|R|+1 \end{cases}$$

(b) If G is a thick spider, then

$$dom_G[i] \;=\; \begin{cases} k + dom_{G[R]}[i-k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ \min\{k, 4k - 2i + 2|R|\}, & \text{if } k + |R| + 1 \leq i \leq 2k + |R|, \\ 0, & \text{if } i > 2k + |R| \end{cases}$$

Using these lemmas, Bonomo et al. proved the theorem below.

Theorem 10 (Bonomo et al. [10])**.** *The dominance vector and the b-chromatic number of a cograph or P_4-sparse graph can be computed in $O(n^3)$ time.*

Let $G = (V, E)$ be a graph and M be a module of G. Let $G_M = G[M]$ and let $N(M)$ be the neighborhood of a vertex in M. Let H, H_1 and H_2 be the subgraphs of G induced by $V \setminus M$, $N(M)$ and $V(H) \setminus N(M)$, respectively. If H has less than q vertices, G is obtained by applying p-component(q) operation over $(G_M, H = (H_1, H_2))$.

To calculate $dom_G[t]$, auxiliary lemma below shows us that there exists a good coloring such that: (a) all colors appears in M or H_1 or (b) vertices of M have distinct colors. Given a coloring c of G and a subgraph G' of G, let $n(C)$ be the number of colors used in C and let (C, G') be the restriction of the coloring C to G'.

Lemma 11. *If $\chi(G) \leq t \leq |V(G)|$, then there is a proper coloring C of G with t colors that maximizes the number of color classes with dominant vertices such that $n(C) = n(C, H_1) + n(C, G_M)$ or $n(C, G_M) = |V(G_M)|$.*

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices and then maximizes $n(C, G_M)$. Since M is a module, each vertex in G_M is adjacent to all vertices in H_1. Thus, $n(C) \geq n(C, H_1) + n(C, G_M)$. Suppose that C does not satisfy the lemma. Since $n(C) > n(C, H_1) + n(C, G_M)$, then there is a color c that appears only in vertices of H_2 and thus no vertex of G_M is dominant in C. Since $n(C, G_M) < |V(G_M)|$, then there are two vertices v and v' of G_M that have the same color in C. Consider the coloring C' obtained from C by coloring v with color c. Note that any dominant vertex in C is also a dominant vertex in C' and thus C' also has a maximum number of color classes with dominant vertices among colorings with t colors. Note that $n(C', G_M) > n(C, G_M)$. Suppose again that C' does not satisfy the lemma. So, we can repeat this argument until we obtain a coloring C^* such that all vertices of G_M are colored with distinct colors. Thus, $n(C^*, G_M) = |V(G_M)|$ as desired. □

Applying this lemma, we have four possible cases:

- (a) all colors appears in M or H_1
  - (a.1) There is no dominant vertex in H_2
  - (a.2) There is a dominant vertex in H_2
- (b) Vertices of M have distinct colors
  - (b.1) There are colors in M that are not in H
  - (b.2) Every color in M appears in H

Case (b.2) is easy to handle because it implies that $|M| \leq |V(H)|$. Since we will force that $|V(H)| \leq q$, we can obtain all colorings of G with t colors in constant time. To deal with cases (a.1), (a.2) and (b.1), we have to define some parameters.

Let $\mathcal{C}(t)$ be the set of all colorings of H with t colors and let $\mathcal{C}(t,t')$ be subset of $\mathcal{C}(t)$ with colorings of H such that H_1 uses t' colors. Let $C \in \mathcal{C}(t,t')$. For $H' \subseteq H$, let $c(C,H')$ denote the set of colors used in H'. We say that a vertex v in H_1 is partially dominant if v is adjacent to at least one vertex receiving each color in $c(C,H_1)$. Let $d_1(C)$ be the number of colors classes of C with partially dominant vertices in H_1. Let $d_2(C)$ be the number of color classes of $c(C,H_2) \setminus c(C,H_1)$ with a dominant vertex. Let $d_3(C)$ be the number of color classes in $c(C,H_1)$ with either a dominant vertex in H_2 or a partially dominant vertex in H_1. Let $J \subseteq c(C,H_2) \setminus c(C,H_1)$. We say that a vertex v in H_1 is $\bar{J}$-dominant if v is adjacent to at least one vertex receiving each color in $c(C,H) \setminus J$. Let $d_4(C,J)$ be the number of color classes of C with either a dominant vertex in H_2 or a $\bar{J}$-dominant vertex in H_1 and $d_5(C,j) = \sup\{d_4(C,J) \mid J \subseteq c(C,H_2) \setminus c(C,H_1), |J| = j\}$.

Let $\chi(G) \leq t \leq |V|$, let $t_1 = \max\{t - |V(G_M)|, 0\}$, let $t_2 = \min\{|V(H_1)|, t - \chi(G_M)\}$, let $t_3 = \min\{|V(H)|, t\}$, let $t_4 = \min\{t - |V(G_M)|, |V(H_1)|\}$ and let

$$
\begin{aligned}
\tau_1(t) &= \sup_{\substack{t_1 \leq t' \leq t_2 \\ t' \leq \hat{t} \leq t_3}} \{dom_{G_M}[t-t'] + d_1(C) \mid C \in \mathcal{C}(\hat{t},t')\} \\
\tau_2(t) &= \sup_{t_1 \leq t' \leq t_2} \{\min\{t-t', d_2(C) + dom_{G_M}[t-t']\} + d_3(C) \mid C \in \mathcal{C}(t,t')\} \\
\tau_3(t) &= \sup_{\substack{t_1 \leq \hat{t} \leq t_3 \\ 0 \leq t' \leq t_4}} \{d_5(C, \hat{t} + |V(G_M)| - t) \mid C \in \mathcal{C}(\hat{t},t')\}
\end{aligned}
$$

Excluding case (b.2) by forcing that $|V(G)| > 2|V(H)|$, we have the important lemma below.

Lemma 12. *If $\chi(G) \leq t \leq |V(G)|$ and $|V(G)| > 2|V(H)|$, then*

$$dom_G[t] = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}.$$

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices. According to Lemma 11, suppose that either $n(C,H_1) + n(C,G_M) = t$ or $n(C,H_1) + n(C,G_M) < t$ and $n(C,G_M) = |V(G_M)|$. Let $\hat{t} = n(C,H)$.

The first case considered is (a) when $n(C,H_1) + n(C,G_M) = t$. Note that if v is a dominant vertex in C, then v is dominant in (C,G_M) if $v \in V(G_M)$ and v is partially dominant in (C,H) if $v \in V(H_1)$. Let $t' = n(C,H_1)$. Since $\chi(G_M) \leq n(C,G_M) = t - n(C,H_1) \leq |V(G_M)|$, then $t - |V(G_M)| \leq t' \leq t - \chi(G_M)$. We also get that $t' \leq |V(H_1)|$ and, thus, $t_1 \leq t' \leq t_2$.

Now, consider (a.1) that there is no dominant vertex in H_2. In this case, $t' \leq \hat{t} \leq \min\{|V(H)|, t\}$. We also have that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_1(C,H)$ and the with dominant vertices of colors that appear in G_M is at most $dom_{G_M}[t-t']$. Thus, if $n(C,H_1) + n(C,G_M) = t$ and there is no dominant vertex of C in H_2, then $dom_G[t] \leq \tau_1(t)$.

Now, consider (a.2) that there is at least one dominant vertex u in H_2. Since u is adjacent to every other color of C and every neighbour of u is in H, then $\hat{t} = t$. Note that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_3(C,H)$. The number of color classes of C with dominant vertices of colors that appear in G_M is at most $\min\{t-t', d_2(C) + dom_G[t-t']\}$. Thus, if $n(C,H_1) + n(C,G_M) = t$ and there is at least one dominant vertex of C in H_2, then $dom_G[t] \leq \tau_2(t)$.

The second case considered is (b) when $n(C,H_1) + n(C,G_M) < t$ and $n(C,G_M) = |V(G_M)|$. If (b.2) $c(C,G_M) \subseteq c(C,H)$, then $n(C,G_M) = |V(G_M)|$ implies that $|M| \leq |V(H)|$ and $|V(G)| \leq |V(H)|$, a contradiction. Thus, (b.1) there is a color unique to vertices in G_M. Note also that $n(C,H_1) + n(C,G_M) < t$ implies that there is a color unique to vertices in H_2. Thus, all dominant vertices of C are in H_1. Note that $\hat{t} \leq |V(H)|$ and $\hat{t} \leq t$ and, thus, $\hat{t} \leq t_3$. We also have that $t \leq \hat{t} + |V(G_M)|$ which implies that $\hat{t} \geq t_1$. Let $J = c(C,G_M) \cap c(C,H)$. Note that $t = \hat{t} + |V(G_M)| - |J|$ which implies that

$|J| = \hat{t} + |V(G_M)| - t$. Since J is a subset of $c(C,H) \setminus c(C,H_1)$, then $|J| = \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$ which implies that $t' \leq t - |V(G_M)|$. Since H_1 has at most $V(H_1)$ colors, then $t' \leq t_4$. Now, note that every dominant vertex of C is a $\bar{J}$-dominant vertex of H_1 in (C,H). Thus, the number of color classes with dominant vertices in C is $d_4((C,H),J)$, which is at most $d_5((C,H),\hat{t} + |V(G_M)| - t)$. Thus, if $n(C,H_1) + n(C,G_M) < t$ and $|V| > 2|V(H)|$, then $dom_G[t] \leq \tau_3(t)$.

We can conclude from the previous paragraphs that if $|V| > 2|V(H)|$, then $dom_G[t] \leq \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}$. To conclude this proof, it remains to prove that $dom_G[t] \geq \tau_i(t)$, for $i \in \{1,2,3\}$. Let C_H be a coloring of H with $\hat{t}$ colors and $t' = n(C_H, H_1)$. We break into cases depending on C_H being related to each of the parameters $\tau_i(t)$. To do so, let C_M be a coloring of G_M with $t - t'$ colors and $dom_{G_M}[t-t']$ color classes with dominant vertices and C'_M be a coloring of G_M with $|V(G_M)|$ colors.

Suppose that $t_1 \leq t' \leq t_2$. If $\hat{t} \leq t$, then rename the colors in $c(C_H,H_2) \setminus c(C_H,H_1)$ to colors in the set $c(C_M)$ and let C be the coloring of G obtained by piecing together this coloring with C_M. Note that C has precisely t colors and there are $dom_{G_M}[t-t']$ color classes with dominant vertices in colors of $c(C,G_M)$ and $d_1(C_H)$ color classes with dominant vertices in colors of $c(C,H_1)$. Since $c(C,G_M) \cap c(C,H_1) = \emptyset$, then C has at least $dom_{G_M}[t-t'] + d_1(C)$ color classes with dominant vertices. This implies that $dom_G[t] \geq \tau_1(t)$.

Now, suppose that $\hat{t} = t$. Let $c(C_M) = \{c_1, \ldots, c_{t-t'}\}$ and suppose that the color classes with indices in $\{1, \ldots, dom_{G_M}[t-t']\}$ have dominant vertices in C_M. Now let C'_H be obtained from C_H by renaming the colors in $c(C_H,H_2) \setminus c(C_H,H_1)$ to colors in $c(C_M)$ in such a way that the color classes with the highest indices have dominant vertices. Note that this is possible as $c(C_H,H_2) \setminus c(C_H,H_1)$ has size precisely $t-t'$. Let C be obtained by piecing together the colorings C_M and C'_H. Note that C has precisely t colors, $d_3(C_H)$ color classes in $c(C,H_1)$ with dominant vertices and $\min\{t-t', d_2(C) + dom_G[t-t']\}$ color classes in $c(C,G_M)$ with dominant vertices. This implies that $dom_G[t] \geq \tau_2(t)$.

Now, suppose that $0 \leq t' \leq t_4$ and $t_1 \leq \hat{t} \leq t_3$. Note that $0 \leq \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$, as $\hat{t} \geq t_1 = t - |V(G_M)|$ and $t' \leq t_4 \leq t - |V(G_M)|$. Thus, let J be a subset of $c(C_H,H_2) \setminus c(C_H,H_1)$ such that $d_4(C_H,J) = d_5(C_H, \hat{t} + |V(G_M)| - t)$. Let C'_H be obtained by renaming the colors of C_H in the set J to colors in C'_M so that $|c(C'_H) \cap c(C'_M)| = |J| = \hat{t} + |V(G_M)| - t$. Let C be obtained by piecing together the colorings C'_H and C'_M. Note that $n(C) = n(C,H) + n(C,G_M) - |J| = \hat{t} + |V(G_M)| - |J| = t$. This implies that $dom_G[t] \geq \tau_3(t)$. □

Lemma 13. *Let $q > 0$ be a fixed integer, let H be a graph with less then q vertices and let H_1 and H_2 be induced subgraphs of H such that $V(H_1)$ and $V(H_2)$ are a vertex partition of H. Given a graph G_M with n vertices, let G be the graph obtained by applying p-component operation over $(G_M, H = (H_1,H_2))$ (just join all edges between G_M and H_1). Then, given the chromatic number $\chi(G_M)$ and the dominance vector dom_M of G_M, we can calculate the chromatic number $\chi(G)$ in time $\Theta(n)$ and the dominance vector dom_G of G in time $\Theta(n^2)$.*

Proof. Since $|V(H)| \leq q$, where q is an integer fixed, we have that parameters $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ can be obtained in linear time (once fixed t' and $\hat{t}$, the value in sup can be obtained in constant time that depends only on q). If $|V(G)| \leq 2|V(H)| \leq 2q$, then we can calculate $dom_G[t]$ in constant time. If $|V(G)| > 2|V(H)|$, then, applying Lemma 12, we have $dom_G[t]$ in linear time. So we can obtain the dominance vector dom_G of G in time $\Theta(n^2)$ for all possible values of t. □

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