(Circular) backbone colouring: forest backbones in planar graphs

Frédéric Havet, Andrew D. King, Mathieu Liedloff, Ioan Todinca

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Abstract

Consider an undirected graph G and a subgraph H of G, on the same vertex set. The *q-backbone chromatic number* $\mathrm{BBC}_q(G, H)$ is the minimum k such that G can be properly coloured with colours from $\{1, \ldots, k\}$, and moreover for each edge of H, the colours of its ends differ by at least q. In this paper we focus on the case when G is planar and H is a forest. We give a series of NP-hardness results as well as upper bounds for $\mathrm{BBC}_q(G, H)$, depending on the type of the forest (matching, galaxy, spanning tree). Eventually, we discuss a circular version of the problem.

1 Introduction

All the graphs considered in this paper are simple. Let $G = (V, E)$ be a graph, and let $H = (V, E(H))$ be a spanning subgraph of G, called the *backbone*. A *k-colouring* of G is a mapping $f : V \to \{1, 2, \ldots, k\}$. Let f be a k-colouring of G. It is a *proper colouring* if $|f(u) - f(v)| \geq 1$ for all edges $uv \in E(G)$. It is a *q-backbone colouring* for (G, H) if f is a proper colouring of G and $|f(u) - f(v)| \geq q$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k-colouring of G. The *q-backbone chromatic number* $\mathrm{BBC}_q(G, H)$ is the smallest integer k for which there exists a q-backbone k-colouring of (G, H).

If f is a proper k-colouring of G, then g defined by $g(v) = q \cdot f(v) - q + 1$ is a q-backbone colouring of (G, H) for any spanning subgraph H of G. Moreover it is well-known that if $G = H$, this q-backbone colouring of (G, H) is optimal. Therefore, since $\mathrm{BBC}_q(H, H) \leq \mathrm{BBC}_q(G, H) \leq \mathrm{BBC}_q(G, G)$, we have

$$q \cdot \chi(H) - q + 1 \leq \mathrm{BBC}_q(G, H) \leq q \cdot \chi(G) - q + 1. \tag{1}$$

If H is empty (i.e. $E(H) = \emptyset$), then $\mathrm{BBC}_q(G, H) = \chi(G)$. Hence for any $k \geq 3$, deciding if $\mathrm{BBC}_q(G, H) \leq k$ is NP-complete because deciding if a graph is k-colourable is NP-complete (See [7]). However, when we impose G or H to belong to certain graph classes, the problem sometimes become polynomial-time solvable. A trivial example is when we consider H with chromatic number at least $r > (k + q - 1)/q$. Then $\mathrm{BBC}_q(G, H) \geq rq - q + 1$, and so deciding if $BBC_q(G, H) \leq k$

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can be done instantly by always returning 'no'. A less trivial example is when we impose H to have minimum degree 1. For such an H, deciding if $BBC_q(G, H) \leq q + 1$ is also polynomial-time solvable, because $\mathrm{BBC}_q(G, H) = q + 1$ if and only if G is bipartite. This simple observation was already made by Broersma et al. [5] when H is a 1-*factor* (a spanning subgraph in which every vertex has degree exactly 1). Furthermore, if we also impose H to be connected, we show in Theorem 17 that deciding if $BBC_q(G, H) \leq q + 2$ can be done in polynomial time. In contrast, if the condition of H being connected is removed, then it is NP-complete (Theorem 18).

In this paper, we will focus on the particular case when G is a planar graph and H is a forest (i.e. an acyclic graph). Inequality (1) and the Four-Colour Theorem imply that for any planar graph G and spanning subgraph H, $\mathrm{BBC}_q(G, H) \leq 3q + 1$. However, for $q = 2$, Broersma et al. [4] conjectured that this is not best possible if the backbone is a forest.

Conjecture 1. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_2(G, F) \leq 6$.

If true, Conjecture 1 would be best possible. Broersma et al. [4] gave an example of a graph $\hat{G}$ with a forest $\hat{F}$ such that $\mathrm{BBC}_2(\hat{G}, \hat{F}) = 6$. See Figure 1. It is then natural to ask how large $\mathrm{BBC}_q(G, F)$

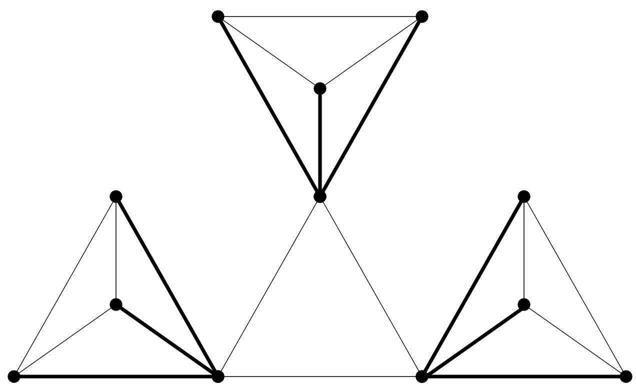

Figure 1: A planar graph $\hat{G}$ with a forest $\hat{F}$ (bold edges) such that $\mathrm{BBC}_q(\hat{G}, \hat{F}) = q + 4$.

could be when G is planar and F is a forest for larger values of q. We prove the following.

Theorem 2. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_q(G, F) \leq q + 6$.

In fact, we prove a more general result in Proposition 13 : for any pair (G, H) with H a subgraph of G,

$$\mathrm{BBC}_q(G, H) \leq (\chi(G) + q - 2)\chi(H) - q + 2.$$

For $q \geq 4$, Theorem 2 is best possible. Indeed, we show a planar graph G^* together with a spanning tree T^* such that $\mathrm{BBC}_q(G^*, T^*) = q + 6$ for all $q \geq 4$. See Figure 2 and Proposition 15.

Furthermore, we show in Theorem 31, that for any fixed $q \geq 4$, given a planar graph G and a spanning tree T of G, it is NP-complete to decide if $\mathrm{BBC}_q(G, T) \leq q + 5$.

On the other hand, we believe that if $q = 3$, Theorem 2 is not best possible.

Conjecture 3. *If G is a planar graph and F a forest in G, then* $\mathrm{BBC}_3(G, F) \leq 8$.

If true, Conjecture 3 would be tight. The pair (G^*, F^*) of Figure 2 satisfies $\mathrm{BBC}_3(G^*, F^*) = 8$. We show in Proposition 16 that Conjecture 1 implies Conjecture 3.

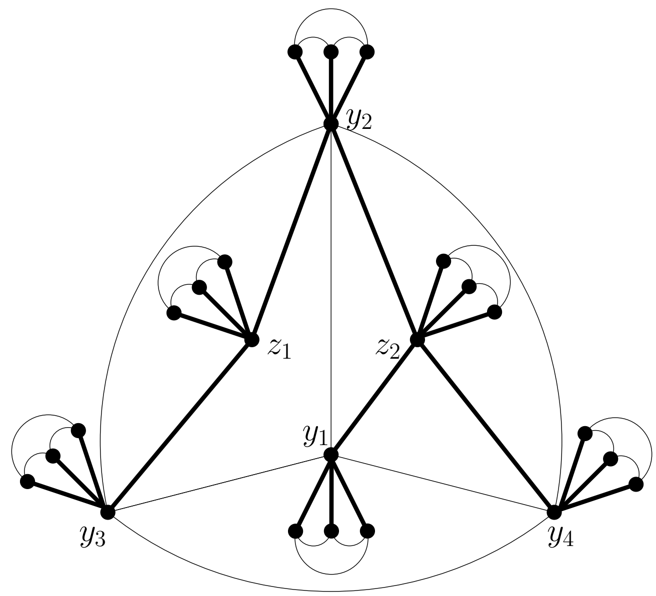

Figure 2: A planar graph G^* and a tree T^* (bold edges) such that $\mathrm{BBC}_q(G^*, T^*) = q + 6$ for $q \geq 4$.

A *star* is a tree in which a vertex v, called the *center* is adjacent to every other. A *galaxy* is a forest of stars. As evidence in support of Conjectures 1 and 3, Broersma et al. [5] showed that if F is a galaxy in a planar graph G, then $\mathrm{BBC}_q(G, F) \leq q + 4$. This result is best possible even if F has maximum degree 3 as shown by the example of Figure 1. Furthermore, we show in Theorems 21 and 28 that, for any $q \geq 2$, it is NP-complete to decide if $\mathrm{BBC}_q(G, F) \leq q + 3$ given a planar graph G and a galaxy of maximum degree 3.

However, if the backbone is a *matching*, i.e. a galaxy with maximum degree 1, then fewer colours are needed. Indeed, Broersma et al. [5] showed that if M is a matching in a planar graph G, then for any $q \geq 3$, $\mathrm{BBC}_q(G, M) \leq q + 3$. They conjectured that the same holds for $q = 2$.

Conjecture 4 (Broersma et al. [5])**.** *If G is a planar graph G and M a matching in G, then* $\mathrm{BBC}_2(G, M) \leq 5$.

It is natural to ask the same question for galaxies with maximum degree at least 2. When $q = 2$, we answer in the negative by showing that there are pairs of planar graphs and spanning forests of maximum degree 2 whose 2-backbone chromatic number is 6. Furthermore, we show that given a planar graph G and a spanning forest F of maximum degree 2, it is NP-complete to decide whether $BBC_2(G, F) \leq 5$ (Theorem 22). We also show that given a planar graph G with a hamiltonian path P, it is NP-complete to decide whether $BBC_2(G, F) \leq 5$. This result refines a result of Broersma et al. [3, 4] who proved it for a general graph G.

For $q = 3$, the problem remains open.

Problem 5. *If G is a planar graph G and F a galaxy of maximum degree* 2*, is is true that* $\mathrm{BBC}_q(G, F) \leq q + 3$*, for all $q \geq 3$?*

Broersma et al. [5] proved that deciding if $\mathrm{BBC}_q(G, M) \leq q+2$ for a given graph G and matching M is NP-complete. We prove in Subection 2.2 that it remains NP-complete even if we impose G to be

	G planar			
	H forest	H spanning tree	H 1-factor	H Hamilton path
$BBC_q(G,H) \leq q+1$?	NP-C	poly	poly	poly
$BBC_q(G,H) \leq q+2$?	NP-C	poly	NP-C	poly
$BBC_q(G,H) \leq q+3$?	NP-C	NP-C	$q \geq 3$: Yes $q = 2$: ?Yes?	$q = 2$: NP-C
$BBC_q(G,H) \leq q+5$?	$q \geq 4$: NP-C $q = 3$: ?Yes? $q = 2$: Yes			
$BBC_q(G,H) \leq q+6$?	Yes			

Table 1: Complexity of deciding if $BBC_q(G,H) \leq q+k$ for $k \in \{1,\ldots,6\}$, when G is a planar graph and H a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.

planar. In contrast, we prove that deciding if $\mathrm{BBC}_q(G,T) \leq q+2$ for a given graph G and spanning tree T is polynomial-time solvable.

Many of the complexity results on backbone colouring of planar graphs with a forest backbone are summarized in Table 1.

One can generalize the notion of backbone colouring by allowing a more complicated structure of the frequency space. The most natural one is to consider a circular metric. A *circular k-colouring* of G or *$\mathbb{Z}_k$-colouring* is a mapping $f : V \to \mathbb{Z}_k$. The notions of *circular q-backbone colouring* and *circular q-backbone chromatic number* are defined similarly to those of *q-backbone colouring* and *circular q-backbone chromatic number* by replacing colouring by circular colouring. The circular q-backbone chromatic number of a graph pair (G,H) is denoted $\mathrm{CBC}_q(G,H)$.

If f is a circular q-backbone k-colouring, then the mapping f^* defined by $f(v) = f(v)+1$ for all vertex v is trivially a q-backbone k-colouring. On the other hand, a q-backbone k-colouring yields a circular q-backbone $(k+q-1)$-colouring. Hence for every graph pair (G,H), where H is a spanning subgraph of G, we have

$$\mathrm{BBC}_q(G,H) \leq \mathrm{CBC}_q(G,H) \leq \mathrm{BBC}_q(G,H) + q - 1. \tag{2}$$

Also,

$$q \cdot \chi(H) \leq \mathrm{CBC}_q(G,H) \leq q \cdot \chi(G). \tag{3}$$

Observe that if G is bipartite and H is non-empty, Equation (3) implies that $CBC_q(G,H) = 2q$. More generally, if $\chi(G) = \chi(H)$, then $\mathrm{CBC}_q(G,H) = q \cdot \chi(G)$. However if $2 \leq \chi(H) < \chi(G)$, one can improve the upper bound. We show in Proposition 32 that, for any pair (G,H) with H a subgraph of G,

$$\mathrm{CBC}_q(G,H) \leq (\chi(G) + q - 2)\chi(H). \tag{4}$$

Since $\mathrm{CBC}_q(G,H) = \chi(G)$ when H is empty and k-COLOURABILITY is NP-complete, for any fixed $k \geq 3$, given a graph G and a subgraph H it is NP-complete to decide if $\mathrm{CBC}_q(G,H) \leq k$. But if insist that H is not empty, then $\mathrm{CBC}_q(G,H) \geq 2q$ by Proposition 3. Hence deciding if $\mathrm{CBC}_q(G,H)$ is at most k with $k \leq 2q-1$ can be done instantly by always returning 'no'. Less trivially, Proposition 35 shows that if H is a connected spanning subgraph of G, then $\mathrm{CBC}_q(G,H) = 2q$ if and only if G is bipartite. Hence deciding if $\mathrm{CBC}_q(G,H) = 2q$ can be done in polynomial time.

Inequality (4) implies that $\mathrm{CBC}_q(G,F) \leq 2q+4$ for any planar graph G and forest F in G. We believe that this upper bound can be reduced by at least one.

Conjecture 6. *If G is a planar graph and F a spanning forest of G, then* $\mathrm{CBC}_q(G,F) \leq 2q+3$.

A natural question is to ask whether this conjecture would be best possible.

Problem 7. *For any $q \geq 2$, does there exist a planar graph G_q and a spanning forest F_q of G_q such that* $\mathrm{CBC}_q(G_q,F_q) = 2q+3$*?*

Conjecture 6 holds if the backbone F is a galaxy. It follows directly from (2) and the fact that $\mathrm{BBC}_q(G,F) \leq q+4$ in such a case, as mentioned earlier. We believe however that one can use one colour less.

Conjecture 8. *Let G be a planar graph and F a galaxy in G, then* $\mathrm{CBC}_q(G,F) \leq 2q+2$.

If true, this conjecture would be tight, since the circular q-backbone chromatic number of a K_4 with backbone $K_{1,3}$ is $2q+2$. As evidence in support of Conjecture 8, Broersma et al. [5] deduced from the Four-Colour Theorem that if G is a planar graph and M a matching in G then $\mathrm{CBC}_q(G,M) \leq 2q+2$.

Broersma et al. also give an example of a planar graph G and a matching M such that (G,M) has no 2-backbone $\mathbb{Z}_5$-colouring. We show in Theorems 36 and 40 that for any fixed $k \in \{4,5\}$, it is NP-complete to decide if $BBC_2(G,M) \leq k$ for given planar graph G and matching M. For larger values of q, the following questions are still open.

Problem 9. *Let G be a planar graph and let M be a matching M in G. For any $q \geq 3$, is it true that* $\mathrm{CBC}_q(G,M) \leq 2q+1$ *?*

Problem 10. *Is it NP-complete to decide if* $\mathrm{CBC}_2(G,F) \leq 6$ *for a planar graph G and spanning forest F?*

Problem 11. *For any $g \geq 5$, is it NP-complete to decide if* $\mathrm{CBC}_2(G,M) \leq 4$ *for a planar graph G of girth at least g and matching M?*

We prove in Theorem 33 that if G has girth at least 5, then $\mathrm{CBC}_q(G,M) \leq 2q+1$. We wonder if the same holds for planar graph of girth 4.

Problem 12. *Let G be a planar graph of girth 4 and let M be a matching in G. Is it true that* $\mathrm{CBC}_q(G,M) \leq 2q+1$ *?*

Many of the complexity results on circular backbone colouring of planar graphs with a forest backbone are summarized in Table 2.

2 Backbone colouring

2.1 About Conjectures 1 and 3

Proposition 13. *Let G be a graph and let H be a subgraph of G. Then* $\mathrm{BBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H)-q+2$.

	G planar			
	H forest	H spanning tree	H 1-factor	H Hamilton path
$CBC_q(G,H) \leq 2q$?	q=2: NP-C	poly	q=2: NP-C	poly
$CBC_q(G,H) \leq 2q+1$?	NP-C	poly	q=2: NP-C	q=2: NP-C
$CBC_q(G,H) \leq 2q+2$?				
$CBC_q(G,H) \leq 2q+3$?	?Yes?			
$CBC_q(G,H) \leq 2q+4$?	Yes			

Table 2: Complexity of deciding if $CBC_q(G,H) \leq 2q+k$ for $k \in \{0,\ldots,4\}$, when G is a planar graph and H a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.

Proof. Let g be a proper $\chi(G)$-colouring of G and h a proper $\chi(H)$-colouring of H. Let f be the colouring defined by:

$$f(v) \;=\; \begin{cases} (h(v)-1)(q-2+\chi(G))+g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v)-1)(q-2+\chi(G))+\chi(G)+1-g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

Let us check that f is a q-backbone $((\chi(G)+q-2)\chi(H)-q+2)$-colouring of (G,H).

Let $uv \in E(G)$. Without loss of generality, $h(u) \geq h(v)$. If $h(u)=h(v)$, then $uv \notin E(H)$ and $|f(u)-f(v)| = |g(u)-g(v)| \neq 0$.

Assume now that $h(u) > h(v)$. If $h(u)$ and $h(v)$ are both odd or both even, then $f(u)-f(v) \geq 2(q-2+\chi(G)) - |(g(u)-g(v)| \geq q$. If $h(u)$ is odd and $h(v)$ is even, then $f(u)-f(v) \geq q-3+g(u)+g(v)$, which is at least q because $g(u)+g(v) \geq 3$ for $g(u) \neq g(v)$. If $h(u)$ is even and $h(v)$ is odd, then $f(u)-f(v) \geq q-2+2\chi(G)+1-g(u)-g(v)$, which is at least q because $g(u)+g(v) \leq 2\chi(G)-1$ for $g(u) \neq g(v)$. $\square$

A *parachute on* v or a *parachute with harness* v is a complete graph on four vertices whose three edges incident to v are in the backbone.

Proposition 14. *(i) For $q \geq 2$, in a q-backbone $(q+3)$-colouring of a parachute, the harness is coloured in $\{1, q+3\}$.*

(ii) For $q \geq 3$, in a q-backbone $(q+4)$-colouring of a parachute, the harness is coloured in $\{1,2,q+3,q+4\}$.

(iii) For $q \geq 4$, in a q-backbone $(q+5)$-colouring of a parachute, the harness is coloured in $\{1,2,3,q+3,q+4,q+5\}$.

Proof. Let y be the harness.

(ii) If $3 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in \{1,2,q+3,q+4\}$.

(iii) If $4 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in \{1,2,3,q+3,q+4,q+5\}$. $\square$

Proposition 15. *Let G^* and T^* be the graph and its spanning tree depicted in Figure 2. For any $q \geq 4$, $BBC_q(G^*,T^*) \geq q+6$.*

Proof. Assume for a contradiction that there is a q-backbone $(q+5)$-colouring ϕ of (G^*, T^*). By Proposition 14-(iii), the vertices $y_1, y_2, y_3, y_4, z_1, z_2$ are coloured in $\{1, 2, 3, q+3, q+4, q+5\}$. Without loss of generaltiy, we may assume that $\phi(y_2) \in \{1, 2, 3\}$. But then $\phi(z_1)$ and $\phi(z_2)$ must be in $\{q+3, q+4, q+5\}$, because y_2z_1 and y_2z_2 are in $E(T^*)$. And $\phi(y_1)$, $\phi(y_2)$ and $\phi(y_3)$ are in $\{1, 2, 3\}$ because y_3z_1 and y_1z_2 and y_4z_2 are in $E(T^*)$. But $\{y_1, y_2, y_3, y_4\}$ is a clique in G^*, so they must all get different colours, a contradiction. □

Proposition 16. *Conjecture 1 implies Conjecture 3.*

Proof. Assume that Conjecture 1 holds. Let G be a planar graph and F a forest in G. Then (G, F) admits a 2-backbone 6-colouring ϕ. Let ψ be defined by $\psi(v) = \phi(v)$ if $\phi(v) \in \{1, 2\}$, $\psi(v) = \phi(v) + 1$ if $\phi(v) \in \{3, 4\}$, and $\psi(v) = \phi(v) + 2$ if $\phi(v) \in \{5, 6\}$. One easily check that ψ is a 3-backbone 8-colouring of (G, F). □

2.2 q-backbone $(q+2)$-colouring

Theorem 17. *Given a connected graph G and a spanning connected subgraph H, one can decide in polynomial time if* $\mathrm{BBC}_q(G, H) \leq q+2$.

Proof. Observe first that if H is not bipartite, then $\mathrm{BBC}(H, H) \geq 2q+1$ by (1), and so $\mathrm{BBC}_q(G, H) \geq q+3$. So we first check if H is bipartite. If not, we return 'no'. If it is, we get a bipartition (A, B) of H.

Observe that if (G, H) has a q-backbone $(q+2)$-colouring, then (free to rename A and B) all the vertices of A are coloured in $\{1, 2\}$ and all the vertices of B in $\{q+1, q+2\}$, because H is connected. We then can transform our instance into an instance $I(G, H)$ of 2SAT as follows. For each vertex v, we create a variable x_v. Intuitively, for a vertex $x \in A$ (resp. $x \in B$), the variable x_v will be true if and only if v is coloured 1 (resp. $q+2$) and false if and only if v is coloured 2 (resp. $q+1$). Now for each edge uv, we create the following clauses.

- If u and v are both in A or both in B, we create the clauses $x_u \vee x_v$ and $\bar{x}_u \vee \bar{x}_v$;
- if $u \in A$ and $v \in B$, we create the clause $x_u \vee x_v$.

It is easy to check that (G, H) has a q-backbone $(q+2)$-colouring if and only if $I(G, H)$ is satisfiable.

Since 2SAT is well-known to be polynomial-time solvable, we can decide in polynomial time if $\mathrm{BBC}_q(G, H) \leq q+2$. □

Theorem 18. *For any $q \geq 2$, the following problem is NP-complete.*
Input: A planar graph G and a 1-factor F of G.
Question: $\mathrm{BBC}_q(G, F) \leq q+2$*?*

Proof. The problem is trivially in NP since a q-backbone $(q+2)$-colouring of (G, F) is clearly a certificate.

Reduction from NOT-ALL-EQUAL 3SAT, which is defined as follows:
Input: A set of clauses each having three literals.
Question: Does there exists a *suitable* truth assignment, that is such that each clause has at least one true and at least one false literal?

This problem was shown NP-complete by Schaefer [11].

Let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a collection of clauses of size three over a set U of variables. We will construct a graph pair (G, F) such that F is a 1-factor of G. Since $V(F) = V(G)$, we only precise which edges are in $E(F)$.

The following gadget will be useful. A *forcing gadget at* v or a *forcing gadget with head* v is the graph depicted in Figure 3.

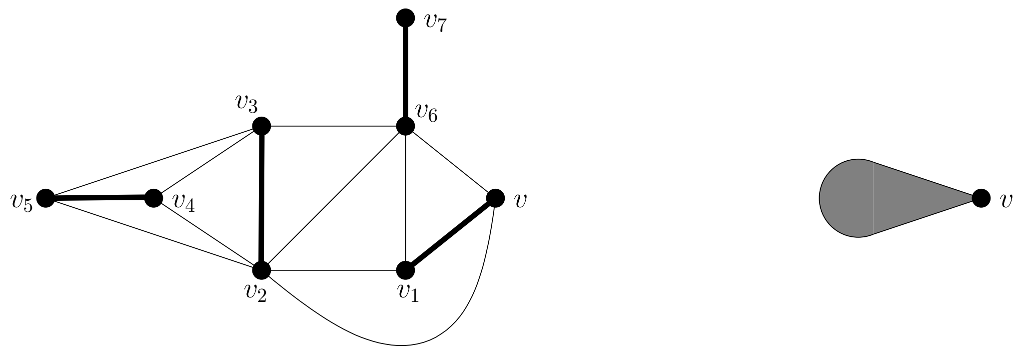

Figure 3: A forcing gadget with head v (left) and its symbol (right) (Edges of $E(F)$ are in bold.)

A key point in the reduction will be the following claim.

Claim 19. *In any q-backbone $(q+2)$-colouring of a forcing gadget, its head is coloured in* $\{1, q+2\}$.

Proof. Consider a forcing gadget, whose vertices are named as in Figure 3, and ϕ a q-backbone $(q+2)$-colouring of it. Since all the vertices are matched in F, there all must be coloured in $\{1, 2, q+1, q+2\}$.

Assume for a contradiction that $\phi(v) = 2$. Then $\phi(v_1) = q + 2$. Thus $\phi(v_2) \in \{1, q + 1\}$. Now if $\phi(v_2) = q + 1$, then necessarily $\phi(v_3) = 1$. Therefore, whatever the colouring may be, v_4 and v_5 are both adjacent to a vertex coloured 1. Hence $\{\phi(v_4), \phi(v_5)\} = \{2, q + 2\}$. Therefore $\{\phi(v_2), \phi(v_3)\} = \{1, q + 1\}$. But then v_6 cannot be coloured.

Similarly, we get a contradiction if $\phi(v) = q + 1$. $\square$

For every variable $u \in U$, create a *variable subgraph* P_u which is obtained from the path $(a_1(u), b_1(u), a_2(u), b_2(u), \dots, a_n(u), b_n(u))$ by adding a forcing gadget on each of its vertex.

For every clause $C_i = \ell_1 \vee \ell_2 \vee \ell_3$, create a clause gadget D_i as shown Figure 4.

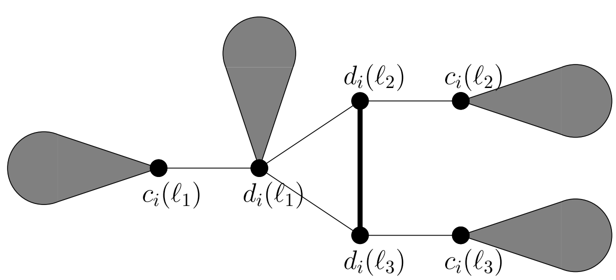

Figure 4: The clause D_i. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Then for each clause C_i and each literal ℓ of C_i, we add a path of length three $(c_i(l), c_i'(l), c_i''(l), a_i(u))$ if ℓ is the non-negated variable u, and $(c_i(l), c_i'(l), c_i''(l), b_i(u))$ if ℓ is the negated variable $\bar{u}$. We also add two forcing gadgets with heads $c_i'(l)$ and $c_i''(l)$.

It is easy to see that the resulting graph G' may be drawn in the plane such that the crossed edges are those of type $c_i'(\ell)c_i''(\ell)$ for some literal ℓ. In particular, the two endvertices of a crossed edge are heads of forcing gadgets.

As long as there is a crossing C between two edges $t(C)u(C)$ and $v(C)w(C)$, we replace these two edges by the crossing gadget $CG(C)$ depicted in Figure 5, so that the only edges that are possibly crossed (if there were several crossings on tu or uv) are $t(C)t'(C)$, $u(C)u'(C)$, $v(C)v'(C)$ and $w(C)w'(C)$. After this process, there is no more crossing so the resulting graph G is planar.

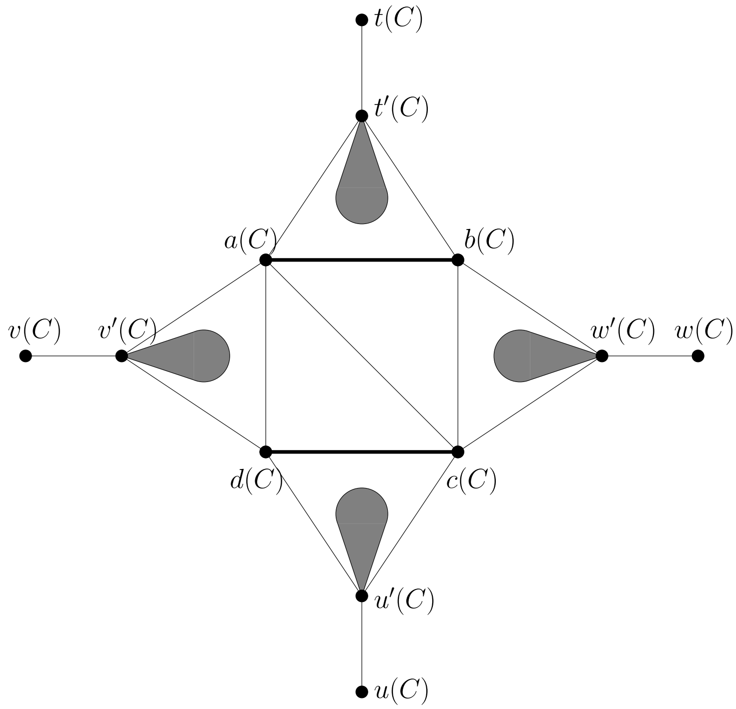

Figure 5: The crossing gadget $CG(C)$. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Claim 20. *Let ϕ be a q-backbone $(q+2)$-colouring of (G, F). For every crossing C in G', we have $\{\phi(t(C)), \phi(u(C))\} = \{1, q+2\}$ and $\{\phi(v(C)), \phi(w(C))\} = \{1, q+2\}$.*

Subproof. By induction on the reverse order of creation of the crossing gadget.

By construction, $t(C)$, $u(C)$, $v(C)$, $w(C)$, $t'(C)$, $u'(C)$, $v'(C)$, and $w'(C)$ are heads of forcing gadgets. So, by Claim 19, they are coloured 1 or $q+2$. Without loss of generality, we may assume that $\phi(t(C)) = 1$.

If the edge $t(C)t'(C)$ was crossed and then replaced by a series of crossing gadget, by induction, $\phi(t'(C)) = q+2$. It is also trivially the case if $t(C)t'(C)$ still exists. Hence $\{\phi(a(C)), \phi(b(C))\} = \{1, q+1\}$.

Assume for a contradiction that $\phi(u(C)) \neq q+2$. Then, as above, $\{\phi(c(C)), \phi(d(C))\} = \{1, q+1\}$. This is a contradiction, because $a(C)c(C)$ and $a(C)d(C)$ are edges. Hence $\phi(u(C)) = q+2$, and so $\phi(u'(C)) = 1$ and $\{\phi(c(C)), \phi(d(C))\} = \{2, q+2\}$.

In particular, one vertex of $\{a(C), b(C), c(C), d(C)\}$ is coloured 1 and another is coloured $q+2$. Now assume for a contradiction that $\{\phi(v(C)), \phi(w(C))\} \neq \{1, q+2\}$. Then $v(C)$ and $w(C)$ are both coloured 1 or both coloured $q+2$, and so $v'(C)$ and $w'(C)$ are both coloured $q+2$ or both coloured 1, respectively. This is a contradiction, as all vertices of $\{a(C), b(C), c(C), d(C)\}$ are adjacent to some vertex in $\{v'(C), w'(C)\}$. $\Diamond$

Let us now prove that (G, F) admits a q-backbone $(q+2)$-colouring if and only if $\mathcal{C}$ has a suitable truth assignment.

Assume first that (G, F) admits a q-backbone $(q+2)$-colouring ϕ. Let u be a variable. Since there are heads of forcing gadgets, by Claim 19, all the $a_i(u)$ and $b_i(u)$ are coloured in $\{1, q+2\}$. Moreover, since they form a path, all the $a_i(u)$ are coloured with the same colour and all the $b_i(u)$ are coloured with the other. Hence one can define the truth assignment ψ by $\psi(u) = true$ if $\phi(a_i(u)) = 1$ for $1 \leq i \leq n$, and $\psi(u) = false$ if $\phi(a_i(u)) = q+2$ for $1 \leq i \leq n$.

We shall prove that ψ is suitable.

Let $C_i = \ell_1 \vee \ell_2 \vee \ell_3$ be a clause. Claim 20 implies that for $j \in \{1, 2, 3\}$, $\phi(c_i(\ell_j)) = 1$ if $\psi(\ell_j) = false$ and $\phi(c_i(\ell_j)) = q+2$ if $\psi(\ell_j) = true$. Now the three $c_i(\ell_j)$, $1 \leq j \leq 3$, cannot be all coloured 1 (resp. $q+2$), for otherwise $\{\phi(d_i(\ell_2)), \phi(d_i(\ell_3))\}$ must be $\{2, q+2\}$ (resp. $\{1, q+1\}$) and so $d_i(\ell_1)$ cannot be coloured, because it must be coloured in $\{1, q+2\}$ as head of a forcing gadget. Thus at least one of the $c_i(l_j)$ is coloured 1 and at least one is coloured $q+2$, and so C_i has at least one true and at least one false literal.

Hence ψ is suitable.

Reciprocally, assume that $\mathcal{C}$ has a suitable truth assignment ψ. For all $u \in U$ and all $1 \leq i \leq n$, let us define $\phi(a_i(u)) = 1$ and $\phi(b_i(u)) = q+2$ if $\psi(u) = true$, and $\phi(a_i(u)) = q+2$ and $\phi(b_i(u)) = 1$ if $\psi(u) = false$. Similarly, for every literal ℓ, we set $\phi(c_i(\ell)) = 1$, $\phi(c'_i(\ell)) = q+2$, $\phi(c''_i(\ell)) = 1$, if ℓ is false, and $\phi(c_i(\ell)) = q+2$, $\phi(c'_i(\ell)) = 1$, $\phi(c''_i(\ell)) = q+2$, if ℓ is true.

One can extend ϕ into a q-backbone $(q+2)$-colouring of (G, F). Indeed, it is sufficient to show that we can extend it to forcing, clause and crossing gadgets.

If v is the head of a forcing gadget and $\phi(v) = 1$, we can set $\phi(v_1) = q+2$, $\phi(v_2) = q+1$, $\phi(v_3) = 1$, $\phi(v_4) = q+2$, $\phi(v_5) = 2$, $\phi(v_6) = 2$, and $\phi(v_7) = q+2$. Similarly, we can extend the colouring to the forcing gadget if $\phi(v) = q+2$.

Consider a clause gadget D_i. Since C_i has at least one true and at least one false literal, at least one vertex of $c_i(\ell_1)$, $c_i(\ell_2)$ $c_i(\ell_3)$ is coloured 1 and at least one is coloured $q+2$. If $c_i(\ell_1)$ is coloured $q+2$, and $c_i(\ell_2)$ and $c_i(\ell_2)$ are assigned 1, then we can set $\phi(d_i(\ell_1)) = 1$, $\phi(d_i(\ell_2)) = 2$, and $\phi(d_i(\ell_3)) = q+2$. If $c_i(\ell_1)$ and $c_i(\ell_2)$ are coloured 1, and $c_i(\ell_3)$ is assigned $q+2$, then we can set $\phi(d_i(\ell_1)) = q+2$, $\phi(d_i(\ell_2)) = q+1$, and $\phi(d_i(\ell_3)) = 1$.

Finally consider a crossing gadget such that $\{\phi(t(C)), \phi(u(C))\} = \{\phi(v(C)), \phi(w(C))\} = \{1, q+2\}$. By symmetry, we may assume that $\phi(t(C)) = \phi(v(C)) = 1$ and $\phi(u(C)) = \phi(w(C)) = q+2$. Then we can set $\phi(t'(C)) = \phi(v'(C)) = q+2$, $\phi(u'(C)) = \phi(w'(C)) = 2$, $\phi(a(C)) = 1$, $\phi(b(C) = q+1$, $\phi(c(C)) = q+2$, and $\phi(d(C)) = 2$. $\square$

2.3 2-backbone 5-colouring

2.3.1 Galaxy backbone

Theorem 21. *The following problem is NP-complete.*
Input: A planar graph G and a galaxy F in G with maximum degree 3.
Question: Is $\mathrm{BBC}_2(G, F) \leq 5$*?*

Proof. Reduction from PLANAR 3-COLOURABILITY, which consists of deciding if a given connected planar graph is 3-colourable. This problem was shown to be NP-complete by Stockmeyer [13]. Clearly, it remains NP-complete when restricted to 2-connected planar graphs.

Let H be a 2-connected planar graph. We shall construct a planar graph G and a galaxy F with maximum degree 3 in G such that $\mathrm{BBC}_2(G, F) \leq 5$ if and only if H is 3-colourable.

As a forcing gadget at v, we will use the parachute with harness v. It is easy to see that in a 2-backbone 5-colouring of a parachute, its harness is coloured in $\{1, 5\}$.

We consider any embedding of H. For each face $(x_1, x_2, \dots, x_k, x_1)$ of H, we put a cycle $(z_1, z_2, \dots, z_{2k}, z_1)$, inside which we put parachutes on every vertex z_i for every $1 \leq i \leq 2k$. We then add the edges $x_i z_{2i}$ $x_i z_{2i+1}$ for all $1 \leq i \leq k$.

Assume that (G, F) has a 2-backbone 5-colouring ϕ, then, because of the parachutes, all the vertices in the cycles added inside faces must be coloured in $\{1, 5\}$. Moreover consecutive vertices on one such cycles get different colours, so one is coloured 1 and the other is coloured 5. Hence all the vertices in H are coloured in $\{2, 3, 4\}$. Hence ϕ induces a proper 3-colouring on H with colours $\{2, 3, 4\}$.

Reciprocally, assume that H is 3-colourable. Then there exists a proper 3-colouring c of H into $\{2, 3, 4\}$. One can then colour all the cycles inside faces with 1 and 5. The colouring can then easily be extended into a 2-backbone 5-colouring of (G, F). □

Theorem 22. *The following problem is NP-complete.*
Input: A planar graph G and a galaxy F in G with maximum degree 2.
Question: Is $\mathrm{BBC}_2(G, F) \leq 5$*?*

Proof. The proof is identical to the one of Theorem 21. The only difference comes from the forcing gadget, which is more complicated because it cannot contains stars of degree 3 in F.

To construct the forcing gadget, we need an auxiliary gadget, called *no-3-gadget*. It is depicted in Figure 6.

Claim 23. *In any 2-backbone 5-colouring of a no-3-gadget, its roof is not coloured in* 3.

Proof. We will denote the vertices of the no-3-gadget by their names in Figure 6. Assume for a contradiction that there is a 2-backbone 5-colouring ϕ of a no-3-gadget such that $\phi(x) = 3$.

Assume first that $\phi(a) \in \{4, 5\}$, then $\phi(b) \in \{1, 2\}$ and $\{\phi(a), \phi(c)\} = \{4, 5\}$. Hence $\phi(d) \in \{1, 2\}$ and so $\{\phi(f), \phi(c)\} = \{4, 5\}$. Therefore $\phi(e) = 3$ and so $\phi(d) = 1$. Similarly, if $\phi(a) \in \{1, 2\}$, we obtain that $\phi(d) = 5$. Hence, $\phi(d) \in \{1, 5\}$.

Similarly, $\phi(d') \in \{1, 5\}$. Free to consider $6 - \phi$ instead of ϕ, we may assume that $\phi(d) = 1$ and $\phi(d') = 5$. Thus $\phi(f') = 2$.

Now $\phi(g) \in \{3, 4\}$. If $\phi(g) = 3$, then $\{\phi(i), \phi(h)\} = \{1, 5\}$, and if $\phi(g) = 4$, then $\{\phi(i), \phi(h)\} = \{1, 2\}$. In both cases, one of h and i is coloured 1, which is impossible because $\phi(d) = 1$. □

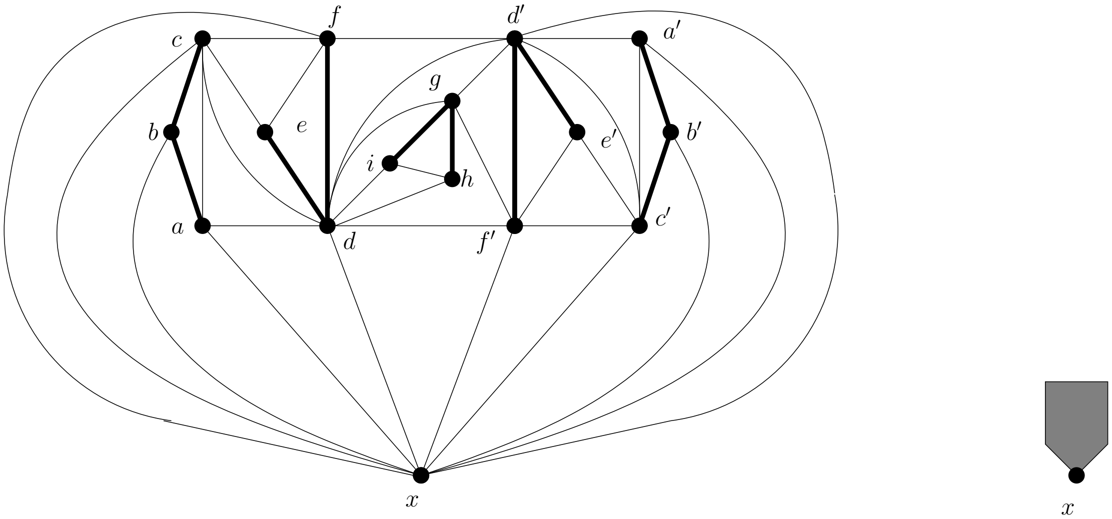

Figure 6: The no-3-gadget with roof x and its symbol.

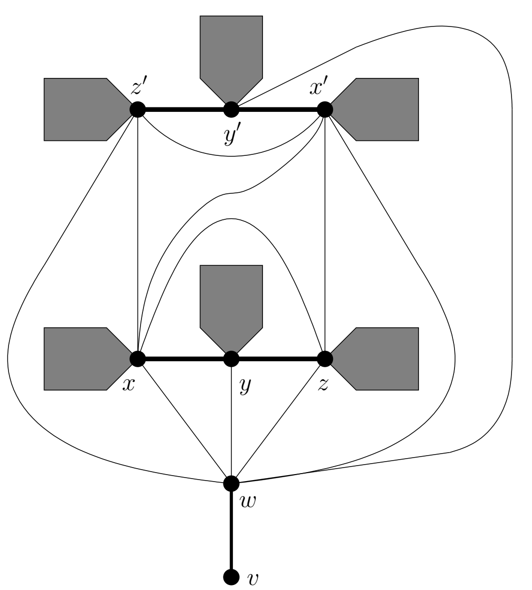

Figure 7: The forcing gadget with head v. (Edges of $E(F)$ are in bold, no-3-gadgets are represented by their symbols.)

The forcing gadget is the one depicted in Figure 7.

Claim 24. *In any* 2*-backbone* 5*-colouring of a forcing gadget, its head is coloured in* $\{1, 5\}$.

Proof. Consider a forcing gadget, whose vertices are named as in Figure 7, and ϕ a 2-backbone 5-colouring of it.

Let us prove that $\phi(w) = 3$ and so that $\phi(v) \in \{1, 5\}$. Assume for a contradiction that $\phi(w) \neq 3$. Without loss of generality, we may assume that $\phi(w) \in \{1, 2\}$.

Observe that the vertices x, y, z, x', y', z' are not assigned 3 because they are roofs of no-3-gadgets.

If $\phi(w) = 1$, then $(\phi(x), \phi(y), \phi(z))$ and $(\phi(x'), \phi(y'), \phi(z'))$ is either $(4, 2, 5)$ or $(5, 2, 4)$. Hence the vertices x, x' and z are all coloured in $\{4, 5\}$, which is impossible, since they form a triangle.

If $\phi(w) = 2$, then $(\phi(x), \phi(y), \phi(z))$ and $(\phi(x'), \phi(y'), \phi(z'))$ is either $(4, 1, 5)$ or $(5, 1, 4)$. Hence the vertices x, x' and z are all coloured in $\{4, 5\}$, which is impossible, since they form a triangle. □

To get the equivalence between the 3-colourability of the original graph H and the existence of a 2-backbone 5-colouring of (G, F), it remains to prove that for any $\alpha \in \{1, 5\}$, there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured α.

We denote the vertices by their names in Figure 7. Set $\phi(w) = 3$, $\phi(x) = \phi(y') = 1$, $\phi(y) = \phi(z') = 5$, $\phi(z) = 2$ and $\phi(x') = 4$.

Observe that no vertex in $\{x, y, z, x', y', z'\}$ has been coloured 3. Hence, it remains to prove that for any $\beta \in \{1, 2, 4, 5\}$, there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured β. By the symmetry $\phi \to 6 - \phi$, it suffices to prove that one exists for $\beta \in \{1, 2\}$. We denote the vertices by their names in Figure 6. Let us denote by $\bar{\beta}$ the colour of $\{1, 2\} \setminus \{\beta\}$.

$\phi(a) = 3$, $\phi(b) = \bar{\beta}$, $\phi(c) = 5$, $\phi(d) = 4$, $\phi(e) = \beta$, $\phi(f) = \bar{\beta}$, $\phi(a') = 3$, $\phi(b') = 5$, $\phi(c') = \bar{\beta}$, $\phi(d') = 5$, $\phi(e') = \beta$, $\phi(f') = 3$, $\phi(g) = 1$, $\phi(h) = 5$, $\phi(i) = 3$. □

2.3.2 Hamiltonian-path backbone

Theorem 25. *The following problem is NP-complete.*
Input: A planar graph G *with a hamiltonian path* P*.*
Question: $\mathrm{BBC}_2(G, P) \leq 5$*?*

To prove this theorem, we shall use a reduction similar to the one of Theorem 21. However, we do not reduce directly from PLANAR 3-COLOURABILITY but use an intermediate problem whose NP-completeness is proven by reducing PLANAR 3-COLOURABILITY to it.

This intermediate problem is the following:
TRACEABLE PLANAR 3-COLOURABILITY
Input: A planar graph G with a hamiltonian path P.
Question: Is G 3-colourable?

Lemma 26. TRACEABLE PLANAR 3-COLOURABILITY *is NP-complete.*

Proof. Reduction from PLANAR 3-COLOURABILITY. Let H be a connected planar graph. We will construct a planar graph G having a hamiltonian path P such that $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

To do so, we shall construct a sequence of pairs (G_i, P_i) for $1 \leq i \leq |V(H)|$ such that P_i is a path in the planar connected graph G_i, $|V(P_i)| = |V(G_i)| - |V(H)| + i$, and $\chi(G_i) \leq 3$ if and only if $\chi(H) \leq 3$. Then the path $P := P_{V(H)}$ will be a hamiltonian path of $G := G_{V(H)}$ and $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

Let x be a vertex of H. We set $G_1 := H$ and $P_1 := (x)$. Trivially, (G_1, P_1) verifies the above property.

Assume now that $i \geq 1$ and let us construct (G_{i+1}, P_{i+1}) from (G_i, P_i). Let $P_i = (v_1, v_2, \dots, v_\ell)$ be a path. Since G_i is connected, there exists j such that v_j is adjacent to a vertex y in $V(G_i) \setminus V(P_i)$. If $j = 1$, then let $P_{i+1} := (y, v_1, v_2, \dots, v_\ell)$, and $G_{i+1} := G_i$; if $j = p$, then let $P_{i+1} := (v_1, v_2, \dots, v_\ell, y)$, and $G_{i+1} := G_i$; if y is also incident to v_{j+1}, let $P_{i+1} := (v_1, \dots, v_j, y, v_{j+1}, \dots, v_\ell))$. In those three cases, (G_{i+1}, P_{i+1}) has trivially the desired property.

So we may assume that $1 < j < \ell$ and y is not adjacent to v_{j+1}. Let $y_1, y_2, \dots y_r$ be the neighbours of v_j in their order around it such that $v_{j+1} = y_r$, $y_k = y$ and $v_{j-1} = y_q$ for $q < r$.

Let G_{i+1} be the graph obtained from G_i as follows. For all $1 \leq s \leq k-1$, remove the edge $v_j y_s$, add three vertices a_s, b_s, c_s and the edges $a_s b_s, b_s c_s, c_s a_s, v_j a_s, v_j b_s, b_s y_s$; Add the edges $c_s a_{s+1}$ for all $1 \leq s \leq k-2$, and $v_{j+1} a_1$. Finally add a vertex y' and the edges yy' and $y' c_{k-1}$. Let P_{i+1} be the path obtained from P_i by replacing the edge $v_j v_{j+1}$ by the subpath $(v_j, y, c_{k-1}, b_{k-1}, a_{k-1}, \dots, c_1, b_1, a_1, v_{j+1})$. See Figure 8, which illustrates the construction when $k = 5$.

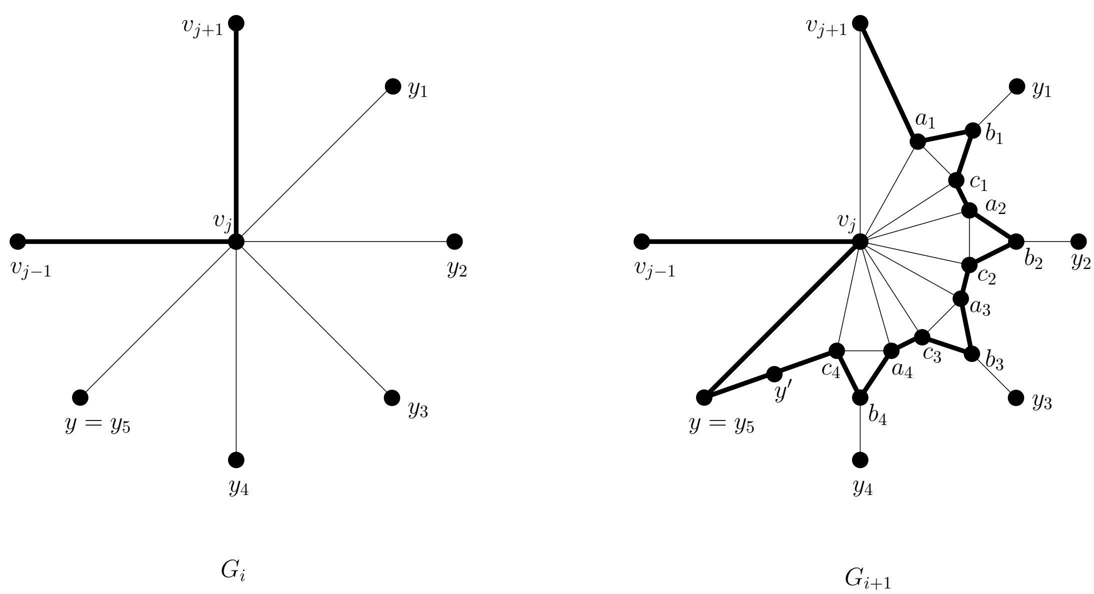

Figure 8: Constructing (G_{i+1}, P_{i+1}) from (G_i, P_i) (Egdes of the paths are in bold.)

Clearly, the number of vertices not covered by P_{i+1} in G_{i+1} is one less than the number of vertices not covered by P_i in G_i. So, since $|V(P_i)| = |V(G_i)| - |V(H)| + i$, we have $|V(P_{i+1})| = |V(G_{i+1})| - |V(H)| + i + 1$.

It remains to prove that G_{i+1} is 3-colourable if and only if G_i is.

Assume first that G_{i+1} admits a proper 3-colouring ϕ in $\{1, 2, 3\}$. We claim that it also induces a proper 3-colouring of G_i. Indeed, without loss of generality, we may assume that $\phi(v_j) = 1$ and $\phi(v_{j+1}) = 2$. Then for all $1 \leq s \leq k-1$, $\phi(a_s) = 3$ and $\phi(c_s) = 2$, so $\phi(b_s) = 1$. Hence $\phi(y_s) \neq 1$. Therefore, for all $1 \leq s \leq k-1$, $\phi(y_s) \neq \phi(v_j)$. Since the $v_j y_s$, $1 \leq s \leq k-1$, are the only edges of G_i which are not in G_{i+1}, ϕ is a proper 3-colouring of G_i.

Conversely, assume that G_i admits a 3-colouring ϕ in $\{1, 2, 3\}$. It induces a partial proper 3-colouring of G, such that $\phi(v_j) \neq \phi(y_s)$ for all $1 \leq s \leq k-1$. Let us extend it. Without loss of generality, $\phi(v_j) = 1$ and $\phi(v_{j+1}) = 2$. For all $1 \leq s \leq k-1$, set $\phi(a_s) = 3$, $\phi(b_s) = 1$, and

$\phi(c_s) = 2$. Finally, colour y' with the colour in $\{1,2,3\} \setminus \{\phi(y), \phi(c_{k-1})\}$. This gives a proper 3-colouring of G_{i+1}. □

Proof of Theorem 25. Reduction from TRACEABLE PLANAR 3-COLOURABILITY. Let (H, Q) be an instance of this problem. We shall construct a graph G and a hamiltonian path P of G such that $\mathrm{BBC}_2(G, P) \leq 5$ if and only if $\chi(H) \leq 3$. To do so we start from H and for each edge xy of Q, we will plug in an edge gadget $E(xy)$ containing a hamiltonian path $P(xy)$ from x to y. The union of all the $P(xy)$, $xy \in E(Q)$, will then be a hamiltonian path P of the resulting graph G.

To construct the edge gadget, we use an auxiliary forcing gadget depicted in Figure 9. The *head* of such a gadget is the vertex denoted by v in the figure. Its *fringes* are the vertices denoted by a and e.

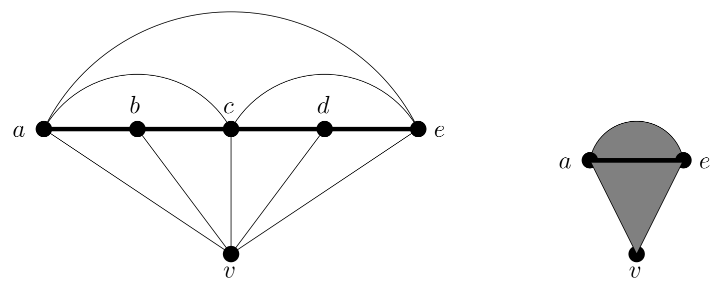

Figure 9: The forcing gadget with head v and fringes a and e (left) and its symbol (right)

Claim 27. *In any 2-backbone 5-colouring of a forcing gadget, the head is coloured in $\{2, 4\}$.*

Proof. We denote the vertices by their names in Figure 9. Suppose for a contradiction that there is a 2-backbone 5-colouring ϕ such that $\phi(v) \notin \{2, 4\}$. By the symmetry $\phi \to 6 - \phi$, we may assume that $\phi(v) \in \{1, 3\}$.

If $\phi(v) = 3$, then all the vertices a, b, c, d, e are coloured in $\{1, 2, 4, 5\}$. On the path (a, b, c, d, e), vertices coloured $\{1, 2\}$ alternate with vertices coloured $\{4, 5\}$. Hence a, c, and e are all coloured in $\{1, 2\}$, or all coloured in $\{4, 5\}$, which is a contradiction as they form a clique.

If $\phi(v) = 1$, then all the vertices a, b, c, d, e are coloured in $\{2, 3, 4, 5\}$. Now $\phi(b)$ is at distance 2 from the two distinct colours $\phi(a)$ and $\phi(c)$, hence $\phi(b) \in \{2, 5\}$. Similarly, $\phi(d) \in \{2, 5\}$. But $\phi(c)$ is at distance 2 from $\phi(b)$ and $\phi(d)$, so $\phi(b) = \phi(d)$. Then the three vertices a, c, and e are all coloured in $\{2, 3, 4, 5\} \setminus \{\phi(b) - 1, \phi(b), \phi(b) + 1\}$, which has cardinality 2. This is a contradiction as those three vertices form a clique. □

Now the edge gadget is the one depicted in Figure 10.

Let us now prove that $\mathrm{BBC}_2(G, P) \leq 5$ if and only if $\chi(H) \leq 3$.

Assume first that (G, P) admits a 2-backbone 5-colouring ϕ. Since H is a subgraph of G, ϕ induces a proper colouring on H. We shall prove that every vertex of H is coloured in $\{1, 3, 5\}$, thus proving that this proper colouring uses (at most) 3 colours.

Every vertex v of H is contained in an edge xy of Q, so it is contained in the edge gadget $E(xy)$ in G. So it is adjacent to two vertices (namely v_1 and v_2 if $v = x$, and v_2 and v_3 if $v = y$), which are heads of forcing gadgets and adjacent. Hence by Claim 27, one of these vertices is coloured 2 and the other is coloured 4. Hence v must be coloured in $\{1, 3, 5\}$.

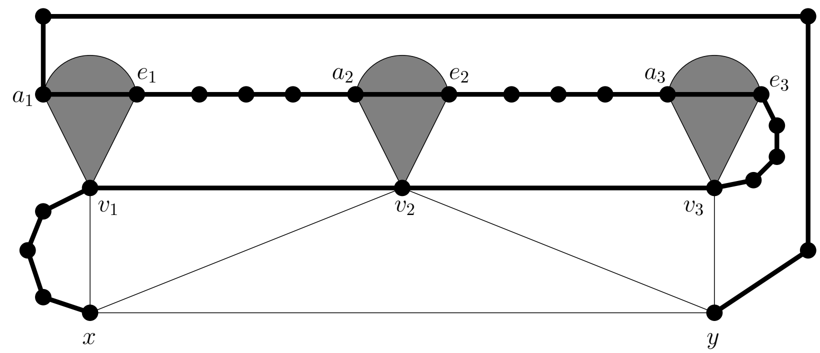

Figure 10: The edge gadget $E(xy)$ and its hamiltonian path $P(xy)$ in bold (Forcing gadgets are represented by their symbols.)

Let us now assume that H is 3-colourable. Then there exists a proper colouring ϕ of H with $\{1, 3, 5\}$. Let us now extend into a 2-backbone 5-colouring of (G, P). It is sufficient to prove that we can extend it to every edge-gadget.

To extend it to the edge-gadget $E(xy)$ (we use the names of Figure 10), set $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = 4$. Now, since for any pair $(\alpha, \beta) \in \{1, 2, 3, 4, 5\}^2$, there is a 2-backbone 5-colouring of the path of length 4 such that the first vertex is coloured α and the last vertex is coloured β, it suffices to prove that we can extend ϕ to the forcing gadget.

Consider such a forcing gadget (with vertex names as in Figure 9). Then $\phi(v) \in \{2, 4\}$. By the symmetry $\phi \to 6 - \phi$, we may assume that $\phi(v) = 2$. Then setting $\phi(a) = 4$, $\phi(b) = \phi(d) = 1$, $\phi(c) = 3$ and $\phi(e) = 5$, we obtain the desired extension.

Hence, $\mathrm{BBC}_2(G, P) \leq 5$. □

2.4 q-backbone $(q + 3)$-colouring for $q \geq 3$

Theorem 28. *For any $q \geq 3$, the following problem is NP-complete.*
Input: A planar graph G and a galaxy F in G with maximum degree 3.
Question: Is $\mathrm{BBC}_q(G, F) \leq q + 3$?

Proof. Reduction from PLANAR 3-COLOURABILITY.

We shall need the graph, which we call a *kite*, depicted in Figure 11. The vertex named t in the figure is the *tip* of the kite, and the one named u is its *corner*.

Claim 29. *If ϕ is a q-backbone $(q+3)$-colouring of a kite such that $\phi(t) \in \{1, 2, 3, q+1, q+2, q+3\}$, then either $\phi(t) \in \{1, 2, 3\}$ and $\phi(u) = q + 3$, or $\phi(t) \in \{q + 1, q + 2, q + 3\}$ and $\phi(u) = 1$.*

Proof. Observe that the vertices v, z_1, z_2, z_3 are harnesses of parachutes. Thus, by Proposition 14-(i), they must be assigned 1 or $q + 3$.

Assume that $\phi(v) = 1$, then $\phi(z_1) = \phi(z_2) = \phi(z_3) = q+3$. Thus $\{\phi(s_1), \phi(s_2)\} = \{q+1, q+2\}$ and so $\phi(u) = q + 3$ and $\phi(t) \in \{1, 2, 3\}$.

Similarly if $\phi(v) = q + 3$, we obtain $\phi(u) = 1$ and $\phi(t) \in \{q + 1, q + 2, q + 3\}$. □

Let H be a planar graph. Let (G, F) be the graph pair obtained from H as follows. Firstly, for each face f of H, we create a parachute P_f with harness v_f, and for each vertex x incident to f, we

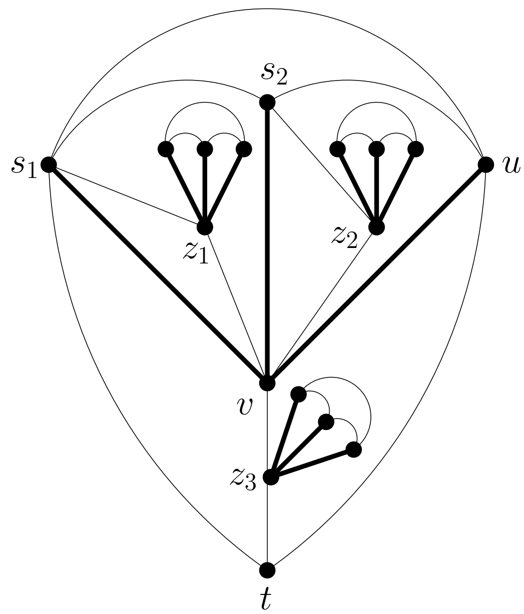

Figure 11: The kite

create a kite $K_f(x)$ with tip x and corner $u_f(x)$. We then link the vertex v_f to all the $u_f(x)$. Secondly, for every vertex $x \in V(H)$, we add a vertex y_x and the edge xy_x in the backbone.

Clearly, the resulting graph G is planar and the resulting backbone F is a galaxy with maximum degree 3.

Let us now prove that $BBC_q(G, F) \leq q + 3$ if and only if H is 3-colourable.

Assume first that (G, F) admits a q-backbone $(q + 3)$-colouring ϕ. Observe that each vertex x in $V(H)$ is coloured in $\{1, 2, 3, q + 1, q + 2, q + 3\}$, because it is adjacent to y_x in F.

Let x be a vertex in $V(H)$. Free to consider $q + 4 - \phi$, we may assume that $\phi(x) \in \{1, 2, 3\}$. Consider a face f incident to x in H. By Claim 29, the kite $K_f(x)$ has its corner coloured $q + 3$. Together with Proposition 14-(i), this implies that $\phi(v_f) = 1$. Thus, the corner of the kites in f cannot be coloured 1, therefore there are coloured $q + 3$ and so by Claim 29, all the vertices incident to f in H are all coloured in $\{1, 2, 3\}$. Applying this reasoning to each face of H, we obtain that all vertices of H are coloured in $\{1, 2, 3\}$. Hence, ϕ induces a proper 3-colouring on H.

Conversely, assume that H admits a proper 3-colouring c. One can extend it into a q-backbone $(q + 3)$-colouring of (G, F) as follows. For every $x \in V(H)$, we colour y_x with $q + 3$; for every face f, we colour the vertex v_f with 1 and the corners of the kites by $q + 3$. One can then extend the colouring to each kite (as in the proof of Claim 29) to obtain a q-backbone $(q + 3)$-colouring of (G, F). $\square$

The reduction above can be modified to have a spanning tree T for the backbone in place of the galaxy F. It suffices consider a spanning tree U of H and do the following: add a path of length two in the backbone along each edge of the tree U; for each kite, add tz_3 and vz_3 in the backbone and add paths of length two in the backbone along edges z_1v and z_2v. This will prove the following statement.

Theorem 30. *The following problem is NP-complete.*
Input: A planar graph G and a spanning tree T of G.

Question: *Is* $\mathrm{BBC}_q(G,T) \leq q+3$*?*

2.5 q-backbone $(q+5)$-colouring

Theorem 31. *For any* $q \geq 4$*, the following problem is NP-complete.*
Input: A planar graph G and a spanning tree T *of G.*
Question: *Is* $\mathrm{BBC}_q(G,T) \leq q+5$*?*

Proof. Reduction from PLANAR 3-COLOURABILITY.

Let H be a planar graph. We shall construct a planar graph G together with a spanning tree T such that H is 3-colourable if and only if $BBC_q(G,T) \leq q+5$. Take U be a spanning tree of H.

We first construct a graph G' from H by adding for every edge $e = uv$ of U a vertex x_e linked to u and v. We let T' be the spanning tree of G' induced by the new edges. The pair (G,T) is then obtained from (G',T') by adding a parachute on every vertex. Clearly G is planar as for each edge $e = uv$ the path ux_ev can be drawn along the edge uv.

Suppose that (G,T) admits a q-backbone $(q+5)$-colouring. Then by Proposition 14-(iii), every vertex in G' is coloured in $\{1,2,3,q+3,q+4,q+5\}$. Note that the vertices of H form one of the part of the bipartition of T'. Hence, the colours of the vertices of H are either all in $\{1,2,3\}$ or all in $\{q+3,q+3,q+5\}$. In both cases, ϕ induces a proper 3-colouring on H.

Conversely, it is straightforward to extend a a proper 3-colouring of H into a q-backbone $(q+5)$-colouring of (G,T). □

3 Circular backbone colouring

The following Proposition is an analogue to Proposition 13 and its proof is similar.

Proposition 32. *Let G be a graph and let H be a subgraph of G such that* $2 \leq \chi(H) < \chi(G)$*. Then* $\mathrm{CBC}_q(G,H) \leq (\chi(G)+q-2)\chi(H)$*.*

Proof. Let g be a proper $\chi(G)$-colouring of G and h a proper $\chi(H)$-colouring of H.

Assume first that $\chi(H)$ is even. Let f be the colouring defined by:

$$f(v) \ = \ \begin{cases} (h(v)-1)(q-2+\chi(G))+g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v)-1)(q-2+\chi(G))+\chi(G)+1-g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

Let us check that f is a circular q-backbone $((\chi(G)+q-2)\chi(H))$-colouring of (G,H). For $1 \leq i \leq \chi(H)$, let $I_i = \{(i-1)(q-2+\chi(G))+1, \ldots, (i-1)(q-2+\chi(G))+\chi(G)\}$. Observe that if $h(v)=i$, then $f(v) \in I_i$. The I_i form intervals of $\mathbb{Z}_{(\chi(G)+q-2)\chi(H)}$. These intervals do not intersect and two consecutives intervals are separated by $q-2$ elements. In particular, if $h(u) \neq h(v)$, then $|f(u)-f(v)| \geq q-1$. Moreover $|f(u)-f(v)| = q-1$ only if $g(u)=g(v)$.

Consider an edge $uv \in E(G)$. By the previous remark, if $h(u) \neq h(v)$, then $f(u) \neq f(v)$. If $h(u)=h(v)$, then $|f(u)-f(v)| = |g(u)-g(v)| \neq 0$, because g is proper.

Consider now an edge $uv \in E(H)$. Then $h(u) \neq h(v)$, so $f(u)$ and $f(v)$ are in different I_i. If they are in non-consecutive I_i (modulo $\chi(H)$), then $|f(u)-f(v)| \geq 2q-2+\chi(G) \geq q$. Assume now that they are in consecutive intervals, then $|f(u)-f(v)| \geq q$ because $g(u) \neq g(v)$.

Assume now that $\chi(H)$ is odd. Let f be the colouring defined by:

$$f(v) = \begin{cases} 1, & \text{if } h(v) = 1 \text{ and } g(v) = \chi(G), \\ g(v) + 1, & \text{if } h(v) = 1 \text{ and } g(v) < \chi(G), \\ \chi(G) + q - 1, & \text{if } h(v) = 2 \text{ and } g(v) = \chi(G) - 1, \\ \chi(G) + q, & \text{if } h(v) = 2 \text{ and } g(v) = \chi(G), \\ 2\chi(G) + q - 1 - g(v), & \text{if } h(v) = 2 \text{ and } g(v) < \chi(G) - 1, \\ (h(v) - 1)(q - 2 + \chi(G)) + g(v), & \text{if } h(v) \text{ is odd and } h(v) > 2, \\ (h(v) - 1)(q - 2 + \chi(G)) + \chi(G) + 1 - g(v) & \text{if } h(v) \text{ is even and } h(v) > 2. \end{cases}$$

Similarly to the even case, one can check that f is a circular q-backbone $((\chi(G) + q - 2)\chi(H))$-colouring of (G, H). □

3.1 Planar graphs of girth at least 5

Theorem 33. *Let G be a planar graph of girth at least 5 and M a matching in G. Then $\mathrm{CBC}_q(G, M) \leq 2q + 1$.*

Proof. Our proof is based on a structural result of Borodin and Glebov [1]. See also [9].

Theorem 34 (Borodin and Glebov [1])**.** *The vertex set of every planar graph of girth at least 5 can be partitioned into an independent set and a set which induces a forest.*

Let (S, F) be a partition of $V(G)$ such that S is stable and F induces a forest. Let us colour every vertex of S with 1. Now since F is a forest, it has an ordering $v_1, \ldots, v_p$ such that for every i, v_i has at most one neighbour in $\{v_1, \ldots, v_{i-1}\}$. We colour the vertices of F according to this ordering as follows. If v_i has no neighbour in $\{v_1, \ldots, v_{i-1}\}$, then colour it with $q + 1$. If v_i has a neighbour u in $\{v_1, \ldots, v_{i-1}\}$ and $uv_i \notin E(M)$, then colour it with a colour of $\{q + 1, q + 2\}$ not assigned to u. If v_i has a neighbour u in $\{v_1, \ldots, v_{i-1}\}$ and $uv_i \in E(M)$, then assign $2q + 1$ (resp. 2 to v_i) if u is coloured $q + 1$ (resp. $q + 2$). It is easy to check that the obtained colouring is a q-backbone $\mathbb{Z}_{2q+1}$-colouring of (G, M). □

3.2 Circular q-backbone $2q$-colouring

Proposition 35. *Let G be a graph and H a spanning connected subgraph of G. Then $\mathrm{CBC}_q(G, H) = 2q$ if and only if G is bipartite.*

Proof. If G is bipartite, then $\chi(G) = \chi(H) = 2$. Thus, by Equation (3), $\mathrm{CBC}_q(G, H) = 2q$.

Assume now that (G, H) admits a circular q-backbone $2q$-colouring f. Let v be a vertex of G. Without loss of generality, we may assume that $f(v) = 1$. Then all the neighbours of v in H must be coloured $q + 1$. And so on, by induction, all the vertices at even distance from v in H are coloured 1 and all the vertices at odd distance from v in H are coloured $q + 1$. Since H is connected and spans G, it follows that all vertices are coloured 1 or $q + 1$, so G is bipartite. □

Proposition 35 implies that given a graph G and a spanning connected subgraph H, deciding if $\mathrm{CBC}_q(G, H) = 2q$ can be done in polynomial time. In contrast, if the condition of G be connected is removed, when $q = 2$, the problem becomes NP-complete, as shown by the following theorem.

Theorem 36. *The following problem is NP-complete.*
Input: A planar graph G and a matching M in G.
Question: Is $\mathrm{CBC}_2(G, M) \leq 4$?

Proof. The problem is trivially in NP since a circular 2-backbone 4-colouring of (G, F) is clearly a certificate.

To prove it is NP-complete, we give a reduction from NOT-ALL-EQUAL 3SAT.

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a collection of clauses of size three over a set U of variables. We will construct a graph pair (G, M) such that M is a matching in G.

To do so we need some definitions and gadgets.

Colours 1 and 3 are said to be *twins* and so do the colours 2 and 4. Trivially two vertices joined by an edge of M receives distinct twin colours. Two colours are *siblings* if they are equal or twins.

A *link* with ends u and v and central edge w_1w_2 is a subgraph with vertex set $\{u, v, w_1, w_2\}$ and edge set $\{uw_1, uw_2, vw_1, vw_2, w_1w_2\}$ with $w_1w_2 \in M$. Two ends of a link are said to be *linked*.

Claim 37. *In a circular* 2*-backbone* 4*-colouring c, the colours of the ends of a link are siblings.*

Proof. The two vertices w_1 and w_2 are joined by an edge of M, so $\{c(w_1), c(w_2)\} \in \{\{1, 3\}; \{2, 4\}\}$. Hence if u is coloured in $\{1, 3\}$ (resp. $\{2, 4\}$), then $\{c(w_1), c(w_2)\}$ is $\{2, 4\}$ (resp. $\{1, 3\}$), and so v is coloured in $\{1, 3\}$ (resp. $\{2, 4\}$). □

For each variable $u \in U$, we create a *variable gadget* G^u which is obtained from the distinct vertices $a^u_1, a^u_2, \ldots, a^u_n$ by linking, from $1 \leq i \leq n-1$, the vertices a^u_i and a^u_{i+1} by an link with central edge $b^u_i c^u_i$.

Claim 37 (and its proof) immediately implies the following.

Claim 38. *In a circular* 2*-backbone* 4*-colouring of G^u, all the a^i_u are coloured with two sibling colours and all the b^i_u with the two other colours (which are also siblings).*

For each clause $C_i = \ell^i_1 \vee \ell^i_2 \vee \ell^i_3$, we create a triangle $z^i_1 z^i_2 z^i_3$. Now for $j = 1, 2, 3$, if ℓ^i_j is the nonnegated literal u, we join z^i_j with a^u_i, and if ℓ^i_j is the negated literal $\bar{u}$, we join z^i_j with b^u_i. Such edges are said to be red. So far, the obtained graph H is not planar, but we can clearly draw it such that only red edges cross. We can now subdivide every red edge into a red path such that every edge is crossed at most once. We then replace the red edges which are not crossed by a link (with the same end) and two red edges uv and xy that cross each other by the crossing gadget depicted in Figure 12. The resulting graph G is planar and it comes with a matching M.

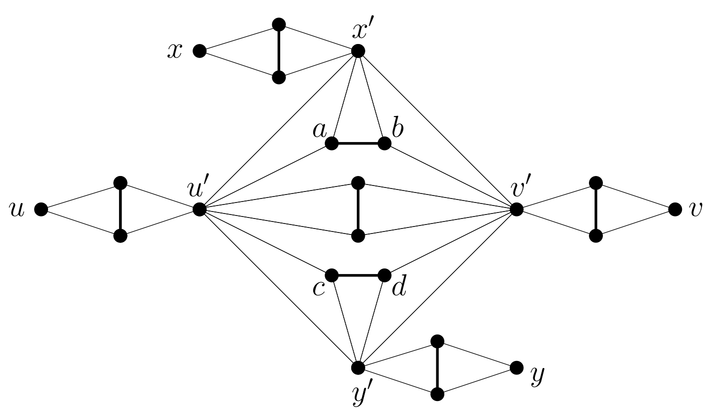

Figure 12: The crossing gadget

Claim 39. *In a circular* 2*-backbone* 4*-colouring of the crossing gadget, the colours of* u *and* v *are siblings and the colours of* x *and* y *are siblings. In addition, for any* 4*-tuple* $\{c_u, c_v, c_x, c_y\}$ *such that* c_u *and* c_v *are siblings and* c_x *and* c_y *are siblings, there is a circular* 2*-backbone* 4*-colouring* c *of the crossing gadget such that* $c(u) = c_u$*,* $c(v) = c_v$*,* $c(x) = c_x$*, and* $c(y) = c_y$*.*

Proof. Consider first a circular 2-backbone 4-colouring of the crossing gadget. u is linked to u', which is linked to v', which in turn is linked to v. Hence, by Claim 37, the colours of u and v are siblings.

Assume that x is coloured in $\{1, 3\}$, then x' is also coloured in $\{1, 3\}$, say 1. The vertices a and b are assigned twin colours, so one is coloured 2 and the other 4. We now distinguish two cases depending on the colour of u'.

1. Assume u' is coloured 3. Then v' must also be coloured 3. The vertices c and d are assigned twin colours, so one is coloured 2 and the other 4. Hence y' is coloured 1.

2. Assume u' is coloured in $\{2, 4\}$. Without loss of generality, we may assume it is coloured 2. Then a is coloured 4 and b is coloured 2, so v' is coloured 4. Hence y' is coloured in $\{1, 3\}$.

In both cases the colour of x and y' are siblings, and so, by Claim 37, the colours of x and y are siblings.

For any 4-tuple $\{c_u, c_v, c_x, c_y\}$ such that c_u and c_v are siblings and c_x and c_y are siblings, finding the desired circular 2-backbone 4-colouring is straightforward and left to the reader. □

We shall now prove that $\mathcal{C}$ admits a suitable truth assignment if and only if $\mathrm{CBC}_2(G, M) \leq 4$.

Assume first that (G, M) admits a circular 2-backbone 4-colouring. Let ϕ be the truth assignment defined by $\phi(u) = true$ if all the a_u^i are coloured in $\{1, 3\}$, and $\phi(u) = false$ if all the a_u^i are coloured in $\{2, 4\}$. Note that is well defined by Claim 38. Now by Claims 37 and 39, for each clause $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$, the vertex z_j^i is coloured in $\{1, 3\}$ if and only if the literal ℓ_j^i is true. But since $z_1^i z_2^i z_3^i$ is a triangle, at least three colours must appear on these vertices, and so at least one from $\{1, 3\}$ and at least one from $\{2, 4\}$. Hence, at least one of the literals of C_i is true and at least one is false. Thus ϕ is suitable.

Reciprocally, assume that $\mathcal{C}$ admits a suitable truth assignment ϕ. If $\phi(u) = true$, then colour all the a_u^i with 1, all the b_u^i with 2 and all the c_u^i with 4. And if $\phi(u) = false$, then colour all the a_u^i with 2, all the b_u^i with 1 and all the c_u^i with 3. Now, for each clause $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$, some literal, say ℓ_1^i, is true and some literal, say ℓ_3^i, is false. Then assign 1 to z_1^i, 2 to z_3^i, and colour z_2^i with 3 if ℓ_2^i is true and 4 otherwise. By Claims 37 and 39, this partial colouring may be extended into a circular 2-backbone 4-colouring of (G, M). □

Theorem 40. *The following problem is NP-complete.*
Input: A planar graph G *and a matching* M *in* G*.*
Question: Is $\mathrm{CBC}_2(G, M) \leq 5$*?*

Proof. The reduction is from PLANAR C_5-COLOURING which is defined as follows:
Input: A planar graph G.
Question: Does G have a homomorphism onto C_5, the cycle of length 5?
This was proved to be NP-complete by MacGillivray and Siggers [10].

To make the reduction we need an *edge gadget*. This gadget is built from the planar graph $H_1(u, v)$ together with the matching $M_1(u, v)$ depicted in Figure 13.

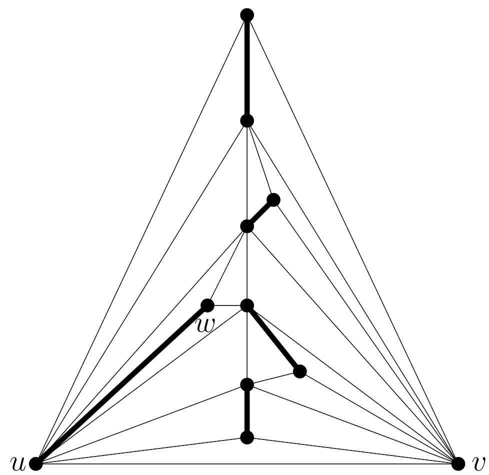

Figure 13: Graph $H_1(u, v)$ with matching $M_1(u, v)$ (in bold)

The graph $H_2(u, v)$ is obtained from $H_1(u, v)$ by replacing the edge uw by $H_1(w, u)$. The matching $M_2(u, v)$ is then the union of $M_1(u, v) \setminus \{uw\}$ and $M_1(w, u)$. Observe that u and v are incident to no edges of $M_2(u, v)$. The pair $(H_2(u, v), M_2(u, v))$ is the edge gadget.

Broersma et al. [5] proved that in any circular 2-backbone 5-colouring of $(H_1(u, v), M_1(u, v))$, vertices u and v receive colours which are cyclically 2 apart. In addition, it is straightforward to see that any precolouring of u and v with colours that are cyclically 2 apart can be extended into a circular 2-backbone 5-colouring of $(H_1(u, v), M_1(u, v))$. These two facts imply the following claim.

Claim 41.

(i) In any circular 2*-backbone* 5*-colouring of* $(H_2(u, v), M_2(u, v))$*, vertices* u *and* v *receive colours which are cyclically 2 apart.*

(ii) Any precolouring of u *and* v *with colours that are cyclically 2 apart can be extended into a circular* 2*-backbone* 5*-colouring of* $(H_2(u, v), M_2(u, v))$*.*

Let H be an instance of PLANAR C_5-COLOURING. Replace each edge $uv \in E(G)$ by an edge gadget $(H_2(u, v), M_2(u, v))$ to obtain a planar graph G and a matching M (the union of the $M_2(u, v)$). By Claim 41-(i), every circular 2-backbone 5-colouring of (G, M) induces a C_5-colouring of H (the vertices of the C_5 are the colours $(1, 3, 5, 2, 4)$). Conversely, by Claim 41-(ii), any C_5-colouring of H can be extended into a circular 2-backbone 5-colouring of (G, M). Hence H admits a C_5-colouring if and only if (G, M) admits a circular 2-backbone 5-colouring. □

Adding long paths along existing edges to transform the matching into a spanning tree, one derives the following:

Theorem 42. *The following problem is NP-complete.*
Input: A planar graph G *and a spanning tree* T *of* G*.*
Question: Is $\mathrm{CBC}_2(G, T) \leq 5$*?*

4 Further research

Campos et al. [6] proved that if G is planar and T has diameter at most 3, then $\mathrm{BBC}_2(G,T) \leq 5$. Hence one can the find the 2-backbone chromatic number of such a pair in polynomial time. One can ask of the complexity for larger diameter.

Problem 43. *For a fixed $d \geq 4$, what is the complexity of finding the 2-backbone chromatic number of (G,T), when G is planar and T a spanning tree of diameter d ?*

Since, for any fixed $k \leq 4$, deciding if the 2-backbone chromatic number of (G,T) is at most k can be done in polynomial time, if Conjecture 1 holds, Problem 43 is equivalent to finding the complexity of deciding if $\mathrm{BBC}_2(G,T) \leq 5$.

If G is a triangle-free planar graph, then, by Grötzsch's Theorem [8], it is 3-colourable, and so $\mathrm{BBC}_q(G,H) \leq 2q+1$ and $\mathrm{CBC}_q(G,H) \leq 3q$ for any subgraph H of G. Hence Conjecture 1 and Conjecture 6 for $q = 2$, hold when G is triangle-free. A natural next step would be to prove Conjecture 6 for values of q larger than 2 when G is triangle-free.

Steinberg's Conjecture (1976) states that every planar graph without 4- and 5-cycles is 3-colourable. Towards this, Erdős (1991) proposed the following relaxation of Steinberg's Conjecture: Determine the smallest value of k, such that every planar graph without cycles of length from 4 to k is 3-colourable. The best known bound for such a k is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [2]. Hence, an evidence to both Conjecture 6 and Steinberg's Conjecture would be to prove the following:

Conjecture 44. *If G is a planar graph without 4- and 5-cycles and F a spanning forest of G, then $\mathrm{CBC}_2(G,F) \leq 7$.*

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