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# (Circular) backbone colouring: forest backbones in planar graphs

Frédéric Havet\*    Andrew D. King†    Mathieu Liedloff‡    Ioan Todinca‡

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## Abstract

Consider an undirected graph  $G$  and a subgraph  $H$  of  $G$ , on the same vertex set. The  $q$ -backbone chromatic number  $\text{BBC}_q(G, H)$  is the minimum  $k$  such that  $G$  can be properly coloured with colours from  $\{1, \dots, k\}$ , and moreover for each edge of  $H$ , the colours of its ends differ by at least  $q$ . In this paper we focus on the case when  $G$  is planar and  $H$  is a forest. We give a series of NP-hardness results as well as upper bounds for  $\text{BBC}_q(G, H)$ , depending on the type of the forest (matching, galaxy, spanning tree). Eventually, we discuss a circular version of the problem.

## 1 Introduction

All the graphs considered in this paper are simple. Let  $G = (V, E)$  be a graph, and let  $H = (V, E(H))$  be a spanning subgraph of  $G$ , called the *backbone*. A  $k$ -colouring of  $G$  is a mapping  $f : V \rightarrow \{1, 2, \dots, k\}$ . Let  $f$  be a  $k$ -colouring of  $G$ . It is a *proper colouring* if  $|f(u) - f(v)| \geq 1$  for all edges  $uv \in E(G)$ . It is a  $q$ -backbone colouring for  $(G, H)$  if  $f$  is a proper colouring of  $G$  and  $|f(u) - f(v)| \geq q$  for all edges  $uv \in E(H)$ . The *chromatic number*  $\chi(G)$  is the smallest integer  $k$  for which there exists a proper  $k$ -colouring of  $G$ . The  $q$ -backbone chromatic number  $\text{BBC}_q(G, H)$  is the smallest integer  $k$  for which there exists a  $q$ -backbone  $k$ -colouring of  $(G, H)$ .

If  $f$  is a proper  $k$ -colouring of  $G$ , then  $g$  defined by  $g(v) = q \cdot f(v) - q + 1$  is a  $q$ -backbone colouring of  $(G, H)$  for any spanning subgraph  $H$  of  $G$ . Moreover it is well-known that if  $G = H$ , this  $q$ -backbone colouring of  $(G, H)$  is optimal. Therefore, since  $\text{BBC}_q(H, H) \leq \text{BBC}_q(G, H) \leq \text{BBC}_q(G, G)$ , we have

$$q \cdot \chi(H) - q + 1 \leq \text{BBC}_q(G, H) \leq q \cdot \chi(G) - q + 1. \quad (1)$$

If  $H$  is empty (i.e.  $E(H) = \emptyset$ ), then  $\text{BBC}_q(G, H) = \chi(G)$ . Hence for any  $k \geq 3$ , deciding if  $\text{BBC}_q(G, H) \leq k$  is NP-complete because deciding if a graph is  $k$ -colourable is NP-complete (See [7]). However, when we impose  $G$  or  $H$  to belong to certain graph classes, the problem sometimes become polynomial-time solvable. A trivial example is when we consider  $H$  with chromatic number at least  $r > (k + q - 1)/q$ . Then  $\text{BBC}_q(G, H) \geq rq - q + 1$ , and so deciding if  $\text{BBC}_q(G, H) \leq k$

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can be done instantly by always returning ‘no’. A less trivial example is when we impose  $H$  to have minimum degree 1. For such an  $H$ , deciding if  $BBC_q(G, H) \leq q + 1$  is also polynomial-time solvable, because  $BBC_q(G, H) = q + 1$  if and only if  $G$  is bipartite. This simple observation was already made by Broersma et al. [5] when  $H$  is a 1-factor (a spanning subgraph in which every vertex has degree exactly 1). Furthermore, if we also impose  $H$  to be connected, we show in Theorem 17 that deciding if  $BBC_q(G, H) \leq q + 2$  can be done in polynomial time. In contrast, if the condition of  $H$  being connected is removed, then it is NP-complete (Theorem 18).

In this paper, we will focus on the particular case when  $G$  is a planar graph and  $H$  is a forest (i.e. an acyclic graph). Inequality (1) and the Four-Colour Theorem imply that for any planar graph  $G$  and spanning subgraph  $H$ ,  $BBC_q(G, H) \leq 3q + 1$ . However, for  $q = 2$ , Broersma et al. [4] conjectured that this is not best possible if the backbone is a forest.

**Conjecture 1.** *If  $G$  is a planar graph and  $F$  a forest in  $G$ , then  $BBC_2(G, F) \leq 6$ .*

If true, Conjecture 1 would be best possible. Broersma et al. [4] gave an example of a graph  $\hat{G}$  with a forest  $\hat{F}$  such that  $BBC_2(\hat{G}, \hat{F}) = 6$ . See Figure 1. It is then natural to ask how large  $BBC_q(G, F)$

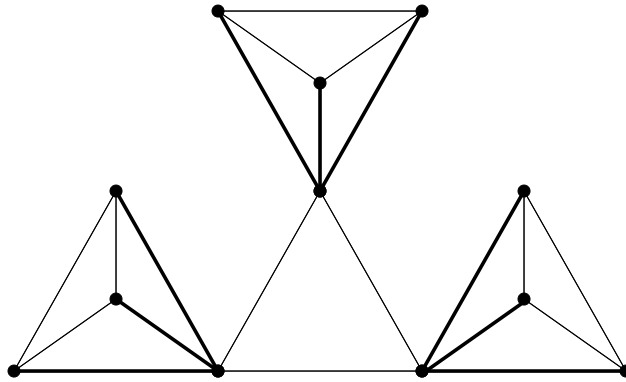


Figure 1: A planar graph  $\hat{G}$  with a forest  $\hat{F}$  (bold edges) such that  $BBC_q(\hat{G}, \hat{F}) = q + 4$ .

could be when  $G$  is planar and  $F$  is a forest for larger values of  $q$ . We prove the following.

**Theorem 2.** *If  $G$  is a planar graph and  $F$  a forest in  $G$ , then  $BBC_q(G, F) \leq q + 6$ .*

In fact, we prove a more general result in Proposition 13 : for any pair  $(G, H)$  with  $H$  a subgraph of  $G$ ,

$$BBC_q(G, H) \leq (\chi(G) + q - 2)\chi(H) - q + 2.$$

For  $q \geq 4$ , Theorem 2 is best possible. Indeed, we show a planar graph  $G^*$  together with a spanning tree  $T^*$  such that  $BBC_q(G^*, T^*) = q + 6$  for all  $q \geq 4$ . See Figure 2 and Proposition 15.

Furthermore, we show in Theorem 31, that for any fixed  $q \geq 4$ , given a planar graph  $G$  and a spanning tree  $T$  of  $G$ , it is NP-complete to decide if  $BBC_q(G, T) \leq q + 5$ .

On the other hand, we believe that if  $q = 3$ , Theorem 2 is not best possible.

**Conjecture 3.** *If  $G$  is a planar graph and  $F$  a forest in  $G$ , then  $BBC_3(G, F) \leq 8$ .*

If true, Conjecture 3 would be tight. The pair  $(G^*, F^*)$  of Figure 2 satisfies  $BBC_3(G^*, F^*) = 8$ . We show in Proposition 16 that Conjecture 1 implies Conjecture 3.

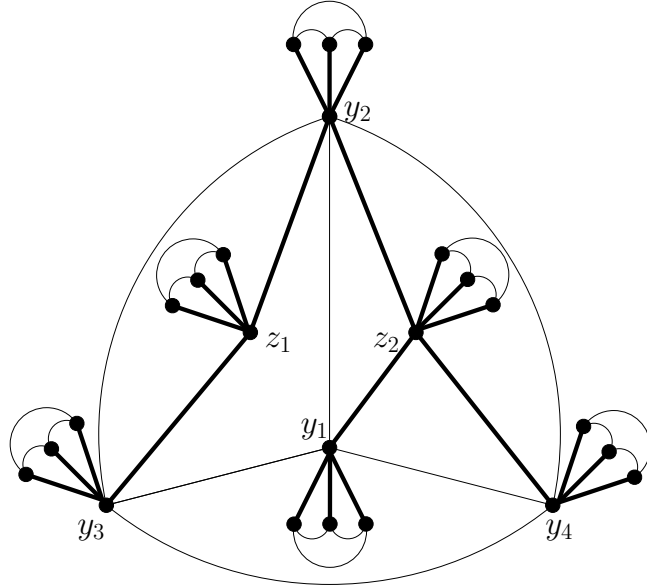


Figure 2: A planar graph  $G^*$  and a tree  $T^*$  (bold edges) such that  $\text{BBC}_q(G^*, T^*) = q + 6$  for  $q \geq 4$ .

A *star* is a tree in which a vertex  $v$ , called the *center* is adjacent to every other. A *galaxy* is a forest of stars. As evidence in support of Conjectures 1 and 3, Broersma et al. [5] showed that if  $F$  is a galaxy in a planar graph  $G$ , then  $\text{BBC}_q(G, F) \leq q + 4$ . This result is best possible even if  $F$  has maximum degree 3 as shown by the example of Figure 1. Furthermore, we show in Theorems 21 and 28 that, for any  $q \geq 2$ , it is NP-complete to decide if  $\text{BBC}_q(G, F) \leq q + 3$  given a planar graph  $G$  and a galaxy of maximum degree 3.

However, if the backbone is a *matching*, i.e. a galaxy with maximum degree 1, then fewer colours are needed. Indeed, Broersma et al. [5] showed that if  $M$  is a matching in a planar graph  $G$ , then for any  $q \geq 3$ ,  $\text{BBC}_q(G, M) \leq q + 3$ . They conjectured that the same holds for  $q = 2$ .

**Conjecture 4** (Broersma et al. [5]). *If  $G$  is a planar graph  $G$  and  $M$  a matching in  $G$ , then  $\text{BBC}_2(G, M) \leq 5$ .*

It is natural to ask the same question for galaxies with maximum degree at least 2. When  $q = 2$ , we answer in the negative by showing that there are pairs of planar graphs and spanning forests of maximum degree 2 whose 2-backbone chromatic number is 6. Furthermore, we show that given a planar graph  $G$  and a spanning forest  $F$  of maximum degree 2, it is NP-complete to decide whether  $\text{BBC}_2(G, F) \leq 5$  (Theorem 22). We also show that given a planar graph  $G$  with a hamiltonian path  $P$ , it is NP-complete to decide whether  $\text{BBC}_2(G, F) \leq 5$ . This result refines a result of Broersma et al. [3, 4] who proved it for a general graph  $G$ .

For  $q = 3$ , the problem remains open.

**Problem 5.** *If  $G$  is a planar graph  $G$  and  $F$  a galaxy of maximum degree 2, is it true that  $\text{BBC}_q(G, F) \leq q + 3$ , for all  $q \geq 3$ ?*

Broersma et al. [5] proved that deciding if  $\text{BBC}_q(G, M) \leq q + 2$  for a given graph  $G$  and matching  $M$  is NP-complete. We prove in Subection 2.2 that it remains NP-complete even if we impose  $G$  to be

	$G$ planar			
	$H$ forest	$H$ spanning tree	$H$ 1-factor	$H$ Hamilton path
$BBC_q(G, H) \leq q + 1?$	NP-C	poly	poly	poly
$BBC_q(G, H) \leq q + 2?$	NP-C	poly	NP-C	poly
$BBC_q(G, H) \leq q + 3?$	NP-C	NP-C	$q \geq 3$ : Yes $q = 2$ : ?Yes?	$q = 2$ : NP-C
$BBC_q(G, H) \leq q + 5?$		$q \geq 4$ : $q = 3$ : $q = 2$ :	NP-C ?Yes? Yes	
$BBC_q(G, H) \leq q + 6?$			Yes	

Table 1: Complexity of deciding if  $BBC_q(G, H) \leq q + k$  for  $k \in \{1, \dots, 6\}$ , when  $G$  is a planar graph and  $H$  a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.

planar. In contrast, we prove that deciding if  $BBC_q(G, T) \leq q + 2$  for a given graph  $G$  and spanning tree  $T$  is polynomial-time solvable.

Many of the complexity results on backbone colouring of planar graphs with a forest backbone are summarized in Table 1.

One can generalize the notion of backbone colouring by allowing a more complicated structure of the frequency space. The most natural one is to consider a circular metric. A *circular  $k$ -colouring* of  $G$  or  $\mathbb{Z}_k$ -*colouring* is a mapping  $f : V \rightarrow \mathbb{Z}_k$ . The notions of *circular  $q$ -backbone colouring* and *circular  $q$ -backbone chromatic number* are defined similarly to those of  *$q$ -backbone colouring* and  *$q$ -backbone chromatic number* by replacing colouring by circular colouring. The circular  $q$ -backbone chromatic number of a graph pair  $(G, H)$  is denoted  $CBC_q(G, H)$ .

If  $f$  is a circular  $q$ -backbone  $k$ -colouring, then the mapping  $f^*$  defined by  $f^*(v) = f(v) + 1$  for all vertex  $v$  is trivially a  $q$ -backbone  $k$ -colouring. On the other hand, a  $q$ -backbone  $k$ -colouring yields a circular  $q$ -backbone  $(k + q - 1)$ -colouring. Hence for every graph pair  $(G, H)$ , where  $H$  is a spanning subgraph of  $G$ , we have

$$BBC_q(G, H) \leq CBC_q(G, H) \leq BBC_q(G, H) + q - 1. \quad (2)$$

Also,

$$q \cdot \chi(H) \leq CBC_q(G, H) \leq q \cdot \chi(G). \quad (3)$$

Observe that if  $G$  is bipartite and  $H$  is non-empty, Equation (3) implies that  $CBC_q(G, H) = 2q$ . More generally, if  $\chi(G) = \chi(H)$ , then  $CBC_q(G, H) = q \cdot \chi(G)$ . However if  $2 \leq \chi(H) < \chi(G)$ , one can improve the upper bound. We show in Proposition 32 that, for any pair  $(G, H)$  with  $H$  a subgraph of  $G$ ,

$$CBC_q(G, H) \leq (\chi(G) + q - 2)\chi(H). \quad (4)$$

Since  $CBC_q(G, H) = \chi(G)$  when  $H$  is empty and  $k$ -COLOURABILITY is NP-complete, for any fixed  $k \geq 3$ , given a graph  $G$  and a subgraph  $H$  it is NP-complete to decide if  $CBC_q(G, H) \leq k$ . But if insist that  $H$  is not empty, then  $CBC_q(G, H) \geq 2q$  by Proposition 3. Hence deciding if  $CBC_q(G, H)$  is at most  $k$  with  $k \leq 2q - 1$  can be done instantly by always returning ‘no’. Less

trivially, Proposition 35 shows that if  $H$  is a connected spanning subgraph of  $G$ , then  $\text{CBC}_q(G, H) = 2q$  if and only if  $G$  is bipartite. Hence deciding if  $\text{CBC}_q(G, H) = 2q$  can be done in polynomial time.

Inequality (4) implies that  $\text{CBC}_q(G, F) \leq 2q + 4$  for any planar graph  $G$  and forest  $F$  in  $G$ . We believe that this upper bound can be reduced by at least one.

**Conjecture 6.** *If  $G$  is a planar graph and  $F$  a spanning forest of  $G$ , then  $\text{CBC}_q(G, F) \leq 2q + 3$ .*

A natural question is to ask whether this conjecture would be best possible.

**Problem 7.** *For any  $q \geq 2$ , does there exist a planar graph  $G_q$  and a spanning forest  $F_q$  of  $G_q$  such that  $\text{CBC}_q(G_q, F_q) = 2q + 3$ ?*

Conjecture 6 holds if the backbone  $F$  is a galaxy. It follows directly from (2) and the fact that  $\text{BBC}_q(G, F) \leq q + 4$  in such a case, as mentioned earlier. We believe however that one can use one colour less.

**Conjecture 8.** *Let  $G$  be a planar graph and  $F$  a galaxy in  $G$ , then  $\text{CBC}_q(G, F) \leq 2q + 2$ .*

If true, this conjecture would be tight, since the circular  $q$ -backbone chromatic number of a  $K_4$  with backbone  $K_{1,3}$  is  $2q + 2$ . As evidence in support of Conjecture 8, Broersma et al. [5] deduced from the Four-Colour Theorem that if  $G$  is a planar graph and  $M$  a matching in  $G$  then  $\text{CBC}_q(G, M) \leq 2q + 2$ .

Broersma et al. also give an example of a planar graph  $G$  and a matching  $M$  such that  $(G, M)$  has no 2-backbone  $\mathbb{Z}_5$ -colouring. We show in Theorems 36 and 40 that for any fixed  $k \in \{4, 5\}$ , it is NP-complete to decide if  $\text{BBC}_2(G, M) \leq k$  for given planar graph  $G$  and matching  $M$ . For larger values of  $q$ , the following questions are still open.

**Problem 9.** *Let  $G$  be a planar graph and let  $M$  be a matching  $M$  in  $G$ . For any  $q \geq 3$ , is it true that  $\text{CBC}_q(G, M) \leq 2q + 1$ ?*

**Problem 10.** *Is it NP-complete to decide if  $\text{CBC}_2(G, F) \leq 6$  for a planar graph  $G$  and spanning forest  $F$ ?*

**Problem 11.** *For any  $g \geq 5$ , is it NP-complete to decide if  $\text{CBC}_2(G, M) \leq 4$  for a planar graph  $G$  of girth at least  $g$  and matching  $M$ ?*

We prove in Theorem 33 that if  $G$  has girth at least 5, then  $\text{CBC}_q(G, M) \leq 2q + 1$ . We wonder if the same holds for planar graph of girth 4.

**Problem 12.** *Let  $G$  be a planar graph of girth 4 and let  $M$  be a matching in  $G$ . Is it true that  $\text{CBC}_q(G, M) \leq 2q + 1$ ?*

Many of the complexity results on circular backbone colouring of planar graphs with a forest backbone are summarized in Table 2.

## 2 Backbone colouring

### 2.1 About Conjectures 1 and 3

**Proposition 13.** *Let  $G$  be a graph and let  $H$  be a subgraph of  $G$ . Then  $\text{BBC}_q(G, H) \leq (\chi(G) + q - 2)\chi(H) - q + 2$ .*

	$G$ planar			
	$H$ forest	$H$ spanning tree	$H$ 1-factor	$H$ Hamilton path
$CBC_q(G, H) \leq 2q?$	q=2: NP-C	poly	q=2: NP-C	poly
$CBC_q(G, H) \leq 2q + 1?$	NP-C	poly	q=2: NP-C	q=2: NP-C
$CBC_q(G, H) \leq 2q + 2?$				
$CBC_q(G, H) \leq 2q + 3?$	?Yes?			
$CBC_q(G, H) \leq 2q + 4?$	Yes			

Table 2: Complexity of deciding if  $CBC_q(G, H) \leq 2q + k$  for  $k \in \{0, \dots, 4\}$ , when  $G$  is a planar graph and  $H$  a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.

*Proof.* Let  $g$  be a proper  $\chi(G)$ -colouring of  $G$  and  $h$  a proper  $\chi(H)$ -colouring of  $H$ . Let  $f$  be the colouring defined by:

$$f(v) = \begin{cases} (h(v) - 1)(q - 2 + \chi(G)) + g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v) - 1)(q - 2 + \chi(G)) + \chi(G) + 1 - g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

Let us check that  $f$  is a  $q$ -backbone  $((\chi(G) + q - 2)\chi(H) - q + 2)$ -colouring of  $(G, H)$ .

Let  $uv \in E(G)$ . Without loss of generality,  $h(u) \geq h(v)$ . If  $h(u) = h(v)$ , then  $uv \notin E(H)$  and  $|f(u) - f(v)| = |g(u) - g(v)| \neq 0$ .

Assume now that  $h(u) > h(v)$ . If  $h(u)$  and  $h(v)$  are both odd or both even, then  $f(u) - f(v) \geq 2(q - 2 + \chi(G)) - |(g(u) - g(v))| \geq q$ . If  $h(u)$  is odd and  $h(v)$  is even, then  $f(u) - f(v) \geq q - 3 + g(u) + g(v)$ , which is at least  $q$  because  $g(u) + g(v) \geq 3$  for  $g(u) \neq g(v)$ . If  $h(u)$  is even and  $h(v)$  is odd, then  $f(u) - f(v) \geq q - 2 + 2\chi(G) + 1 - g(u) - g(v)$ , which is at least  $q$  because  $g(u) + g(v) \leq 2\chi(G) - 1$  for  $g(u) \neq g(v)$ .  $\square$

A *parachute on  $v$*  or a *parachute with harness  $v$*  is a complete graph on four vertices whose three edges incident to  $v$  are in the backbone.

**Proposition 14.** (i) For  $q \geq 2$ , in a  $q$ -backbone  $(q + 3)$ -colouring of a parachute, the harness is coloured in  $\{1, q + 3\}$ .

(ii) For  $q \geq 3$ , in a  $q$ -backbone  $(q + 4)$ -colouring of a parachute, the harness is coloured in  $\{1, 2, q + 3, q + 4\}$ .

(iii) For  $q \geq 4$ , in a  $q$ -backbone  $(q + 5)$ -colouring of a parachute, the harness is coloured in  $\{1, 2, 3, q + 3, q + 4, q + 5\}$ .

*Proof.* Let  $y$  be the harness.

(ii) If  $3 \leq \phi(y) \leq q + 2$ , then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so  $\phi(y) \in \{1, 2, q + 3, q + 4\}$ .

(iii) If  $4 \leq \phi(y) \leq q + 2$ , then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so  $\phi(y) \in \{1, 2, 3, q + 3, q + 4, q + 5\}$ .  $\square$

**Proposition 15.** Let  $G^*$  and  $T^*$  be the graph and its spanning tree depicted in Figure 2. For any  $q \geq 4$ ,  $BBC_q(G^*, T^*) \geq q + 6$ .

*Proof.* Assume for a contradiction that there is a  $q$ -backbone  $(q + 5)$ -colouring  $\phi$  of  $(G^*, T^*)$ . By Proposition 14-(iii), the vertices  $y_1, y_2, y_3, y_4, z_1, z_2$  are coloured in  $\{1, 2, 3, q + 3, q + 4, q + 5\}$ . Without loss of generality, we may assume that  $\phi(y_2) \in \{1, 2, 3\}$ . But then  $\phi(z_1)$  and  $\phi(z_2)$  must be in  $\{q + 3, q + 4, q + 5\}$ , because  $y_2 z_1$  and  $y_2 z_2$  are in  $E(T^*)$ . And  $\phi(y_1), \phi(y_2)$  and  $\phi(y_3)$  are in  $\{1, 2, 3\}$  because  $y_3 z_1$  and  $y_1 z_2$  and  $y_4 z_2$  are in  $E(T^*)$ . But  $\{y_1, y_2, y_3, y_4\}$  is a clique in  $G^*$ , so they must all get different colours, a contradiction.  $\square$

**Proposition 16.** *Conjecture 1 implies Conjecture 3.*

*Proof.* Assume that Conjecture 1 holds. Let  $G$  be a planar graph and  $F$  a forest in  $G$ . Then  $(G, F)$  admits a 2-backbone 6-colouring  $\phi$ . Let  $\psi$  be defined by  $\psi(v) = \phi(v)$  if  $\phi(v) \in \{1, 2\}$ ,  $\psi(v) = \phi(v) + 1$  if  $\phi(v) \in \{3, 4\}$ , and  $\psi(v) = \phi(v) + 2$  if  $\phi(v) \in \{5, 6\}$ . One easily check that  $\psi$  is a 3-backbone 8-colouring of  $(G, F)$ .  $\square$

## 2.2 $q$ -backbone $(q + 2)$ -colouring

**Theorem 17.** *Given a connected graph  $G$  and a spanning connected subgraph  $H$ , one can decide in polynomial time if  $\text{BBC}_q(G, H) \leq q + 2$ .*

*Proof.* Observe first that if  $H$  is not bipartite, then  $\text{BBC}(H, H) \geq 2q + 1$  by (1), and so  $\text{BBC}_q(G, H) \geq q + 3$ . So we first check if  $H$  is bipartite. If not, we return ‘no’. If it is, we get a bipartition  $(A, B)$  of  $H$ .

Observe that if  $(G, H)$  has a  $q$ -backbone  $(q + 2)$ -colouring, then (free to rename  $A$  and  $B$ ) all the vertices of  $A$  are coloured in  $\{1, 2\}$  and all the vertices of  $B$  in  $\{q + 1, q + 2\}$ , because  $H$  is connected. We then can transform our instance into an instance  $I(G, H)$  of 2SAT as follows. For each vertex  $v$ , we create a variable  $x_v$ . Intuitively, for a vertex  $x \in A$  (resp.  $x \in B$ ), the variable  $x_v$  will be true if and only if  $v$  is coloured 1 (resp.  $q + 2$ ) and false if and only if  $v$  is coloured 2 (resp.  $q + 1$ ). Now for each edge  $uv$ , we create the following clauses.

- If  $u$  and  $v$  are both in  $A$  or both in  $B$ , we create the clauses  $x_u \vee x_v$  and  $\bar{x}_u \vee \bar{x}_v$ ;
- if  $u \in A$  and  $v \in B$ , we create the clause  $x_u \vee x_v$ .

It is easy to check that  $(G, H)$  has a  $q$ -backbone  $(q + 2)$ -colouring if and only if  $I(G, H)$  is satisfiable.

Since 2SAT is well-known to be polynomial-time solvable, we can decide in polynomial time if  $\text{BBC}_q(G, H) \leq q + 2$ .  $\square$

**Theorem 18.** *For any  $q \geq 2$ , the following problem is NP-complete.*

Input: A planar graph  $G$  and a 1-factor  $F$  of  $G$ .

Question:  $\text{BBC}_q(G, F) \leq q + 2$ ?

*Proof.* The problem is trivially in NP since a  $q$ -backbone  $(q + 2)$ -colouring of  $(G, F)$  is clearly a certificate.

Reduction from NOT-ALL-EQUAL 3SAT, which is defined as follows:

Input: A set of clauses each having three literals.

Question: Does there exists a *suitable* truth assignment, that is such that each clause has at least one true and at least one false literal?

This problem was shown NP-complete by Schaefer [11].



Let  $\mathcal{C} = \{C_1, \dots, C_n\}$  be a collection of clauses of size three over a set  $U$  of variables. We will construct a graph pair  $(G, F)$  such that  $F$  is a 1-factor of  $G$ . Since  $V(F) = V(G)$ , we only precise which edges are in  $E(F)$ .

The following gadget will be useful. A *forcing gadget at  $v$*  or a *forcing gadget with head  $v$*  is the graph depicted in Figure 3.

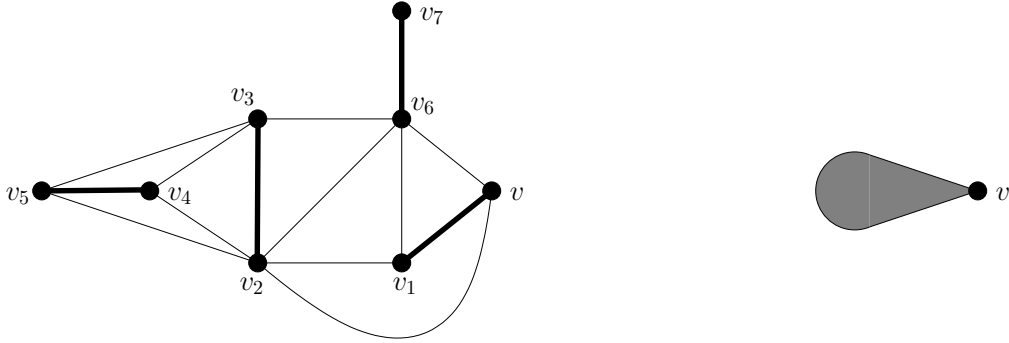


Figure 3: A forcing gadget with head  $v$  (left) and its symbol (right) (Edges of  $E(F)$  are in bold.)

A key point in the reduction will be the following claim.

**Claim 19.** *In any  $q$ -backbone  $(q+2)$ -colouring of a forcing gadget, its head is coloured in  $\{1, q+2\}$ .*

*Proof.* Consider a forcing gadget, whose vertices are named as in Figure 3, and  $\phi$  a  $q$ -backbone  $(q+2)$ -colouring of it. Since all the vertices are matched in  $F$ , there all must be coloured in  $\{1, 2, q+1, q+2\}$ .

Assume for a contradiction that  $\phi(v) = 2$ . Then  $\phi(v_1) = q+2$ . Thus  $\phi(v_2) \in \{1, q+1\}$ . Now if  $\phi(v_2) = q+1$ , then necessarily  $\phi(v_3) = 1$ . Therefore, whatever the colouring may be,  $v_4$  and  $v_5$  are both adjacent to a vertex coloured 1. Hence  $\{\phi(v_4), \phi(v_5)\} = \{2, q+2\}$ . Therefore  $\{\phi(v_2), \phi(v_3)\} = \{1, q+1\}$ . But then  $v_6$  cannot be coloured.

Similarly, we get a contradiction if  $\phi(v) = q+1$ . □

For every variable  $u \in U$ , create a *variable subgraph*  $P_u$  which is obtained from the path  $(a_1(u), b_1(u), a_2(u), b_2(u), \dots, a_n(u), b_n(u))$  by adding a forcing gadget on each of its vertex.

For every clause  $C_i = \ell_1 \vee \ell_2 \vee \ell_3$ , create a clause gadget  $D_i$  as shown Figure 4.

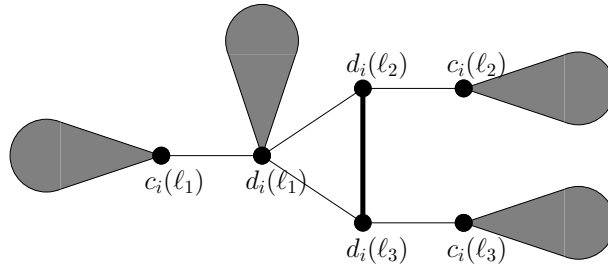


Figure 4: The clause  $D_i$ . (Edges of  $E(F)$  are in bold, forcing gadgets are represented by their symbols.)

Then for each clause  $C_i$  and each literal  $\ell$  of  $C_i$ , we add a path of length three  $(c_i(\ell), c'_i(\ell), c''_i(\ell), a_i(u))$  if  $\ell$  is the non-negated variable  $u$ , and  $(c_i(\ell), c'_i(\ell), c''_i(\ell), b_i(u))$  if  $\ell$  is the negated variable  $\bar{u}$ . We also add two forcing gadgets with heads  $c'_i(\ell)$  and  $c''_i(\ell)$ .

It is easy to see that the resulting graph  $G'$  may be drawn in the plane such that the crossed edges are those of type  $c'_i(\ell)c''_i(\ell)$  for some literal  $\ell$ . In particular, the two endvertices of a crossed edge are heads of forcing gadgets.

As long as there is a crossing  $C$  between two edges  $t(C)u(C)$  and  $v(C)w(C)$ , we replace these two edges by the crossing gadget  $CG(C)$  depicted in Figure 5, so that the only edges that are possibly crossed (if there were several crossings on  $tu$  or  $uv$ ) are  $t(C)t'(C)$ ,  $u(C)u'(C)$ ,  $v(C)v'(C)$  and  $w(C)w'(C)$ . After this process, there is no more crossing so the resulting graph  $G$  is planar.

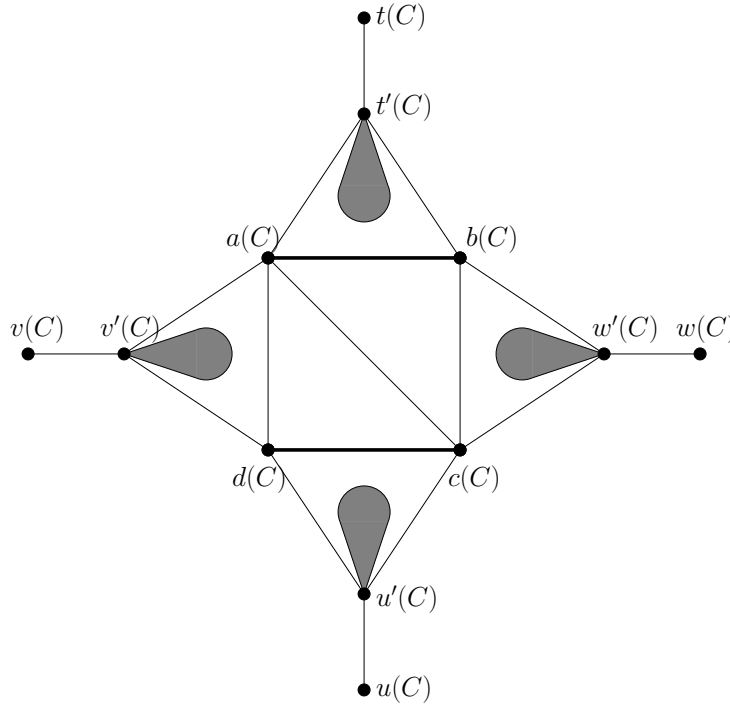


Figure 5: The crossing gadget  $CG(C)$ . (Edges of  $E(F)$  are in bold, forcing gadgets are represented by their symbols.)

**Claim 20.** *Let  $\phi$  be a  $q$ -backbone  $(q + 2)$ -colouring of  $(G, F)$ . For every crossing  $C$  in  $G'$ , we have  $\{\phi(t(C)), \phi(u(C))\} = \{1, q + 2\}$  and  $\{\phi(v(C)), \phi(w(C))\} = \{1, q + 2\}$ .*

*Subproof.* By induction on the reverse order of creation of the crossing gadget.

By construction,  $t(C)$ ,  $u(C)$ ,  $v(C)$ ,  $w(C)$ ,  $t'(C)$ ,  $u'(C)$ ,  $v'(C)$ , and  $w'(C)$  are heads of forcing gadgets. So, by Claim 19, they are coloured 1 or  $q + 2$ . Without loss of generality, we may assume that  $\phi(t(C)) = 1$ .

If the edge  $t(C)t'(C)$  was crossed and then replaced by a series of crossing gadget, by induction,  $\phi(t'(C)) = q + 2$ . It is also trivially the case if  $t(C)t'(C)$  still exists. Hence  $\{\phi(a(C)), \phi(b(C))\} = \{1, q + 1\}$ .

Assume for a contradiction that  $\phi(u(C)) \neq q + 2$ . Then, as above,  $\{\phi(c(C)), \phi(d(C))\} = \{1, q + 1\}$ . This is a contradiction, because  $a(C)c(C)$  and  $a(C)d(C)$  are edges. Hence  $\phi(u(C)) = q + 2$ ,

and so  $\phi(u'(C)) = 1$  and  $\{\phi(c(C)), \phi(d(C))\} = \{2, q+2\}$ .

In particular, one vertex of  $\{a(C), b(C), c(C), d(C)\}$  is coloured 1 and another is coloured  $q+2$ . Now assume for a contradiction that  $\{\phi(v(C)), \phi(w(C))\} \neq \{1, q+2\}$ . Then  $v(C)$  and  $w(C)$  are both coloured 1 or both coloured  $q+2$ , and so  $v'(C)$  and  $w'(C)$  are both coloured  $q+2$  or both coloured 1, respectively. This is a contradiction, as all vertices of  $\{a(C), b(C), c(C), d(C)\}$  are adjacent to some vertex in  $\{v'(C), w'(C)\}$ .  $\diamond$

Let us now prove that  $(G, F)$  admits a  $q$ -backbone  $(q+2)$ -colouring if and only if  $\mathcal{C}$  has a suitable truth assignment.

Assume first that  $(G, F)$  admits a  $q$ -backbone  $(q+2)$ -colouring  $\phi$ . Let  $u$  be a variable. Since there are heads of forcing gadgets, by Claim 19, all the  $a_i(u)$  and  $b_i(u)$  are coloured in  $\{1, q+2\}$ . Moreover, since they form a path, all the  $a_i(u)$  are coloured with the same colour and all the  $b_i(u)$  are coloured with the other. Hence one can define the truth assignment  $\psi$  by  $\psi(u) = \text{true}$  if  $\phi(a_i(u)) = 1$  for  $1 \leq i \leq n$ , and  $\psi(u) = \text{false}$  if  $\phi(a_i(u)) = q+2$  for  $1 \leq i \leq n$ .

We shall prove that  $\psi$  is suitable.

Let  $C_i = \ell_1 \vee \ell_2 \vee \ell_3$  be a clause. Claim 20 implies that for  $j \in \{1, 2, 3\}$ ,  $\phi(c_i(\ell_j)) = 1$  if  $\psi(\ell_j) = \text{false}$  and  $\phi(c_i(\ell_j)) = q+2$  if  $\psi(\ell_j) = \text{true}$ . Now the three  $c_i(\ell_j)$ ,  $1 \leq j \leq 3$ , cannot be all coloured 1 (resp.  $q+2$ ), for otherwise  $\{\phi(d_i(\ell_2)), \phi(d_i(\ell_3))\}$  must be  $\{2, q+2\}$  (resp.  $\{1, q+1\}$ ) and so  $d_i(\ell_1)$  cannot be coloured, because it must be coloured in  $\{1, q+2\}$  as head of a forcing gadget. Thus at least one of the  $c_i(\ell_j)$  is coloured 1 and at least one is coloured  $q+2$ , and so  $C_i$  has at least one true and at least one false literal.

Hence  $\psi$  is suitable.

Reciprocally, assume that  $\mathcal{C}$  has a suitable truth assignment  $\psi$ . For all  $u \in U$  and all  $1 \leq i \leq n$ , let us define  $\phi(a_i(u)) = 1$  and  $\phi(b_i(u)) = q+2$  if  $\psi(u) = \text{true}$ , and  $\phi(a_i(u)) = q+2$  and  $\phi(b_i(u)) = 1$  if  $\psi(u) = \text{false}$ . Similarly, for every literal  $\ell$ , we set  $\phi(c_i(\ell)) = 1$ ,  $\phi(c'_i(\ell)) = q+2$ ,  $\phi(c''_i(\ell)) = 1$ , if  $\ell$  is false, and  $\phi(c_i(\ell)) = q+2$ ,  $\phi(c'_i(\ell)) = 1$ ,  $\phi(c''_i(\ell)) = q+2$ , if  $\ell$  is true.

One can extend  $\phi$  into a  $q$ -backbone  $(q+2)$ -colouring of  $(G, F)$ . Indeed, it is sufficient to show that we can extend it to forcing, clause and crossing gadgets.

If  $v$  is the head of a forcing gadget and  $\phi(v) = 1$ , we can set  $\phi(v_1) = q+2$ ,  $\phi(v_2) = q+1$ ,  $\phi(v_3) = 1$ ,  $\phi(v_4) = q+2$ ,  $\phi(v_5) = 2$ ,  $\phi(v_6) = 2$ , and  $\phi(v_7) = q+2$ . Similarly, we can extend the colouring to the forcing gadget if  $\phi(v) = q+2$ .

Consider a clause gadget  $D_i$ . Since  $C_i$  has at least one true and at least one false literal, at least one vertex of  $c_i(\ell_1), c_i(\ell_2), c_i(\ell_3)$  is coloured 1 and at least one is coloured  $q+2$ . If  $c_i(\ell_1)$  is coloured  $q+2$ , and  $c_i(\ell_2)$  and  $c_i(\ell_3)$  are assigned 1, then we can set  $\phi(d_i(\ell_1)) = 1$ ,  $\phi(d_i(\ell_2)) = 2$ , and  $\phi(d_i(\ell_3)) = q+2$ . If  $c_i(\ell_1)$  and  $c_i(\ell_2)$  are coloured 1, and  $c_i(\ell_3)$  is assigned  $q+2$ , then we can set  $\phi(d_i(\ell_1)) = q+2$ ,  $\phi(d_i(\ell_2)) = q+1$ , and  $\phi(d_i(\ell_3)) = 1$ .

Finally consider a crossing gadget such that  $\{\phi(t(C)), \phi(u(C))\} = \{\phi(v(C)), \phi(w(C))\} = \{1, q+2\}$ . By symmetry, we may assume that  $\phi(t(C)) = \phi(v(C)) = 1$  and  $\phi(u(C)) = \phi(w(C)) = q+2$ . Then we can set  $\phi(t'(C)) = \phi(v'(C)) = q+2$ ,  $\phi(u'(C)) = \phi(w'(C)) = 2$ ,  $\phi(a(C)) = 1$ ,  $\phi(b(C)) = q+1$ ,  $\phi(c(C)) = q+2$ , and  $\phi(d(C)) = 2$ .  $\square$

## 2.3 2-backbone 5-colouring

### 2.3.1 Galaxy backbone

**Theorem 21.** *The following problem is NP-complete.*

Input: A planar graph  $G$  and a galaxy  $F$  in  $G$  with maximum degree 3.

Question: Is  $\text{BBC}_2(G, F) \leq 5$ ?

*Proof.* Reduction from PLANAR 3-COLOURABILITY, which consists of deciding if a given connected planar graph is 3-colourable. This problem was shown to be NP-complete by Stockmeyer [13]. Clearly, it remains NP-complete when restricted to 2-connected planar graphs.

Let  $H$  be a 2-connected planar graph. We shall construct a planar graph  $G$  and a galaxy  $F$  with maximum degree 3 in  $G$  such that  $\text{BBC}_2(G, F) \leq 5$  if and only if  $H$  is 3-colourable.

As a forcing gadget at  $v$ , we will use the parachute with harness  $v$ . It is easy to see that in a 2-backbone 5-colouring of a parachute, its harness is coloured in  $\{1, 5\}$ .

We consider any embedding of  $H$ . For each face  $(x_1, x_2, \dots, x_k, x_1)$  of  $H$ , we put a cycle  $(z_1, z_2, \dots, z_{2k}, z_1)$ , inside which we put parachutes on every vertex  $z_i$  for every  $1 \leq i \leq 2k$ . We then add the edges  $x_i z_{2i} x_i z_{2i+1}$  for all  $1 \leq i \leq k$ .

Assume that  $(G, F)$  has a 2-backbone 5-colouring  $\phi$ , then, because of the parachutes, all the vertices in the cycles added inside faces must be coloured in  $\{1, 5\}$ . Moreover consecutive vertices on one such cycles get different colours, so one is coloured 1 and the other is coloured 5. Hence all the vertices in  $H$  are coloured in  $\{2, 3, 4\}$ . Hence  $\phi$  induces a proper 3-colouring on  $H$  with colours  $\{2, 3, 4\}$ .

Reciprocally, assume that  $H$  is 3-colourable. Then there exists a proper 3-colouring  $c$  of  $H$  into  $\{2, 3, 4\}$ . One can then colour all the cycles inside faces with 1 and 5. The colouring can then easily be extended into a 2-backbone 5-colouring of  $(G, F)$ .  $\square$

**Theorem 22.** *The following problem is NP-complete.*

Input: A planar graph  $G$  and a galaxy  $F$  in  $G$  with maximum degree 2.

Question: Is  $\text{BBC}_2(G, F) \leq 5$ ?

*Proof.* The proof is identical to the one of Theorem 21. The only difference comes from the forcing gadget, which is more complicated because it cannot contains stars of degree 3 in  $F$ .

To construct the forcing gadget, we need an auxiliary gadget, called *no-3-gadget*. It is depicted in Figure 6.

**Claim 23.** *In any 2-backbone 5-colouring of a no-3-gadget, its roof is not coloured in 3.*

*Proof.* We will denote the vertices of the no-3-gadget by their names in Figure 6. Assume for a contradiction that there is a 2-backbone 5-colouring  $\phi$  of a no-3-gadget such that  $\phi(x) = 3$ .

Assume first that  $\phi(a) \in \{4, 5\}$ , then  $\phi(b) \in \{1, 2\}$  and  $\{\phi(a), \phi(c)\} = \{4, 5\}$ . Hence  $\phi(d) \in \{1, 2\}$  and so  $\{\phi(f), \phi(c)\} = \{4, 5\}$ . Therefore  $\phi(e) = 3$  and so  $\phi(d) = 1$ . Similarly, if  $\phi(a) \in \{1, 2\}$ , we obtain that  $\phi(d) = 5$ . Hence,  $\phi(d) \in \{1, 5\}$ .

Similarly,  $\phi(d') \in \{1, 5\}$ . Free to consider  $6 - \phi$  instead of  $\phi$ , we may assume that  $\phi(d) = 1$  and  $\phi(d') = 5$ . Thus  $\phi(f') = 2$ .

Now  $\phi(g) \in \{3, 4\}$ . If  $\phi(g) = 3$ , then  $\{\phi(i), \phi(h)\} = \{1, 5\}$ , and if  $\phi(g) = 4$ , then  $\{\phi(i), \phi(h)\} = \{1, 2\}$ . In both cases, one of  $h$  and  $i$  is coloured 1, which is impossible because  $\phi(d) = 1$ .  $\square$

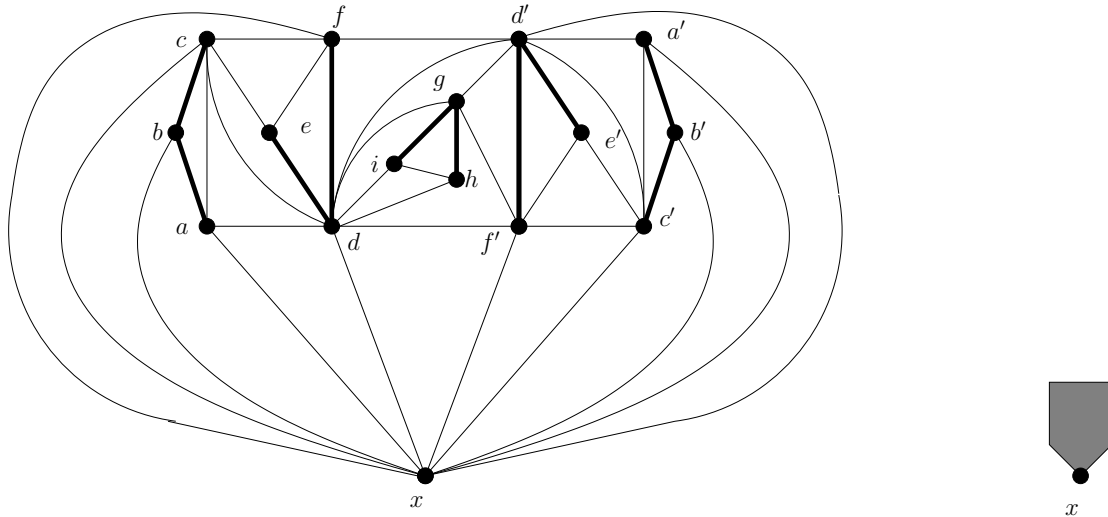


Figure 6: The no-3-gadget with roof  $x$  and its symbol.

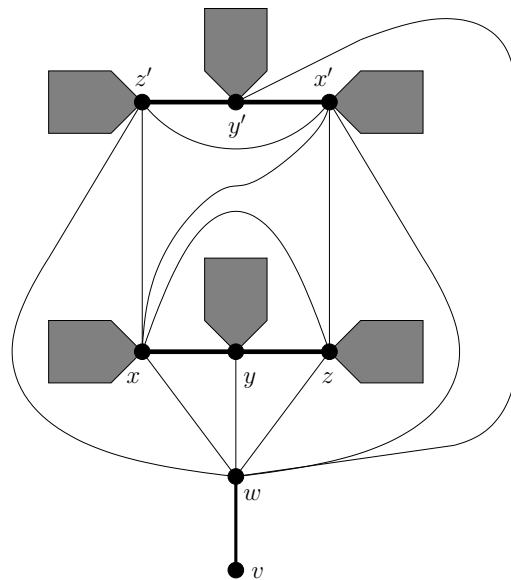


Figure 7: The forcing gadget with head  $v$ . (Edges of  $E(F)$  are in bold, no-3-gadgets are represented by their symbols.)

The forcing gadget is the one depicted in Figure 7.

**Claim 24.** *In any 2-backbone 5-colouring of a forcing gadget, its head is coloured in  $\{1, 5\}$ .*

*Proof.* Consider a forcing gadget, whose vertices are named as in Figure 7, and  $\phi$  a 2-backbone 5-colouring of it.

Let us prove that  $\phi(w) = 3$  and so that  $\phi(v) \in \{1, 5\}$ . Assume for a contradiction that  $\phi(w) \neq 3$ . Without loss of generality, we may assume that  $\phi(w) \in \{1, 2\}$ .

Observe that the vertices  $x, y, z, x', y', z'$  are not assigned 3 because they are roofs of no-3-gadgets.

If  $\phi(w) = 1$ , then  $(\phi(x), \phi(y), \phi(z))$  and  $(\phi(x'), \phi(y'), \phi(z'))$  is either  $(4, 2, 5)$  or  $(5, 2, 4)$ . Hence the vertices  $x, x'$  and  $z$  are all coloured in  $\{4, 5\}$ , which is impossible, since they form a triangle.

If  $\phi(w) = 2$ , then  $(\phi(x), \phi(y), \phi(z))$  and  $(\phi(x'), \phi(y'), \phi(z'))$  is either  $(4, 1, 5)$  or  $(5, 1, 4)$ . Hence the vertices  $x, x'$  and  $z$  are all coloured in  $\{4, 5\}$ , which is impossible, since they form a triangle.  $\square$

To get the equivalence between the 3-colourability of the original graph  $H$  and the existence of a 2-backbone 5-colouring of  $(G, F)$ , it remains to prove that for any  $\alpha \in \{1, 5\}$ , there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured  $\alpha$ .

We denote the vertices by their names in Figure 7. Set  $\phi(w) = 3, \phi(x) = \phi(y') = 1, \phi(y) = \phi(z') = 5, \phi(z) = 2$  and  $\phi(x') = 4$ .

Observe that no vertex in  $\{x, y, z, x', y', z'\}$  has been coloured 3. Hence, it remains to prove that for any  $\beta \in \{1, 2, 4, 5\}$ , there is a 2-backbone 5-colouring of the forcing gadget such that the head is coloured  $\beta$ . By the symmetry  $\phi \rightarrow 6 - \phi$ , it suffices to prove that one exists for  $\beta \in \{1, 2\}$ . We denote the vertices by their names in Figure 6. Let us denote by  $\bar{\beta}$  the colour of  $\{1, 2\} \setminus \{\beta\}$ .

$\phi(a) = 3, \phi(b) = \bar{\beta}, \phi(c) = 5, \phi(d) = 4, \phi(e) = \beta, \phi(f) = \bar{\beta}, \phi(a') = 3, \phi(b') = 5, \phi(c') = \bar{\beta}, \phi(d') = 5, \phi(e') = \beta, \phi(f') = 3, \phi(g) = 1, \phi(h) = 5, \phi(i) = 3.$   $\square$

### 2.3.2 Hamiltonian-path backbone

**Theorem 25.** *The following problem is NP-complete.*

Input: A planar graph  $G$  with a hamiltonian path  $P$ .

Question:  $\text{BBC}_2(G, P) \leq 5$ ?

To prove this theorem, we shall use a reduction similar to the one of Theorem 21. However, we do not reduce directly from PLANAR 3-COLOURABILITY but use an intermediate problem whose NP-completeness is proven by reducing PLANAR 3-COLOURABILITY to it.

This intermediate problem is the following:

TRACEABLE PLANAR 3-COLOURABILITY

Input: A planar graph  $G$  with a hamiltonian path  $P$ .

Question: Is  $G$  3-colourable?

**Lemma 26.** TRACEABLE PLANAR 3-COLOURABILITY is NP-complete.

*Proof.* Reduction from PLANAR 3-COLOURABILITY. Let  $H$  be a connected planar graph. We will construct a planar graph  $G$  having a hamiltonian path  $P$  such that  $\chi(G) \leq 3$  if and only if  $\chi(H) \leq 3$ .

To do so, we shall construct a sequence of pairs  $(G_i, P_i)$  for  $1 \leq i \leq |V(H)|$  such that  $P_i$  is a path in the planar connected graph  $G_i$ ,  $|V(P_i)| = |V(G_i)| - |V(H)| + i$ , and  $\chi(G_i) \leq 3$  if and only if  $\chi(H) \leq 3$ . Then the path  $P := P_{V(H)}$  will be a hamiltonian path of  $G := G_{V(H)}$  and  $\chi(G) \leq 3$  if and only if  $\chi(H) \leq 3$ .

Let  $x$  be a vertex of  $H$ . We set  $G_1 := H$  and  $P_1 := (x)$ . Trivially,  $(G_1, P_1)$  verifies the above property.

Assume now that  $i \geq 1$  and let us construct  $(G_{i+1}, P_{i+1})$  from  $(G_i, P_i)$ . Let  $P_i = (v_1, v_2, \dots, v_\ell)$  be a path. Since  $G_i$  is connected, there exists  $j$  such that  $v_j$  is adjacent to a vertex  $y$  in  $V(G_i) \setminus V(P_i)$ . If  $j = 1$ , then let  $P_{i+1} := (y, v_1, v_2, \dots, v_\ell)$ , and  $G_{i+1} := G_i$ ; if  $j = p$ , then let  $P_{i+1} := (v_1, v_2, \dots, v_\ell, y)$ , and  $G_{i+1} := G_i$ ; if  $y$  is also incident to  $v_{j+1}$ , let  $P_{i+1} := (v_1, \dots, v_j, y, v_{j+1}, \dots, v_\ell)$ . In those three cases,  $(G_{i+1}, P_{i+1})$  has trivially the desired property.

So we may assume that  $1 < j < \ell$  and  $y$  is not adjacent to  $v_{j+1}$ . Let  $y_1, y_2, \dots, y_r$  be the neighbours of  $v_j$  in their order around it such that  $v_{j+1} = y_r, y_k = y$  and  $v_{j-1} = y_q$  for  $q < r$ .

Let  $G_{i+1}$  be the graph obtained from  $G_i$  as follows. For all  $1 \leq s \leq k-1$ , remove the edge  $v_j y_s$ , add three vertices  $a_s, b_s, c_s$  and the edges  $a_s b_s, b_s c_s, c_s a_s, v_j a_s, v_j b_s, b_s y_s$ ; Add the edges  $c_s a_{s+1}$  for all  $1 \leq s \leq k-2$ , and  $v_{j+1} a_1$ . Finally add a vertex  $y'$  and the edges  $y y'$  and  $y' c_{k-1}$ . Let  $P_{i+1}$  be the path obtained from  $P_i$  by replacing the edge  $v_j v_{j+1}$  by the subpath  $(v_j, y, c_{k-1}, b_{k-1}, a_{k-1}, \dots, c_1, b_1, a_1, v_{j+1})$ . See Figure 8, which illustrates the construction when  $k = 5$ .

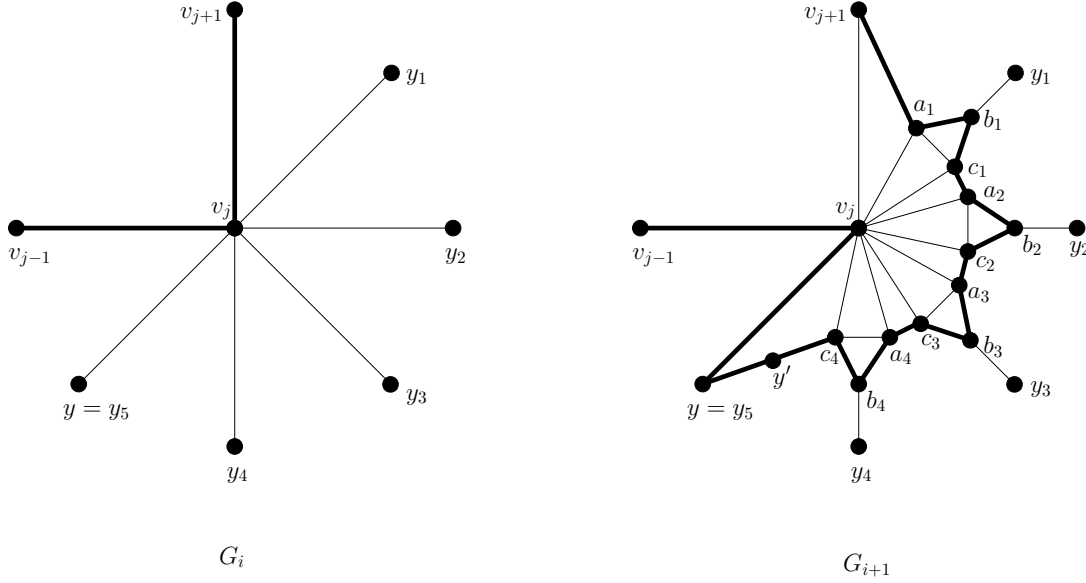


Figure 8: Constructing  $(G_{i+1}, P_{i+1})$  from  $(G_i, P_i)$  (Edges of the paths are in bold.)

Clearly, the number of vertices not covered by  $P_{i+1}$  in  $G_{i+1}$  is one less than the number of vertices not covered by  $P_i$  in  $G_i$ . So, since  $|V(P_i)| = |V(G_i)| - |V(H)| + i$ , we have  $|V(P_{i+1})| = |V(G_{i+1})| - |V(H)| + i + 1$ .

It remains to prove that  $G_{i+1}$  is 3-colourable if and only if  $G_i$  is.

Assume first that  $G_{i+1}$  admits a proper 3-colouring  $\phi$  in  $\{1, 2, 3\}$ . We claim that it also induces a proper 3-colouring of  $G_i$ . Indeed, without loss of generality, we may assume that  $\phi(v_j) = 1$  and  $\phi(v_{j+1}) = 2$ . Then for all  $1 \leq s \leq k-1$ ,  $\phi(a_s) = 3$  and  $\phi(c_s) = 2$ , so  $\phi(b_s) = 1$ . Hence  $\phi(y_s) \neq 1$ . Therefore, for all  $1 \leq s \leq k-1$ ,  $\phi(y_s) \neq \phi(v_j)$ . Since the  $v_j y_s$ ,  $1 \leq s \leq k-1$ , are the only edges of  $G_i$  which are not in  $G_{i+1}$ ,  $\phi$  is a proper 3-colouring of  $G_i$ .

Conversely, assume that  $G_i$  admits a 3-colouring  $\phi$  in  $\{1, 2, 3\}$ . It induces a partial proper 3-colouring of  $G$ , such that  $\phi(v_j) \neq \phi(y_s)$  for all  $1 \leq s \leq k-1$ . Let us extend it. Without loss of generality,  $\phi(v_j) = 1$  and  $\phi(v_{j+1}) = 2$ . For all  $1 \leq s \leq k-1$ , set  $\phi(a_s) = 3$ ,  $\phi(b_s) = 1$ , and

$\phi(c_s) = 2$ . Finally, colour  $y'$  with the colour in  $\{1, 2, 3\} \setminus \{\phi(y), \phi(c_{k-1})\}$ . This gives a proper 3-colouring of  $G_{i+1}$ .  $\square$

*Proof of Theorem 25.* Reduction from TRACEABLE PLANAR 3-COLOURABILITY. Let  $(H, Q)$  be an instance of this problem. We shall construct a graph  $G$  and a hamiltonian path  $P$  of  $G$  such that  $\text{BBC}_2(G, P) \leq 5$  if and only if  $\chi(H) \leq 3$ . To do so we start from  $H$  and for each edge  $xy$  of  $Q$ , we will plug in an edge gadget  $E(xy)$  containing a hamiltonian path  $P(xy)$  from  $x$  to  $y$ . The union of all the  $P(xy)$ ,  $xy \in E(Q)$ , will then be a hamiltonian path  $P$  of the resulting graph  $G$ .

To construct the edge gadget, we use an auxiliary forcing gadget depicted in Figure 9. The *head* of such a gadget is the vertex denoted by  $v$  in the figure. Its *fringes* are the vertices denoted by  $a$  and  $e$ .

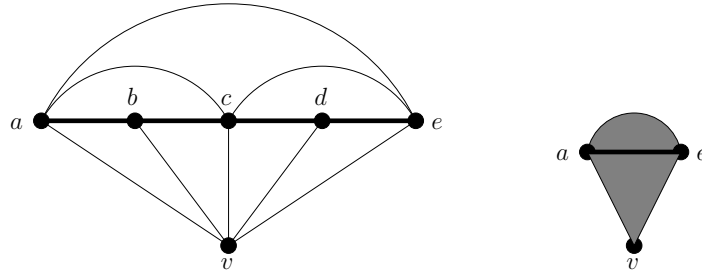


Figure 9: The forcing gadget with head  $v$  and fringes  $a$  and  $e$  (left) and its symbol (right)

**Claim 27.** *In any 2-backbone 5-colouring of a forcing gadget, the head is coloured in  $\{2, 4\}$ .*

*Proof.* We denote the vertices by their names in Figure 9. Suppose for a contradiction that there is a 2-backbone 5-colouring  $\phi$  such that  $\phi(v) \notin \{2, 4\}$ . By the symmetry  $\phi \rightarrow 6 - \phi$ , we may assume that  $\phi(v) \in \{1, 3\}$ .

If  $\phi(v) = 3$ , then all the vertices  $a, b, c, d, e$  are coloured in  $\{1, 2, 4, 5\}$ . On the path  $(a, b, c, d, e)$ , vertices coloured  $\{1, 2\}$  alternate with vertices coloured  $\{4, 5\}$ . Hence  $a, c$ , and  $e$  are all coloured in  $\{1, 2\}$ , or all coloured in  $\{4, 5\}$ , which is a contradiction as they form a clique.

If  $\phi(v) = 1$ , then all the vertices  $a, b, c, d, e$  are coloured in  $\{2, 3, 4, 5\}$ . Now  $\phi(b)$  is at distance 2 from the two distinct colours  $\phi(a)$  and  $\phi(c)$ , hence  $\phi(b) \in \{2, 5\}$ . Similarly,  $\phi(d) \in \{2, 5\}$ . But  $\phi(c)$  is at distance 2 from  $\phi(b)$  and  $\phi(d)$ , so  $\phi(b) = \phi(d)$ . Then the three vertices  $a, c$ , and  $e$  are all coloured in  $\{2, 3, 4, 5\} \setminus \{\phi(b) - 1, \phi(b), \phi(b) + 1\}$ , which has cardinality 2. This is a contradiction as those three vertices form a clique.  $\square$

Now the edge gadget is the one depicted in Figure 10.

Let us now prove that  $\text{BBC}_2(G, P) \leq 5$  if and only if  $\chi(H) \leq 3$ .

Assume first that  $(G, P)$  admits a 2-backbone 5-colouring  $\phi$ . Since  $H$  is a subgraph of  $G$ ,  $\phi$  induces a proper colouring on  $H$ . We shall prove that every vertex of  $H$  is coloured in  $\{1, 3, 5\}$ , thus proving that this proper colouring uses (at most) 3 colours.

Every vertex  $v$  of  $H$  is contained in an edge  $xy$  of  $Q$ , so it is contained in the edge gadget  $E(xy)$  in  $G$ . So it is adjacent to two vertices (namely  $v_1$  and  $v_2$  if  $v = x$ , and  $v_2$  and  $v_3$  if  $v = y$ ), which are heads of forcing gadgets and adjacent. Hence by Claim 27, one of these vertices is coloured 2 and the other is coloured 4. Hence  $v$  must be coloured in  $\{1, 3, 5\}$ .



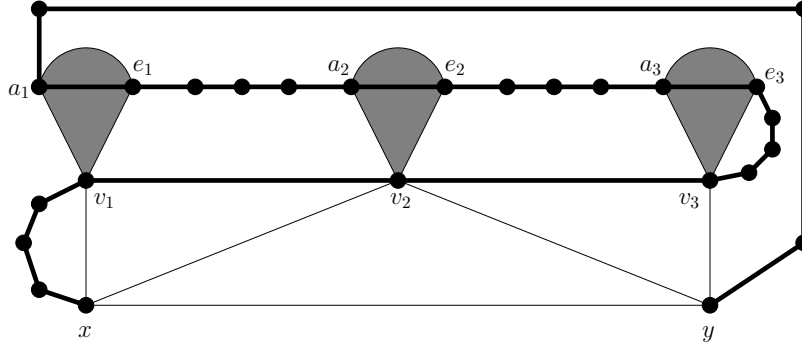


Figure 10: The edge gadget  $E(xy)$  and its hamiltonian path  $P(xy)$  in bold (Forcing gadgets are represented by their symbols.)

Let us now assume that  $H$  is 3-colourable. Then there exists a proper colouring  $\phi$  of  $H$  with  $\{1, 3, 5\}$ . Let us now extend into a 2-backbone 5-colouring of  $(G, P)$ . It is sufficient to prove that we can extend it to every edge-gadget.

To extend it to the edge-gadget  $E(xy)$  (we use the names of Figure 10), set  $\phi(v_1) = \phi(v_3) = 2$  and  $\phi(v_2) = 4$ . Now, since for any pair  $(\alpha, \beta) \in \{1, 2, 3, 4, 5\}^2$ , there is a 2-backbone 5-colouring of the path of length 4 such that the first vertex is coloured  $\alpha$  and the last vertex is coloured  $\beta$ , it suffices to prove that we can extend  $\phi$  to the forcing gadget.

Consider such a forcing gadget (with vertex names as in Figure 9). Then  $\phi(v) \in \{2, 4\}$ . By the symmetry  $\phi \rightarrow 6 - \phi$ , we may assume that  $\phi(v) = 2$ . Then setting  $\phi(a) = 4$ ,  $\phi(b) = \phi(d) = 1$ ,  $\phi(c) = 3$  and  $\phi(e) = 5$ , we obtain the desired extension.

Hence,  $\text{BBC}_2(G, P) \leq 5$ . □

## 2.4 $q$ -backbone $(q + 3)$ -colouring for $q \geq 3$

**Theorem 28.** *For any  $q \geq 3$ , the following problem is NP-complete.*

Input: A planar graph  $G$  and a galaxy  $F$  in  $G$  with maximum degree 3.

Question: Is  $\text{BBC}_q(G, F) \leq q + 3$ ?

*Proof.* Reduction from PLANAR 3-COLOURABILITY.

We shall need the graph, which we call a *kite*, depicted in Figure 11. The vertex named  $t$  in the figure is the *tip* of the kite, and the one named  $u$  is its *corner*.

**Claim 29.** *If  $\phi$  is a  $q$ -backbone  $(q + 3)$ -colouring of a kite such that  $\phi(t) \in \{1, 2, 3, q + 1, q + 2, q + 3\}$ , then either  $\phi(t) \in \{1, 2, 3\}$  and  $\phi(u) = q + 3$ , or  $\phi(t) \in \{q + 1, q + 2, q + 3\}$  and  $\phi(u) = 1$ .*

*Proof.* Observe that the vertices  $v, z_1, z_2, z_3$  are harnesses of parachutes. Thus, by Proposition 14-(i), they must be assigned 1 or  $q + 3$ .

Assume that  $\phi(v) = 1$ , then  $\phi(z_1) = \phi(z_2) = \phi(z_3) = q + 3$ . Thus  $\{\phi(s_1), \phi(s_2)\} = \{q + 1, q + 2\}$  and so  $\phi(u) = q + 3$  and  $\phi(t) \in \{1, 2, 3\}$ .

Similarly if  $\phi(v) = q + 3$ , we obtain  $\phi(u) = 1$  and  $\phi(t) \in \{q + 1, q + 2, q + 3\}$ . □

Let  $H$  be a planar graph. Let  $(G, F)$  be the graph pair obtained from  $H$  as follows. Firstly, for each face  $f$  of  $H$ , we create a parachute  $P_f$  with harness  $v_f$ , and for each vertex  $x$  incident to  $f$ , we

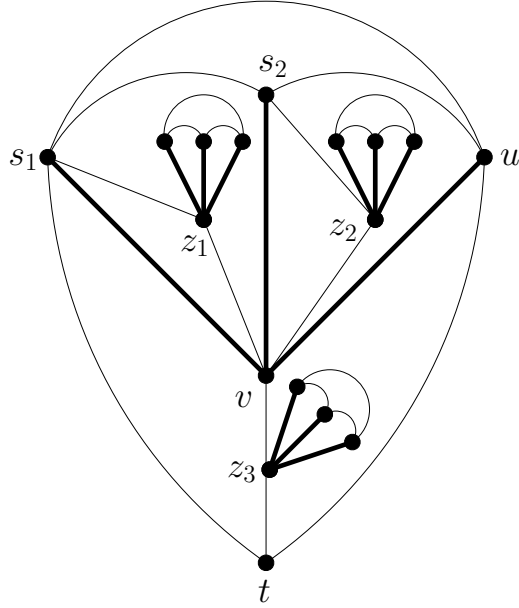


Figure 11: The kite

create a kite  $K_f(x)$  with tip  $x$  and corner  $u_f(x)$ . We then link the vertex  $v_f$  to all the  $u_f(x)$ . Secondly, for every vertex  $x \in V(H)$ , we add a vertex  $y_x$  and the edge  $xy_x$  in the backbone.

Clearly, the resulting graph  $G$  is planar and the resulting backbone  $F$  is a galaxy with maximum degree 3.

Let us now prove that  $BBC_q(G, F) \leq q + 3$  if and only if  $H$  is 3-colourable.

Assume first that  $(G, F)$  admits a  $q$ -backbone  $(q + 3)$ -colouring  $\phi$ . Observe that each vertex  $x$  in  $V(H)$  is coloured in  $\{1, 2, 3, q + 1, q + 2, q + 3\}$ , because it is adjacent to  $y_x$  in  $F$ .

Let  $x$  be a vertex in  $V(H)$ . Free to consider  $q + 4 - \phi$ , we may assume that  $\phi(x) \in \{1, 2, 3\}$ . Consider a face  $f$  incident to  $x$  in  $H$ . By Claim 29, the kite  $K_f(x)$  has its corner coloured  $q + 3$ . Together with Proposition 14-(i), this implies that  $\phi(v_f) = 1$ . Thus, the corner of the kites in  $f$  in  $H$  are all coloured  $q + 3$  and so by Claim 29, all the vertices incident to  $f$  in  $H$  are all coloured in  $\{1, 2, 3\}$ . Applying this reasoning to each face of  $H$ , we obtain that all vertices of  $H$  are coloured in  $\{1, 2, 3\}$ . Hence,  $\phi$  induces a proper 3-colouring on  $H$ .

Conversely, assume that  $H$  admits a proper 3-colouring  $c$ . One can extend it into a  $q$ -backbone  $(q + 3)$ -colouring of  $(G, F)$  as follows. For every  $x \in V(H)$ , we colour  $y_x$  with  $q + 3$ ; for every face  $f$ , we colour the vertex  $v_f$  with 1 and the corners of the kites by  $q + 3$ . One can then extend the colouring to each kite (as in the proof of Claim 29) to obtain a  $q$ -backbone  $(q + 3)$ -colouring of  $(G, F)$ .  $\square$

The reduction above can be modified to have a spanning tree  $T$  for the backbone in place of the galaxy  $F$ . It suffices consider a spanning tree  $U$  of  $H$  and do the following: add a path of length two in the backbone along each edge of the tree  $U$ ; for each kite, add  $tz_3$  and  $vz_3$  in the backbone and add paths of length two in the backbone along edges  $z_1v$  and  $z_2v$ . This will prove the following statement.

**Theorem 30.** *The following problem is NP-complete.*

Input: A planar graph  $G$  and a spanning tree  $T$  of  $G$ .

Question: Is  $\text{BBC}_q(G, T) \leq q + 3$ ?

## 2.5 $q$ -backbone $(q + 5)$ -colouring

**Theorem 31.** For any  $q \geq 4$ , the following problem is NP-complete.

Input: A planar graph  $G$  and a spanning tree  $T$  of  $G$ .

Question: Is  $\text{BBC}_q(G, T) \leq q + 5$ ?

*Proof.* Reduction from PLANAR 3-COLOURABILITY.

Let  $H$  be a planar graph. We shall construct a planar graph  $G$  together with a spanning tree  $T$  such that  $H$  is 3-colourable if and only if  $\text{BBC}_q(G, T) \leq q + 5$ . Take  $U$  be a spanning tree of  $H$ .

We first construct a graph  $G'$  from  $H$  by adding for every edge  $e = uv$  of  $U$  a vertex  $x_e$  linked to  $u$  and  $v$ . We let  $T'$  be the spanning tree of  $G'$  induced by the new edges. The pair  $(G, T)$  is then obtained from  $(G', T')$  by adding a parachute on every vertex. Clearly  $G$  is planar as for each edge  $e = uv$  the path  $ux_e v$  can be drawn along the edge  $uv$ .

Suppose that  $(G, T)$  admits a  $q$ -backbone  $(q + 5)$ -colouring. Then by Proposition 14-(iii), every vertex in  $G'$  is coloured in  $\{1, 2, 3, q + 3, q + 4, q + 5\}$ . Note that the vertices of  $H$  form one of the part of the bipartition of  $T'$ . Hence, the colours of the vertices of  $H$  are either all in  $\{1, 2, 3\}$  or all in  $\{q + 3, q + 4, q + 5\}$ . In both cases,  $\phi$  induces a proper 3-colouring on  $H$ .

Conversely, it is straightforward to extend a proper 3-colouring of  $H$  into a  $q$ -backbone  $(q + 5)$ -colouring of  $(G, T)$ .  $\square$

## 3 Circular backbone colouring

The following Proposition is an analogue to Proposition 13 and its proof is similar.

**Proposition 32.** Let  $G$  be a graph and let  $H$  be a subgraph of  $G$  such that  $2 \leq \chi(H) < \chi(G)$ . Then  $\text{CBC}_q(G, H) \leq (\chi(G) + q - 2)\chi(H)$ .

*Proof.* Let  $g$  be a proper  $\chi(G)$ -colouring of  $G$  and  $h$  a proper  $\chi(H)$ -colouring of  $H$ .

Assume first that  $\chi(H)$  is even. Let  $f$  be the colouring defined by:

$$f(v) = \begin{cases} (h(v) - 1)(q - 2 + \chi(G)) + g(v), & \text{if } h(v) \text{ is odd,} \\ (h(v) - 1)(q - 2 + \chi(G)) + \chi(G) + 1 - g(v) & \text{if } h(v) \text{ is even.} \end{cases}$$

Let us check that  $f$  is a circular  $q$ -backbone  $((\chi(G) + q - 2)\chi(H))$ -colouring of  $(G, H)$ . For  $1 \leq i \leq \chi(H)$ , let  $I_i = \{(i - 1)(q - 2 + \chi(G)) + 1, \dots, (i - 1)(q - 2 + \chi(G)) + \chi(G)\}$ . Observe that if  $h(v) = i$ , then  $f(v) \in I_i$ . The  $I_i$  form intervals of  $\mathbb{Z}_{(\chi(G) + q - 2)\chi(H)}$ . These intervals do not intersect and two consecutive intervals are separated by  $q - 2$  elements. In particular, if  $h(u) \neq h(v)$ , then  $|f(u) - f(v)| \geq q - 1$ . Moreover  $|f(u) - f(v)| = q - 1$  only if  $g(u) = g(v)$ .

Consider an edge  $uv \in E(G)$ . By the previous remark, if  $h(u) \neq h(v)$ , then  $f(u) \neq f(v)$ . If  $h(u) = h(v)$ , then  $|f(u) - f(v)| = |g(u) - g(v)| \neq 0$ , because  $g$  is proper.

Consider now an edge  $uv \in E(H)$ . Then  $h(u) \neq h(v)$ , so  $f(u)$  and  $f(v)$  are in different  $I_i$ . If they are in non-consecutive  $I_i$  (modulo  $\chi(H)$ ), then  $|f(u) - f(v)| \geq 2q - 2 + \chi(G) \geq q$ . Assume now that they are in consecutive intervals, then  $|f(u) - f(v)| \geq q$  because  $g(u) \neq g(v)$ .

Assume now that  $\chi(H)$  is odd. Let  $f$  be the colouring defined by:

$$f(v) = \begin{cases} 1, & \text{if } h(v) = 1 \text{ and } g(v) = \chi(G), \\ g(v) + 1, & \text{if } h(v) = 1 \text{ and } g(v) < \chi(G), \\ \chi(G) + q - 1, & \text{if } h(v) = 2 \text{ and } g(v) = \chi(G) - 1, \\ \chi(G) + q, & \text{if } h(v) = 2 \text{ and } g(v) = \chi(G), \\ 2\chi(G) + q - 1 - g(v), & \text{if } h(v) = 2 \text{ and } g(v) < \chi(G) - 1, \\ (h(v) - 1)(q - 2 + \chi(G)) + g(v), & \text{if } h(v) \text{ is odd and } h(v) > 2, \\ (h(v) - 1)(q - 2 + \chi(G)) + \chi(G) + 1 - g(v) & \text{if } h(v) \text{ is even and } h(v) > 2. \end{cases}$$

Similarly to the even case, one can check that  $f$  is a circular  $q$ -backbone  $((\chi(G) + q - 2)\chi(H))$ -colouring of  $(G, H)$ .  $\square$

### 3.1 Planar graphs of girth at least 5

**Theorem 33.** *Let  $G$  be a planar graph of girth at least 5 and  $M$  a matching in  $G$ . Then  $\text{CBC}_q(G, M) \leq 2q + 1$ .*

*Proof.* Our proof is based on a structural result of Borodin and Glebov [1]. See also [9].

**Theorem 34** (Borodin and Glebov [1]). *The vertex set of every planar graph of girth at least 5 can be partitioned into an independent set and a set which induces a forest.*

Let  $(S, F)$  be a partition of  $V(G)$  such that  $S$  is stable and  $F$  induces a forest. Let us colour every vertex of  $S$  with 1. Now since  $F$  is a forest, it has an ordering  $v_1, \dots, v_p$  such that for every  $i$ ,  $v_i$  has at most one neighbour in  $\{v_1, \dots, v_{i-1}\}$ . We colour the vertices of  $F$  according to this ordering as follows. If  $v_i$  has no neighbour in  $\{v_1, \dots, v_{i-1}\}$ , then colour it with  $q + 1$ . If  $v_i$  has a neighbour  $u$  in  $\{v_1, \dots, v_{i-1}\}$  and  $uv_i \notin E(M)$ , then colour it with a colour of  $\{q + 1, q + 2\}$  not assigned to  $u$ . If  $v_i$  has a neighbour  $u$  in  $\{v_1, \dots, v_{i-1}\}$  and  $uv_i \in E(M)$ , then assign  $2q + 1$  (resp. 2 to  $v_i$ ) if  $u$  is coloured  $q + 1$  (resp.  $q + 2$ ). It is easy to check that the obtained colouring is a  $q$ -backbone  $\mathbb{Z}_{2q+1}$ -colouring of  $(G, M)$ .  $\square$

### 3.2 Circular $q$ -backbone $2q$ -colouring

**Proposition 35.** *Let  $G$  be a graph and  $H$  a spanning connected subgraph of  $G$ . Then  $\text{CBC}_q(G, H) = 2q$  if and only if  $G$  is bipartite.*

*Proof.* If  $G$  is bipartite, then  $\chi(G) = \chi(H) = 2$ . Thus, by Equation (3),  $\text{CBC}_q(G, H) = 2q$ .

Assume now that  $(G, H)$  admits a circular  $q$ -backbone  $2q$ -colouring  $f$ . Let  $v$  be a vertex of  $G$ . Without loss of generality, we may assume that  $f(v) = 1$ . Then all the neighbours of  $v$  in  $H$  must be coloured  $q + 1$ . And so on, by induction, all the vertices at even distance from  $v$  in  $H$  are coloured 1 and all the vertices at odd distance from  $v$  in  $H$  are coloured  $q + 1$ . Since  $H$  is connected and spans  $G$ , it follows that all vertices are coloured 1 or  $q + 1$ , so  $G$  is bipartite.  $\square$

Proposition 35 implies that given a graph  $G$  and a spanning connected subgraph  $H$ , deciding if  $\text{CBC}_q(G, H) = 2q$  can be done in polynomial time. In contrast, if the condition of  $G$  be connected is removed, when  $q = 2$ , the problem becomes NP-complete, as shown by the following theorem.

**Theorem 36.** *The following problem is NP-complete.*

*Input:* A planar graph  $G$  and a matching  $M$  in  $G$ .

*Question:* Is  $\text{CBC}_2(G, M) \leq 4$ ?

*Proof.* The problem is trivially in NP since a circular 2-backbone 4-colouring of  $(G, F)$  is clearly a certificate.

To prove it is NP-complete, we give a reduction from NOT-ALL-EQUAL 3SAT.

Let  $\mathcal{C} = \{C_1, \dots, C_n\}$  be a collection of clauses of size three over a set  $U$  of variables. We will construct a graph pair  $(G, M)$  such that  $M$  is a matching in  $G$ .

To do so we need some definitions and gadgets.

Colours 1 and 3 are said to be *twins* and so do the colours 2 and 4. Trivially two vertices joined by an edge of  $M$  receives distinct twin colours. Two colours are *siblings* if they are equal or twins.

A *link* with ends  $u$  and  $v$  and central edge  $w_1w_2$  is a subgraph with vertex set  $\{u, v, w_1, w_2\}$  and edge set  $\{uw_1, uw_2, vw_1, vw_2, w_1w_2\}$  with  $w_1w_2 \in M$ . Two ends of a link are said to be *linked*.

**Claim 37.** *In a circular 2-backbone 4-colouring  $c$ , the colours of the ends of a link are siblings.*

*Proof.* The two vertices  $w_1$  and  $w_2$  are joined by an edge of  $M$ , so  $\{c(w_1), c(w_2)\} \in \{\{1, 3\}; \{2, 4\}\}$ . Hence if  $u$  is coloured in  $\{1, 3\}$  (resp.  $\{2, 4\}$ ), then  $\{c(w_1), c(w_2)\}$  is  $\{2, 4\}$  (resp.  $\{1, 3\}$ ), and so  $v$  is coloured in  $\{1, 3\}$  (resp.  $\{2, 4\}$ ).  $\square$

For each variable  $u \in U$ , we create a *variable gadget*  $G^u$  which is obtained from the distinct vertices  $a_1^u, a_2^u, \dots, a_n^u$  by linking, from  $1 \leq i \leq n - 1$ , the vertices  $a_i^u$  and  $a_{i+1}^u$  by an link with central edge  $b_i^u c_i^u$ .

Claim 37 (and its proof) immediately implies the following.

**Claim 38.** *In a circular 2-backbone 4-colouring of  $G^u$ , all the  $a_u^i$  are coloured with two sibling colours and all the  $b_u^i$  with the two other colours (which are also siblings).*

For each clause  $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$ , we create a triangle  $z_1^i z_2^i z_3^i$ . Now for  $j = 1, 2, 3$ , if  $\ell_j^i$  is the nonnegated literal  $u$ , we join  $z_j^i$  with  $a_u^u$ , and if  $\ell_j^i$  is the negated literal  $\bar{u}$ , we join  $z_j^i$  with  $b_u^u$ . Such edges are said to be red. So far, the obtained graph  $H$  is not planar, but we can clearly draw it such that only red edges cross. We can now subdivide every red edge into a red path such that every edge is crossed at most once. We then replace the red edges which are not crossed by a link (with the same end) and two red edges  $uv$  and  $xy$  that cross each other by the crossing gadget depicted in Figure 12. The resulting graph  $G$  is planar and it comes with a matching  $M$ .

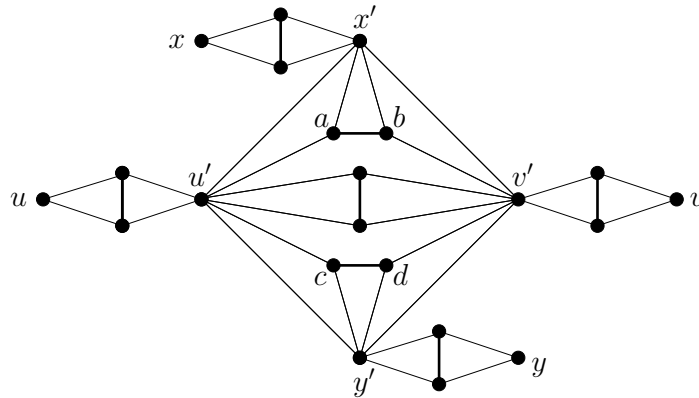


Figure 12: The crossing gadget

**Claim 39.** *In a circular 2-backbone 4-colouring of the crossing gadget, the colours of  $u$  and  $v$  are siblings and the colours of  $x$  and  $y$  are siblings. In addition, for any 4-tuple  $\{c_u, c_v, c_x, c_y\}$  such that  $c_u$  and  $c_v$  are siblings and  $c_x$  and  $c_y$  are siblings, there is a circular 2-backbone 4-colouring  $c$  of the crossing gadget such that  $c(u) = c_u$ ,  $c(v) = c_v$ ,  $c(x) = c_x$ , and  $c(y) = c_y$ .*

*Proof.* Consider first a circular 2-backbone 4-colouring of the crossing gadget.  $u$  is linked to  $u'$ , which is linked to  $v'$ , which in turn is linked to  $v$ . Hence, by Claim 37, the colours of  $u$  and  $v$  are siblings.

Assume that  $x$  is coloured in  $\{1, 3\}$ , then  $x'$  is also coloured in  $\{1, 3\}$ , say 1. The vertices  $a$  and  $b$  are assigned twin colours, so one is coloured 2 and the other 4. We now distinguish two cases depending on the colour of  $u'$ .

1. Assume  $u'$  is coloured 3. Then  $v'$  must also be coloured 3. The vertices  $c$  and  $d$  are assigned twin colours, so one is coloured 2 and the other 4. Hence  $y'$  is coloured 1.
2. Assume  $u'$  is coloured in  $\{2, 4\}$ . Without loss of generality, we may assume it is coloured 2. Then  $a$  is coloured 4 and  $b$  is coloured 2, so  $v'$  is coloured 4. Hence  $y'$  is coloured in  $\{1, 3\}$ .

In both cases the colour of  $x$  and  $y'$  are siblings, and so, by Claim 37, the colours of  $x$  and  $y$  are siblings.

For any 4-tuple  $\{c_u, c_v, c_x, c_y\}$  such that  $c_u$  and  $c_v$  are siblings and  $c_x$  and  $c_y$  are siblings, finding the desired circular 2-backbone 4-colouring is straightforward and left to the reader.  $\square$

We shall now prove that  $\mathcal{C}$  admits a suitable truth assignment if and only if  $\text{CBC}_2(G, M) \leq 4$ .

Assume first that  $(G, M)$  admits a circular 2-backbone 4-colouring. Let  $\phi$  be the truth assignment defined by  $\phi(u) = \text{true}$  if all the  $a_u^i$  are coloured in  $\{1, 3\}$ , and  $\phi(u) = \text{false}$  if all the  $a_u^i$  are coloured in  $\{2, 4\}$ . Note that is well defined by Claim 38. Now by Claims 37 and 39, for each clause  $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$ , the vertex  $z_j^i$  is coloured in  $\{1, 3\}$  if and only if the literal  $\ell_j^i$  is true. But since  $z_1^i z_2^i z_3^i$  is a triangle, at least three colours must appear on these vertices, and so at least one from  $\{1, 3\}$  and at least one from  $\{2, 4\}$ . Hence, at least one of the literals of  $C_i$  is true and at least one is false. Thus  $\phi$  is suitable.

Reciprocally, assume that  $\mathcal{C}$  admits a suitable truth assignment  $\phi$ . If  $\phi(u) = \text{true}$ , then colour all the  $a_u^i$  with 1, all the  $b_u^i$  with 2 and all the  $c_u^i$  with 4. And if  $\phi(u) = \text{false}$ , then colour all the  $a_u^i$  with 2, all the  $b_u^i$  with 1 and all the  $c_u^i$  with 3. Now, for each clause  $C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i$ , some literal, say  $\ell_1^i$ , is true and some literal, say  $\ell_3^i$ , is false. Then assign 1 to  $z_1^i$ , 2 to  $z_3^i$ , and colour  $z_2^i$  with 3 if  $\ell_2^i$  is true and 4 otherwise. By Claims 37 and 39, this partial colouring may be extended into a circular 2-backbone 4-colouring of  $(G, M)$ .  $\square$

**Theorem 40.** *The following problem is NP-complete.*

Input: A planar graph  $G$  and a matching  $M$  in  $G$ .

Question: Is  $\text{CBC}_2(G, M) \leq 5$ ?

*Proof.* The reduction is from PLANAR  $C_5$ -COLOURING which is defined as follows:

Input: A planar graph  $G$ .

Question: Does  $G$  have a homomorphism onto  $C_5$ , the cycle of length 5?

This was proved to be NP-complete by MacGillivray and Siggers [10].

To make the reduction we need an *edge gadget*. This gadget is built from the planar graph  $H_1(u, v)$  together with the matching  $M_1(u, v)$  depicted in Figure 13.

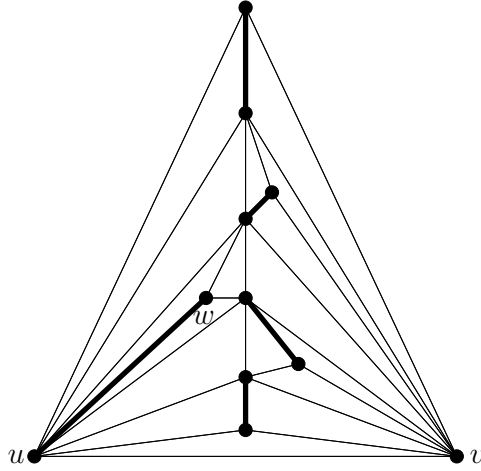


Figure 13: Graph  $H_1(u, v)$  with matching  $M_1(u, v)$  (in bold)

The graph  $H_2(u, v)$  is obtained from  $H_1(u, v)$  by replacing the edge  $uw$  by  $H_1(w, u)$ . The matching  $M_2(u, v)$  is then the union of  $M_1(u, v) \setminus \{uw\}$  and  $M_1(w, u)$ . Observe that  $u$  and  $v$  are incident to no edges of  $M_2(u, v)$ . The pair  $(H_2(u, v), M_2(u, v))$  is the edge gadget.

Broersma et al. [5] proved that in any circular 2-backbone 5-colouring of  $(H_1(u, v), M_1(u, v))$ , vertices  $u$  and  $v$  receive colours which are cyclically 2 apart. In addition, it is straightforward to see that any precolouring of  $u$  and  $v$  with colours that are cyclically 2 apart can be extended into a circular 2-backbone 5-colouring of  $(H_1(u, v), M_1(u, v))$ . These two facts imply the following claim.

**Claim 41.**

- (i) *In any circular 2-backbone 5-colouring of  $(H_2(u, v), M_2(u, v))$ , vertices  $u$  and  $v$  receive colours which are cyclically 2 apart.*
- (ii) *Any precolouring of  $u$  and  $v$  with colours that are cyclically 2 apart can be extended into a circular 2-backbone 5-colouring of  $(H_2(u, v), M_2(u, v))$ .*

Let  $H$  be an instance of PLANAR  $C_5$ -COLOURING. Replace each edge  $uv \in E(G)$  by an edge gadget  $(H_2(u, v), M_2(u, v))$  to obtain a planar graph  $G$  and a matching  $M$  (the union of the  $M_2(u, v)$ ). By Claim 41-(i), every circular 2-backbone 5-colouring of  $(G, M)$  induces a  $C_5$ -colouring of  $H$  (the vertices of the  $C_5$  are the colours  $(1, 3, 5, 2, 4)$ ). Conversely, by Claim 41-(ii), any  $C_5$ -colouring of  $H$  can be extended into a circular 2-backbone 5-colouring of  $(G, M)$ . Hence  $H$  admits a  $C_5$ -colouring if and only if  $(G, M)$  admits a circular 2-backbone 5-colouring.  $\square$

Adding long paths along existing edges to transform the matching into a spanning tree, one derives the following:

**Theorem 42.** *The following problem is NP-complete.*

Input: A planar graph  $G$  and a spanning tree  $T$  of  $G$ .

Question: Is  $\text{CBC}_2(G, T) \leq 5$ ?

## 4 Further research

Campos et al. [6] proved that if  $G$  is planar and  $T$  has diameter at most 3, then  $\text{BBC}_2(G, T) \leq 5$ . Hence one can find the 2-backbone chromatic number of such a pair in polynomial time. One can ask of the complexity for larger diameter.

**Problem 43.** *For a fixed  $d \geq 4$ , what is the complexity of finding the 2-backbone chromatic number of  $(G, T)$ , when  $G$  is planar and  $T$  a spanning tree of diameter  $d$ ?*

Since, for any fixed  $k \leq 4$ , deciding if the 2-backbone chromatic number of  $(G, T)$  is at most  $k$  can be done in polynomial time, if Conjecture 1 holds, Problem 43 is equivalent to finding the complexity of deciding if  $\text{BBC}_2(G, T) \leq 5$ .

If  $G$  is a triangle-free planar graph, then, by Grötzsch's Theorem [8], it is 3-colourable, and so  $\text{BBC}_q(G, H) \leq 2q + 1$  and  $\text{CBC}_q(G, H) \leq 3q$  for any subgraph  $H$  of  $G$ . Hence Conjecture 1 and Conjecture 6 for  $q = 2$ , hold when  $G$  is triangle-free. A natural next step would be to prove Conjecture 6 for values of  $q$  larger than 2 when  $G$  is triangle-free.

Steinberg's Conjecture (1976) states that every planar graph without 4- and 5-cycles is 3-colourable. Towards this, Erdős (1991) proposed the following relaxation of Steinberg's Conjecture: Determine the smallest value of  $k$ , such that every planar graph without cycles of length from 4 to  $k$  is 3-colourable. The best known bound for such a  $k$  is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [2]. Hence, an evidence to both Conjecture 6 and Steinberg's Conjecture would be to prove the following:

**Conjecture 44.** *If  $G$  is a planar graph without 4- and 5-cycles and  $F$  a spanning forest of  $G$ , then  $\text{CBC}_2(G, F) \leq 7$ .*

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